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The Direct Sum Decomposability of ${}^eM$ in Dimension 2

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Dedicated to Professor Melvin Hochster on the occasion of his sixty-fifth birthday

0. Introduction

Unless explicitly stated otherwise, throughout this paper we assume that R is a Noetherian ring of prime characteristic p and that M is a finitely generated R-module. By $(R, \mathfrak{m}, k)$ we indicate that R is local with its maximal ideal $\mathfrak{m}$ and its residue field $k = R/\mathfrak{m}$. We always denote $q := p^e$ for varying $e \in \mathbb{N}$.

For every $e \in \mathbb{N}$, there exists a Frobenius map (which is a ring homomorphism) $F^e : R \to R$ defined by $F^e(r) = r^q = r^{p^e}$ for any $r \in R$. Thus, given M, there is a derived R-module structure, denoted by ${}^eM$, on the same abelian group M but with its scalar multiplication determined by $r \cdot x = r^q x = r^{p^e} x$ for $r \in R$ and $x \in M$. It is routine to verify that $\operatorname{Ann}(M) \subseteq \operatorname{Ann}({}^eM) \subseteq \sqrt{\operatorname{Ann}(M)}$ and that $\operatorname{Ass}(M) = \operatorname{Ass}({}^eM)$ for all $e \in \mathbb{N}$.

When R is reduced it is clear that ${}^eR$ and $R^{1/q} := \{r^{1/p^e} \mid r \in R\}$ are isomorphic as R-modules for every e. Using this terminology, a result of Kunz [K1, Thm. 2.1] states that R is regular if and only if ${}^eR$ is flat over R for some $e \geq 1$ or, equivalently, for all $e \in \mathbb{N}$.

We say that R is *F-finite* if ${}^1R$ is finitely generated over R or, equivalently, if ${}^eR$ is finitely generated over R for all $e \in \mathbb{N}$. By a result of Kunz [K2], every F-finite ring is excellent. If R is F-finite and if M is a finitely generated R-module, then it is easy to see that ${}^eM$ remains finitely generated over R for every $e \in \mathbb{N}$.

Similarly, if ${}^1M$ is finitely generated over R then so is ${}^1(R/\operatorname{Ann}(M))$. This means that $R/\operatorname{Ann}(M)$ is an F-finite ring. In other words, ${}^e(R/\operatorname{Ann}(M))$ is finite over $R/\operatorname{Ann}(M)$ (or, equivalently, over R) for all e, which forces ${}^eM$ to be finitely generated over R for all $e \in \mathbb{N}$.

For any $e \in \mathbb{N}$, the derived R-module ${}^eM$ can be roughly identified as the module structure of M over the subring $R^q := \{r^q = r^{p^e} \mid r \in R\}$. Thus, in general, the "size" of ${}^eM$ should increase as $e \to \infty$. Assuming that ${}^eM$ is finite over R for all $e \in \mathbb{N}$, we are interested in whether it is possible for the derived R-modules ${}^eM$ to remain indecomposable (i.e., not writable as a direct sum of two nontrivial submodules) for all $e \in \mathbb{N}$. Since we can always replace R by $R/\operatorname{Ann}(M)$, we may simply assume that R is F-finite.

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Here is a case where ${}^eM$ remain indecomposable for all $e \in \mathbb{N}$. Suppose R has a maximal ideal $\mathfrak{m}$ such that $k = R/\mathfrak{m}$ is a perfect field, and let $M = k$. Then it is easy to see that ${}^eM \cong M = k$ and hence is indecomposable for all $e \in \mathbb{N}$. Another (trivial) case of nonsplitting is when $M = 0$.

Hochster [H] showed the eventual splitting of ${}^eM$ for $e \gg 0$ in many cases. In particular, he proved that ${}^eM$ decomposes for all $e \gg 0$ if $\dim(M) \le 1$ (except for the cases mentioned in the previous paragraph). Indeed, if $\dim(M) = 0$ then the eventual splitting reduces to a local case in which we see that $\operatorname{Ann}_R(M)$ is an $\mathfrak{m}$-primary ideal of $(R, \mathfrak{m}, k)$. Then $\mathfrak{m}^{[p^{e_0}]} \subseteq \operatorname{Ann}(M)$ for some $e_0 \in \mathbb{N}$, where $\mathfrak{m}^{[p^{e_0}]}$ denotes the ideal of R generated by $\{r^{p^{e_0}} \mid r \in \mathfrak{m}\}$. Thus the derived R-modules ${}^eM$ become vector spaces over $k = R/\mathfrak{m}$ for all $e \ge e_0$, since $\mathfrak{m} \cdot {}^eM = \mathfrak{m}^{[p^e]}M = 0$. As for the 1-dimensional case, we quote what was essentially proved in [H, Thm. 5.16(2)] as follows.

THEOREM 0.1. *Let $(R, \mathfrak{m}, k)$ be an F-finite local Noetherian ring of characteristic p, and let M be a finitely generated R-module with* $\dim(M) = 1$. *Fix any $P \in$* $\operatorname{Ass}(M)$ *with* $\dim(R/P) = 1$ *and let $A = \overline{R/P}$ be the integral closure of R/P in its fraction field $(R/P)_P$. Then, for any $n \in \mathbb{N}$, there exists an $e_0 \in \mathbb{N}$ such that ${}^eM$ has a direct summand isomorphic to A^n for all $e \ge e_0$.*

One of the main ideas in the proof of this theorem is [H, Lemma 5.17], which also plays an important role in this paper. Because we will need a stronger result than the original version of [H, Lemma 5.17], we state the following lemma.

LEMMA 0.2 (cf. [H, Lemma 5.17]). *Consider the short exact sequence*

$$0 \longrightarrow D^{r+1} \oplus B \longrightarrow M \longrightarrow N \longrightarrow 0$$

of finitely generated modules over a Noetherian ring R (not necessarily of characteristic p). Assume that $\mu(E) \le r$ for all submodules $E \subseteq \operatorname{Ext}^1_R(N, D)$, where $\mu(E)$ denotes the least number of generators of E. Then M has a direct summand that is isomorphic to D.

Proof. This can be derived from the proof of [H, Lemma 5.17]. We omit the details. □

In [H, Fact 5.14] it was also observed that, if M is a graded module over an F-finite $\mathbb{N}$-graded Noetherian ring R with R_0 a field of characteristic p and $\dim(M) \ge 1$, then for any $n \in \mathbb{N}$ there exists an e such that ${}^eM$ splits as a direct sum of more than n nonzero R-modules. This splitting property was then used to prove the existence of small Cohen–Macaulay modules (see [H, Prop. 5.11]).

In this paper we study the direct sum decomposability of ${}^eM$ when $\dim(M) \ge 2$. Our approach is similar to that of [H, Thm. 5.16(2)]. We now state the main result, which is proved in Section 1.

MAIN THEOREM (see Theorem 1.8). *Let $(R, \mathfrak{m}, k)$ be an F-finite Noetherian local ring of characteristic p, and let M be a finitely generated R-module with*

$\dim(M) = 2$. *Let A be the integral closure of R/P in some finite algebraic extension field of $(R/P)_P$ for some $P \in \mathrm{Ass}(M)$ with* $\dim(R/P) = 2$. *If A is strongly F-regular then, for any $n \in \mathbb{N}$, A^n is isomorphic to a direct summand of ${}^eM$ for every $e \gg 0$.*

Recall that an F-finite ring R is said to be *strongly F-regular* [HHu2, Def. 5.1] if, for any $c \in R^\circ := R \setminus \bigcup_{P \in \min(R)} P$, the R-linear map $R \to {}^eR$ defined by $1 \mapsto c$ splits for some $e > 0$ (or, equivalently, for all $e \gg 0$). Strong F-regularity can be equivalently defined in terms of *tight closure* (cf. [HHu1]): $(R, \mathfrak{m}, k)$ is strongly F-regular if and only if 0 is *tightly closed* in the injective hull of k. For example, if $(R, \mathfrak{m}, k)$ is an F-finite regular local ring, then ${}^eR$ is free over R for all e by [K1, Thm. 2.1]. Thus, for any $c \neq 0 \in R$, the R-linear map $R \to {}^eR$ sending 1 to c splits as long as e is large enough that $c \notin \mathfrak{m}^{[p^e]} = \mathfrak{m} \cdot {}^eR$. This shows that every F-finite regular ring is strongly F-regular.

In Hochster's result (our Theorem 0.1), R/P is a domain with $\dim(R/P) \leq 1$ and so its integral closure $A = \overline{R/P}$ is regular (and hence strongly F-regular) automatically. However, when $\dim(R/P) = 2$, its integral closure may not be regular. Nevertheless, Theorem 1.8(1) states that if there is a module-finite domain extension of R/P that is strongly F-regular, then the same splitting result for ${}^eM$ still holds. In this sense, Theorem 1.8(1) may be regarded as a generalization of Theorem 0.1.

1. The Eventual Splitting of ${}^eM$ in Dimension 2

We begin this section with an easy remark.

REMARK 1.1. Let R be a ring, and let $M_1 \to M \to M_2$ be an exact sequence. Then

$$\sup\{\mu(E) \mid E \subseteq M\} \leq \sup\{\mu(E_1) \mid E_1 \subseteq M_1\} + \sup\{\mu(E_2) \mid E_2 \subseteq M_2\}.$$

Throughout this paper, $\mu(E)$ denotes the minimal number of generators for any R-module E.

Let us next recall a familiar and useful fact about 1-dimensional R-modules. We use $\lambda_R(\cdot)$ to denote the length of an R-module.

LEMMA 1.2. *Let M be a finitely generated module over a local ring* $(R, \mathfrak{m}, k)$ (*not necessarily of characteristic p*) *with* $\dim(M) \leq 1$. *Then*

$$\sup\{\mu(E) \mid E \subseteq M\} \leq \lambda(\mathrm{H}^0_{\mathfrak{m}}(M)) + e(M) < \infty,$$

where $\mathrm{H}^0_{\mathfrak{m}}(M) := \bigcup_{n \in \mathbb{N}} (0 :_M \mathfrak{m}^n)$ *and* $e(M) := \lim_{n \to \infty} \lambda(M/\mathfrak{m}^n M)/n$, *the Hilbert multiplicity of M* (*as a module of dimension* 1).

Proof. We sketch a proof. Let E be an arbitrary submodule of M; then

$$\mathrm{H}^0_{\mathfrak{m}}(E) = \mathrm{H}^0_{\mathfrak{m}}(M) \cap E$$

and so there exists an exact sequence $0 \to E/\mathrm{H}^0_{\mathfrak{m}}(E) \to M/\mathrm{H}^0_{\mathfrak{m}}(M)$. Notice that $E/\mathrm{H}^0_{\mathfrak{m}}(E)$ is either zero or Cohen–Macaulay of dimension 1. Thus, by Remark 1.1 (and other considerations) we have

$$\begin{aligned}\mu(E) &\le \mu(\mathrm{H}^0_{\mathfrak{m}}(E)) + \mu(E/\mathrm{H}^0_{\mathfrak{m}}(E)) \le \lambda(\mathrm{H}^0_{\mathfrak{m}}(E)) + e(E/\mathrm{H}^0_{\mathfrak{m}}(E)) \\ &\le \lambda(\mathrm{H}^0_{\mathfrak{m}}(M)) + e(M/\mathrm{H}^0_{\mathfrak{m}}(M)) = \lambda(\mathrm{H}^0_{\mathfrak{m}}(M)) + e(M).\end{aligned}$$

Here we have used that $\mu(E/\mathrm{H}^0_{\mathfrak{m}}(E)) \le e(E/\mathrm{H}^0_{\mathfrak{m}}(E))$, which holds because $N := E/\mathrm{H}^0_{\mathfrak{m}}(E)$ is a 1-dimensional Cohen–Macaulay R-module. To prove this, we assume without loss of generality that $\dim(R) = 1$ and $|k| = \infty$. Then there exists an $x \in \mathfrak{m}$ such that x is N-regular and xR is a reduction of $\mathfrak{m}$. Consequently,

$$e(N) = \lim_{n\to\infty} \frac{\lambda(N/\mathfrak{m}^n N)}{n} = \lim_{n\to\infty} \frac{\lambda(N/x^n N)}{n} = \lambda_R(N/xN) \ge \mu(N). \qquad \square$$

The next result plays an important role in the proof of our main theorem. Before stating the result, we first explain some notation and terminology that we shall use.

NOTATION 1.3. Let $L, D \neq 0$ be finitely generated modules over an F-finite ring R.

(1) We denote by $\#_R(L, D)$, or by $\#(L, D)$ if the ring R is clearly understood, the maximal integer n such that $L \cong D^n \oplus N$ for some R-module N.
(2) When $(R, \mathfrak{m}, k)$ is local, we denote $\alpha(R) = \log_p[k : k^p]$ (i.e., $p^{\alpha(R)}$ is the rank of ${}^1k$ as a k-vector space).
(3) Assuming that the ring $(R, \mathfrak{m}, k)$ is local, we say that D is an *F-contributor* of L if $\limsup_{e\to\infty} \#({}^eL, D)/q^{\alpha(R)+\dim(L)} > 0$, where $q = p^e$. (See [Y1] for some properties of F-contributors.)
(4) Given functions $f, g\colon \mathbb{N} \to \mathbb{N}$, we say that $f(e) = O(g(e))$ if there exists an $a \in \mathbb{N}$ such that $f(e) \le a g(e)$ for all $e \in \mathbb{N}$.

Recall that, for any $P \in \operatorname{Spec}(R)$ and $e \in \mathbb{N}$, the derived module ${}^e(R/P)$ has torsion-free rank $q^{\alpha(R)+\dim(R/P)}$ over R/P [K2, Prop. 2.3]. The next lemma gives a criterion for when the eventual splitting of ${}^eM$ occurs.

LEMMA 1.4. *Let $(R, \mathfrak{m}, k)$ be an F-finite local Noetherian ring of prime characteristic p, and let M be a finitely generated R-module. For some $e_0 \ge 0$, suppose there exists a short exact sequence*

$$0 \longrightarrow L \longrightarrow {}^{e_0}M \longrightarrow N \longrightarrow 0$$

such that $\dim(N) = d \le 1$ *and* $\limsup_{e\to\infty} \#({}^eL, D)/q^{\alpha(R)+d} = \infty$ *for some finitely generated R-module $D \neq 0$ (e.g.,* $\dim(L) > d$ *and D is a F-contributor of L). Then, for any $n \in \mathbb{N}$, there exists an $e \in \mathbb{N}$ such that ${}^eM$ has a direct summand isomorphic to D^n.*

Proof. Because the assumption implies also that $\limsup_{e\to\infty} \#({}^eL, D^n)/q^{\alpha(R)+d} = \infty$ for any $n \in \mathbb{N}$, we need only prove the lemma for the case $n = 1$. Also, since ${}^e({}^{e_0}M) = {}^{e+e_0}M$ for all $e \in \mathbb{N}$, we may relabel ${}^{e_0}M$ as M and thus assume $e_0 = 0$ without loss of generality.

We can filter N by finitely many submodules such that successive quotients are isomorphic to either $k = R/\mathfrak{m}$ or R/P with $P \in \operatorname{Spec}(R)$ and $\dim(R/P) = 1$. For each such P, denote by $\overline{R/P}$ the integral closure of R/P in its fraction field. Then $\overline{R/P}$ is regular and finitely generated over R/P since R is excellent. Hence there exists an exact sequence $0 \to \overline{R/P} \to R/P \to U \to 0$ with $\lambda_R(U) < \infty$. This shows that N may be filtered by finitely many submodules with successive quotients isomorphic to either $k = R/\mathfrak{m}$ or $\overline{R/P}$ with $P \in \operatorname{Spec}(R)$ and $\dim(R/P) = 1$. Fix such a filtration, say

$$0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_r = N,$$

and let $\Lambda_0 \subseteq \{1, 2, \dots, r\}$ and $\Lambda_1 = \{1, 2, \dots, r\} \setminus \Lambda_0$ such that $N_i/N_{i-1} \cong k = R/\mathfrak{m}$ when $i \in \Lambda_0$ and $N_i/N_{i-1} \cong \overline{R/P_i}$ with $P_i \in \operatorname{Spec}(R)$ and $\dim(R/P_i) = 1$ when $i \in \Lambda_1$. Since the integral closure $\overline{R/P_i}$ is a 1-dimensional regular semilocal domain for each $i \in \Lambda_1$, we have ${}^e(\overline{R/P_i}) \cong (\overline{R/P_i})^{q^{\alpha(R)+1}}$ (cf. [K2, Prop. 2.3] and Lemma 1.10). Hence, for any $e \in \mathbb{N}$, the derived R-module ${}^eN$ may be filtered correspondingly as

$$0 = {}^eN_0 \subsetneq {}^eN_1 \subsetneq \cdots \subsetneq {}^eN_r = {}^eN,$$

where

$${}^eN_i/{}^eN_{i-1} \cong \begin{cases} {}^ek \cong k^{q^{\alpha(R)}} & \text{if } i \in \Lambda_0, \\ {}^e(\overline{R/P_i}) \cong (\overline{R/P_i})^{q^{\alpha(R)+1}} & \text{if } i \in \Lambda_1. \end{cases}$$

Thus, by induction on r (we omit the details) and iteration of Remark 1.1, for all $e \in \mathbb{N}$ we have

$$\begin{aligned} &\sup\{\mu(E) \mid E \subseteq \operatorname{Ext}^1_R({}^eN, D)\} \\ &\quad \le \sum_{i=1}^{r} \sup\{\mu(E) \mid E \subseteq \operatorname{Ext}^1_R({}^eN_i/{}^eN_{i-1}, D)\} \\ &\quad = q^{\alpha(R)} \sum_{i \in \Lambda_0} \sup\{\mu(E) \mid E \subseteq \operatorname{Ext}^1_R(k, D)\} \\ &\qquad + q^{\alpha(R)+1} \sum_{i \in \Lambda_1} \sup\{\mu(E) \mid E \subseteq \operatorname{Ext}^1_R(\overline{R/P_i}, D)\}. \end{aligned}$$

We therefore conclude that $\sup\{\mu(E) \mid E \subseteq \operatorname{Ext}^1_R({}^eN, D)\} = O(q^{\alpha(R)+d})$. Observe that if $\dim(N) = d = 0$ then $\Lambda_1 = \emptyset$.

Put $\mu(e) = \sup\{\mu(E) \mid E \subseteq \operatorname{Ext}^1_R({}^eN, D)\}$ for every $e \in \mathbb{N}$. Because $\mu(e) = O(q^{\alpha(R)+d})$ and $\limsup_{e\to\infty} \#({}^eL, D)/q^{\alpha(R)+d} = \infty$, there exists a large enough e such that $\#({}^eL, D) \ge \mu(e) + 1$. That is, ${}^eL \cong D^{\mu(e)+1} \oplus B$ for some R-module B, and hence we have the exact sequence

$$0 \longrightarrow D^{\mu(e)+1} \oplus B \longrightarrow {}^eM \longrightarrow {}^eN \longrightarrow 0.$$

By Lemma 0.2, we see that D is isomorphic to a direct summand of ${}^eM$. □

Remark 1.5. We may sketch another proof of Lemma 1.4; again, it suffices to prove the case where $n = 1$. The assumption $\limsup_{e\to\infty} \#({}^eL, D)/q^{\alpha(R)+d} = \infty$ implies that D has depth at least $d + 1 = \dim(N) + 1$ (see the proof of [Y1, Lemma 2.2]). Hence there exists an $x \in \mathrm{Ann}(N) \subseteq \mathrm{Ann}({}^eN)$ for all $e \in \mathbb{N}$ such that x is D-regular. Let $\bar{R} = R/\mathrm{Ann}_R(N)$ and $\bar{D} = D/xD$. Then, for all $e \in \mathbb{N}$,

$$\mathrm{Ext}^1_R({}^eN, D) \cong \mathrm{Hom}_R({}^eN, D/xD) \subseteq \mathrm{Hom}_R(\bar{R}^{\mu({}^eN)}, \bar{D}) \cong \mathrm{Hom}_R(\bar{R}, \bar{D})^{\mu({}^eN)}.$$

Because $\mathrm{Hom}_R(\bar{R}, \bar{D})$ has dimension at most 1, Remark 1.1 and Lemma 1.2 imply that

$$\begin{aligned}\sup\{\mu(E) \mid E \subseteq \mathrm{Ext}^1_R({}^eN, D)\} &\leq \sup\{\mu(E) \mid E \subseteq \mathrm{Hom}_R(\bar{R}, \bar{D})^{\mu({}^eN)}\} \\ &\leq \mu({}^eN) \sup\{\mu(E) \mid E \subseteq \mathrm{Hom}_R(\bar{R}, \bar{D})\} \\ &= O(\mu({}^eN)).\end{aligned}$$

On the other hand, $\mu({}^eN) = \lambda({}^eN/\mathfrak{m} \cdot {}^eN) = q^{\alpha(R)}\lambda(N/\mathfrak{m}^{[q]}N) = O(q^{\alpha(R)+d})$ by the existence of Hilbert–Kunz multiplicity (see [Mo]), where $d = \dim(N)$. Hence $\sup\{\mu(E) \mid E \subseteq \mathrm{Ext}^1_R({}^eN, D)\} = O(q^{\alpha(R)+d})$, and from this point the proof proceeds as in the original proof of Lemma 1.4.

REMARK 1.6. By Lemma 1.4 we see that if $\lim_{e\to\infty} \#({}^eL, D)/q^{\alpha(R)+d} = \infty$ then, for any given $n \in \mathbb{N}$, there exists an $e_1 \in \mathbb{N}$ such that ${}^eM$ has a direct summand isomorphic to D^n for all $e \geq e_1$.

Next, we use the criterion of Lemma 1.4 to produce a situation where ${}^eM$ splits for $e \gg 0$. For any finitely generated R-module M, set

$$\mathrm{Assh}(M) = \{P \in \mathrm{Ass}(M) \mid \dim(R/P) = \dim(M)\},$$

which is the same as $\{P \in \min(M) \mid \dim(R/P) = \dim(M)\}$. We remark that some of the arguments in the proof of the following proposition are similar to those outlined in the proof of [H, Thm. 5.16(2)].

PROPOSITION 1.7. *Let $(R, \mathfrak{m}, k)$ be an F-finite local Noetherian ring of characteristic p, and let M, L, D be finitely generated nonzero R-modules such that* $\dim(M) = 2, \mathrm{Ass}(L) \subseteq \mathrm{Assh}(M)$ (*so that* $\dim(L) = 2$), *and* $\limsup_{e\to\infty} \#({}^eL, D)/q^{\alpha(R)+1} = \infty$ (*e.g., D is an F-contributor of L*). *Then, for any $n \in \mathbb{N}$, there exists an $e \in \mathbb{N}$ such that D^n is isomorphic to a direct summand of ${}^eM$.*

Proof. Choose a primary decomposition of 0 in M, say

$$0 = Q_1 \cap Q_2 \cap \cdots \cap Q_s,$$

such that $\mathrm{Ass}(M/Q_i) = \{P_i\}$. Assume the primary decomposition is minimal so that $\mathrm{Ass}(M) = \{P_1, P_2, \ldots, P_s\}$. Let $\mathrm{Assh}(M) = \{P_1, P_2, \ldots, P_r\}$ for some $1 \leq r \leq s$ and let $S = R \setminus \bigcup_{i=1}^r P_i$. Then, over the localization ring $S^{-1}R$, we obtain a primary decomposition of 0 in $S^{-1}M$,

$$0 = S^{-1}Q_1 \cap S^{-1}Q_2 \cap \cdots \cap S^{-1}Q_r,$$

which shows that $S^{-1}\big(\bigoplus_{i=1}^r M/Q_i\big) \cong S^{-1}M$ by the Chinese remainder theorem. Lifting the isomorphism back to R, we have the short exact sequence

$$0 \longrightarrow \bigoplus_{i=1}^{r} M/Q_i \longrightarrow M \longrightarrow N \longrightarrow 0$$

for some finitely generated R-module N with $\dim(N) \leq 1$. Then, for every $e \in \mathbb{N}$, there is a short exact sequence

$$0 \longrightarrow \bigoplus_{i=1}^{r} {}^e(M/Q_i) \longrightarrow {}^eM \longrightarrow {}^eN \longrightarrow 0 \qquad (1.7.1)$$

with $\dim({}^eN) = \dim(N) \leq 1$. Since $\mathrm{Ass}(M/Q_i) = \{P_i\}$, it follows that ${}^e(M/Q_i) \neq 0$ are finitely generated torsion-free (R/P_i)-modules for all $e \gg 0$. (Indeed, because $\sqrt{\mathrm{Ann}_R(M/Q_i)} = P_i$, there exists an $e_0 \in \mathbb{N}$ such that $(\mathrm{Ann}_R(M/Q_i))^{[p^{e_0}]} \subseteq P_i$, which implies that ${}^e(M/Q_i)$ is annihilated by P_i for every $e \geq e_0$; moreover, for any $x \in R \setminus P_i$, x is a nonzero divisor on M/Q_i and thus it remains so on ${}^e(M/Q_i)$ for every $e \geq 0$.) For any $e \geq e_0$ and any $i = 1, \dots, r$, let $n(e,i)$ denote the torsion-free rank of ${}^e(M/Q_i)$ over R/P_i. Then $n(e,i) > 0$ and there exists a short exact sequence

$$0 \longrightarrow (R/P_i)^{n(e,i)} \longrightarrow {}^e(M/Q_i) \longrightarrow N_{(e,i)} \longrightarrow 0 \qquad (1.7.2)$$

such that $N_{(e,i)}$ is finitely generated over R/P_i with $\dim(N_{(e,i)}) < \dim(M/Q_i) = 2$ for every $i = 1, \dots, r$. Combining (1.7.1) and (1.7.2), we obtain the short exact sequence

$$0 \longrightarrow \bigoplus_{i=1}^{r} (R/P_i)^{n(e,i)} \longrightarrow {}^eM \longrightarrow N_e \longrightarrow 0 \qquad (*_e)$$

for each $e \geq e_0$, where N_e is a finitely generated R-module with $\dim(N_e) \leq 1$. Notice that $n(e+1,i) = p^{\alpha(R)+2} n(e,i)$ for each $e \geq e_0$ and each $i \in \{1,2,\dots,r\}$ (cf. [K2, Prop. 2.3]). In particular, we see that $n(e,i) \to \infty$ as $e \to \infty$.

Now we carry out a similar procedure on L. Let $\mathrm{Ass}(L) = \{P_1, P_2, \dots, P_t\}$ for some $1 \leq t \leq r$. Fix a primary decomposition of 0 in L, say $0 = Q_1' \cap Q_2' \cap \cdots \cap Q_t'$ with $\mathrm{Ass}(L/Q_i') = \{P_i\}$. Let $U = R \setminus \bigcup_{i=1}^{t} P_i$. Then $U^{-1}\big(\bigoplus_{i=1}^{t} L/Q_i'\big) \cong U^{-1}L$, which gives the short exact sequence

$$0 \longrightarrow L \longrightarrow \bigoplus_{i=1}^{t} L/Q_i' \longrightarrow N' \longrightarrow 0 \qquad (1.7.3)$$

for some finitely generated R-module N' with $\dim(N') \leq 1$. Similarly, we can find a large enough $e' \in \mathbb{N}$ such that, for each $i = 1, 2, \dots, t$, ${}^{e'}(L/Q_i')$ is torsion-free over R/P_i (say, with torsion-free rank $n(i)$); hence there exists a short exact sequence

$$0 \longrightarrow {}^{e'}(L/Q_i') \longrightarrow (R/P_i)^{n(i)} \longrightarrow N_i' \longrightarrow 0 \qquad (1.7.4)$$

for some finitely generated (R/P_i)-module N_i' with $\dim(N_i') \leq 1$. Together, (1.7.3) and (1.7.4) produce the short exact sequence

$$0 \longrightarrow {}^{e'}L \longrightarrow \bigoplus_{i=1}^{t} (R/P_i)^{n(i)} \longrightarrow N'' \longrightarrow 0 \qquad (**)$$

for some finitely generated R-module N'' with $\dim(N'') \leq 1$.

Now fix a sufficiently large $e_1 \in \mathbb{N}$ such that $n(e_1, i) \geq n(i)$ for all $i = 1, \dots, t$. Then the exact sequences $(*_{e_1})$ and $(**)$ generate the short exact sequence

$$0 \longrightarrow {}^{e'}L \oplus B \longrightarrow {}^{e_1}M \longrightarrow N''' \longrightarrow 0,$$

where

$$B = \left(\bigoplus_{i=1}^{t} (R/P_i)^{n(e_1,i)-n(i)}\right) \oplus \left(\bigoplus_{i=t+1}^{r} (R/P_i)^{n(e_1,i)}\right)$$

and N''' is a finitely generated R-module with $\dim(N''') \leq 1$. Moreover, it is clear that

$$\limsup_{e\to\infty} \frac{\#({}^{e}L, D)}{q^{\alpha(R)+1}} = \infty \implies \limsup_{e\to\infty} \frac{\#({}^{e}({}^{e'}L \oplus B), D)}{q^{\alpha(R)+1}} = \infty.$$

Now the desired result follows from Lemma 1.4. □

Proposition 1.7 can be applied to the following case, which proves our main theorem. First, recall that Aberbach and Leuschke [AL] have proved the following result concerning strong F-regularity: *An F-finite local ring $(R, \mathfrak{m})$ is strongly F-regular if and only if* $\liminf_{e\to\infty} \#({}^{e}R, R)/q^{\alpha(R)+\dim(R)} > 0$. See [SV; Y1] for the definition and properties of modules of finite F-representation type (abbreviated FFRT).

THEOREM 1.8. *Let $(R, \mathfrak{m}, k)$ be an F-finite local Noetherian ring of characteristic p, and let M be a finitely generated R-module with* $\dim(M) = 2$. *Let A be a domain that is a module-finite extension of R/P for some $P \in \operatorname{Assh}(M)$.*

(1) *If A is strongly F-regular then, for any $n \in \mathbb{N}$, there exists an $e_1 \in \mathbb{N}$ such that ${}^{e}M$ has a direct summand isomorphic to A^n for all $e \geq e_1$.*
(2) *If there is a finitely generated torsion-free A-module $L \neq 0$ that has FFRT, then there exists an R-module $D \neq 0$ such that, for any $n \in \mathbb{N}$, there is an $e_1 \in \mathbb{N}$ such that ${}^{e_1}M$ has a direct summand isomorphic to D^n.*

Proof. (1) Evidently $\dim(A) = 2$, $\operatorname{Ass}_R(A) = \{P\}$, and A is a semilocal F-finite ring, say with maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_c$. Observe that $\dim(A_{\mathfrak{m}_i}) = \dim(A) = 2$ and $\alpha(A_{\mathfrak{m}_i}) = \alpha(R)$ for each $i = 1, 2, \dots, c$. Indeed, the equation $\alpha(A_{\mathfrak{m}_i}) = \alpha(R)$ holds because $A/\mathfrak{m}_i$ is a finite field extension of $R/\mathfrak{m}$; then $\dim(A_{\mathfrak{m}_i}) = \dim(R)$ follows from [K2, Prop. 2.3]. It also follows from Ratliff's dimension formula (see [M, Thm. 15.6]).

We must show that $\liminf_{e\to\infty} \#_R({}^{e}A, A)/q^{\alpha(R)+\dim(A)} > 0$ in order to apply Proposition 1.7 and Remark 1.6. It suffices to show $\liminf_{e\to\infty} \#_A({}^{e}A, A)/q^{\alpha(R)+\dim(A)} > 0$ by considering A as an A-module. Because A is strongly F-regular, so is $A_{\mathfrak{m}_i}$ for each $i = 1, 2, \dots, c$. Therefore, $\liminf_{e\to\infty} \#_{A_{\mathfrak{m}_i}}({}^{e}A_{\mathfrak{m}_i}, A_{\mathfrak{m}_i})/q^{\alpha(A_{\mathfrak{m}_i})+\dim(A_{\mathfrak{m}_i})} > 0$ for each $i = 1, 2, \dots, c$ (see [AL]). Finally, by Lemma 1.10 (to follow) we have

$$\liminf_{e\to\infty} \frac{\#_A({}^{e}A, A)}{q^{\alpha(R)+\dim(A)}} = \min\left\{\liminf_{e\to\infty} \frac{\#_{A_{\mathfrak{m}_i}}({}^{e}A_{\mathfrak{m}_i}, A_{\mathfrak{m}_i})}{q^{\alpha(R)+\dim(A)}} \;\middle|\; 1 \leq i \leq c\right\} > 0,$$

which finishes the proof of part (1).

(2) Clearly, we have $\mathrm{Ass}_R(L) = \{P\} \subseteq \mathrm{Assh}_R(M)$ and so $\dim(L) = 2$. Also, our assumption that L has FFRT as an A-module implies that L has FFRT as an R-module by definition. By [Y1, Lemma 2.1], there is a nonzero F-contributor D of L over R. Now Proposition 1.7 applies. $\square$

REMARK 1.9. Because every strongly F-regular ring A is normal, we see that if A is a module-finite extension of R/P (as in Theorem 1.8) then A is the integral closure of R/P in some finite field extension of $(R/P)_P$.

In general, for any (F-finite) local ring $(R, \mathfrak{m}, k)$ of prime characteristic p, the invariant $s(R) = \lim_{e\to\infty} \#({}^eR, R)/q^{\alpha(R)+\dim(R)}$, if it exists, is called the *F-signature* of R, which was first defined and studied in [HuL]. For related work on the F-signature, see for example [AE1; AE2; AL; Si; SV; Y1; Y2].

Although Lemma 1.10 may be well known, we state and prove it for the completeness of the proof of Theorem 1.8. (Lemma 1.10 was also referred to in the proof of Lemma 1.4.) Before stating the lemma we remark that, for any Noetherian local ring $(R, \mathfrak{m})$ and any finitely generated R-modules $N, D \neq 0$,

$$\#_R(N \oplus D^n, D) = \#_R(N, D) + n \tag{$\dagger$}$$

for every $n \in \mathbb{N}$, which follows from $\#_{\hat{R}}(\hat{M}, \hat{D}) = \#_R(M, D)$ and the Krull–Schmidt property of $\hat{R}$, the $\mathfrak{m}$-adic completion of R.

LEMMA 1.10. *Let A be a semilocal Noetherian ring (not necessarily with prime characteristic p) with maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_c$ exactly and with $M, D \neq 0$ finitely generated A-modules. Then*

$$\#_A(M, D) = \min\{\#_{A_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i}, D_{\mathfrak{m}_i}) \mid 1 \le i \le c\}.$$

Proof. One could prove this lemma by using that (a) $\#_A(M, D) = \#_{\hat{A}}(\hat{M}, \hat{D})$, where $\hat{\ } = \hat{\ }^{\mathfrak{m}}$ denotes the completion with respect to the $\mathfrak{m}$-adic topology for $\mathfrak{m} = \bigcap_{i=1}^c \mathfrak{m}_i$ the Jacobson radical, and (b) $\hat{A} = \prod_{i=1}^c (\hat{A}_{\mathfrak{m}_i})^{\mathfrak{m}_i}$, where $(\hat{\cdot})^{\mathfrak{m}_i}$ denotes the $(\mathfrak{m}_i A_{\mathfrak{m}_i})$-adic completion.

For an alternative proof, let $\#_A(M, D) = n$ and $M \cong N \oplus D^n$. Then $\#_A(N, D) = 0$ and, by ($\dagger$), we have

$$\min\{\#_{A_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i}, D_{\mathfrak{m}_i}) \mid 1 \le i \le c\} = n + \min\{\#_{A_{\mathfrak{m}_i}}(N_{\mathfrak{m}_i}, D_{\mathfrak{m}_i}) \mid 1 \le i \le c\}.$$

It suffices to prove that $\min\{\#_{A_{\mathfrak{m}_i}}(N_{\mathfrak{m}_i}, D_{\mathfrak{m}_i}) \mid 1 \le i \le c\} = 0$. Suppose, to the contrary, that $\min\{\#_{A_{\mathfrak{m}_i}}(N_{\mathfrak{m}_i}, D_{\mathfrak{m}_i}) \mid 1 \le i \le c\} > 0$. Then, for each $1 \le i \le c$, there exist homomorphisms

$$\phi_i/s_i \in (\mathrm{Hom}_A(N, D))_{\mathfrak{m}_i} = \mathrm{Hom}_{A_{\mathfrak{m}_i}}(N_{\mathfrak{m}_i}, D_{\mathfrak{m}_i}),$$
$$\psi_i/s_i \in (\mathrm{Hom}_A(D, N))_{\mathfrak{m}_i} = \mathrm{Hom}_{A_{\mathfrak{m}_i}}(D_{\mathfrak{m}_i}, N_{\mathfrak{m}_i})$$

such that $(\phi_i/s_i) \circ (\psi_i/s_i) = 1_{D_{\mathfrak{m}_i}}$, where $\phi_i \in \mathrm{Hom}_A(N, D)$, $\psi_i \in \mathrm{Hom}_A(D, N)$, and $s_i \in A \setminus \mathfrak{m}_i$. Choose $r_i \in \bigcap_{j \neq i} \mathfrak{m}_j \setminus \mathfrak{m}_i$ for each $i = 1, 2, \dots, c$. Then it is routine to verify that

$$\left(\sum_{i=1}^c r_i \phi_i\right) \circ \left(\sum_{i=1}^c r_i \psi_i\right) \in \mathrm{Hom}_A(D, D)$$

is surjective (by Nakayama's lemma) and hence is an isomorphism, which implies that N has a direct summand isomorphic to D—a contradiction. $\square$

Note that Proposition 1.7 and Theorem 1.8 apply only to 2-dimensional cases. For higher dimensions we have the following result, which was obtained during a discussion with Melvin Hochster.

THEOREM 1.11. *Let* $(R, \mathfrak{m}, k)$ *be an F-finite local domain of prime characteristic p with* $\dim(R) \geq 2$ *such that* R_P *is integrally closed in its fraction field for all* $P \in \operatorname{Spec}(R)$ *with* $\dim(R/P) \geq 2$. *Let* $A := \bar{R}$ *be the integral closure of R in its fraction field. If A is strongly F-regular then, for any finitely generated faithful R-module M and any* $n \in \mathbb{N}$, *there exists an* e_1 *such that* ${}^eM$ *has a direct summand isomorphic to* A^n *(as an R-module) for all* $e \geq e_1$.

Proof. Let $\mathfrak{C} = \{r \in R \mid rA \subseteq R\}$ be the conductor, which is the largest common ideal of R and A. Then we see $\dim(R/\mathfrak{C}) \leq 1$ by assumption. Consider the short exact sequence

$$0 \longrightarrow \mathfrak{C}M \longrightarrow M \longrightarrow M/\mathfrak{C}M \longrightarrow 0,$$

where $\dim(M/\mathfrak{C}M) \leq 1$ and $\mathfrak{C}M$ is a finitely generated faithful A-module. Thus there exist $h \in \operatorname{Hom}_A(\mathfrak{C}M, A)$ and $x \in \mathfrak{C}M$ such that $h(x) = c \in A^\circ$. Then, since A is strongly F-regular, there exist $e_0 \in \mathbb{N}$ and $g \in \operatorname{Hom}_A({}^{e_0}A, A)$ such that $g(c) = 1$. Consequently, we obtain an A-linear homomorphism $g \circ h \colon {}^{e_0}(\mathfrak{C}M) \to {}^{e_0}A \to A$ that maps x to 1, showing that A is a direct summand of ${}^{e_0}(\mathfrak{C}M)$ as an A-module.

The strong F-regularity of A also implies $\liminf_{e\to\infty} \#_A({}^eA, A)/q^{\alpha(R)+\dim(A)} > 0$ (cf. [AL] and the proof of Theorem 1.8(1)). As a result,

$$\liminf_{e\to\infty} \frac{\#_R({}^e(\mathfrak{C}M), A)}{q^{\alpha(R)+\dim(R)}} \geq \liminf_{e\to\infty} \frac{\#_A({}^e(\mathfrak{C}M), A)}{q^{\alpha(R)+\dim(A)}} > 0.$$

Now the claim follows from Lemma 1.4 and Remark 1.6. $\square$

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