Michigan Math. J. 56 (2008)

Mass Flow for Noncompact Manifolds

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1. Introduction

The group of homeomorphisms acts on the space of measures on a manifold M in such a way that, for each measure μ on M and each homeomorphism $h: M \to M$, the action $h_*\mu$ of h on μ is defined by $h_*\mu(E) = \mu(h^{-1}(E))$ for each Borel set $E \subseteq M$. The group of homeomorphisms preserving a given measure is just the stabilizer of that measure under the action. In 1941, Oxtoby and Ulam [12] characterized the orbit of standard Lebesgue measure on the unit cube under this action. Since then, Oxtoby and Ulam's result has been of enormous importance for the study of groups of measure-preserving homeomorphisms.

The aim of our work is to generalize and reformulate previous research of Fathi [9] on the definition and properties of the so-called mass flow homomorphism to the σ-compact case. Our "noncompact" methods allow us to replace Fathi's use of handles and tubular neighborhoods by neighborhoods alone. Therefore, we do not depend on the existence of a combinatorial structure on the base manifold to proceed with the argumentation. As a by-product, simplification of Fathi's arguments is gained.

Let M be a connected, second-countable manifold without boundary equipped with a "good" measure μ_o (see Section 2). Let $\mathcal{H}_c(M, \mu_o)$ be the group of μ_o-measure-preserving homeomorphisms of M with compact support endowed with the Whitney topology, and let $\mathcal{H}_{c,o}(M, \mu_o)$ be the path component of the identity. The *mass flow homomorphism* is a group homomorphism from the universal covering space of $\mathcal{H}_{c,o}(M, \mu_o)$, thought of as a space of paths modulo homotopy (rel. ∂I), to the first homology group $H_1(M, \mathbb{R})$.

Fathi's approach to the mass flow, in which homology is viewed as a set of homotopy classes of maps into the circle, is suitable only for compact manifolds. In order to extend Fathi's theory to noncompact manifolds we define, in Section 4, a new version of the mass flow homomorphism that relies on a real homology theory based on measures due to W. Thurston. In Section 5, both approaches are compared (see Proposition 5.4 and the subsequent comment). Via a quotient process, in Section 6 the mass flow homomorphism on $\mathcal{H}_{c,o}(M, \mu_o)$ is effectively defined, giving rise to an important commutative diagram that is studied in some detail.

Received March 16, 2006. Revision received April 24, 2008.

In Section 7 it is established that the mass flow is surjective and its kernel, Ker Θ, is generated by its elements supported in topological n-balls. If the dimension n of M is at least three, then Ker Θ is shown to be a simple group equal to the commutator subgroup of $\mathcal{H}_{c,o}(M, \mu_o)$. In [9], Fathi uses the condition $n \geq 3$ in an essential way in order to prove that $\mathcal{H}_c(\text{Int } I^n, \text{Lebesgue})$ is perfect. Since the perfectness of $\mathcal{H}_c(\text{Int } I^n, \text{Lebesgue})$ is needed to imply the simplicity of Ker Θ, our final result is left unsolved for 2-manifolds.

The cases of a manifold with boundary and homeomorphisms fixing ∂M pointwise are considered jointly. Also, the basic constructions that lead to the definition of the mass flow homomorphism and its first properties are valid for a locally compact, locally connected, second-countable Hausdorff space X. Hence, for simplicity, these constructions are stated in this topological setting. Before proceeding to our main topic, some preliminary concepts are discussed in Section 2 and a necessary "extension of isotopies theorem" is proved afterward in Section 3.

2. Preliminaries

Let Y, Z be topological spaces and let $C(Y, Z)$ denote the set of continuous functions from Y to Z. The *Whitney topology* on $C(Y, Z)$ is the topology having for a basis intersections of the form $\bigcap_{i \in \Lambda} [K_i, U_i]$, where $\{K_i\}_{i \in \Lambda}$ is a locally finite family of compact sets in Y, $\{U_i\}_{i \in \Lambda}$ is an open family in Z, and $[K_i, U_i]$ is the set of continuous functions $f : Y \to Z$ such that $f(K_i) \subset U_i$. If we further require the set of indices Λ to be finite then we get the *compact-open topology*. The *compactly generated space* of a given Hausdorff space S_τ is the space $\mathsf{k}S$ with the largest topology making the inclusions $\{K \hookrightarrow S \mid K \subset S_\tau \text{ compact}\}$ continuous. In this manner, the injections $\mathsf{k}(C(Y, Z)_m) \hookrightarrow C(Y, Z)_m \hookrightarrow C(Y, Z)_\kappa$ are continuous, where the indices m and κ denote the Whitney topology and the compact-open topology, respectively.

Let X be a locally compact Hausdorff space. Denote by $\mathcal{H}(X)$ the group of homeomorphisms of X. For h in $\mathcal{H}(X)$, define its *support* as the closure of $\{x \in X \mid h(x) \neq x\}$. Define $\mathcal{H}_c(X)$ to be the group of all homeomorphisms of X with compact support.

A *Radon measure* μ on X is a locally finite positive measure defined on the σ-algebra of all Borel subsets. The *support* of μ is the complement of the largest open set in X that has μ-measure zero. We say that μ is a *good measure* if it has no *atoms* (i.e., points of positive measure) and its support is the whole of X. Let $\mathcal{M}_g(X)$ be the set of good Radon measures on X.

For $\mu \in \mathcal{M}_g(X)$ and $h \in \mathcal{H}(X)$, $h_*\mu$ is the good measure in $\mathcal{M}_g(X)$ defined by $h_*\mu(B) = \mu(h^{-1}(B))$ for all $B \subset X$ Borel. Define the group of *μ-measure-preserving homeomorphisms* $\mathcal{H}(X, \mu)$ as the set $\{h \in \mathcal{H}(X) \mid h_*\mu = \mu\}$. Let $\mathcal{H}_c(X, \mu) = \mathcal{H}_c(X) \cap \mathcal{H}(X, \mu)$. Denote by $\mathcal{H}(X, \mu\text{-reg})$ the group of all homeomorphisms h in $\mathcal{H}(X)$ such that $h_*\mu$ and μ have the same sets of measure zero. Let $\mathcal{H}_c(X, \mu\text{-reg}) = \mathcal{H}_c(X) \cap \mathcal{H}(X, \mu\text{-reg})$.

Let M be a manifold, possibly with nonempty boundary ∂M. Then it is straightforward to define the group $\mathcal{H}^\partial(M)$ of homeomorphisms fixing ∂M pointwise and $\mathcal{H}_c^\partial(M) = \mathcal{H}_c(M) \cap \mathcal{H}^\partial(M)$. Let $\mathcal{M}_g^\partial(M)$ be the set of good Radon measures on M having ∂M as a null set. For $\mu \in \mathcal{M}_g^\partial(M)$ define, similarly, the groups $\mathcal{H}^\partial(M,\mu)$, $\mathcal{H}_c^\partial(M,\mu)$, $\mathcal{H}^\partial(M,\mu\text{-reg})$, and $\mathcal{H}_c^\partial(M,\mu\text{-reg})$.

If A is a subset of a topological space X, denote its *interior, closure,* and *frontier* by $\operatorname{Int} A$, $\operatorname{Cl} A$, and $\operatorname{Fr} A$, respectively. Denote the unit interval by both I and $\triangle^1$. Call any subset $K \subset X$ a (*closed*) *n-cell* if it is homeomorphic to the unit n-cube $I^n = [0,1]^n$. Call $K \subset X$ a *relative n*-cell if there is a continuous surjection $\phi\colon I^n \to K$ such that, when restricted to $\operatorname{Int} I^n$, ϕ is a homeomorphism on its image and $\phi^{-1} \circ \phi(\partial I^n) = \partial I^n$.

3. Extension of Isotopies

We apply here the Černavskii–Edwards-Kirby–Rogalski theorem and a parameterized version of the von Neumann–Oxtoby–Ulam theorem to show that, under certain circumstances, measure-preserving perturbations and measure-preserving isotopies of a compact subset in a manifold can be extended to measure-preserving homeomorphisms and measure-preserving ambient isotopies, respectively.

Let M be a (second-countable) manifold and let $\mu_o \in \mathcal{M}_g^\partial(M)$. Let A be a subset of M. By a *proper embedding* ι of A into M we mean an injective (continuous) map $\iota\colon A \hookrightarrow M$ such that ι is a homeomorphism of A onto $\iota(A)$ and $\iota^{-1}(\partial M) = A \cap \partial M$. Denote by $\mathcal{I}(A,M)$ the space of proper embeddings of A into M. If $\iota \in \mathcal{I}(A,M)$ and A is a Borel subset of M, we can define a measure $\iota^*\mu_o$ on A such that $\iota^*\mu_o(B) = \mu_o(\iota(B))$ for each Borel subset $B \subset A$. Let $\mathcal{I}(A,M;\,\mu_o) = \{\iota \in \mathcal{I}(A,M) \mid \iota^*\mu_o = \mu_o|_A\}$.

We say that a proper embedding $\iota\colon A \hookrightarrow M$ is *biregular* (*with respect to μ_o*) if $\iota^*\mu_o$ and $\mu_o|_A$ have the same sets of measure zero. Denote by $\mathcal{I}(A,M;\,\mu_o\text{-reg})$ the set of all proper biregular embeddings of A into M. Suppose B is a subset of M. We define $\mathcal{I}(A,B,M) = \{\iota \in \mathcal{I}(A,M) \mid \iota|_{(B\cap A)} = \mathrm{Id}\}$ and $\mathcal{I}(A,B,M;\,\mu_o\text{-reg}) = \mathcal{I}(A,B,M) \cap \mathcal{I}(A,M;\,\mu_o\text{-reg})$. All spaces of proper embeddings will be endowed with the compact-open topology.

Suppose M is a manifold with subsets Q and S. A *deformation of Q into S* is a continuous map $\phi\colon Q \times I \to M$ such that $\phi|_{(Q\times\{0\})} = \mathrm{Id}_Q$ and $\phi(Q \times \{1\}) \subset S$. If $T \subset M$ and $\phi(Q \times I) \subset T$, we say that ϕ *takes place in* T. Let $\mathcal{P}$ be a subset of $\mathcal{I}(A,M)$ and W a subset of A. A deformation $\phi\colon \mathcal{P} \times I \to \mathcal{I}(A,M)$ of $\mathcal{P}$ is *modulo* W if $\phi(\iota,t)|_W = \iota|_W$ for all $\iota \in \mathcal{P}$ and $t \in I$.

The following theorem, where no measures intervene, is due to Černavskii [5]. A much more readable and elegant approach is due to Edwards and Kirby [7]. The measure-theoretic version stated next is taken from Fathi [9]. Fathi gives the credit for this result to M. Rogalski.

THEOREM 3.1. *Let U be a neighborhood of a compact C in a manifold M. Let μ_o be a measure in $\mathcal{M}_g^\partial(M)$. Given any neighborhood $\mathcal{N}$ of the inclusion $\zeta : U \hookrightarrow M$ in* $\mathcal{I}(U,M;\,\mu_o\text{-reg})$, *there is a neighborhood $\mathcal{P}$ of ζ in* $\mathcal{I}(U,M;\,\mu_o\text{-reg})$ *and a deformation $\phi\colon \mathcal{P} \times I \to \mathcal{N}$ into* $\mathcal{I}(U,C,M;\,\mu_o\text{-reg})$ *such that*:

(1) *ϕ is modulo the complement of a compact neighborhood of C in* $\operatorname{Int} U$;
(2) *$\phi(\zeta,t) = \zeta$ for all $t \in I$*;
(3) $\phi|_{[\mathcal{P}\cap\mathcal{I}(U,\partial M,M;\,\mu_o\text{-reg})]} \times I$ *takes place in* $\mathcal{I}(U,\partial M,M;\,\mu_o\text{-reg})$.

Furthermore, suppose in addition to these hypotheses that a closed set D in M (respectively ∂M) and a neighborhood V of D in M (respectively ∂M) are given. Then ϕ can be chosen so that the deformation $\phi|_{[\mathcal{P}\cap\mathcal{I}(U,V,M;\,\mu_o\text{-reg})]}\times I$ takes place in $\mathcal{I}(U,D,M;\,\mu_o\text{-reg})$.

PROPOSITION 3.2. *Let M be a second-countable manifold and let $\mu_0\in\mathcal{M}_g^{\partial}(M)$. Let C and U be subsets of M such that C is compact and U is a neighborhood of C. Then there is a neighborhood $\mathcal{P}$ of the inclusion $\zeta: U\hookrightarrow M$ in $\mathcal{I}(U,M;\,\mu_o\text{-reg})$ and a continuous map $\mathcal{P}\to\mathcal{H}_c(M,\mu_o\text{-reg})$, $\iota\mapsto\bar\iota$, such that:*

(1) $\bar\iota|_C=\iota|_C$;
(2) *there is a compact neighborhood F of C in* Int U, *independent of ι, such that the support of $\bar\iota$ is contained in F;*
(3) $\bar\zeta=\mathrm{Id}_M$;
(4) *if ι fixes $U\cap\partial M$ pointwise, then $\bar\iota$ fixes ∂M pointwise.*

Furthermore, suppose in addition to these hypotheses that a closed set D in M (respectively ∂M) and a neighborhood V of D in M (respectively ∂M) are given. Then the correspondence $\iota\mapsto\bar\iota$ can be chosen so that, for each ι fixing $U\cap V$ pointwise, its extension $\bar\iota$ fixes D pointwise.

Proof (cf. [9]). Let ϕ be the deformation given in Theorem 3.1. Then $\iota\circ\phi(\iota,1)^{-1}$: $\phi(\iota,1)(U)\to M$ is equal to ι on C and is the identity outside $\phi(\iota,1)(F)$; hence it can be extended by the identity to a homeomorphism of M. □

PROPOSITION 3.3. *Let M be a second-countable manifold and let $\mu_0\in\mathcal{M}_g^{\partial}(M)$. Let C and U be subsets of M such that C is compact and U is a neighborhood of C. If $M\setminus C$ is connected, then there is a neighborhood $\mathcal{P}_{\mu_o}$ of the inclusion $\zeta: U\hookrightarrow M$ in $\mathcal{I}(U,M;\,\mu_o)$ and a compact neighborhood F of C in M (not necessarily in* Int U*) such that, for each $\iota\in\mathcal{P}_{\mu_o}$, there is a measure-preserving homeomorphism $\tilde\iota\in\mathcal{H}_c(M,\mu_o)$ with the following properties:*

(1) *$\tilde\iota$ depends continuously on ι;*
(2) $\tilde\iota|_C=\iota|_C$;
(3) $\operatorname{supp}\tilde\iota\subset F$;
(4) $\tilde\zeta=\mathrm{Id}_M$;
(5) *if ι fixes $U\cap\partial M$ pointwise, then $\tilde\iota$ fixes ∂M pointwise.*

(6) *Furthermore, suppose in addition to these hypotheses that a closed set D in ∂M and a neighborhood V of D in ∂M are given. Then the correspondence $\iota\mapsto\tilde\iota$ can be chosen so that, for each ι fixing $U\cap V$ pointwise, its extension $\tilde\iota$ fixes D pointwise.*

Proof. Let C^+ be a compact neighborhood of C in M such that $C^+\subset\operatorname{Int}U$. By applying Proposition 3.2 to the pair (U,C^+) we get a neighborhood $\mathcal{P}$ of μ_o-biregular embeddings ι: $U\hookrightarrow M$, a compact set $F^-\subset M$, and a continuous function $\iota\mapsto\bar\iota$ on $\mathcal{P}$ satisfying certain properties.

Lemma 7 in Berlanga and Epstein [4] implies that there is a relative n-cell L contained in M such that $L\cap C=\emptyset$, $F^-\setminus\operatorname{Int}C^+\subset L$, and $\mu_o(\operatorname{Fr}L)=0$. It is not difficult to verify that $b=\mu_o(\bar\iota(L))$ is independent of $\iota\in\mathcal{P}_{\mu_o}$, where $\mathcal{P}_{\mu_o}=\mathcal{P}\cap\mathcal{I}(U,M;\,\mu_o)$.

Define $\mathcal{M}_g^B(L, (\mu_o|_L)\text{-reg})$ to be the set of all measures $\nu \in \mathcal{M}_g(L)$ with total mass equal to b and having the same sets of measure zero as $\mu_o|_L$. In particular, $(\partial M \cap L) \cup \operatorname{Fr} L$ is a set of zero ν-measure.

Define the continuous function $\mathcal{P}_{\mu_o} \to \mathcal{M}_g^B(L, (\mu_o|_L)\text{-reg})$, $\iota \mapsto (\bar{\iota}^{-1}{}_* \mu_o)|_L$, where $\mathcal{M}_g^B(L, (\mu_o|_L)\text{-reg})$ is endowed with the weak topology that makes the functionals $\nu \mapsto \int f\, d\nu$ continuous for each continuous $f : L \to \mathbb{R}$.

Let $\mathcal{H}^\partial(L, M, \mu_o\text{-reg})$ be the set of homeomorphisms h in $\mathcal{H}^\partial(M, \mu_o\text{-reg})$ supported in L with the compact-open topology. The fact that L is a relative n-cell implies that we can find a continuous map

$$\sigma : \mathcal{M}_g^B(L, (\mu_o|_L)\text{-reg}) \to \mathcal{H}^\partial(L, M, \mu_o\text{-reg})$$

(see [1]) such that $\sigma(\nu)_*(\mu_o|_L) = \nu$. Let $F = C^+ \cup L$. Then $\mathcal{P}_{\mu_o} \to \mathcal{H}_c(M, \mu_o)$, $\iota \mapsto \bar{\iota} \circ \sigma(\bar{\iota}^{-1}{}_*(\mu_o|_L))$ is the required μ_o-measure-preserving extension function. □

Now we want to generalize Proposition 3.3 to the case in which the set $\mathcal{C}(M \setminus C)$ of connected components of $M \setminus C$ is finite and $\mu_o(M) < \infty$. Let $S^1 \subset S^2$ be the equator included in the 2-sphere. Then, the reflection of S^2 along S^1 fixes S^1 pointwise and sends one hemisphere onto the other. The next remark states that such homeomorphisms are always far from the identity and that under some mild hypotheses they do not exist at all.

Remark 3.4. Let M be a manifold and let C be a compact subset of M such that $\mathcal{C}(M \setminus C)$ is finite. Then there exists a neighborhood $\mathcal{N}$ of the identity in $\mathcal{H}(M)$, in the compact-open topology, such that if $f \in \mathcal{N}$ fixes C pointwise then $f(A) = A$ for each $A \in \mathcal{C}(M \setminus C)$. More precisely, the set $\mathcal{S}$ of homeomorphisms $f : M \to M$ such that $f|_C = \operatorname{Id}_C$ and $F(A) = A$ for each $A \in \mathcal{C}(M \setminus C)$ is both open and closed (in the compact-open topology) when considered as a subset of the space of all homeomorphisms of M fixing C pointwise.

Furthermore, suppose that C is such that $\operatorname{Fr} A_1 \neq \operatorname{Fr} A_2$ for each two distinct components of $M \setminus C$. Then the set $\mathcal{S}$ of homeomorphisms just defined is equal to the full group of homeomorphisms of M fixing C pointwise. As an example of this situation, let M be connected and let C be a locally flat codimension-zero submanifold of M. Then the frontiers of any two distinct components of $M \setminus C$ are disjoint and nonempty.

Remark 3.5. Let M be a connected manifold, C a compact subset with $\mathcal{C}(M \setminus C)$ finite, and $\mu_o \in \mathcal{M}_g^\partial(M)$ such that $\mu_o(M) < \infty$. Let $C^+ \subset U \subset M$ be neighborhoods of C with C^+ compact. By applying Proposition 3.2 to the pair (U, C^+) we get a neighborhood $\mathcal{P}$ of μ_o-biregular embeddings $\iota : U \hookrightarrow M$ and a continuous function $\iota \mapsto \bar{\iota}$ on $\mathcal{P}$ satisfying certain properties. Observe that $\bar{\iota}(A)$ ($A \in \mathcal{C}(M \setminus C)$) is independent of the particular extension of $\iota|_{C^+}$ to M. We now generalize Proposition 3.3.

Proposition 3.6. *Consider the situation of Remark 3.5 and let*

$$\mathcal{P}_{\mu_o} = \mathcal{P} \cap \mathcal{I}(U, M; \mu_o).$$

Then there exists a compact neighborhood F of C in M such that, for each $\iota \in \mathcal{P}_{\mu_o}$ satisfying $\mu_o(\bar{\iota}(A)) = \mu_o(A)$ ($\forall A \in \mathcal{C}(M \setminus C)$), there is a measure-preserving *homeomorphism* $\tilde{\iota} \in \mathcal{H}_c(M, \mu_o)$ *for which properties* (1)–(6) *of Proposition 3.3 hold.*

Proof. The proof is the same as that of Proposition 3.3. □

We conclude this section with an extension of the isotopies theorem.

THEOREM 3.7. *Let* M *be a connected second-countable manifold, and let* $\mu_0 \in \mathcal{M}_g^{\partial}(M)$ *with* $\mu_o(M) < \infty$. *Let* $C \subset M$ *be a compact subset such that* $\mathcal{C}(M \setminus C)$ *is finite. Let* $U \subset M$ *be a neighborhood of* C *and let* $\iota \colon I \to \mathcal{I}(U, M; \mu_o)$, $\tau \mapsto \iota_\tau$, *be an isotopy of embeddings preserving* μ_o *such that* ι_0 *is the inclusion* $\zeta : U \hookrightarrow M$. *Then the following conditions are equivalent.*

(1) $\tau \mapsto \iota_\tau|_C$ *extends to a compactly supported isotopy* $\tilde{\iota} \colon I \to \mathcal{H}_c(M, \mu_o)$ *of measure-preserving homeomorphisms such that* $\tilde{\iota}_0 = \mathrm{Id}_M$.
(2) *If* $\tilde{\iota} \colon I \to \mathcal{H}_c(M, \mu_o\text{-reg})$ *is any extension of* $\tau \mapsto \iota_\tau|_C$ *to a biregular homeomorphism such that* $\tilde{\iota}_0 = \mathrm{Id}_M$, *then* $\mu_o(\tilde{\iota}_\tau(A)) = \mu_o(A)$ *for each* $A \in \mathcal{C}(M \setminus C)$.

Proof. Observe that if condition (2) is satisfied by some extension of $\tau \mapsto \iota_\tau|_C$ then, by Remark 3.4, it is satisfied by all extensions. Therefore, (1) implies (2).

Let $C^+ \subset \operatorname{Int} U$ be a compact neighborhood of C. Then, from Theorem 3.1, it follows that a compactly supported extension of $\tau \mapsto \iota_\tau|_{C^+}$ to biregular homeomorphisms does exist (see [7, proof of Cor. 1.2, p. 79]). Then we can modify this biregular extension to a measure-preserving extension of $\tau \mapsto \iota_\tau|_C$ as in Proposition 3.3. □

REMARK 3.8. Suppose $\tau \mapsto \iota_\tau$ can be extended to a measure-preserving isotopy of homeomorphisms. Furthermore, suppose that $D \subset \partial M$ is closed and a neighborhood V of D in ∂M is given in such a way that ι_τ fixes $U \cap V$ pointwise; then $\tilde{\iota}_\tau$ can be chosen to fix D pointwise for each $\tau \in I$.

4. The Mass Flow

Through the rest of this work X will denote a locally compact, locally connected, second-countable Hausdorff space. These properties imply that X is metrizable and locally path connected (see [10]). Also, μ_0 will represent a fixed good measure on X.

Call $S(\triangle^1, X) = C(I, X)$ the *space of singular 1-simplices* and endow it with the compact-open topology (since I is compact, this topology coincides with the Whitney topology). Note that $S(\triangle^1, X)$ is second countable if X is second countable (see [6]). Endow $C(X, S(\triangle^1, X))$, $\mathcal{H}_c(X)$, and $\mathcal{H}_c(X, \mu_o)$ with the Whitney topology.

It is not difficult to verify that, given a compact subset $\mathcal{K}$ in $\mathcal{H}_c(X)$, there is a compact $K \subset X$ such that every element in $\mathcal{K}$ has support in K [2, Lemma 2.1]. This remark is certainly false if $\mathcal{H}_c(X)$ is endowed with the compact-open topology instead.

Let $\mathcal{P}(\mathcal{H}_c(X)) = \{h \in C(I, \mathcal{H}_c(X)) \mid h_0 = \mathrm{Id}_X\}$ be the *space of paths* (i.e., compactly supported isotopies) in $\mathcal{H}_c(X)$ based at the identity. Endow $\mathcal{P}(\mathcal{H}_c(X))$ with the compact-open topology. Observe that the (*exponential*) *map*

$$E\colon \mathcal{P}(\mathcal{H}_c(X)) \to C(X, S(\triangle^1, X)),$$
$$h = \{h_t\}_{t\in I} \mapsto (x \mapsto (t \mapsto h_t(x)))$$

is a topological embedding.

Let $\eta \in \mathcal{P}(\mathcal{H}_c(X))$ be the constant path such that $\eta_t = \mathrm{Id}\colon X \to X$ for each $t \in I$. In what follows we shall reserve the letter E for the exponential map and the letter η for the constant path just defined.

If $f\colon X \to S(\triangle^1, X)$ is continuous and a Borel measure ν on X is given then $f_*\nu$, defined by $f_*\nu(B) = \nu(f^{-1}(B))$ for each Borel subset B in $S(\triangle^1, X)$, is a Borel measure on $S(\triangle^1, X)$. Note that $\operatorname{supp} f_*\nu \subset f(\operatorname{supp}\nu)$, where $\operatorname{supp}\nu$ denotes the support of ν.

ASSERTION 4.1. *If $h \in \mathcal{P}(\mathcal{H}_c(X, \mu_o))$ and K is a compact subset of X containing* $\operatorname{supp} h = \mathrm{Cl}\big(\bigcup_{t\in I} \operatorname{supp} h_t\big)$, *then*

$$E(h)_*(\mu_0|_K) - E(\eta)_*(\mu_0|_K) = E(h)_*(\mu_0) - E(\eta)_*(\mu_0).$$

Proof. We give an informal proof. The map $E(\eta)\colon X \to S(\triangle^1, X)$ is an embedding of X as a set of trivial simplices in $S(\triangle^1, X)$, which assigns to each $x \in X$ the constant singular 1-simplex $t \mapsto x$. Call the image of $E(\eta)$ the *trivial copy* of X in $S(\triangle^1, X)$.

The map $E(h)\colon X \to S(\triangle^1, X)$ is another embedding of X, whose image agrees with the trivial copy of X in $S(\triangle^1, X)$ except for a "bubble" lying over the trivial copy of $\operatorname{supp} h$ in $S(\triangle^1, X)$.

It follows that the measures $E(h)_*(\mu_0)$ and $E(\eta)_*(\mu_0)$ agree in the complement of the union of the trivial copy of $\operatorname{supp} h$ and the "bubble". □

Let C_1X denote the real vector space of *finite signed Borel measures on* $S(\triangle^1, X)$ *with compact support*; that is, $\vartheta \in C_1X$ if and only if ϑ is a σ-additive, real-valued function defined on all Borel subsets of $S(\triangle^1, X)$ such that there exists a $\mathcal{K} \subset S(\triangle^1, X)$ compact with the property that the complement of $\mathcal{K}$ in $S(\triangle^1, X)$ has zero ϑ-measure.

The linear space C_1X becomes a locally convex Hausdorff topological vector space if it is given the weakest topology such that, for each continuous λ: $S(\triangle^1, X) \to \mathbb{R}$, the functional $C_1X \to \mathbb{R}$, given by $\vartheta \mapsto \int \lambda\, d\vartheta$, is continuous.

Denote by $\mathcal{H}_{c,o}(X, \mu_o)$ the group of measure-preserving homeomorphisms of X, with compact support, compactly isotopic to the identity in $\mathcal{H}_c(X, \mu_o)$. That is, $\mathcal{H}_{c,o}(X, \mu_o)$ is the path component of the identity in $\mathcal{H}_c(X, \mu_o)$.

Define the function

$$\Phi = \Phi_{(X,\mu_o)}\colon \mathcal{P}(\mathcal{H}_{c,o}(X, \mu_o)) \to C_1X$$

such that

$$\Phi(h) = E(h)_*(\mu_0|_K) - E(\eta)_*(\mu_0|_K),$$

where K is any compact subset of X containing the support of the path h.

ASSERTION 4.2. *The map Φ is continuous.*

Proof. Given a continuous $\lambda\colon S(\triangle^1, X) \to \mathbb{R}$ we have to check that the functional

$$\begin{aligned} &\mathcal{P}(\mathcal{H}_{c,o}(X,\mu_o)) \to \mathbb{R}, \\ &h \mapsto \int \lambda\, d\Phi(h) = \int [\lambda \circ E(h) - \lambda \circ E(\eta)]\, d\mu_o \end{aligned} \tag{4.2.1}$$

is continuous.

Recall that $E(h)\colon X \to S(\triangle^1, X)$ is an embedding of X in $S(\triangle^1, X)$. The map $S(\triangle^1, X) \to X$, such that $\sigma \mapsto \sigma(0)$, shows that X is actually embedded as a retract of $S(\triangle^1, X)$. Hence $E(h)$ is a closed embedding, so it is proper.

Let $C_c(X, \mathbb{R})$ be the space of real-valued functions on X with compact support endowed with the Whitney topology. Then the fact that $E(h)$ is proper for each path h implies that the map

$$\begin{aligned} &\mathcal{P}(\mathcal{H}_{c,o}(X,\mu_o)) \to C_c(X, \mathbb{R}), \\ &h \mapsto \lambda \circ E(h) - \lambda \circ E(\eta) \end{aligned} \tag{4.2.2}$$

is continuous. Clearly, the functional

$$\begin{aligned} &C_c(X, \mathbb{R}) \to \mathbb{R}, \\ &f \mapsto \int_X f\, d\mu_o \end{aligned} \tag{4.2.3}$$

is continuous.

Since the composite of the maps given in (4.2.2) and (4.2.3) is the map given in (4.2.1), the assertion follows. $\square$

In [3] a homology theory based on measures, due to W. Thurston, is discussed in some detail. For topological manifolds this theory is isomorphic to the standard singular homology theory with real coefficients. Let us now briefly recall the definition of Thurston's measure-theoretic first homology group.

Let $\triangle^2 = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x, y \text{ and } x + y \le 1\}$ and let its vertices be denoted by $e_0 = (0, 0)$, $e_1 = (1, 0)$, and $e_2 = (0, 1)$. Let $S(\triangle^2, X) = C(\triangle^2, X)$ be the space of *singular 2-simplices* with the compact-open topology. Let C_2X and C_0X be, respectively, the real vector space of finite signed Borel measures with compact support on $S(\triangle^2, X)$ and X.

Define the *faces* of $\triangle^2$, $F_2^0 = \langle e_1, e_2\rangle$, $F_2^1 = \langle e_0, e_2\rangle$, and $F_2^2 = \langle e_0, e_1\rangle$, as the affine maps from $\triangle^1$ to $\triangle^2$ given in vertices by $F_2^0(0) = e_1$, $F_2^0(1) = e_2$; $F_2^1(0) = e_0$, $F_2^1(1) = e_2$; and $F_2^2(0) = e_0$, $F_2^2(1) = e_1$. For $i = 0, 1, 2$, there is a continuous ith *face map* on singular 2-simplices, say $(F_2^i)^*\colon S(\triangle^2, X) \to S(\triangle^1, X)$, defined by $\sigma \mapsto \sigma \circ F_2^i$, which induces a linear transformation $\partial_2^i\colon C_2X \to C_1X$ such that $\partial_2^i(\vartheta) = [(F_2^i)^*]_*\vartheta$ for each $\vartheta \in C_2X$, where $\partial_2^i(\vartheta)(B) = \vartheta([(F_2^i)^*]^{-1}(B))$ for each Borel subset B contained in $S(\triangle^1, X)$. Let $\partial_2 = \partial_2^0 - \partial_2^1 + \partial_2^2$.

For $\alpha = 0, 1$, let $ev_\alpha\colon S(\triangle^1, X) \to X$ be such that $\sigma \mapsto \sigma(\alpha)$. Define $\partial_1^\alpha = (ev_\alpha)_*\colon C_1X \to C_0X$ and let $\partial_1 = \partial_1^0 - \partial_1^1$. It is readily verified that $\partial_1 \circ \partial_2 = 0$. Hence the quotient $\mathfrak{A} = \text{kernel}(\partial_1) \setminus \text{Image}(\partial_2)$ is well-defined.

The vector space $\mathfrak{A}$ is then isomorphic to the first singular homology group with real coefficients when X is an absolute neighborhood retract (ANR). We shall write simply $\mathfrak{A} = H_1(X, \mathbb{R})$. As a quotient of a subspace of the locally convex Hausdorff space C_1X, $H_1(X, \mathbb{R})$ is Hausdorff as well (see [3]). In particular, finite-dimensional subspaces of $H_1(X, \mathbb{R})$ inherit the natural Euclidean topology.

ASSERTION 4.3. *If $h \in \mathcal{P}(\mathcal{H}_{c,o}(X, \mu_o))$, then $\Phi(h)$ is a 1-cycle; that is,*

$$\partial_1(\Phi(h)) = 0.$$

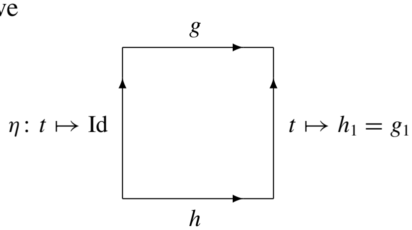
Proof. Let $K \subset X$ be a compact set such that $\operatorname{supp} h \subset K$. It is enough to prove that $\partial_1(E(h)_*(\mu_o|_K)) = 0$. But then,

$$\begin{aligned}
\partial_1(E(h)_*(\mu_o|_K)) &= \partial_1^0(E(h)_*(\mu_o|_K)) - \partial_1^1(E(h)_*(\mu_o|_K)) \\
&= (ev_1)_*(E(h)_*(\mu_o|_K)) - (ev_0)_*(E(h)_*(\mu_o|_K)) \\
&= (ev_1 \circ E(h))_*(\mu_o|_K) - (ev_0 \circ E(h))_*(\mu_o|_K) \\
&= (h_1)_*(\mu_o|_K) - (h_0)_*(\mu_o|_K) \\
&= \mu_o|_{h_1(K)} - \mu_o|_K \\
&= \mu_o|_K - \mu_o|_K \\
&= 0,
\end{aligned}$$

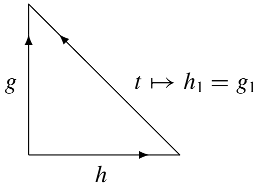
because $\operatorname{supp} h_1 \subset K$ and h_1 preserves measure. □

ASSERTION 4.4. *Let $h, g \in \mathcal{P}(\mathcal{H}_{c,o}(X, \mu_o))$ be homotopic paths relative to ∂I. Then $\Phi(h)$ is homologous to $\Phi(g)$.*

Proof. Let $F\colon I \times I \to \mathcal{H}_{c,o}(X, \mu_o)$ be a homotopy from h to g(rel. ∂I). Diagrammatically, we have



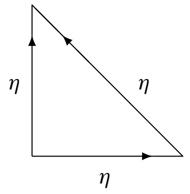
Since F restricted to the edge $\{0\} \times I$ is constant, we can collapse $\{0\} \times I$ into a point and get a map from the standard 2-simplex Δ^2 into $\mathcal{H}_{c,o}(X, \mu_o)$ given by the following diagram on the edges:



This map induces a continuous function from X into the space of 2-singular simplices on X, say $E_2\colon X \to S(\Delta^2, X)$. Let K be a compact subset of X containing $\operatorname{supp} F$. A calculation shows that

$$\begin{aligned}\partial_2(E_2(F)_*(\mu_o|_K)) &= (E(\eta)\circ h_1)_*(\mu_o|_K) - E(g)_*(\mu_o|_K) + E(h)_*(\mu_o|_K)\\ &= \Phi(h) - \Phi(g) + E(\eta)_*(h_{1*}(\mu_o|_K))\\ &= \Phi(h) - \Phi(g) + E(\eta)_*(\mu_o|_K).\end{aligned}$$

By performing the same calculation using the constant map $\triangle^2 \to \{\mathrm{Id}\} \subset \mathcal{H}_{c,o}(X,\mu_o)$, defined also by the diagram



we see that $E(\eta)_*(\mu_o|_K)$ is a boundary itself. □

Let $\widetilde{\mathcal{H}}_{c,o}(X,\mu_o)$ be the quotient space of $\mathcal{P}(\mathcal{H}_{c,o}(X,\mu_o))$ under the equivalence relation of "homotopy relative to ∂I". The continuous map $ev\colon \mathcal{P}(\mathcal{H}_{c,o}(X,\mu_o)) \to \mathcal{H}_{c,o}(X,\mu_o)$ defined by $ev(h) = h_1$ for each $h \in \mathcal{P}(\mathcal{H}_{c,o}(X,\mu_o))$ naturally induces a continuous projection $\rho\colon \widetilde{\mathcal{H}}_{c,o}(X,\mu_o) \to \mathcal{H}_{c,o}(X,\mu_o)$ on equivalence classes. Observe that $\rho^{-1}(\{\mathrm{Id}\})$ is the (underlying set of the) fundamental group of $\mathcal{H}_{c,o}(X,\mu_o)$. When X is a manifold, the compactly generated space of $\mathcal{H}_{c,o}(X,\mu_o)$ is locally path connected and locally contractible, so $\mathcal{H}_{c,o}(X,\mu_o)$ is semilocally simply connected (and of course connected and locally path connected) in the Whitney topology. Hence, $\widetilde{\mathcal{H}}_{c,o}(X,\mu_o)$ is (a model of) the universal covering space of $\mathcal{H}_{c,o}(X,\mu_o)$ in both the compactly generated topology and the Whitney topology [2, Thm. 4.3; 13, Chap. 2].

With these preliminaries we make the following definition.

DEFINITION 4.5. Define the *mass flow*

$$\widetilde{\Theta} = \widetilde{\Theta}_X = \widetilde{\Theta}_{(X,\mu_0)}\colon \widetilde{\mathcal{H}}_{c,o}(X,\mu_o) \to H_1(X,\mathbb{R})$$

by the formula

$$\widetilde{\Theta}([h]) = [E(h)_*(\mu_o|_{\operatorname{supp} h})],$$

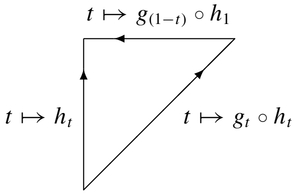
where $[h] \in \widetilde{\mathcal{H}}_{c,o}(X,\mu_o)$ is the homotopy class (relative to ∂I) of the path $h \in \mathcal{P}(\mathcal{H}_{c,o}(X,\mu_o))$ and $[E(h)_*(\mu_o|_{\operatorname{supp} h})]$ is the homology class of $\Phi(h)$ in $H_1(X,\mathbb{R})$.

REMARK 4.6. Since Φ is continuous (Assertion 4.2), it follows that the mass flow $\widetilde{\Theta}$ is continuous in the corresponding quotient topologies.

5. First Consequences

PROPOSITION 5.1. *The mass flow is a group homomorphism.*

Proof. Let $\triangle_1^2$ be the geometric 2-simplex in I^2 having $P_0 = (0,0)$, $P_1 = (1,1)$, and $P_2 = (0,1)$ as vertices. Let $H\colon \triangle_1^2 \to \mathcal{H}_{c,o}(X,\mu_o)$ be defined by the formula $H(s,t) = g_s \circ h_t$ for each $(s,t) \in \triangle_1^2$. Diagrammatically,



Certainly H induces a continuous function $E_2(H)\colon X \to S(\triangle_1^2, X)$.

Identifying $\triangle_1^2$ with $\triangle^2$ and computing $\partial_2(E_2(H)_*(\mu_o|_K))$, where K is any compact set in X containing $\operatorname{supp} h \cup \operatorname{supp} g$, we get that

$$\partial_2(E_2(H)_*(\mu_o|_K)) = (\delta^* \circ E(g) \circ h_1)_*(\mu_o|_K) - E(h)_*(\mu_o|_K) + E(g \circ h)_*(\mu_o|_K),$$

where $\delta^*\colon S(\triangle^1, X) \to S(\triangle^1, X)$ is the map induced by the reflection $t \mapsto (1-t)$ in $\triangle^1$. Hence

$$\partial_2(E_2(H)_*(\mu_o|_K)) = (\delta^*)_*(E(g)_*(\mu_o|_K)) - E(h)_*(\mu_o|_K) + E(g \circ h)_*(\mu_o|_K).$$

Since $(\delta^*)_*$ induces multiplication by -1 in homology, the proposition follows. □

The proof of the following assertion is immediate.

ASSERTION 5.2. *Let f be an arbitrary homeomorphism of X. Then there is a commutative diagram*

$$\begin{array}{ccc} \widetilde{\mathcal{H}}_{c,o}(X,\mu_o) & \xrightarrow{\widetilde{\Theta}_{(X,\mu_o)}} & H_1(X,\mathbb{R}) \\ \big\downarrow {\scriptstyle [f(\cdot)f^{-1}]^{\sim}} & & \big\downarrow {\scriptstyle H_1(f)} \\ \widetilde{\mathcal{H}}_{c,o}(X,f_*\mu_o) & \xrightarrow{\widetilde{\Theta}_{(X,f_*\mu_o)}} & H_1(X,\mathbb{R}) \end{array}$$

where

$$[f(\cdot)f^{-1}]^{\sim}([t \mapsto h_t]) = [t \mapsto f \circ h_t \circ f^{-1}]$$

for each $t \mapsto h_t$ in $\mathcal{P}(\mathcal{H}_{c,o}(X,\mu_o))$.

Both $[f(\cdot)f^{-1}]^{\sim}$ and $H_1(f)$ are isomorphisms of topological groups and linear spaces, respectively. In particular, if f is isotopic to the identity, then $H_1(f)$ is the identity on $H_1(X,\mathbb{R})$.

ASSERTION 5.3. *Let T^1 denote the topological group $\mathbb{R}/\mathbb{Z}$ isomorphic to the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Let $\lambda\colon S(\triangle^1, T^1) \to \mathbb{R}$ be the continuous map defined by $\lambda(\sigma) = [\overline{\sigma - \sigma(0)}](1)$, where $\overline{\sigma - \sigma(0)}\colon \triangle^1 \to \mathbb{R}$ is the lifting of $\sigma - \sigma(0)$ such that $[\overline{\sigma - \sigma(0)}](0) = 0$. Define $D\colon C_1T^1 \to \mathbb{R}$ such that $D(\vartheta) = \int \lambda\, d\vartheta$ for all $\vartheta \in C_1T^1$. Then D induces an isomorphism $\overline{D}$ of $H_1(T^1,\mathbb{R})$ onto $\mathbb{R}$* (see [3]).

PROPOSITION 5.4. *Let $f\colon X \to T^1$ be a continuous map and $h \in \mathcal{P}(\mathcal{H}_{c,o}(X,\mu_o))$. Define $f \circ h - f\colon X \times I \to T^1$ such that $(x,t) \mapsto f(h_t(x)) - f(x)$. Since*

$f(h_0(x)) - f(x) = 0$, there is a unique lifting $\overline{f \circ h - f}\colon X \times I \to \mathbb{R}$ identically zero on $X \times \{0\} \cup (X \times I \setminus \operatorname{supp} h \times I)$. Now, the induced functional $H_1(f)\colon H_1(X,\mathbb{R}) \to H_1(T^1,\mathbb{R})$ is such that

$$\bar{D} \circ H_1(f) \circ \widetilde{\Theta}([h]) = \int_X \overline{f \circ h_1 - f}\, d\mu_o. \tag{F}$$

Proof. Let $S(f)\colon S(\triangle^1, X) \to S(\triangle^1, T^1)$ be such that $\sigma \mapsto f \circ \sigma$ and let $S(f)_*$ be the map induced in measures. Then,

$$\begin{aligned}\bar{D}(H_1(f)(\widetilde{\Theta}([h]))) &= \int_X \lambda\, d(S(f)_* \circ E(h)_* \mu_o)\\ &= \int_X (\lambda \circ S(f) \circ E(h))\, d\mu_o = \int_X \overline{f \circ h_1 - f}\, d\mu_o. \qquad \square\end{aligned}$$

COMMENT 5.5. Formula (F) is exactly the expression Fathi uses in [9] to define the mass flow, in the case where X is compact, as a group homomorphism from $\widetilde{\mathcal{H}}_o(X,\mu_o)$ into $\operatorname{Hom}_{\mathbb{Z}}([X,T^1],\mathbb{R})$.

6. Properties

Let $\operatorname{Ker}\widetilde{\Theta}$ be the kernel of the mass flow homomorphism, and let Π denote the fundamental group of $\mathcal{H}_{c,o}(X,\mu_o)$. Then, the diagram

$$\begin{array}{ccccc} & & \Pi & & \\ & & \downarrow & & \\ \operatorname{Ker}\widetilde{\Theta} & \hookrightarrow & \widetilde{\mathcal{H}}_{c,o}(X,\mu_o) & \xrightarrow{\widetilde{\Theta}} & H_1(X,\mathbb{R}) \\ & & \downarrow{\scriptstyle \rho} & & \\ & & \mathcal{H}_{c,o}(X,\mu_o) & & \end{array}$$

induces a commutative diagram with exact columns

$$\begin{array}{ccccc} \Pi \cap \operatorname{Ker}\widetilde{\Theta} & \hookrightarrow & \Pi & \xrightarrow{\widetilde{\Theta}|_\Pi} & \Gamma_X \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Ker}\widetilde{\Theta} & \hookrightarrow & \widetilde{\mathcal{H}}_{c,o}(X,\mu_o) & \xrightarrow{\widetilde{\Theta}} & H_1(X,\mathbb{R}) \\ \downarrow{\scriptstyle \rho|_{\operatorname{Ker}\widetilde{\Theta}}} & & \downarrow{\scriptstyle \rho} & & \downarrow \\ \operatorname{Ker}\Theta & \hookrightarrow & \mathcal{H}_{c,o}(X,\mu_o) & \xrightarrow{\Theta} & H_1(X,\mathbb{R})/\Gamma_X \end{array} \tag{♠}$$

where Γ_X is defined to be the image of Π under $\widetilde{\Theta}$ and Θ is the homomorphism induced by $\widetilde{\Theta}$ in the quotient. It is easy to see that the equalities $\operatorname{Ker}(\rho|_{\operatorname{Ker}\widetilde{\Theta}}) = \operatorname{Ker}(\widetilde{\Theta}|_\Pi) = \Pi \cap \operatorname{Ker}\widetilde{\Theta}$ hold. Also, the fact that $\widetilde{\Theta}|_\Pi$ is a surjection implies that ρ maps $\operatorname{Ker}\widetilde{\Theta}$ onto $\operatorname{Ker}\Theta$. This shows that the first column in diagram (♠) is exact.

If $\mathcal{H}_{c,o}(X, \mu_o)$ is locally connected and semilocally simply connected, then the map Θ is continuous. This follows immediately from the continuity of $\widetilde{\Theta}$ and the fact that in this case ρ is a covering map.

Let us consider the general group $\mathcal{H}(X)$ of homeomorphisms of X with the compact-open topology and let $\Omega(\mathcal{H}(X)) = \{h \in C(I, \mathcal{H}(X)) \mid h(0) = h(1) = \mathrm{Id}_X\}$ be the *space of loops* of homeomorphisms based on the identity. If x_o is a point in X, then $ev_o \colon \mathcal{H}(X) \to X$ given by $h \mapsto h(x_o)$ induces a group homomorphism $\Pi(ev_o) \colon \Pi(\mathcal{H}(X)) \to \Pi(X, x_o)$ that assigns to each class $[h]$ of a loop h the class $[E(h)x_o]$ of the loop $t \mapsto h_t(x_o)$ in X. Observe that $\Pi(\mathcal{H}(X))$ is abelian (consider the homotopy $(s,t) \mapsto \gamma_1(s) \circ \gamma_2(t)$).

ASSERTION 6.1. *If $h \in \Omega(\mathcal{H}(X))$ then $\Pi(ev_o)([h])$ lies in the group-theoretic center $\mathbb{Z}(\Pi(X, x_o))$ of the fundamental group of X; that is, $\Pi(ev_o)([h])$ commutes with every element in $\Pi(X, x_o)$.*

Proof. Let γ be a loop in X based at x_o. Then $G \colon I \times I \to X$ given by $G(s,t) = h_s(\gamma(t))$, for $(s,t) \in I \times I$, is such that $G(s,0) = E(h)x_o$, $G(s,1) = E(h)x_o$, $G(0,t) = \gamma(t)$, and $G(1,t) = \gamma(t)$. This proves that $\Pi(ev_o)([h])$ commutes with $[\gamma]$ in $\Pi(X, x_o)$. □

REMARK 6.2. $\mathbb{Z}(\Pi(X, x_o))$ is an invariant of X that does not depend on the base point x_o. For if $x_1 \in X$, then any two curves from x_o to x_1, say γ_o and γ_1, define the same isomorphism from $\mathbb{Z}(\Pi(X, x_o))$ onto $\mathbb{Z}(\Pi(X, x_1))$; that is, $[\gamma_o^{-1}\zeta\gamma_o] = [\gamma_1^{-1}\zeta\gamma_1]$ for each loop ζ in X with $[\zeta] \in \mathbb{Z}(\Pi(X, x_o))$. Hence $\mathbb{Z}(\Pi(X))$ is well-defined. If $h \in \Omega(\mathcal{H}(X))$ then these comments and the homotopy $(s,t) \mapsto h_s(\gamma_o(t))$ show that $[E(h)x_o]$ and $[E(h)x_1]$ define the same element in $\mathbb{Z}(\Pi(X))$.

REMARK 6.3. If (Y, y_o) is a topological pointed space and if $f \colon (Y, y_o) \to (\mathcal{H}(X), \mathrm{Id})$ is continuous, then the composite $ev_o \circ f$ induces an homomorphism $\Pi(Y, y_o) \to \mathbb{Z}(\Pi(X))$ such that $[\gamma] \mapsto [E(f \circ \gamma)x_o]$. In particular, we can apply this to the inclusion $\mathcal{H}_{c,o}(X, \mu_o) \hookrightarrow \mathcal{H}(X)$.

If X is not compact, then $\Pi(\mathcal{H}_{c,o}(X, \mu_o)) \to \mathbb{Z}(\Pi(X))$ is trivial; for if $h \in \Omega(\mathcal{H}_{c,o}(X, \mu_o))$ then some point $x \in X$ is fixed, so $E(h)x$ is the constant loop at x. The same is true for the case where X is a manifold with nonempty boundary and we apply the preceding remark to the inclusion $\mathcal{H}^{\partial}_{c,o}(X, \mu_o) \hookrightarrow \mathcal{H}(X)$, where $\mathcal{H}^{\partial}_{c,o}(X, \mu_o)$ denotes the group of measure-preserving homeomorphisms of X fixing ∂X pointwise, with compact support and compactly isotopic to the identity in $\mathcal{H}^{\partial}_{c}(X, \mu_o)$.

PROPOSITION 6.4. *Let X be a connected n-dimensional manifold, possibly with a nonempty boundary, such that $\mu_o(\partial X) = 0$. Let $x_o \in X$ be given. If X is compact, then the following diagram commutes*:

$$\begin{array}{ccc} \Pi(\mathcal{H}_{c,o}(X, \mu_o)) & \xrightarrow{\Pi(ev_o)} & \mathbb{Z}(\Pi(X)) \\ \downarrow & & \downarrow{\scriptstyle Hwz} \\ \widetilde{\mathcal{H}}_{c,o}(X, \mu_o) & \xrightarrow{\widetilde{\Theta}} & H_1(X, \mathbb{R}) \end{array}$$

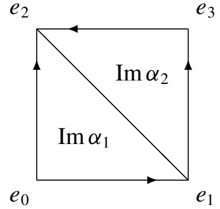
where $\mathrm{Hwz}\colon \Pi(X,x_o) \to H_1(X,\mathbb{R})$ *is the map that sends the homotopy class of a loop* γ *on* X *at* x_o *into the homology class of the atomic measure on* $S(\triangle^1, X)$ *concentrated at* $\{\gamma\}$ *with mass* $\mu_o(X)$.

If X *is noncompact or* $\partial X \neq \emptyset$ *then, for any loop* h *on* $\mathcal{H}_{c,o}(X,\mu_o)$ *based at the identity,* $\widetilde{\Theta}([h])$ *is zero.*

Proof. Let h be a loop on $\mathcal{H}_{c,o}(X,\mu_o)$ based at the identity. Now there is a continuous function on the unit n-cube, say $\varphi\colon I^n \to X$ and a measure ν_o on I^n such that $\operatorname{supp} h \subset \varphi(I^n)$ and $\varphi_*\nu_o = \mu_o|_{\varphi(I^n)}$ (see [4, Prop. 1 and Lemma 7]). Assume that φ maps the origin $\bar{0} \in \mathbb{R}^n$ to x_o.

Let $c\colon I^n \times I \to I^n$ be the contraction of I^n defined by $c(x,t) = (1-t)x$ for all $(x,t) \in I^n \times I$. Let $H\colon I \times I \times I^n \to X$ be defined by $H(s,t)x = h_s(\varphi(c(x,t)))$ for all $(s,t) \in I \times I$ and $x \in I^n$. Then $H(s,0)x = h_s(\varphi(x)) = E(h)\varphi(x)$, $H(s,1)x = h_s(x_o) = E(h)x_o$, and $H(0,t)x = H(1,t)x = \varphi(c(x,t)) = \gamma_x(t)$ for all $(s,t) \in I \times I$ and $x \in I^n$.

Define the affine simplices $\alpha_j\colon \triangle^2 \to I \times I$ $(j = 1,2)$, where $\alpha_1 = \langle e_0, e_1, e_2\rangle$, $\alpha_2 = \langle e_1, e_3, e_2\rangle$, and e_0, e_1, e_2, e_3 are the vertices of the unit square.



For $j = 1,2$, define $E_2(H \circ \alpha_j)\colon I^n \to S(\triangle^2, X)$ such that $x \mapsto ((s,t) \mapsto H(\alpha_j(s,t))x)$. Let $\delta\colon \triangle^1 \to \triangle^1$ be such that $\delta(t) = 1 - t$ for each $t \in \triangle^1$. Then

$$\begin{aligned}
\partial_2(E_2(H \circ \alpha_1)_*\nu_o &+ E_2(H \circ \alpha_2)_*\nu_o)\\
&= (E(h) \circ \varphi)_*\nu_o + (\delta^* \circ (x \mapsto E(h)x_o))_*\nu_o\\
&= E(h)_*(\varphi_*\nu_o) + (\delta^*)_*(\nu_o(I^n)\lfloor E(h)x_o\rfloor)\\
&= E(h)_*(\mu_o|_{\varphi(I^n)}) + (\delta^*)_*(\nu_o(I^n)\lfloor E(h)x_o\rfloor),
\end{aligned}$$

where $\lfloor E(h)x_o\rfloor$ represents the atomic probability supported at $\{E(h)x_o\}$. Hence $\widetilde{\Theta}([h]) = \mu_o(\varphi(I^n))[\lfloor E(h)x_o\rfloor]$.

In particular, if $\operatorname{supp} h \neq X$ then $E(h)x\colon s \mapsto h_s(x)$ $(0 \leq j \leq 1)$ is a constant loop for some $x \in X$. Therefore $E(h)x_o$ is homologically trivial because it is homotopic to $E(h)x$. This is always the case when X is noncompact or we are in the case when homeomorphisms are to fix ∂X pointwise. If X is compact and $\operatorname{supp} h = X = \varphi(I^n)$ then $\widetilde{\Theta}([h]) = \mu_o(X))[\lfloor E(h)x_o\rfloor]$. ☐

ASSERTION 6.5. *Let* X *be a connected manifold. The map* $\rho|_{\operatorname{Ker}\widetilde{\Theta}}\colon \operatorname{Ker}\widetilde{\Theta} \to \operatorname{Ker}\Theta$, *defined on diagram* (♠), *is a covering projection.*

Proof. We consider two cases.

Case 1: X is noncompact. In this case Γ_X is trivial, so $\operatorname{Ker}\widetilde{\Theta}$ is the full inverse image of $\operatorname{Ker}\Theta$ under ρ.

Case 2: X is compact. Assume, without loss of generality, that $\mu_o(X) = 1$. Therefore Γ_X is contained in the integral part of the finite-dimensional space $H_1(X, \mathbb{R})$ (a compact metric ANR is of the same homotopy type of a compact polyhedron and hence it has a finitely generated integral homology; see [11]). Hence Γ_X is discrete in the Euclidean topology of $H_1(X, \mathbb{R})$.

Since the mass flow homomorphism is continuous, choose a neighborhood $\mathfrak{A}$ of the identity in $\widetilde{\mathcal{H}}_{c,o}(X, \mu_o)$ with $\widetilde{\Theta}(\mathfrak{A}) \cap \Gamma_X = \{0\}$. It follows that $\rho(\operatorname{Ker} \widetilde{\Theta} \cap \mathfrak{A}) = \operatorname{Ker} \Theta \cap \rho(\mathfrak{A})$, proving that $\rho|_{\operatorname{Ker} \widetilde{\Theta}} \colon \operatorname{Ker} \widetilde{\Theta} \to \operatorname{Ker} \Theta$ is an open map. Hence $\operatorname{Ker} \Theta$ is the quotient space obtained from $\operatorname{Ker} \widetilde{\Theta}$ under the action of the discrete group $\Pi(\mathcal{H}_{c,o}(X, \mu_o)) \cap \operatorname{Ker} \widetilde{\Theta}$. Thus $\rho|_{\operatorname{Ker} \widetilde{\Theta}}$ is a covering projection. □

7. The Kernel of the Mass Flow

DISCUSSION 7.1. The following discussion is taken from Fathi [9, pp. 73–74]. Let A and B be locally connected, second-countable, Hausdorff spaces. Assume further that A is compact and that B is locally compact (Fathi needs B compact only).

Now let $\phi \colon (A \times \{0\} \cup A \times \{1\}) \to B$ be some embedding. Define $\mathcal{W} = A \times I \sqcup_\phi B = (A \times I \cup B)/\{\phi(a,0) \sim (a,0), \phi(a,1) \sim (a,1)\}$ by *glueing* $A \times I$ *to* B *using* ϕ. Define $f \colon \mathcal{W} \to T^1$ by $f(x) = t \bmod 1$ if $x = (a,t) \in A \times I$ and by $f(x) = 0$ if $x \in B$. Consider the pull-back of the natural covering $\mathbb{R} \to T^1$ by f to obtain a covering projection $\widetilde{\mathcal{W}} \to \mathcal{W}$.

Suppose that μ is a measure on $\mathcal{W}$ such that $\mu|_{A\times I} = \nu \times dt$, where ν is a good measure on A and dt is Lebesgue measure on I. Lift μ to a measure $\tilde{\mu}$ in $\widetilde{\mathcal{W}}$. Let h be a measure-preserving, compactly supported isotopy in $\mathcal{P}(\mathcal{H}_{c,o}(\mathcal{W}, \mu_o))$. Then we can lift h in a unique way to $\tilde{h} \in \mathcal{P}(\mathcal{H})_{c,o}(\widetilde{\mathcal{W}}, \mu_o)$. Observe that $\operatorname{supp} \tilde{h}$ is not compact if $h_t \neq \operatorname{Id}$ for some $t \in I$. Moreover, $\tilde{h}$ commutes with the covering transformations of $\widetilde{\mathcal{W}} \to \mathcal{W}$. Suppose that h is close enough to the identity; then we can define a region $\mathcal{R}(h) \subset \widetilde{\mathcal{W}}$ that consists of the points "to the right" of $A \times \{0\}$ and $\tilde{h}_1(A \times \{1/2\})$.

Now, if $H_1(T^1, \mathbb{R})$ is identified with $\mathbb{R}$ via the isomorphism found in Assertion 5.3, then

$$H_1(f)(\widetilde{\Theta}([h])) = \tilde{\mu}(\mathcal{R}(h)) - \tilde{\mu}(A \times [0, 1/2]).$$

We omit the proof of this fact, which certainly explains the name of the mass flow homomorphism. The figures in Fathi [9, pp. 73–74] illustrate the preceding formula.

REMARK 7.2. Let a, b, c, d be real numbers such that $0 < c < a < b < d < 1$. Suppose further that A is a connected manifold. If we define $\mathcal{C}_0(h)$ as the connected component of $A \times [0,1] \setminus (h_1(A \times [a,b])$ that contains $A \times \{0\}$, then the foregoing discussion shows that $H_1(f)(\widetilde{\Theta}([h])) = \mu(\mathcal{C}_0(h)) - \mu(A \times [0,a])$.

In particular, if the space $\mathcal{W} = (A \times I) \sqcup_\phi B$ is such that $\phi(A \times \{0\})$ and $\phi(A \times \{1\})$ lie on different connected components of B, then $f \colon \mathcal{W} \to T^1$ is homotopically trivial. Hence we are just asserting that $\mu(\mathcal{C}_0(h)) = \mu(A \times [0,a])$.

EXAMPLE 7.3. Suppose that σ is an embedded circle in X representing a nontrivial element of $H_1(X,\mathbb{R})$ and suppose that $N \subset X$ is a compact tubular neighborhood of σ. Using the previous discussion, Fathi shows how to circulate mass inside N in order to produce homeomorphisms with nonzero mass flow along σ. An important consequence of this construction is the surjectivity of the mass flow homomorphism when $H_1(X,\mathbb{R})$ has a basis represented by embedded curves having tubular neighborhoods. In Proposition 7.5 we prove that the mass flow is onto without the aid of tubular neighborhoods. The present example illustrates that, in a particular setting, $\widetilde{\Theta}$ is not trivial.

Let $A = I^{n-1}$ and let B be an n-dimensional connected manifold with boundary. Let $\phi\colon A \times \{0,1\} \to \partial B$ be an embedding that extends to an open embedding of $\mathbb{R}^{n-1} \times \{0,1\}$ into ∂B. Then $\mathcal{W} = A \times I \sqcup_\phi B$ is a manifold with boundary $\partial\mathcal{W} = \partial A \times I \cup (\partial B \setminus \phi((\operatorname{Int} A) \times \{0,1\}))$. By definition, $\mathcal{W}$ is the result of *adding an n-handle of index 1 to B*.

Suppose that μ is a measure in $\mathcal{M}_g^\partial(\mathcal{W})$ such that $\mu|_{A\times I} = (\alpha_o m) \times dt$, where m is Lebesgue measure on A and $\alpha_o > 0$. Let $f\colon \mathcal{W} \to T^1$ be given by $f(x) = u \bmod 1$ if $x = (a,u) \in A \times I$ and by $f(x) = 0$ if $x \in B$. It is not difficult to see that $H_1(\mathcal{W},\mathbb{R}) \cong H_1(B,\mathbb{R}) \oplus \mathbb{R}$ by the Mayer–Vietoris theorem and that the linear map $H_1(f)\colon H_1(\mathcal{W},\mathbb{R}) \to H_1(T^1,\mathbb{R})$ may be interpreted as the projection of $H_1(B,\mathbb{R}) \oplus \mathbb{R}$ onto its second factor. Choose some continuous function $\delta\colon A \to [0,1/3]$ such that $\alpha_o \int_A \delta\, dm = \alpha_o/4$ and $\delta|_{\partial A} = 0$. Define, for each $t \in I$, the embedding $A \times [1/4,1/2] \hookrightarrow \mathcal{W}$, $(a,u) \mapsto (a, u + t\delta(a))$. Then Theorem 3.7 on the extension of isotopies says that we can find a path $h\colon I \to \mathcal{H}_{c,o}^\partial(\mathcal{W},\mu)$ such that $h_t(a,u) = (a, u + t\delta(a))$ for each $(a,u) \in A \times [1/4,1/2]$. By Remark 7.2, it follows that $\widetilde{\Theta}_\mathcal{W}([h]) = (\beta, \alpha_o/4)$ in $H_1(B,\mathbb{R}) \oplus \mathbb{R}$. Furthermore, by adding an extra parameter $s \in [0,\alpha_o/4]$, we can construct a continuous map $\gamma\colon [0,\alpha_o/4] \to \widetilde{\mathcal{H}}_{c,o}^\partial(\mathcal{W},\mu)$ (e.g., $\gamma(s)_t = [h_{4st/\alpha_o}]$ for each $s \in [0,\alpha_o/4]$ and each $t \in I$) with the property that, for each $s \in [0,\alpha_o/4]$, $\widetilde{\Theta}_\mathcal{W}(\gamma(s)) = (\beta_s, s)$ in $H_1(B,\mathbb{R}) \oplus \mathbb{R}$. Finally, extend γ to $\hat{\gamma}\colon \mathbb{R} \to \widetilde{\mathcal{H}}_{c,o}^\partial(\mathcal{W},\mu)$ satisfying $\widetilde{\Theta}_\mathcal{W}(\hat{\gamma}(s)) = (\beta_s, s)$ for each $s \in \mathbb{R}$.

DEFINITION 7.4. Let W be a real vector space. Then, the ℓ-topology on W is the topology that makes the inclusions $\iota\colon E \hookrightarrow W$ continuous for each finite-dimensional linear subspace $E \subset W$ that is endowed with its standard Euclidean topology.

If the dimension of W is at most countable, then W_ℓ is a locally convex vector space. Otherwise the (affine) space W_ℓ is not a topological vector space (see [6]).

Recall that $\widetilde{\Theta}\colon \widetilde{\mathcal{H}}_{c,o}(X,\mu_o) \to H_1(X,\mathbb{R})$ is a continuous group homomorphism and that $H_1(X,\mathbb{R})$ is Hausdorff [3]. Consequently, finite-dimensional subspaces in $H_1(X,\mathbb{R})$ inherit their natural topology making the inclusion $H_1(X,\mathbb{R})_\ell \hookrightarrow H_1(X,\mathbb{R})$ continuous.

Since $\rho\colon \widetilde{\mathcal{H}}_{c,o}(X,\mu_o) \to \mathcal{H}_{c,o}(X,\mu_o)$ is a covering map it follows that, for $\mathcal{K} \subset \widetilde{\mathcal{H}}_{c,o}(X,\mu_o)$ compact, there is a compact $K \subset X$ such that $\widetilde{\Theta}(\mathcal{K})$ is contained in the (finite-dimensional) image of the canonical map $H_1(K,\mathbb{R}) \to H_1(X,\mathbb{R})$.

Therefore, $\widetilde{\Theta}\colon \mathsf{k}(\widetilde{\mathcal{H}}_{c,o}(X,\mu_o)) \to H_1(X,\mathbb{R})_\ell$ is continuous ($\mathsf{k}S$ is the compactly generated space associated with S).

PROPOSITION 7.5. *Let X be an n-manifold. Then* $\widetilde{\Theta}\colon \mathsf{k}(\widetilde{\mathcal{H}}_{c,o}(X,\mu_o)) \to H_1(X,\mathbb{R})_\ell$ *is surjective. Moreover,* $\widetilde{\Theta}$ *has a continuous section.*

Proof. Let z be a nontrivial element in $H_1(X,\mathbb{R})$ represented by an embedding $\sigma\colon T^1 \hookrightarrow X \setminus \partial X$. By covering the image of σ with n-cells, we can construct a neighborhood $\mathcal{Z}$ of $\sigma(T^1)$ in the interior of X such that $\mathcal{Z}$ can be expressed as the result of adding an n-handle of index 1 to a submanifold of dimension n in the interior of X. Let ψ be a homeomorphism from $\mathcal{Z}$ onto some $\mathcal{W} = A \times I \sqcup_\phi B$.

By the von Neumann–Oxtoby–Ulam theorem on manifolds (see [4]), assume that $\psi_*(\mu_o|_{\mathcal{Z}}) = \mu$ where μ is the measure in $\mathcal{M}_g^\partial(\mathcal{W})$ considered in Example 7.3. Since T^1 is an ANR, we can take $\mathcal{Z}$ small enough so that the canonical map $H_1(B,\mathbb{R}) \to H_1(X,\mathbb{R})$ is trivial. Using Example 7.3 and the commutative diagram

$$\begin{array}{ccccc}
\widetilde{\mathcal{H}}^\partial_{c,o}(\mathcal{Z},\mu_o|_{\mathcal{Z}}) & \xrightarrow{\widetilde{\Theta}_{\mathcal{Z}}} & H_1(\mathcal{Z},\mathbb{R}) & = & H_1(B,\mathbb{R}) \oplus \mathbb{R}z \\
\downarrow & & \downarrow & & \downarrow \pi_2 \\
\widetilde{\mathcal{H}}^\partial_{c,o}(X,\mu_o) & \xrightarrow{\widetilde{\Theta}_X} & H_1(X,\mathbb{R}) & \hookleftarrow & \mathbb{R}z
\end{array}$$

where π_2 is the projection on $\mathbb{R}z$, we can find a continuous mapping $\gamma_\sigma\colon \mathbb{R}z \to \mathsf{k}(\widetilde{\mathcal{H}}^\partial_{c,o}(X,\mu_o))$ such that $\widetilde{\Theta}_X \circ \gamma_\sigma$ is the identity on $\mathbb{R}z$. Now choose an at most countable family $\{\sigma\colon T^1 \hookrightarrow X\}$ of embeddings representing a base for $H_1(X,\mathbb{R})$.

Note that any element in $H_1(X,\mathbb{R})$ is a finite linear combination of basic vectors. Hence, we can multiply the corresponding family of sections $\{\gamma_\sigma\colon \mathbb{R}z \to \mathsf{k}(\widetilde{\mathcal{H}}^\partial_{c,o}(X,\mu_o))\}$ in some fixed order, for $\gamma_{\sigma_1} \circ \gamma_{\sigma_2} \neq \gamma_{\sigma_2} \circ \gamma_{\sigma_1}$ in general, to obtain a continuous section $\gamma\colon H_1(X,\mathbb{R})_\ell \to \mathsf{k}(\widetilde{\mathcal{H}}^\partial_{c,o}(X,\mu_o))$. □

REMARK 7.6. If the dimension of X is greater than two, then there is a locally finite family $\{\sigma\colon T^1 \hookrightarrow X\}$ of continuous disjoint embeddings representing a base for $H_1(X,\mathbb{R})$. Suppose now that each $\sigma\colon T^1 \hookrightarrow X$ has a tubular neighborhood. Then, in this situation, a section that at the same time is a group homomorphism can be constructed.

COROLLARY 7.7. *There is a homeomorphism*

$$\mathsf{k}(\widetilde{\mathcal{H}}_{c,o}(X,\mu_o)) \cong \mathsf{k}(\operatorname{Ker}\widetilde{\Theta}) \times H_1(X,\mathbb{R})_\ell.$$

Furthermore, $\mathsf{k}(\operatorname{Ker}\widetilde{\Theta}) \to \mathsf{k}(\operatorname{Ker}\Theta)$ *is a universal covering projection of a connected, locally contractible space.*

Proof. First note that the subspace topology of (the underlying set of) $\operatorname{Ker}\widetilde{\Theta}$ in $\mathsf{k}(\widetilde{\mathcal{H}}_{c,o}(X,\mu_o))$ is the topology of the k-space $\mathsf{k}(\operatorname{Ker}\widetilde{\Theta})$ [2]. Now, the existence of the stated homeomorphism is a consequence of the existence of a continuous section. The fact that $\mathsf{k}(\mathcal{H}_{c,o}(X,\mu_o))$ is connected and locally contractible (see [2]) implies that its universal covering has these same properties. From Assertion 6.5 and the fact that $H_1(X,\mathbb{R})_\ell$ is contractible, the second statement of the corollary follows. □

REMARK 7.8. Proposition 7.5 and its corollary hold for the group of homeomorphisms fixing ∂X pointwise.

LEMMA 7.9. *Let ν_o be a ∂-good measure in I^n. Then $\mathcal{H}^\partial(I^n,\nu_o)$ is contractible.*

Proof. The lemma is a consequence of "Alexander's trick" and the von Neumann–Oxtoby–Ulam theorem (see [9]). □

LEMMA 7.10. *Let X be an n-manifold. Let K be a closed n-cell in X. Suppose $h \in \mathcal{H}(X,\mu_o)$ is isotopic to the identity by a μ_o-preserving isotopy supported in K. Then $\Theta(h)=0$. If $h \in \mathcal{H}(X,\mu_o)$ has its support in* Int K, *then h is isotopic to the identity in* Int K *and $\Theta(h)=0$.*

Proof. Any homeomorphism in $\mathcal{H}(K,\mu_o|_K)$ that fixes Fr K pointwise can be extended to a homeomorphism in $\mathcal{H}(X,\mu_o)$ by defining such an extension to be the identity out of K (note that Fr K may be different from ∂K). Let $\mathcal{H}^{\mathrm{Fr}}(K,\mu_o|_K)$ denote the group of homeomorphisms of K preserving $\mu_o|_K$ and fixing Fr K pointwise endowed with the compact-open topology. Let $\mathcal{H}_o^{\mathrm{Fr}}(K,\mu_o|_K)$ be the path-connected component of the identity. There is a commutative diagram

$$\begin{array}{ccc} \mathcal{H}_o^{\mathrm{Fr}}(K,\mu_o|_K) & \xrightarrow{\Theta} & H_1(K,\mathbb{R}) \equiv 0 \\ \downarrow & & \downarrow \\ \mathcal{H}_{c,o}(X,\mu_o) & \xrightarrow{\Theta} & H_1(X,\mathbb{R})/\Gamma_X \end{array}$$

This implies the first part of the lemma. The rest is a consequence of Lemma 7.9. □

PROPOSITION 7.11. *Let X be an n-manifold. If $h \in \operatorname{Ker}\Theta_X$, then h can be written as a composition $h = h_1 \circ h_2 \circ \cdots \circ h_q$ such that, for each i, h_i is in $\mathcal{H}_{c,o}(X,\mu_o)$ with* supp h_i *contained in an n-cell.*

We first prove a lemma.

LEMMA 7.12. *Let N be an n-dimensional manifold and let S be a compact, connected, bicollared $(n-1)$-submanifold of N. Let μ be a ∂-good measure on N such that*

(1) *$\mu|_{(S\times\langle -1,2\rangle)} = \nu \times dt$, where $S \times \langle -1,2\rangle$ is a bicollar of $S = S \times \{1/2\}$, $\nu \in \mathcal{M}_g^\partial(S)$, and dt is Lebesgue measure on $\langle -1,2\rangle$.*

Let $h \in \mathcal{P}(\mathcal{H}_{c,o}(N,\mu))$ satisfy

(2) *$\Theta_N(h_t) = 0$ for each $t \in I$ and*

(3) *$h_t(S \times [1/4,3/4]) \subset S \times \langle 0,1\rangle$ for each $t \in I$.*

Then there is a b in $\mathcal{P}(\mathcal{H}_{c,o}(N,\mu))$ such that

(4) $\operatorname{supp} b \subset S \times \langle 0,1\rangle$ *and*
(5) $b_t|_{(S\times[1/4,3/4])} = h_t|_{(S\times[1/4,3/4])}$ *for each* $t \in I$.

Furthermore, if D *is a closed set in* ∂N *and* W *is a neighborhood of* D *in* ∂N *such that* $h_t|_W = \mathrm{Id}_W$ *for each* $t \in I$, *then* b *can be chosen in such a way that* $b_t|_D = \mathrm{Id}_D$ *for each* $t \in I$.

Proof. Define, for each $t \in I$, $\mathcal{C}_{0,t}$ (respectively $\mathcal{C}_{1,t}$) as the connected component of $S \times [0,1] \setminus h_t(S \times [1/4, 3/4])$ that contains $S \times \{0\}$ (respectively $S \times \{1\}$). By Theorem 3.7 and Remark 3.8, it is enough to prove that, for each $i = 1, 2$, $\mu(\mathcal{C}_{i,t}) = \mu(\mathcal{C}_{i,0})$ if $t \in I$. But now this follows from Remark 7.2 and the assumption that h is a path in $\operatorname{Ker}\Theta$. □

Proof of Proposition 7.11. Let $\mathbb{E}^n$ denote either $\mathbb{R}^n$ or the upper half-space $\mathbb{H}^n = \mathbb{R}^{n-1} \times [0,\infty\rangle$, and let $B_0(r)$ denote the standard Euclidean closed ball of radius r centered at the origin.

Let U be an open subset of X with compact closure satisfying $h \in \mathcal{H}_{c,o}(U, \mu_o|_U)$ and $\Theta_U(h) = 0$. Since $\operatorname{Cl} U$ is compact, we can find a finite number of coordinate charts, say $\phi_j : \mathbb{E}^n \hookrightarrow X$ $(1 \le j \le l)$, such that $\bigcup\{\phi_j(B_0(1/2) \cap \mathbb{E}^n) \mid 1 \le j \le l\}$ is an open set containing $\operatorname{Cl} U$. By induction on the covering number l of charts it is enough to prove that, for a given chart $\phi\colon \mathbb{E}^n \hookrightarrow X$, any sufficiently small $h \in \operatorname{Ker}\Theta_U$ can be written as a product $g \circ f$ such that, for $rA = \phi(B_0(r) \cap \mathbb{E}^n)$:

(1) $g \in \mathcal{H}_{c,o}(\operatorname{Int} 2A, \mu_o|_{\operatorname{Int} 2A})$;
(2) $f \in \mathcal{H}_{c,o}(U \setminus (1/2)A, \mu_o|_{U\setminus(1/2)A})$;
(3) $f \in \operatorname{Ker}\Theta_{U\setminus(1/2)A}$.

Using the differentiable structure that $\phi\colon \mathbb{E}^n \hookrightarrow X$ imposes on $\phi(\mathbb{E}^n)$, we can construct an open subset N of X such that

(4) $\operatorname{Cl} N \subset U$,
(5) $h \in \mathcal{H}_{c,o}(N, \mu_o|_N)$,
(6) $h \in \operatorname{Ker}\Theta_N$, and
(7) $\operatorname{Cl}(N \cap \operatorname{Fr} A)$ is an $(n-1)$-dimensional manifold such that there is an embedding
$$\operatorname{Cl}(N \cap \operatorname{Fr} A) \times [-1,2] \hookrightarrow \operatorname{Cl}(N \cap 2A) \setminus (1/2)A$$
that satisfies the following property: $(N \cap \operatorname{Fr} A) \times [-1,2]$ is mapped into N in such a way that $(N \cap \operatorname{Fr} A) \times \langle -1,2\rangle \hookrightarrow N$ is a bicollar of $(N \cap \operatorname{Fr} A) \times \{1/2\} = N \cap \operatorname{Fr} A$ in N, and $(N \cap \operatorname{Fr} A) \times \{0\} \subset A$ (so the bicollar goes from A to the outside).

In other words, we have engineered U into N in order to canalize the flow of mass in and out of $\operatorname{Fr} A$ through channels with reasonable embankments. Assume, without loss of generality, that $\mu_o|_{\operatorname{Cl}(N\cap\operatorname{Fr} A)\times[-1,2]} = \nu \times dt$, where ν is a good measure in $\operatorname{Cl}(N \cap \operatorname{Fr} A)$ and dt is Lebesgue measure on $[-1,2]$.

By Corollary 7.7, $\operatorname{Ker}\Theta_N$ is connected and hence is generated, as a group, by any neighborhood of the identity. Expressing h as a product of small homeomorphisms and replacing it by one of its factors, we can suppose further that $h((N \cap \operatorname{Fr} A) \times [1/4, 3/4]) \subset (N \cap \operatorname{Fr} A) \times \langle 0,1\rangle$. Now define the sets

$$Y_1 = (N \cap A) \cup (N \cap \operatorname{Fr} A) \times [1/2, 1\rangle,$$
$$Y_2 = (N \setminus A) \cup (N \cap \operatorname{Fr} A) \times \langle 0, 1/2];$$
$$L = Y_1 \cap Y_2 = (N \cap \operatorname{Fr} A) \times \langle 0, 1\rangle.$$

Applying Lemma 7.12 to each component of $\operatorname{Cl}(N \cap \operatorname{Fr} A)$ and using a small isotopy of h in $\operatorname{Ker} \Theta_N$, we can find an homeomorphism b of X in $\mathcal{H}_{c,o}(L, \mu_o|_L)$ such that b agrees with h on $(N \cap \operatorname{Fr} A) \times [1/4, 3/4]$. Since the composite $b^{-1} \circ h$ is the identity on $(N \cap \operatorname{Fr} A) \times [1/4, 3/4]$, we can find two homeomorphisms, say g'' and f', such that

(8) $b^{-1} \circ h = g'' \circ f'$,
(9) $g'' \in \mathcal{H}_{c,o}(Y_1, \mu_o|_{Y_1})$, and
(10) $f' \in \mathcal{H}_{c,o}(Y_2, \mu_o|_{Y_2})$.

Let $g' = b \circ g''$ and observe that, although h is equal to the product $g' \circ f'$ and conditions (1) and (2) are satisfied, it may well happen that $\Theta_{Y_2}(f')$ is not zero. To remedy this situation, we want to perturb f' and g'. For this purpose consider the commutative diagram

$$\begin{array}{ccccc}
\widetilde{\mathcal{H}}^{\partial}_{c,o}(L, \mu_o|_L) & \xrightarrow{\Psi_1} & \widetilde{\mathcal{H}}_{c,o}(Y_1, \mu_o|_{Y_1}) \times \widetilde{\mathcal{H}}_{c,o}(Y_2, \mu_o|_{Y_2}) & \xrightarrow{\Phi_1} & \widetilde{\mathcal{H}}_{c,o}(N, \mu_o|_N) \\
\downarrow \widetilde{\Theta}_L & & \downarrow \widetilde{\Theta}_{Y_1} \times \widetilde{\Theta}_{Y_2} & & \downarrow \widetilde{\Theta}_N \\
H_1(L, \mathbb{R}) & \xrightarrow{\Psi_2} & H_1(Y_1, \mathbb{R}) \oplus H_1(Y_2, \mathbb{R}) & \xrightarrow{\Phi_2} & H_1(N, \mathbb{R})
\end{array}$$

where $\Psi_1(P) = (P, P^{-1})$, $\Phi_1(G, F) = G \circ F$, $\Psi_2(z) = (z, -z)$, and $\Phi_2(v, w) = v + w$. The Mayer–Vietoris theorem implies that $\operatorname{Image} \Psi_2 = \operatorname{Ker} \Phi_2$.

Suppose now that $G \in \widetilde{\mathcal{H}}_{c,o}(Y_1, \mu_o|_{Y_1})$ and $F \in \widetilde{\mathcal{H}}_{c,o}(Y_2, \mu_o|_{Y_2})$ are given such that $\widetilde{\Theta}_N(G \circ F) = 0$. Then, by the exactness of the Mayer–Vietoris sequence and the fact that $\widetilde{\Theta}_L$ is surjective, we can find a $P \in \widetilde{\mathcal{H}}^{\partial}_{c,o}(L, \mu_o|_L)$ with $\Psi_2(\widetilde{\Theta}_L(P)) = (\widetilde{\Theta}_{Y_1}(G), \widetilde{\Theta}_{Y_2}(F))$. Therefore $P \circ F \in \operatorname{Ker} \widetilde{\Theta}_{Y_2}$ and $(G \circ P^{-1}) \circ (P \circ F) = G \circ F$. □

DEFINITION 7.13. Let $\mathcal{A}$ be an open covering of the manifold X. A homeomorphism of X is said to be $\mathcal{A}$-*small* if its support is contained in some element of $\mathcal{A}$.

REMARK 7.14. Let $\mathcal{A}$ be an open covering of X. By using balls of small diameter in the proof of Proposition 7.11, we can add in its statement that each h_i in the decomposition of h is $\mathcal{A}$-small. Collecting the results in Propositions 7.5 and 7.11, we can state the following theorem.

THEOREM 7.15. *Let X be a second-countable connected n-manifold, let μ_o be a ∂-good measure in X, and let $\mathcal{A}$ be an open cover of X. Then the map Θ: $\mathcal{H}_{c,o}(X, \mu_o) \to H_1(X, \mathbb{R})/\Gamma_X$ is surjective. Moreover, the kernel of Θ is generated as a group by its $\mathcal{A}$-small μ_o-preserving elements supported in n-cells.*

REMARK 7.16. The same result holds for $\Theta\colon \mathcal{H}^{\partial}_{c,o}(X, \mu_o) \to H_1(X, \mathbb{R})$.

THEOREM 7.17. *Let X be an n-dimensional connected manifold without boundary such that $n \geq 3$. Then the kernel of the mass flow homomorphism is a simple* *group. Furthermore, it is the smallest nontrivial normal subgroup of* $\mathcal{H}_{c,o}(X, \mu_o)$ *and it is equal to the commutator subgroup* $\mathcal{H}_{c,o}(X, \mu_o)$.

Theorem 7.17 is a consequence of the following three results.

ASSERTION 7.18. *Let* Y *be a* k*-dimensional connected manifold with* $k \geq 2$, *let* $\nu_o \in \mathcal{M}_g^\partial(Y)$, *and let* x, y *be two points in* $Y \setminus \partial Y$. *Then there exists a homeomorphism in* $\mathcal{H}_{c,o}^\partial(Y, \mu_o)$ *supported in a topological ball and sending* x *into* y.

Proof (cf. [12, p. 895]). Let B be a ball in Y such that $\{x, y\} \in \operatorname{Int} B$. Assume, without loss of generality, that $B = I^k$ and $\nu_o|_B$ is a multiple of Lebesgue measure. By sliding a small cube centered at x onto a small cube centered at y and then applying Theorem 3.7 (extension of isotopies) we get the desired homeomorphism. □

In particular, $\operatorname{Ker}\Theta$ acts transitively on X.

ASSERTION 7.19. *Let* m *be Lebesgue measure on* $\operatorname{Int} I^n$ $(n \geq 3)$. *Then the group* $\mathcal{H}_c(\operatorname{Int} I^n, m)$ *is perfect.*

REMARK 7.20. This nontrivial result is [9, Thm. 7.4]. Assertion 7.19, the von Neumann–Oxtoby–Ulam theorem (see [12]), and Theorem 7.15 imply that $\operatorname{Ker}\Theta$ is perfect. Note that it is only here where the restriction $n \geq 3$ is needed.

ASSERTION 7.21. *Let* $\mathcal{K}, \mathcal{N}$ *be two subgroups of the full group of homeomorphisms* $\mathcal{H}(X)$ *of* X *satisfying the following properties.*

(1) $\mathcal{N}$ *is not trivial.*
(2) $\mathcal{K}$ *acts on* $\mathcal{N}$ *by conjugation* (*i.e.,* $\mathcal{K}$ *is contained in the normalizer of* $\mathcal{N}$ *in* $\mathcal{H}(X)$).
(3) $\mathcal{K}$ *acts transitively on* X.
(4) *For any open covering* $\mathcal{A}$ *of* X, $\mathcal{K}$ *is generated by its* $\mathcal{A}$*-small elements.*

Then, the commutator subgroup of $\mathcal{K}$ *is contained in* $\mathcal{N}$. *In particular, if* $\mathcal{K}$ *is perfect, then it is simple.*

REMARK 7.22. This is just a restatement of a well-known argument due to Epstein [8].

Proof of Assertion 7.2.1 (cf. [9, p. 90]). Let f_0 be a nontrivial element in $\mathcal{N}$. Then we can find a nonempty open subset $V_0 \subset X$ such that

(5) $V_0 \cap f_0(V_0) = \emptyset$.

Let $\mathcal{V} = \{b(V_0) \mid b \in \mathcal{K}\}$. By (3), $\mathcal{V}$ is a covering of X. Let $\mathcal{A}$ be an open baricentric refinement of $\mathcal{V}$. Hence, if $U_1, U_2 \in \mathcal{A}$ and $U_1 \cap U_2 \neq \emptyset$, then $U_1 \cup U_2 \subset V$ for some $V \in \mathcal{V}$. By (4) it is enough to show that, if $h, g \in \mathcal{K}$ are $\mathcal{A}$-small, then the commutator $[h, g] = hgh^{-1}g^{-1}$ belongs to $\mathcal{N}$. Let $U_1, U_2 \in \mathcal{A}$ be such that $\operatorname{supp} h \subset U_1$ and $\operatorname{supp} g \subset U_2$. If $U_1 \cap U_2 = \emptyset$, then $[h, g] = \mathrm{Id}$ and we are done. Otherwise let $b \in \mathcal{K}$ with $U_1 \cup U_2 \subset b(V_0) = V$. By (2) and (5), we have that by putting $f = bf_0b^{-1}$ then $f \in \mathcal{N}$ and $V \cap f(V) = \emptyset$. Using (2) we see that

$[h, f] \in \mathcal{N}$ because $[h, f] = hfh^{-1}f^{-1} = (hfh^{-1})f^{-1}$. By (2) again, the commutator $[[h, f], g]$ lies also in $\mathcal{N}$. Since $\operatorname{supp}(fh^{-1}f^{-1}) \subset f(V)$, it follows that $fh^{-1}f^{-1}$ commutes with g. Finally,

$$\begin{aligned}[[h, f], g] &= h(fh^{-1}f^{-1})g(fhf^{-1})h^{-1}g^{-1} \\ &= hg(fh^{-1}f^{-1})(fhf^{-1})h^{-1}g^{-1} = hgh^{-1}g^{-1} = [h, g],\end{aligned}$$

proving that $[h, g] \in \mathcal{K}$. □

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