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Corona Theorem for H^∞ on Coverings of Riemann Surfaces of Finite Type

Alexander Brudnyi

1. Introduction

1.1

Let X be a complex manifold and let $H^\infty(X)$ be the Banach algebra of bounded holomorphic functions on X equipped with the supremum norm. We assume that X is Carathéodory hyperbolic; that is, the functions in $H^\infty(X)$ separate the points of X. The maximal ideal space $\mathcal{M} = \mathcal{M}(H^\infty(X))$ is the set of all nonzero linear multiplicative functionals on $H^\infty(X)$. Since the norm of each $\phi \in \mathcal{M}$ is ≤ 1, $\mathcal{M}$ is a subset of the closed unit ball of the dual space $(H^\infty(X))^*$. It is a compact Hausdorff space in the Gelfand topology (i.e., in the weak-$*$ topology induced by $(H^\infty(X))^*$). Further, there is a continuous embedding $i\colon X \hookrightarrow \mathcal{M}$ taking $x \in X$ to the evaluation homomorphism $f \mapsto f(x)$, $f \in H^\infty(X)$. The complement to the closure of $i(X)$ in $\mathcal{M}$ is called the *corona*. The *corona problem* is: Given X, determine whether the corona is empty. For example, according to Carleson's celebrated corona theorem [C], this is true for X the open unit disk in $\mathbb{C}$. (This was conjectured by Kakutani in 1941.) Also, there are nonplanar Riemann surfaces for which the corona is nontrivial (see e.g. [JM; Ga; BD; L] and references therein). The general problem for planar domains is still open, as is the problem in several variables for the ball and polydisk. (In fact, there are no known examples of domains in $\mathbb{C}^n$, $n \ge 2$, without corona.) At present, the strongest corona theorem for planar domains is due to Garnett and Jones [GJ]. It states that the corona is empty for any Denjoy domain—that is, a domain of the form $\bar{\mathbb{C}} \setminus E$ where $E \subset \mathbb{R}$.

The corona problem has the following analytic reformulation (see e.g. [G]): A collection $f_1, \dots, f_n$ of functions from $H^\infty(X)$ satisfies the *corona condition* if

$$1 \ge \max_{1\le j\le n} |f_j(x)| \ge \delta > 0 \quad \text{for all } x \in X. \tag{1.1}$$

The corona problem being solvable (i.e., the corona is empty) means that the Bezout equation

$$f_1 g_1 + \dots + f_n g_n \equiv 1 \tag{1.2}$$

has a solution $g_1, \dots, g_n \in H^\infty(X)$ for any $f_1, \dots, f_n$ satisfying the corona condition. We refer to $\max_{1\le j\le n} \|g_j\|_\infty$ as a "bound on the corona solutions". (Here $\|\cdot\|_\infty$ is the norm on $H^\infty(X)$.)

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This paper is concerned with the corona problem for coverings of Riemann surfaces of finite type. Let us recall that a Riemann surface Y is of finite type if the fundamental group $\pi_1(Y)$ is finitely generated. Our main result extends the class of Riemann surfaces for which the corona theorem is true.

THEOREM 1.1.1. *Let $r: X \to Y$ be an unbranched covering of a Carathéodory hyperbolic Riemann surface of finite type Y. Then X is Carathéodory hyperbolic and for any $f_1, \dots, f_n \in H^\infty(X)$ satisfying* (1.1) *there are solutions $g_1, \dots, g_n \in H^\infty(X)$ of* (1.2) *with the bound* $\max_{1\le j\le n}\|g_j\|_\infty \le C(Y, n, \delta)$.

This result, in a sense, completes our work started in [Br1; Br2; Br3] on the corona problems on coverings of certain Riemann surfaces. Similarly to [Br1; Br2; Br3], the methods used in this paper are based on an L^2 cohomology technique on complete Kähler manifolds and Cartan's A and B theorems for coherent Banach sheaves on Stein manifolds.

REMARKS 1.1.2. (1) Note that the assumption of the Carathéodory hyperbolicity of Y cannot be removed: It follows from the results of Lárusson [L] and the author [Br1] that for any integer $n \ge 2$ there are a compact Riemann surface S_n and its regular covering $r_n : \tilde{S}_n \to S_n$ such that:

(a) $\tilde{S}_n$ is a complex submanifold of an open Euclidean ball $\mathbb{B}_n \subset \mathbb{C}^n$;
(b) the embedding $i: \tilde{S}_n \hookrightarrow \mathbb{B}_n$ induces an isometry $i^*: H^\infty(\mathbb{B}_n) \to H^\infty(\tilde{S}_n)$.

In particular, (b) implies that the maximal ideal spaces of $H^\infty(\tilde{S}_n)$ and $H^\infty(\mathbb{B}_n)$ coincide. Thus the corona problem is not solvable for $H^\infty(\tilde{S}_n)$.

(2) Under the assumptions of the theorem, let $U \hookrightarrow X$ be a domain such that the embedding induces an injective homomorphism of the corresponding fundamental groups and $r(U) \subset\subset Y$. Then, as was shown in [Br2; Br3], in this case the following extension of Theorem 1.1.1 is valid.

THEOREM 1.1.3. *Let $A = (a_{ij})$ be an $n \times k$ matrix, $k < n$, with entries in $H^\infty(U)$. Assume that the family of minors of order k of A satisfies the corona condition. Then there is an $n \times n$ matrix $\tilde{A} = (\tilde{a}_{ij})$, $\tilde{a}_{ij} \in H^\infty(U)$, such that $\tilde{a}_{ij} = a_{ij}$ for $1 \le j \le k$, $1 \le i \le n$, and* $\det \tilde{A} = 1$.

The proof of the theorem is based on a Forelli-type theorem on projections in H^∞ discovered in [Br3] and a Grauert-type theorem for "holomorphic" vector bundles on maximal ideal spaces (which are not usual manifolds) of certain Banach algebras proved in [Br2]. In the forthcoming paper [Br5] we prove a result similar to Theorem 1.1.3 for matrices with entries in $H^\infty(X)$ with X satisfying the assumptions of Theorem 1.1.1. The techniques are necessarily more complicated than those used in this paper.

(3) The remarkable class of Riemann surfaces X for which a Forelli-type theorem is valid was introduced by Jones and Marshall [JM]. The definition is in terms of an interpolating property for the critical points of the Green function on X. For such X the corona problem is solvable, as well. Moreover, every X from this class is of *Widom type*; see [W] for the corresponding definition. (Roughly speaking,

this means that the topology of X grows slowly as measured by the Green function.) It is an interesting open question whether the surfaces X in Theorem 1.1.1 are also of Widom type.

(4) Similarly to [JM] and [Br3], our proof of Theorem 1.1.1 uses the Carleson corona theorem for the open unit disk.

1.2

In this part we formulate some results used in the proof of Theorem 1.1.1. First, we recall the following definition.

DEFINITION 1.2.1. Let X be a complex manifold. A sequence $\{x_j\}_{j\in\mathbb{N}} \subset X$ is called *interpolating for $H^\infty(X)$* if for every bounded sequence of complex numbers $a = \{a_j\}_{j\in\mathbb{N}}$ there is an $f \in H^\infty(X)$ such that $f(x_j) = a_j$ for all j. The *constant of interpolation* for $\{x_j\}_{j\in\mathbb{N}}$ is defined as

$$\sup_{\|a\|_{l^\infty}\le 1} \inf\{\|f\|_\infty \mid f \in H^\infty(X),\ f(x_j) = a_j,\ j \in \mathbb{N}\}, \tag{1.3}$$

where $\|a\|_{l^\infty} := \sup_{j\in\mathbb{N}}|a_j|$.

Let $r : X \to Y$ be an unbranched covering of a Carathéodory hyperbolic Riemann surface of finite type Y. Let $K \subset\subset Y$ be a compact subset.

THEOREM 1.2.2. *For every $x \in K$, the sequence $r^{-1}(x) \subset X$ is interpolating for $H^\infty(X)$ with the constant of interpolation bounded by a number depending on K and Y only.*

To formulate our next result used in the proof, we will assume that Y is equipped with a hermitian metric h_Y with the associated $(1,1)$-form ω_Y. Then we equip X with the hermitian metric h_X induced by the pullback $r^*\omega_Y$ of ω_Y to X. Now, if η is a smooth differential $(0,1)$-form on X, by $|\eta|_z$, $z \in X$, we denote the norm of η at z defined by the hermitian metric h_X^* on the fibres of the cotangent bundle T^*X on X. We say that η is *bounded* if

$$\|\eta\| := \sup_{z\in X}|\eta|_z < \infty. \tag{1.4}$$

THEOREM 1.2.3. *Let η be a smooth bounded $(0,1)$-form on X with support* $\operatorname{supp}\eta$ *satisfying $r(\operatorname{supp}\eta) \subset K$ for some compact subset $K \subset\subset Y$. Then the equation $\bar\partial f = \eta$ has a smooth bounded solution f on X such that*

$$\|f\|_{L^\infty} := \sup_{z\in X}|f(z)| \le C\|\eta\| \tag{1.5}$$

with C depending on K, Y, and h_Y only.

REMARK 1.2.4. The methods used in the proofs of Theorems 1.2.2 and 1.2.3 can be applied also to prove similar results for holomorphic L^p-functions on unbranched coverings of certain Stein manifolds. We present these results in a forthcoming paper.

2. Auxiliary Results

In this section we collect some auxiliary results used in the proofs.

2.1

Let Y be a Carathéodory hyperbolic Riemann surface of finite type. According to the theorem of Stout [S, Thm. 8.1], there exist a compact Riemann surface R and a holomorphic embedding $\phi\colon Y \to R$ such that $R \setminus \phi(Y)$ consists of finitely many closed disks with analytic boundaries together with finitely many isolated points. Since Y is Carathéodory hyperbolic, the set of the disks in $R \setminus \phi(Y)$ is not empty. Also, without loss of generality we may and will assume that the set of isolated points in $R \setminus \phi(Y)$ is not empty as well. (For otherwise, $\phi(Y)$ is a bordered Riemann surface and the required results follow from [Br3].) We will naturally identify Y with $\phi(Y)$. Also, we set

$$R \setminus Y := \Bigg(\bigsqcup_{1\le i\le k} \bar{D}_i\Bigg) \cup \Bigg(\bigcup_{1\le j\le l}\{x_j\}\Bigg) \quad \text{and} \quad \tilde{Y} := Y \cup \Bigg(\bigcup_{1\le j\le l}\{x_j\}\Bigg), \qquad (2.1)$$

where each D_i is biholomorphic to the open unit disk $\mathbb{D} \Subset \mathbb{C}$ and these biholomorphisms are extended to diffeomorphisms of the closures $\bar{D}_i \to \bar{\mathbb{D}}$.

According to these definitions, $\tilde{Y}$ is a bordered Riemann surface, and there is a bordered Riemann surface $\tilde{Y}_1 \subset\subset R$ with $\tilde{Y} \subset\subset \tilde{Y}_1$ and $\pi_1(\tilde{Y}_1) \cong \pi_1(\tilde{Y})$.

2.2

Next, we introduce a complete Kähler metric on $Y_1 := \tilde{Y}_1 \setminus \big(\bigcup_{1\le j\le l}\{x_j\}\big)$.

To this end we consider an open cover $\mathcal{U} = (U_j)_{0\le j\le l}$ of R such that for $1 \le j \le l$, $U_j \subset\subset \tilde{Y}$ is an open coordinate disk centered at x_j and U_0 is a bordered Riemann surface that intersects each U_j, $1 \le j \le l$, by a set biholomorphic to an open annulus. Let $\{\rho_j\}_{0\le j\le l}$ be a smooth partition of unity on R subordinate to the cover $\mathcal{U}$. By z we denote a complex coordinate in U_j, $1 \le j \le l$, such that $z(x_j) = 0$ and $|z| < 1$. We set $f_j := \rho_j|z|^2$, $1 \le j \le l$. Then f_j is a smooth nonnegative function on R with $\operatorname{supp} f_j \subset U_j$.

Now, we consider the positive smooth function

$$f := \frac{\rho_0}{2} + \sum_{j=1}^{l} f_j \qquad (2.2)$$

on R and determine the $(1,1)$-form $\tilde{\omega}$ on R by

$$\tilde{\omega} := -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(\log f)^2. \qquad (2.3)$$

By the definition $0 < f < 1$, the form ω is well-defined. Also, in an open neighborhood of x_j, $1 \le j \le l$, the form ω is equal to

$$\omega_P := \frac{\sqrt{-1}}{\pi}\,\frac{dz \wedge d\bar{z}}{|z|^2(\log|z|^2)^2} = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(\log|z|^2)^2. \qquad (2.4)$$

In the natural identification $U_j \setminus \{x_j\} = \mathbb{D} \setminus \{0\}$, ω_P coincides with the $(1,1)$-form of the Poincaré metric on the punctured disk.

Let ω_R be a Kähler $(1,1)$-form on the compact Riemann surface R from Section 2.1. Since $\tilde{Y}_1$ is a bordered Riemann surface in R, there is a smooth plurisubharmonic function f_R defined in a neighborhood of the closure of $\tilde{Y}_1$ such that

$$\omega_R = \sqrt{-1} \cdot \partial\bar{\partial} f_R \quad \text{on } \tilde{Y}_1. \tag{2.5}$$

Further, since $\tilde{Y}_1$ is a Stein manifold, by the Narasimhan theorem [N] there is a holomorphic embedding $i\colon \tilde{Y}_1 \hookrightarrow \mathbb{C}^3$ of $\tilde{Y}_1$ as a closed complex submanifold of $\mathbb{C}^3$. By ω_e we denote the $(1,1)$-form on $\tilde{Y}_1$ obtained as the pullback by i of the Euclidean Kähler form $\sqrt{-1}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3)$ on $\mathbb{C}^3$ (here z_1, z_2, z_3 are complex coordinates on $\mathbb{C}^3$). Clearly, ω_e is a Kähler form on $\tilde{Y}_1$ (i.e., it is positive on $\tilde{Y}_1$ and d-closed).

Lemma 2.2.1. *There is a positive number c_1 depending on Y_1, ω_R, and $\tilde{\omega}$ such that the $(1,1)$-form*

$$\omega := \tilde{\omega} + c_1(\omega_R + \omega_e) \tag{2.6}$$

is a complete Kähler form on Y_1.

Proof. Since ω_R is a Kähler form on R, by the definition of $\tilde{\omega}$ there is a constant $c_1 > 0$ depending on Y_1, ω_R, and $\tilde{\omega}$ such that

$$\tilde{\omega} > -c_1\omega_R \quad \text{on } Y_1.$$

Thus the form $\omega = \tilde{\omega} + c_1(\omega_R + \omega_e)$ is positive (and d-closed) on Y_1. Its completeness means that the path metric d on Y_1 induced by ω is complete. Let us check this fact.

Assume, to the contrary, that d is not complete. This means that there is a sequence $\{w_n\} \subset Y_1$ convergent either to the boundary of $\tilde{Y}_1$ or to one of the points x_j, $1 \le j \le l$, such that $\{d(o, w_n)\}$ is bounded (for a fixed point $o \in Y_1$). Then, since $\omega \ge \omega_e$, the sequence $\{i(w_n)\} \subset \mathbb{C}^3$ is bounded. This implies that $\{w_n\}$ cannot converge to the boundary of $\tilde{Y}_1$. Thus it converges to one of x_j. But since $\omega \ge \omega_P$ near x_j, the latter is impossible because the Poincaré metric on the punctured disk is complete. □

2.3

According to our construction, the embedding $Y \hookrightarrow Y_1$ induces an isomorphism of the corresponding fundamental groups. By the covering homotopy theorem this implies that for any unbranched covering $r\colon X \to Y$ there are an unbranched covering $r_1\colon X_1 \to Y_1$ and an embedding $j\colon X \hookrightarrow X_1$ such that $r_1 \circ j = r$ and $j_*\colon \pi_1(X) \to \pi_1(X_1)$ is an isomorphism. Without loss of generality we identify $j(X)$ with X. Then $r := r_1|_X$. Now the form $r_1^*\omega$ with ω determined by (2.6) is a complete Kähler form on X_1.

Let TY_1 be the complex tangent bundle on Y_1 equipped with the hermitian metric induced by the Kähler form ω_R. Since TY_1 is the restriction to Y_1 of the tangent bundle TR on R with the hermitian metric defined by ω_R, the curvature Θ_{Y_1} of TY_1 satisfies

$$\Theta_{Y_1} \geq -c_2\omega_R \tag{2.7}$$

for some $c_2 > 0$ depending on Y_1 and ω_R. In turn, the curvature $\Theta_{X_1} := r_1^*\Theta_{Y_1}$ of the tangent bundle TX_1 on X_1 equipped with the hermitian metric induced by $r_1^*\omega_R$ satisfies

$$\Theta_{X_1} \geq -c_2 r_1^*\omega_R. \tag{2.8}$$

Next, by the definition of ω_e there is a smooth plurisubharmonic function g on $\tilde{Y}_1$ such that $\omega_e = \sqrt{-1}\cdot\partial\bar{\partial}g$ (for such g, one takes the pullback by i of the function $\|z\|^2 := |z_1|^2 + |z_2|^2 + |z_3|^2$ on $\mathbb{C}^3$).

Let $E_0 := X_1 \times \mathbb{C}$ be the trivial holomorphic line bundle on X_1. We equip E_0 with the hermitian metric $e^{h_1-(c_1+c_2)g_1}$ where $h_1 = r_1^*h := \log(\log r_1^*f)^2/2\pi$ with f from (2.2) and $g_1 := r_1^*(g + f_R)$; see (2.5). (This means that for $z \times v \in E$ the square of its norm in this metric equals $e^{h_1(z)-(c_1+c_2)g_1(z)}|v|^2$ where $|v|$ is the modulus of $v \in \mathbb{C}$.) Then by (2.8) the curvature Θ_E of the bundle $E := E_0 \otimes TX_1$ satisfies

$$\Theta_E := -\sqrt{-1}\cdot\partial\bar{\partial}\log e^{h_1-(c_1+c_2)g_1} + \Theta_{X_1} \geq r_1^*\omega. \tag{2.9}$$

2.4

Let X be a complete Kähler manifold of dimension n with a Kähler form ω and let E be a hermitian holomorphic vector bundle on X with curvature Θ. Let $L_2^{p,q}(X, E)$ be the space of L^2 E-valued (p,q)-forms on X with the L^2 norm, and let $W_2^{p,q}(X, E)$ be the subspace of forms such that $\bar{\partial}\eta$ is L^2. (The forms η may be taken to be either smooth or just measurable, in which case $\bar{\partial}\eta$ is understood in the distributional sense.) The cohomology of the resulting L^2 Dolbeault complex $(W_2^{\cdot,\cdot}, \bar{\partial})$ is the L^2 cohomology

$$H_{(2)}^{p,q}(X, E) = Z_2^{p,q}(X, E)/B_2^{p,q}(X, E),$$

where $Z_2^{p,q}(X, E)$ and $B_2^{p,q}(X, E)$ are the spaces of $\bar{\partial}$-closed and $\bar{\partial}$-exact forms in $L_2^{p,q}(X, E)$, respectively.

If $\Theta \geq \varepsilon\omega$ for some $\varepsilon > 0$ in the sense of Nakano, then the L^2 Kodaira–Nakano vanishing theorem (see [De; O]) states that

$$H_{(2)}^{n,r}(X, E) = 0 \quad \text{for } r > 0. \tag{2.10}$$

Moreover, for $\eta \in Z_2^{n,r}(X, E)$, $r > 0$, there is a form $\xi \in W_2^{n,r-1}(X, E)$ such that $\bar{\partial}\xi = \eta$ and

$$\|\xi\|_2 \leq \frac{1}{\varepsilon}\|\eta\|_2; \tag{2.11}$$

see [De, Rem. 4.2]. Here $\|\cdot\|_2$ denotes the corresponding L^2 norms.

We can apply this result to the bundle E from Section 2.3 with X_1 equipped with the complete Kähler form $r_1^*\omega$. Then from (2.9) we obtain the following statement.

PROPOSITION 2.4.1. *For every $\eta \in W_2^{1,1}(X_1, E)$ there is $\xi \in W_2^{1,0}(X_1, E)$ such that $\bar\partial\xi = \eta$ and*

$$\|\xi\|_2 \le \|\eta\|_2. \tag{2.12}$$

2.5

Let T^*X_1 be the cotangent bundle on X_1 equipped with the hermitian metric induced by $r_1^*\omega$. We consider the hermitian line bundle $V := E \otimes T^*X_1$ equipped with the tensor product of the corresponding hermitian metrics. Then from Proposition 2.4.1 we obtain

$$H_{(2)}^{0,1}(X_1, V) \cong H_{(2)}^{1,1}(X_1, E) = 0. \tag{2.13}$$

Moreover, for every $\eta \in W_2^{0,1}(X_1, V)$ there is $F \in W_2^{0,0}(X_1, V)$ such that $\bar\partial F = \eta$ and

$$\|F\|_2 \le \|\eta\|_2. \tag{2.14}$$

Further, there is a canonical isomorphism $I\colon X_1 \times \mathbb{C} \to V$ defined in local coordinates z on X_1 by the formula

$$I(z \times v) := v \cdot 1 \otimes \frac{\partial}{\partial z} \otimes dz. \tag{2.15}$$

(Clearly this definition does not depend on the choice of local coordinates.) In what follows we identify V with $X_1 \times \mathbb{C}$ by I.

3. Proof of Theorem 1.2.2

In what follows, by $A, B, C, c, \dots$ we denote constants depending on characteristics related to the sets Y, $x \in Y$, and $K \subset\subset Y$ but not on coverings X of Y. (We will briefly say that they depend on Y, x, and K only.) These constants may change from line to line and even in a single line.

3.1

Let $x \in K \subset\subset Y$. We must check that the sequence $r^{-1}(x) \subset X$ is interpolating for $H^\infty(X)$ with the constant of interpolation bounded by a number depending on K and Y only. Fix a neighborhood $\hat Y$ of the closure of Y in Y_1 such that $\hat Y$ is relatively compact in $\tilde Y_1$; see Section 2.1 for the corresponding definitions. First we will prove that $r^{-1}(x)$ is interpolating for the space $L^2_{\mathcal{O}}(\hat X_1; r_1^*\omega_R)$ of holomorphic L^2-functions on the covering $\hat X_1 := r_1^{-1}(\hat Y)$ of $\hat Y$ with norm defined by integration with respect to $r_1^*\omega_R$ (recall that $X \subset X_1$ and $r_1|_X = r$).

PROPOSITION 3.1.1. *Let a be an l^2-function on $r^{-1}(x)$ with norm $\|a\|_{l^2}$. Then there is a function $f \in L^2_{\mathcal{O}}(\hat X_1; r_1^*\omega_R)$ such that $f|_{r^{-1}(x)} = a$ and*

$$\|f\|_2 \le c\|a\|_{l^2}$$

with c depending on K and Y only.

Proof. Using the fact that the closure of $\tilde{Y}_1$ in R possesses a Stein neighborhood and applying some basic results of the theory of Stein manifolds (see e.g. [GrR]) one obtains easily that there are a holomorphic function ϕ_x with a simple zero at x defined in a fixed neighborhood of the closure of $\tilde{Y}_1$, a simply connected coordinate neighborhood $U_x \subset\subset Y$ of x with a complex coordinate z, $z(x) = 0$, and $|z| < 1$ on U_x, and positive numbers A and r, $0 < r < 1$, depending on x and Y only such that

(1) $\sup_{y \in \tilde{Y}_1} |\phi_x(y)| \le A$;
(2) $|\phi_x(y)| \ge 1/A$ for all $y \in U_{x;r} := \{z \in U_x : |z| \ge r\}$;
(3) $\phi_x(y) \ne 0$ for all $y \in U_x \setminus \{x\}$.

(For a construction of such ϕ_x, see e.g. [Br2, Cor. 1.8].)

Further, there is a C^∞-function ρ_x, $0 \le \rho_x \le 1$, on U_x such that ρ_x is equal to 1 on $U_x \setminus U_{x;r}$ and 0 outside U_x, and

$$|d\rho_x|_{z;\omega} \le B \quad \text{for all } z \in Y_1, \tag{3.1}$$

where $\{|\cdot|_{z;\omega} : z \in Y_1\}$ is the hermitian metric on the fibres of the cotangent bundle T^*Y_1 on Y_1 determined by the form ω (see (2.6)), and the constant B depends on x and Y only.

Since U_x is simply connected, $r_1^{-1}(U_x)$ is biholomorphic to $U_x \times S$ where S is the fibre of r_1. In what follows, without loss of generality we will identify these sets. Then a is an l^2-function of $\{x\} \times S$. We extend a to a locally constant function $\hat{a}$ on $U_x \times S$ by the formula

$$\hat{a}(z,s) := a(x,s) \quad \text{for all } (z,s) \in U_x \times S. \tag{3.2}$$

Let us consider a $(0,1)$-form $\hat{\eta}$ on X_1 determined by

$$\hat{\eta}(w) = \begin{cases} \hat{a}(z,s)d\rho_x(z)/\phi_x(z) & \text{if } w = (z,s) \in U_x \times S, \\ 0 & \text{if } w \notin U_x \times S. \end{cases} \tag{3.3}$$

Next, we fix a noncompact neighborhood O of the closure $\tilde{Y}_1$ in R. Because O is a one-dimensional Stein manifold, it is homotopically equivalent to a one-dimensional CW-complex (see e.g. [GrR]). In particular, any continuous vector bundle on O is topologically trivial. Then, by the Grauert theorem [Gr], any holomorphic vector bundle on O is also trivial. Applying this to the bundle $T^*R|_O$ we find a nowhere vanishing holomorphic section λ of $T^*R|_O$. (By the definition, λ is a holomorphic 1-form on O.) Then $\lambda^{-1} = 1/\lambda$ is a nowhere vanishing holomorphic section of the tangent bundle TO on O. Moreover, there is a positive constant C depending on $\tilde{Y}_1$ such that

$$\frac{1}{C} \le |\lambda|_{z;\omega_R} \le C, \quad \frac{1}{C} \le |\lambda^{-1}|_{z;\omega_R} \le C \quad \text{for all } z \in \tilde{Y}_1, \tag{3.4}$$

where $|\cdot|_{z;\omega_R}$ denotes the corresponding norms on $T^*\tilde{Y}_1$ and $T\tilde{Y}_1$ at $z \in \tilde{Y}_1$ determined by the form ω_R.

Continuing the proof of the proposition, consider the V-valued $(0,1)$-form $\eta := \hat{\eta} \wedge r_1^*\lambda \otimes r_1^*\lambda^{-1}$ on X_1. By N_w and N'_w we denote the hermitian norms on V and E (see Section 2.3) at $w \in X_1$. Then we have

$$N_w(\eta) := N'_w(\hat{\eta}\otimes r_1^*\lambda^{-1})\cdot |r_1^*\lambda|_{w;r_1^*\omega}$$
$$= |\hat{\eta}|_{w;r_1^*\omega}\cdot |r_1^*\lambda^{-1}|_{w;r_1^*\omega_R}\cdot |r_1^*\lambda|_{w;r_1^*\omega}\cdot e^{h_1(w)-(c_1+c_2)g_1(w)}. \quad (3.5)$$

Here $|\hat{\eta}|_{w;r_1^*\omega}$ and $|r_1^*\lambda|_{w;r_1^*\omega}$ are determined by the form $r_1^*\omega$ and $|r_1^*\lambda^{-1}|_{w;r_1^*\omega_R}$ is determined by the form $r_1^*\omega_R$. Since ω_R is equivalent to ω on $U_x \subset\subset Y$, from (3.4) we obtain for all $w\in r_1^{-1}(U_x)$:

$$|r_1^*\lambda|_{w;r_1^*\omega} := |\lambda|_{r_1(w);\omega}\le C_1, \qquad |r_1^*\lambda^{-1}|_{w;r_1^*\omega_R} := |\lambda^{-1}|_{r_1(w);\omega_R}\le C, \quad (3.6)$$

where C_1 depends on C and on the constant of the equivalence of ω_R and ω on U_x. Also, by the definition of h_1 and g_1 (see Section 2.3), there is a constant $C_2 > 0$ depending on h_1, g_1, U_x, and Y_1 such that

$$e^{h_1(w)-(c_1+c_2)g_1(w)}\le C_2 \quad \text{for all } \ w\in r_1^{-1}(U_x). \quad (3.7)$$

From here, (3.6), (3.5), (3.1)–(3.3), and the definition of ϕ_x (see conditions (1)–(3)) we obtain

$$N_w(\eta)\le \tilde{C}|\tilde{a}(w)|, \quad (3.8)$$

where $\tilde{C} := ABCC_1C_2$, $\tilde{a} := \hat{a}\cdot\chi_{r_1^{-1}(U_x)}$, and $\chi_{r_1^{-1}(U_x)}$ is the characteristic function of $r_1^{-1}(U_x)$. Using (3.8) we estimate the L^2 norm of η as follows:

$$\|\eta\|_2 := \left(\int_{w\in X_1} N_w^2(\eta)(r_1^*\omega)(w)\right)^{1/2}\le \tilde{C}\|a\|_{l^2}\left(\int_{y\in U_x}\omega(y)\right)^{1/2}\le \hat{C}\|a\|_{l^2}. \quad (3.9)$$

Here $\hat{C}$ depends on x and Y only.

Next, according to (2.14) there is $F\in W_2^{0,0}(X_1, V)$ such that $\bar{\partial}F = \eta$ and

$$\|F\|_2\le\|\eta\|_2\le\hat{C}\|a\|_{l^2}. \quad (3.10)$$

We regard F as a function on X_1; see Section 2.5.

Observe that since $\hat{Y}$ ($\subset Y_1$) is relatively compact in $\tilde{Y}_1$, g_1 is bounded on $\hat{X}_1$ and h_1 is bounded from below on $\hat{X}_1$; see Section 2 for the corresponding definitions. Moreover, $\omega_R\le\tilde{c}\cdot\lambda\wedge\bar{\lambda}$ on $\tilde{Y}_1$ for some $\tilde{c}$ depending on $\tilde{Y}_1$. These facts, (3.4), and (3.5) imply

$$\begin{aligned}\left(\int_{\hat{X}_1}|F|^2\cdot r_1^*\omega_R\right)^{1/2} &\le C\left(\int_{w\in\hat{X}_1}[N'_w(F\otimes r_1^*\lambda^{-1})]^2(r_1^*\lambda\wedge r_1^*\bar{\lambda})(w)\right)^{1/2}\\ &= C\left(\int_{w\in\hat{X}_1}N_w^2(F\cdot r_1^*\lambda\otimes r_1^*\lambda^{-1})(r_1^*\omega)(w)\right)^{1/2}\\ &\le C\left(\int_{w\in X_1}N_w^2(F\cdot r_1^*\lambda\otimes r_1^*\lambda^{-1})(r_1^*\omega)(w)\right)^{1/2}\\ &:= C\|F\|_2\end{aligned} \quad (3.11)$$

with C depending on Y. Let us consider the function $f := \tilde{a}\cdot\rho_x - F\cdot r_1^*\phi_x$ on $\hat{X}_1$. Then, according to (3.10) and (3.11),

$$\left(\int_{\hat{X}_1}|f|^2 r_1^*\omega_R\right)^{1/2}\le c\|a\|_{l^2}$$

where c depends on Y and x only. Moreover, from (3.3) it follows that f is holomorphic and $f|_{r_1^{-1}(x)} = a$.

This shows that $r^{-1}(x)$ is interpolating for $L^2_{\mathcal{O}}(\hat{X}_1; r_1^*\omega_R)$.

To complete the proof of the proposition it remains to prove that for $x \in K \subset\subset Y$ the constant of interpolation of $r^{-1}(x)$ with respect to $L^2_{\mathcal{O}}(\hat{X}_1; r_1^*\omega_R)$ is bounded by a number depending on K and Y only.

Let us consider the restriction map $R_x\colon L^2_{\mathcal{O}}(\hat{X}_1; r_1^*\omega_R) \to L_{\mathcal{O}}(\hat{X}_1; r_1^*\omega_R)|_{r^{-1}(x)}$.

LEMMA 3.1.2. *R_x maps $L^2_{\mathcal{O}}(\hat{X}_1; r_1^*\omega_R)$ continuously onto $l^2(r^{-1}(x))$. Moreover, the norm $\|R_x\|$ of R_x is bounded by a constant depending on Y and x.*

Proof. We will consider the coordinate neighborhood $U_x \subset\subset Y$ from the proof of Proposition 3.1.1 with a complex coordinate z such that $z(x) = 0$, $|z| < 1$ on U_x. Also, we naturally identify $r_1^{-1}(U_x)$ with $U_x \times S$ where S is the fibre of r_1. By definition there is a constant C depending on Y, U_x, and ω_R such that

$$\sqrt{-1}\cdot dz \wedge d\bar{z} \le C\omega_R \ \text{ on } U_x. \tag{3.12}$$

Let $f \in L^2_{\mathcal{O}}(\hat{X}_1; r_1^*\omega_R)$. Then on $r^{-1}(x) = \{x\} \times S$ we have, by the mean value property for subharmonic functions,

$$\begin{aligned}\sum_{s\in S}|f(x,s)|^2 &\le \sum_{s\in S}\left(\frac{1}{\pi}\int_{U_x}|f(z,s)|^2\sqrt{-1}\cdot dz\wedge d\bar{z}\right)\\ &\le \frac{C}{\pi}\int_{U_x}\left(\sum_{s\in S}|f(z,s)|^2\right)\omega_R(z)\\ &= \frac{C}{\pi}\int_{r_1^{-1}(U_x)}|f|^2 r_1^*\omega_R \le \frac{C}{\pi}\|f\|_2^2.\end{aligned} \tag{3.13}$$

This shows that R_x maps $L^2_{\mathcal{O}}(\hat{X}_1; r_1^*\omega_R)$ continuously into $l^2(r^{-1}(x))$. Also, R_x is surjective according to the first part of Proposition 3.1.1. □

Now, since $R_x\colon L^2_{\mathcal{O}}(\hat{X}_1; r_1^*\omega_R) \to l^2(r^{-1}(x))$ is a linear continuous surjective map of Hilbert spaces, there is a linear continuous map $T_x\colon l^2(r^{-1}(x)) \to L^2_{\mathcal{O}}(\hat{X}_1; r_1^*\omega_R)$ such that $R_x \circ T_x = \mathrm{id}$. Let $\{e_s\}_{s\in S}$, $e_s(x,t) = 0$ for $t \ne s$, and $e_s(x,s) = 1$ be an orthonormal basis of $l^2(r^{-1}(x))$. We set

$$h_s := T_x(e_s) \in L^2_{\mathcal{O}}(\hat{X}_1; r_1^*\omega_R).$$

Then for a sequence $a = \{a_s\}_{s\in S} \in l^2(S)$ we have

$$h_a := \sum_{s\in S} a_s h_s \in L^2_{\mathcal{O}}(\hat{X}_1; r_1^*\omega_R) \quad \text{and} \quad \|h_a\|_2 \le c\|a\|_{l^2}. \tag{3.14}$$

Further, for each $y \in U_x$, by $L_y\colon l^2(r^{-1}(y)) \to l^2(r^{-1}(x))$ we denote the natural isomorphism that sends $a(y,s) \in l^2(r^{-1}(y))$ to $a(x,s) \in l^2(r^{-1}(x))$. Let us consider the map $S_y := R_y \circ T_x \circ L_y\colon l^2(r^{-1}(y)) \to l^2(r^{-1}(y))$ determined by the formula

$$[S_y(a)](y,t) := \sum_{s\in S} a_s h_s(y,t), \quad (y,t) \in r^{-1}(y). \tag{3.15}$$

Here $a(y,t) = \sum_{s\in S} a_s e_s(y,t)$, $t \in S$, and $e_s(y,\cdot)$ are determined similarly to $e_s(x,\cdot)$. By identifying $a \in l^2(r^{-1}(y))$ with $\{a_s\}_{s\in S} \in l^2(S)$ we can regard, according to (3.15), $\{S_y\}_{y\in U_x}$ as a family of bounded linear operators $l^2(S) \to l^2(S)$ depending holomorphically on $y \in U_x$. According to (3.14) and Lemma 3.1.2 there is a constant c' depending on U_x and Y such that

$$\|S_y\| \le c' \quad \text{for all } \ y \in U_x.$$

Moreover, by our construction, $S_x = I$ where $I : l^2(S) \to l^2(S)$ is the identity operator. Identifying U_x with $\mathbb{D}$ by the coordinate z, we obtain by the Cauchy integral formula for bounded holomorphic on $\mathbb{D}$ functions:

$$S_z := I + \sum_{k=1}^{\infty} S_k z^k \quad \text{for some } \ S_k : l^2(S) \to l^2(S),\ \ \|S_k\| \le c'.$$

In particular, for $|z| < \frac{1}{2c'+4}$ we have

$$\left\| \sum_{k=1}^{\infty} S_k z^k \right\| \le c' \frac{|z|}{1-|z|} < \frac{2}{3}.$$

Thus for every $y \in U_x$, $|z(y)| < \frac{1}{2c'+4}$, the inverse operator S_y^{-1} exists, and its norm is bounded by $\frac{1}{1-2/3} = 3$.

Finally we set

$$\hat{T}_y := T_x \circ L_y \circ S_y^{-1}, \qquad y \in \hat{U}_x := \left\{ y \in U_x : |z(y)| < \frac{1}{2c'+4} \right\}. \tag{3.16}$$

Then by the definition we have

$$R_y \circ \hat{T}_y = \mathrm{id} \quad \text{for all } \ y \in \hat{U}_x.$$

This shows that $\{\hat{T}_y : l^2(r^{-1}(y)) \to L^2_{\mathcal{O}}(\hat{X}_1; r_1^*\omega_R) : y \in \hat{U}_x\}$ is a family of interpolation operators depending holomorphically on y such that $\|\hat{T}_y\| \le 3c$. Taking a finite open cover of $K \subset\subset Y$ by the sets $\hat{U}_x$, $x \in K$, and considering on these sets the interpolation operators $\hat{T}_y$, $y \in \hat{U}_x$, we obtain that for every $x \in K$ the constant of interpolation of $r^{-1}(x)$ with respect to $L^2_{\mathcal{O}}(\hat{X}_1; r_1^*\omega_R)$ is bounded by a number depending on K and Y only.

This completes the proof of Proposition 3.1.1. □

3.2

Let us prove now that $r^{-1}(x)$ is interpolating for $H^\infty(X)$ with the constant of interpolation bounded by a number depending on K and Y only.

We will use the interpolation operators $\hat{T}_y$, $y \in \hat{U}_x \subset\subset U_x$, of the previous section. As before we set

$$h_{s,y} := \hat{T}_y(e_s(y,\cdot)) \in L^2_{\mathcal{O}}(\hat{X}_1; r_1^*\omega_R).$$

Then the family of functions $\{h_{s,y} : s \in S,\ y \in \hat{U}_x\}$ depends holomorphically on y.

Now for a sequence $a = \{a_s\}_{s\in S} \in l^2(S)$ we have

$$h_{a,y} := \sum_{s\in S} a_s h_{s,y} \in L^2_{\mathcal{O}}(\hat{X}_1; r_1^*\omega_R) \quad \text{and} \quad \|h_{a,y}\|_2 \le 3c\|a\|_{l^2}. \tag{3.17}$$

From here and Lemma 3.1.2 it follows that

$$\sum_{z\in r^{-1}(w)} |h_{a,y}(z)|^2 \le c\|a\|_{l^2}^2, \quad w \in Y,\ y \in \hat{U}_x, \tag{3.18}$$

with c depending on w, x, and Y.

To continue the proof we require an extension of Lemma 3.1.2. For its formulation we fix a holomorphic function ϕ defined in a neighborhood of the closure of $\tilde{Y}_1 \subset\subset R$ having simple zeros at all points x_j, $1 \le j \le l$ (see (2.1)), and nonzero outside these points. (Such ϕ exists, e.g., by [Br2, Cor. 1.8].) As before, by $R_w : L^2_{\mathcal{O}}(\hat{X}_1; r_1^*\omega_R) \to l^2(r^{-1}(w))$, $w \in Y$, we denote the restriction map (here $r := r_1|_X$).

LEMMA 3.2.1. *There is a constant $A > 0$ depending on Y such that*

$$\|R_w\| \le \frac{A}{|\phi(w)|}.$$

Proof. Let $U_j \subset\subset \tilde{Y}$ be a coordinate neighborhood of x_j with a complex coordinate z such that $z(x_j) = 0$, $|z| < 1$, on U_j. We have

$$\sqrt{-1} \cdot dz \wedge d\bar{z} \le C\omega_R \ \text{ on } U_j$$

for some C depending on ω_R, z, and Y. Thus for any $f \in L^2_{\mathcal{O}}(\hat{X}_1; r_1^*\omega_R)$ its restriction $f|_{r_1^{-1}(U_j)}$ belongs to the L^2 space on $r_1^{-1}(U_j)$ defined by integration with respect to the form $r_1^*\big(\sqrt{-1} \cdot dz \wedge d\bar{z}\big)$, and the L^2 norm of the restriction is bounded by $C\|f\|_2$.

Next, for a point $w \in \tilde{U}_j \setminus \{x_j\}$, $\tilde{U}_j := \{z \in U_j : |z| < 1/2\}$, we set $d := |z(w)|$ and $D_w = \{y \in U_j : |z(y) - z(w)| < d\}$. Then $D_w \subset U_j \setminus \{x_j\}$ and so $r_1^{-1}(D_w)$ is naturally identified with $D_w \times S$. In this identification we have, by the mean value property for subharmonic functions,

$$\begin{aligned}\sum_{s\in S} |f(w,s)|^2 &\le \frac{1}{\pi d^2} \int_{D_w} \Big(\sum_{s\in S} |f(z,s)|^2\Big) \sqrt{-1} \cdot dz \wedge d\bar{z} \\ &\le \frac{C}{\pi d^2} \int_{r_1^{-1}(D_w)} |f|^2 r_1^*\omega_R \le \frac{C}{\pi d^2} \|f\|_2^2.\end{aligned}$$

From here and the fact that the function ϕ/z is bounded on U_j we obtain that there is a constant $c_j > 0$ depending on Y such that

$$\|f|_{r_1^{-1}(w)}\|_{l^2} \le \frac{c_j}{|\phi(w)|} \|f\|_2 \quad \text{for all } \ w \in \tilde{U}_j. \tag{3.19}$$

This proves the required inequality for $w \in \bigcup_{1\le j\le l} \tilde{U}_j$.

The remaining part $K := Y \setminus \left(\bigcup_{1\le j\le l} \tilde{U}_j\right)$ is a relatively compact subset of $\hat{Y} \setminus \left(\bigcup_{1\le j\le l}\{x_j\}\right)$; see Section 3.1 for the definition of $\hat{Y}$. Then the required estimate on K follows from (3.13) and the fact that $|\phi|$ is bounded on K. We leave the details to the reader. □

From this lemma we obtain the following improvement of (3.18):

$$\sum_{z\in r^{-1}(w)} |h_{a,y}(z)|^2 \le \frac{C}{|\phi(w)|^2}\|a\|_{l^2}^2, \quad w \in Y,\ y \in \hat{U}_x, \tag{3.20}$$

with C depending on Y and x only. This and the definition of $h_{a,y}$ (see (3.17)) imply

$$\sum_{s\in S}|h_{s,y}(z)|^2 \le \frac{C}{|\phi(w)|^2}, \quad z \in r^{-1}(w),\ w \in Y,\ y \in \hat{U}_x. \tag{3.21}$$

Let us continue the proof of Theorem 1.2.2. Consider the holomorphic function

$$f_y := \frac{\phi^2}{\phi^2(y)}, \quad y \in \hat{U}_x,$$

defined in a neighborhood of the closure of $\tilde{Y}_1$. Here ϕ is the same as in Lemma 3.2.1 and $\hat{U}_x \subset\subset Y$ is the coordinate neighborhood of $x \in K$ defined by (3.16). Then f_y has double zeros at all x_j, $1 \le j \le l$, is nonzero outside these points, and $f_y(y) = 1$.

Finally, we introduce

$$F_{s,y}(w) := h_{s,y}^2(w)\cdot(r_1^* f_y)(w) \quad \text{for all } w \in X,\ s \in S,\ y \in \hat{U}_x. \tag{3.22}$$

According to (3.21) we have

$$\sum_{s\in S}|F_{s,y}(z)| \le C' \quad \text{for all } z \in X,\ y \in \hat{U}_x, \tag{3.23}$$

with $C' := C/|\phi(y)|^2$ depending on x and Y only. Moreover,

$$F_{s,y}(y,t) = \delta_{st} \tag{3.24}$$

where $\delta_{st} = 0$ for $s \ne t$ and $\delta_{ss} = 1$.

Using the functions $F_{s,y}$, $s \in S$, let us prove that $r^{-1}(y)$ is interpolating for $H^\infty(X)$ for all $y \in \hat{U}_x$.

In fact, for $a = \{a_s\}_{s\in S} \in l^\infty(S)$ and $y \in \hat{U}_x$, consider the function

$$[L_y(a)](z) := \sum_{s\in S} a_s F_{s,y}(z), \quad z \in X. \tag{3.25}$$

According to (3.23) we have

$$\sup_{z\in X}|[L_y(a)](z)| \le \|a\|_{l^\infty}\cdot\sup_{z\in X}\left(\sum_{s\in S}|F_{s,y}(z)|\right) \le C'\|a\|_{l^\infty}.$$

Thus L_y is a linear continuous operator from $l^\infty(r^{-1}(y))$ to $H^\infty(X)$ depending holomorphically on $y \in \hat{U}_x$ with the norm bounded by a number depending on x and Y only. Also, from (3.24) we obtain

$$[L_y(a)](y,t) := \sum_{s\in S} a_s F_{s,y}(y,t) = \sum_{s\in S} a_s\delta_{st} = a_t =: a(y,t). \tag{3.26}$$

That is, $L_y(a)|_{r^{-1}(y)} = a$. Therefore $r^{-1}(y)$, $y \in \hat{U}_x$, is an interpolating sequence for $H^\infty(X)$ with the constant of interpolation depending on Y and x only. Taking a finite open cover of K by sets $\hat{U}_x$ and considering the corresponding interpolation operators L_y on $\hat{U}_x$ we obtain that the constant of interpolation of each $r^{-1}(x)$, $x \in K$, is bounded by a number depending on K and Y only.

The proof of Theorem 1.2.2 is complete.

4. Proof of Theorem 1.2.3

Let η be a smooth bounded $(0,1)$-form on X with $r(\operatorname{supp}\eta) \subset K$ for some compact $K \subset\subset Y$. We must find a smooth function f on X such that

$$\bar{\partial} f = \eta \quad \text{and} \quad \|f\|_{L^\infty} := \sup_{z\in X}|f(z)| \le C\|\eta\| \tag{4.1}$$

with C depending on K, Y, and a hermitian metric h_Y used in the definition of the norm of η; see (1.4).

Without loss of generality we may assume that η has compact support. Indeed, let $\{X_i\}_{i\in\mathbb{N}}$, $X_i \subset\subset X$, be an exhaustion of X by relatively compact open domains. Let $\{\chi_i\}_{i\in\mathbb{N}}$ be a family of smooth functions on X such that χ_i equals 1 on a subdomain $Z_i \subset\subset X_i$ and equals 0 outside X_i with $0 \le \chi_i \le 1$, $i \in \mathbb{N}$. Assume also that $\{Z_i\}_{i\in\mathbb{N}}$ forms an exhaustion of X. Now, we set $\eta_i := \chi_i \cdot \eta$. Then $\{\eta_i\}$ converges to η uniformly on compact subsets of X and $\|\eta_i\| \le \|\eta\|$ for all i. If we find smooth functions f_i on X satisfying the corresponding conditions (4.1), then a standard normal family argument will give us a subsequence $\{f_{i_k}\}_{k\in\mathbb{N}}$ of $\{f_i\}$ converging uniformly on compact subsets of X to a smooth function f satisfying (4.1). Thus it suffices to prove the theorem for the forms η with compact supports.

Next, consider a finite open cover $(U_i)_{1\le i\le n}$ of $K \subset\subset Y$ by sets $U_i := \hat{U}_{x_i}$, $x_i \in K$, defined by (3.16). By definition, $(U_i)_{1\le i\le n}$ also covers a neighborhood $N \subset\subset Y$ of K. Now we consider a finite open cover $(U_i)_{n+1\le i\le m}$ of $R \setminus N$ (where R is a compact Riemann surface from (2.1) containing Y) by coordinate disks U_i such that $U_i \cap K = \emptyset$ for all $n+1 \le i \le m$. Let $\{\rho_i\}_{1\le i\le m}$ be a smooth partition of unity subordinate to the cover $(U_i)_{1\le i\le m}$ of R. Then, since $r(\operatorname{supp}\eta) \subset K$, $U_i \cap K = \emptyset$ for all $n+1 \le i \le m$, and $\operatorname{supp}\rho_i \subset U_i$ for $1 \le i \le n$,

$$\eta = \sum_{i=1}^{m}(r_1^*\rho_i)\eta = \sum_{i=1}^{n}(r_1^*\rho_i)\eta.$$

By the definition, each $\eta_i := (r_1^*\rho_i)\eta$ is a smooth $(0,1)$-form with compact support such that $r(\operatorname{supp}\eta_i) \subset U_i$. It suffices to prove the theorem for such forms η_i—that is, to find smooth functions f_i such that $\bar{\partial} f_i = \eta_i$ and $\|f_i\|_{L^\infty} \le C_i\|\eta_i\|$ $(\le C_i\|\eta\|)$ with C_i depending on U_i, Y, and h_Y. Then $f := \sum_{i=1}^{n} f_i$ satisfies the required statement of the theorem.

Thus without loss of generality we may assume that $r(\operatorname{supp}\eta) \subset\subset \hat{U}_x := U$ for some $x \in K$ and that η has compact support. As before, we identify $r^{-1}(U)$ with $U \times S$ where S is the fibre of r. Then there is a finite subset $S_\eta \subset S$ such that $\operatorname{supp}\eta \subset\subset U \times S_\eta$. In a complex coordinate z on U the form η is written as

$$\eta(z,s) = g(z,s)d\bar{z}, \quad (z,s) \in U \times S_\eta, \quad \eta = 0 \ \text{ outside } U \times S_\eta.$$

By the hypothesis of the theorem we have

$$|\eta| := \sup_{z\in U, s\in S_\eta} |g(z,s)| \le c\|\eta\| \tag{4.2}$$

for some c depending on U and h_Y only.

The remaining part of the proof repeats literally the proof of [Br4, Prop. 5.1]. We refer to that paper for details.

Consider the family of interpolation operators $L_z\colon l^\infty(r^{-1}(y)) \to H^\infty(X)$ holomorphic in $z \in U$ with norms bounded by a number C' depending on U and Y only; see (3.25). Let us define the $(0,1)$-form λ on R with values in $H^\infty(X)$ by the formula

$$\lambda(z) := L_z(g(z,\cdot))d\bar{z}, \quad z \in U, \quad \lambda = 0 \ \text{ outside } U. \tag{4.3}$$

Since $\operatorname{supp}\eta \subset\subset U \times S_\eta$ and S_η is a finite subset of S, the definition of L_z in (3.26) implies that λ is smooth. Using an integral formula we can solve the equation $\bar\partial F = \lambda$ on a fixed neighborhood of the closure of Y in R to get a smooth solution $F\colon Y \to H^\infty(X)$ satisfying

$$\sup_{z\in Y, w\in X} |[F(z)](w)| \le c'|\lambda| := c' \cdot \sup_{z\in Y, w\in X} |[L_z(g(z,\cdot))](w)|, \tag{4.4}$$

with c' depending on Y only. Finally, we set

$$f(w) := [F(r(w))](w), \quad w \in X. \tag{4.5}$$

Since $\{L_z\}$ are interpolation operators holomorphic in $z \in U$ one has

$$\bar\partial f(w) := [\lambda(r(w))](w) = \eta(w), \quad w \in X.$$

Moreover, from estimates (4.2), (4.4), and $\|L_z\| \le C'$ we get

$$\sup_{w\in X} |f(w)| \le C\|\eta\|$$

where $C := c' \cdot C' \cdot c$.

This completes the proof of Theorem 1.2.3.

5. Proof of Theorem 1.1.1

Let $r\colon X \to Y$ be an unbranched covering of a Carathéodory hyperbolic Riemann surface of finite type Y. The fact that X is Carathéodory hyperbolic follows easily from Theorem 1.2.2 and the Carathéodory hyperbolicity of Y. Let us prove now the corona theorem for $H^\infty(X)$.

First we consider a finite open cover $\mathcal{U} = (U_j)_{0\le j\le l}$ of $\tilde{Y} := Y \cup \left(\bigcup_{1\le j\le l}\{x_j\}\right)$ such that for $1 \le j \le l$ the set $U_j \subset\subset \tilde{Y}$ is an open coordinate disk centered at x_j and U_0 is a bordered Riemann surface that intersects each U_j, $1 \le j \le l$, by a set biholomorphic to an open annulus; see (2.1). By $\{\rho_j\}_{0\le j\le l}$ we denote a smooth partition of unity on $\tilde{Y}$ subordinate to the cover $\mathcal{U}$. We set $U_j^* := U_j \setminus \{x_j\}$, $1 \le j \le l$. Then U_j^* is biholomorphic to a punctured open disk in $\mathbb{C}$. Now, $r^{-1}(U_j^*)$ is a disjoint union of sets biholomorphic to $\mathbb{D}$ or to the punctured disk $\mathbb{D}^*$ (because the fundamental group of U_j^* is $\mathbb{Z}$). Moreover, according to our construction, $\pi_1(U_0) \cong \pi_1(Y)$. Hence $r^{-1}(U_0)$ is an open connected subset of X.

Suppose now that a collection $f_1,\dots,f_n$ of functions from $H^\infty(X)$ satisfies the corona condition (1.1). Since each connected component of $r^{-1}(U_j^*)$ is biholomorphic to $\mathbb{D}$ or $\mathbb{D}^*$, according to the Carleson corona theorem (see e.g. [G, Chap. VIII, Thm. 2.1]), there are a constant $C_1(n,\delta)$ (with δ from (1.1)) and functions $g_1^j,\dots,g_n^j$ from $H^\infty(r^{-1}(U_j^*))$, $1\le j\le l$, such that

$$\begin{aligned} &f_1g_1^j+\cdots+f_ng_n^j\equiv 1 \ \text{ on } r^{-1}(U_j^*) \quad \text{and}\\ &\|g_k^j\|\le C_1(n,\delta),\quad 1\le j\le l,\ 1\le k\le n. \end{aligned} \tag{5.1}$$

Also, since $U_0\subset\subset Y$ is a bordered Riemann surface, according to [Br3, Cor. 1.6] there are a constant $C_2(Y,n,\delta)$ and functions $g_1^0,\dots,g_n^0$ from $H^\infty(r^{-1}(U_0))$ such that

$$\begin{aligned} &f_1g_1^0+\cdots+f_ng_n^0\equiv 1 \ \text{ on } r^{-1}(U_0) \quad \text{and}\\ &\|g_k^0\|\le C_2(Y,n,\delta),\quad 1\le k\le n. \end{aligned} \tag{5.2}$$

We set

$$h_k:=\sum_{j=0}^{l}(r^*\rho_j)g_k^j,\quad 1\le k\le n,\ 0\le j\le l. \tag{5.3}$$

Since $\operatorname{supp}\rho_j\subset\subset U_j$, $0\le j\le l$, h_k are smooth functions on X such that

$$\begin{aligned} &f_1h_1+\cdots+f_nh_n\equiv 1 \ \text{ on } X \quad \text{and}\\ &\|h_k\|_{L^\infty}\le C_3(Y,n,\delta),\quad 1\le k\le n. \end{aligned} \tag{5.4}$$

Next we will use a standard construction based on the Koszul complex; see [G, Chap. VIII]. Namely, we write

$$\begin{aligned} &g_j(z)=h_j(z)+\sum_{k=1}^{n}a_{j,k}(z)f_k(z),\\ &a_{j,k}(z)=b_{j,k}(z)-b_{k,j}(z),\quad \text{and}\\ &\bar\partial b_{j,k}=h_j\cdot\bar\partial h_k=:\eta_{j,k},\quad j\ne k. \end{aligned} \tag{5.5}$$

According to (5.3) and (5.4), the smooth $(0,1)$-forms $\eta_{j,k}$ on X satisfy

$$r(\operatorname{supp}\eta_{j,k})\subset\subset U_0 \quad \text{and} \quad \|\eta_{j,k}\|\le C_4(Y,n,\delta) \quad \text{for all } j,k,$$

where $\|\cdot\|$ is defined with respect to a fixed hermitian metric h_Y on Y; see (1.5). Therefore, by Theorem 1.2.3 there are smooth functions $b_{j,k}$ on X satisfying equations (5.5) such that

$$\|b_{j,k}\|_{L^\infty}\le C_5(Y,n,\delta) \quad \text{for all } j,k.$$

Then the functions g_j on X belong to $H^\infty(X)$ and satisfy

$$\begin{aligned} &f_1g_1+\cdots+f_ng_n\equiv 1 \quad \text{and}\\ &\|g_j\|\le C(Y,n,\delta) \quad \text{for all } j. \end{aligned} \tag{5.6}$$

This completes the proof of Theorem 1.1.1.

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Department of Mathematics and Statistics
University of Calgary
Calgary, Alberta
Canada