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Noncommutative Resolutions and Rational Singularities

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Dedicated to Mel Hochster on the occasion of his 65th birthday

1. Introduction

Throughout the paper, k will denote a fixed algebraically closed field of characteristic 0 and, unless otherwise specified, all rings will be k-algebras. Our aim is to show that the center of a homologically homogeneous, finitely generated k-algebra has rational singularities; in particular, if a finitely generated normal commutative k-algebra has a noncommutative crepant resolution (as introduced by the second author), then it has rational singularities.

We begin by setting this result in context and defining the relevant terms. Suppose that $X = \operatorname{Spec} R$ for an affine (i.e., finitely generated) normal Gorenstein k-algebra R. The nicest form of resolution of singularities $f : Y \to X$ occurs when f is *crepant* in the sense that $f^*\omega_X = \omega_Y$. Even when they exist, crepant resolutions need not be unique, but they are related—indeed, Bondal and Orlov conjectured in [BoO2] (see also [BoO1]) that two such resolutions should be derived equivalent.

Bridgeland [Bri] proved the Bondal–Orlov conjecture in dimension 3. The second author observed in [V3] that Bridgeland's proof could be explained in terms of a *third* crepant resolution of X that is now noncommutative (the definition will be given in what follows). This and similar observations by others have led to a number of different approaches to the Bondal–Orlov conjecture and related topics (see e.g. [Be; BeKa; Ch; IR; Ka2; Kaw]).

It is therefore natural to ask how the existence of a noncommutative crepant resolution affects the original commutative singularity. It is well known, and follows easily from [KoMo, Thm. 5.10], that if a Gorenstein singularity has a crepant resolution then it has rational singularities. So it is logical to ask, as raised in [V2, Ques. 3.2], *is this true for a noncommutative crepant resolution?* Here we answer this question affirmatively.

Let Δ be a prime affine k-algebra that is finitely generated as a module over its center $Z(\Delta)$. Mimicking [BH], we say that Δ is *homologically homogeneous of*

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dimension d if all simple Δ-modules have the same projective dimension d. By [Ra] and [BH], such a ring Δ has global and Krull dimensions equal to d and, as has been shown in [BH], the properties of homologically homogeneous rings closely resemble those of commutative regular rings. So the idea is to use such a ring Δ as a noncommutative analogue of a crepant resolution. Formally, following [V2] we define a *noncommutative crepant resolution* of R to be any homologically homogeneous ring of the form $\Delta = \mathrm{End}_R(M)$, where M is a reflexive and finitely generated R-module. We refer the reader to [V2, Sec. 4] for the logic behind this definition.

Our main result is as follows.

THEOREM 1.1 (Theorem 4.3). *Let Δ be a homologically homogeneous k-algebra. Then the center $Z(\Delta)$ has rational singularities.*

In particular, if a normal affine k-domain R has a noncommutative crepant resolution then it has rational singularities.

In Section 5 we give two examples related to the theorem. The first example shows that if $\Delta = \mathrm{End}_R(M)$ has finite global dimension then it need not be homologically homogeneous even under reasonable hypotheses on M and R. The second shows that Theorem 1.1 can fail in positive characteristic.

NOTATION. Throughout the paper, R will be a normal commutative Noetherian k-domain and Δ will be a k-algebra, with center $Z = Z(\Delta)$ containing R, such that Δ is a finitely generated R-module. We say that R is *essentially affine* if it is a localization of an affine k-algebra. The dimension function used in this paper will be the *Gelfand–Kirillov dimension* of Δ as a k-algebra, written $\mathrm{GKdim}\,\Delta$. By [McRo, Prop. 8.2.9(ii) and Thm. 8.2.14(ii)], $\mathrm{GKdim}\,\Delta = \mathrm{GKdim}\,R$ and $\mathrm{GKdim}\,R$ is just the transcendence degree of R over k.

2. Homologically Homogeneous Rings

In this section we introduce homologically homogeneous rings and prove some basic facts about their structure and their dualizing complexes. Many of these results use the machinery of tame orders, so we start by discussing this concept.

TAME ORDERS. Assume that Δ is *a prime R-order in A*, by which we mean that Δ is a prime ring with simple Artinian ring of fractions A. We write $\mathfrak{P}_1(R)$ for the set of height-1 prime ideals of R and say that *a property $\mathcal{P}$ holds for Δ in codimension 1* if it holds for all $\Delta_{\mathfrak{p}} = \Delta \otimes_R R_{\mathfrak{p}} : \mathfrak{p} \in \mathfrak{P}_1(R)$. Following [Si], the prime R-order Δ is called a *tame R-order* if Δ is a finitely generated and reflexive R-module that is hereditary in codimension 1.

In [Si] it is implicitly assumed that $R = Z(\Delta)$, but we prefer not to make this assumption. However, by the following standard result, the question of whether Δ is a tame R-order is independent of the choice of normal central subring R.

LEMMA 2.1. *Let Δ be a tame R-order. Then a finitely generated Δ-module is reflexive as an R-module if and only if it is reflexive as a Δ-module.*

Proof. By [Si, Cor. 1.6] (which does not require $R = Z(\Delta)$), a Δ-reflexive module is R-reflexive. Conversely, suppose that M is a finitely generated Δ-module that is R-reflexive. Since M is therefore torsion-free as a Δ-module, $M_{\mathfrak{p}} = M \otimes_R R_{\mathfrak{p}}$ is torsion-free and hence projective over the hereditary prime ring $\Delta_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathfrak{P}_1(R)$. Thus, by [Si, Lemma 1.1],

$$\begin{aligned} M = \bigcap_{\mathfrak{p}\in\mathfrak{P}_1(R)} M_{\mathfrak{p}} &= \bigcap_{\mathfrak{p}\in\mathfrak{P}_1(R)} \operatorname{Hom}_{\Delta_p}(\operatorname{Hom}_{\Delta_p}(M_{\mathfrak{p}}, \Delta_{\mathfrak{p}}), \Delta_{\mathfrak{p}}) \\ &\supseteq \operatorname{Hom}_{\Delta}(\operatorname{Hom}_{\Delta}(M, \Delta), \Delta). \end{aligned}$$

Thus, $M = \operatorname{Hom}_{\Delta}(\operatorname{Hom}_{\Delta}(M, \Delta), \Delta)$, as required. □

Let Δ be a tame R-order in A. A *divisorial* fractional Δ-ideal is any reflexive fractional Δ-ideal in A that is invertible in codimension 1. By [Si, Thm. 2.3], divisorial fractional ideals form a free abelian group $\operatorname{Div}(\Delta)$ with product $I \cdot J = (IJ)^{**}$, where $K^* = \operatorname{Hom}_R(K, R)$ denotes the R-dual of a fractional ideal K. The nth power $(I^n)^{**}$ of I under this dot operation is called the nth *symbolic power of I*, written $I^{(n)}$. In particular, $I^{(-n)} = (I^n)^*$ for all $n > 0$. Write $\operatorname{rad} S$ for the Jacobson radical of a ring S.

Homologically Homogeneous Rings. Homologically homogeneous rings, as defined in the Introduction, have a particularly pleasant structure. The next theorem provides some of the properties we will need. We start with a observation that will be used several times.

Lemma 2.2. *Let Δ be an R-order in a simple Artinian ring A.*

(1) *Let Z have field of fractions F and write $\tau : A \to F$ for the reduced trace map. If Z is normal then $\tau(\Delta) = Z$ and $\Delta = Z \oplus \ker\tau$ as Z-modules.*
(2) *If Δ is a tame R-order then $Z = Z(\Delta)$ is a tame R-order and is normal. Moreover, Δ is a tame Z-order.*

Proof. (1) By construction and the fact that $\operatorname{char} k = 0$, $\tau|_Z$ is a nonzero scalar multiple of the identity map. By [McRo, (13.9.3) and Prop. 13.9.8], $\tau(\Delta)$ is integral over Z and so, because Z is normal, $\tau(\Delta) = Z$. Since τ is a Z-module map, it therefore splits.

(2) The identity $\Delta = \bigcap_{\mathfrak{p}\in\mathfrak{P}_1(R)} \Delta_{\mathfrak{p}}$ restricts to give $Z = \bigcap_{\mathfrak{p}\in\mathcal{R}_1(R)} Z_{\mathfrak{p}}$. Since each $\Delta_{\mathfrak{p}}$ is hereditary, it follows from [McRo, Thm. 13.9.16] that each $Z_{\mathfrak{p}}$ is a Dedekind domain and hence that Z is normal. That Z_R is reflexive and hence tame now follows from part (1). The final assertion follows from Lemma 2.1. □

Theorem 2.3. *Assume that Δ is homologically homogeneous of dimension d.*

(1) *Δ is CM (Cohen–Macaulay) as a module over its center Z.*
(2) *Both* $\operatorname{GKdim}\Delta$ *and the global homological dimension* $\operatorname{gl\,dim}\Delta$ *of Δ equal d.*
(3) *Z is an affine CM normal domain.*
(4) *Δ is a tame Z-order.*

Proof. By [Ra, Thm. 8], $\operatorname{gl\,dim}\Delta = d$. The rest of parts (1) and (2) follow from [BH, Thm. 2.5].

(3) By hypothesis, Δ is finitely generated as both a Z-module and a k-algebra. Thus the Artin–Tate lemma [McRo, Lemma 13.9.10] implies that Z is an affine k-algebra. Since Δ is prime it follows that Z is a domain, while Z is normal by [BH, Thm. 6.1]. Thus Lemma 2.2(1) implies that Z is a Z-module summand of Δ, and so it is CM by part (1).

(4) Because Δ is CM as a Z-module, it is certainly reflexive. By [BH, Cor. 2.2 and Thm. 2.5], Δ is hereditary in codimension 1. □

The standing assumption that k has characteristic 0 is crucial for the proof of part (3) of the theorem. Indeed, [BHMa, Ex. 7.3] shows that the center $Z(\Gamma)$ of a homologically homogeneous ring Γ need not be CM in bad characteristic.

The following criterion for a ring to be homologically homogeneous will be useful.

LEMMA 2.4. *Suppose that R is an affine k-algebra and that Δ is a prime ring. If Δ is a CM R-module with* $\operatorname{GKdim}\Delta = \operatorname{gl\,dim}\Delta$, *then Δ is homologically homogeneous.*

Proof. This is, in essence, [BH, Prop. 7.2], but here is a direct proof. Suppose that S is a simple Δ-module with projective dimension $u < d = \operatorname{gl\,dim}\Delta$, and consider a projective Δ-resolution of S:

$$0 \to P_u \to \cdots \to P_1 \to P_0 \to S \to 0.$$

Viewed as a complex over R, this is a resolution of length $< d$ of a finite-length R-module by CM modules of dimension d. An easy depth argument shows that this is impossible. □

DUALIZING MODULES AND COMPLEXES. In order to relate properties of a homologically homogeneous ring to those of its center we use the machinery of dualizing complexes, whose structure we discuss next. Most of the background material comes from [V1; Y2; YZ1; YZ2], and the reader is referred to those papers for more details. Throughout this discussion, in addition to our standing assumptions we also assume that R is essentially affine.

Write $\Delta^e = \Delta \otimes_k \Delta^{\mathrm{op}}$ and denote the derived category of left Δ^e-modules by $D(\Delta^e)$. Following [Y1], a *dualizing complex* for Δ is a complex of Δ-bimodules D, with finite injective dimension on both sides, such that

(1) the cohomology of D is given by Δ-bimodules that are finitely generated on both sides, and
(2) in $D(\Delta^e)$ the pair of natural morphisms $\Phi\colon \Delta \to \operatorname{RHom}_\Delta(D, D)$ and Φ^o: $\Delta \to \operatorname{RHom}_{\Delta^{\mathrm{op}}}(D, D)$ are isomorphisms.

Following [V1, Def. 8.1], the dualizing complex D_Δ is called *rigid* if there is an isomorphism $\chi : D_\Delta \cong \operatorname{RHom}_{\Delta^e}(\Delta, D_\Delta \otimes D_\Delta)$ in $D(\Delta^e)$. The significance of rigidity is that, although dualizing complexes are not unique, rigid dualizing complexes are—in the sense that the pair (D_Δ, χ) is unique up to a unique isomorphism [V1, Prop. 8.2; YZ1, Thm. 3.2].

Although dualizing complexes (rigid or otherwise) do not exist for all finitely generated noncommutative Noetherian rings [KRS, p. 529], by [Y2, Prop. 5.7] and [YZ2, Thm. 3.8] they do exist for our rings R and Δ.

Write $d = \operatorname{GKdim} \Delta = \operatorname{GKdim} R$. The cohomology of D_R and D_Δ lies in degrees $\geq -d$, and we define $\omega_R = H^{-d}(D_R)$ and $\omega_\Delta = H^{-d}(D_\Delta)$. An important fact [YZ1, Cor. 3.6] is that the cohomology of D_Δ is *Z-central* in the sense that the left and right actions of Z agree. In particular, ω_Δ is Z-central.

The following results give some basic properties that we will need about these objects. If M is Δ-bimodule, then $Z(M) = \{w \in M : \delta w = w\delta \text{ for all } \delta \in \Delta\}$ is called the *center* of M.

LEMMA 2.5. *Assume that R is an essentially affine k-algebra. Then*:

(1) $D_\Delta \cong \operatorname{RHom}_R(\Delta, D_R)$ *in* $D(\Delta^e)$;
(2) $\omega_\Delta \cong \operatorname{Hom}_R(\Delta, \omega_R)$ *as Δ^e-modules*;
(3) *if $\mathcal{C} \subset Z$ is multiplicatively closed, then $\omega_{\Delta_\mathcal{C}} \cong (\omega_\Delta)_\mathcal{C}$ as Δ-bimodules.*

Assume in addition that Δ is a tame R-order. Then:

(4) *ω_Δ is reflexive as a left or right Δ-module and is invertible in codimension 1*;
(5) *there is a canonical isomorphism $Z(\omega_\Delta) = \omega_Z$.*

Proof. (1) The proof of [Y2, Prop. 5.7] shows that $\operatorname{RHom}_R(\Delta, D_R)$ is a rigid dualizing complex for Δ and so the result follows by the uniqueness of D_Δ.

(2) Take cohomology of (1).

(3) By [YZ2, Thm. 3.8], $D_{\Delta_\mathcal{C}} \cong \Delta_\mathcal{C} \overset{L}{\otimes}_\Delta D_\Delta \overset{L}{\otimes} \Delta_\mathcal{C}$ as Δ-bimodules. Now take cohomology, using that, as already mentioned, each $\mathrm{H}^q(D_\Delta)$ is Z-central.

(4) By part (2) and [Si, Lemma 1.5], it suffices to prove the result for ω_R. This case is well known, but here is an easy proof. By part (3) we may assume that R is a normal affine k-algebra. By Noether normalization, R is a finitely generated module over some polynomial subring R_0 and is a tame R_0-order because it is normal. It is standard [YZ2, Ex. 3.13] that $\omega_{R_0} \cong R_0$ as bimodules, so [Si, Lemma 1.5] and Lemma 2.1 imply that $\omega_R \cong \operatorname{Hom}_{R_0}(R, \omega_{R_0})$ is a reflexive R-module. That ω_Δ is invertible in codimension 1 follows, for example, from [CuR, Cor. 37.9] combined with part (2).

(5) By part (2), $\omega_\Delta \cong \operatorname{Hom}_R(\Delta, \omega_R)$ and so

$$Z(\omega_\Delta) = \{\theta \in \omega_\Delta : \theta(\delta_1\delta_2 - \delta_2\delta_1) = 0 \text{ for all } \delta_j \in \Delta\}. \tag{2.6}$$

Let $\tau\colon \Delta \to Z$ be the reduced trace map; thus $\Delta = Z \oplus \ker\tau$ by Lemma 2.2. We have maps $\alpha\colon Z(\omega_\Delta) \to Z(\omega_Z) = \omega_Z$ given by restriction of functions and $\beta\colon \omega_Z \to Z(\omega_\Delta)$ defined by $\beta(\phi)(a \oplus b) = \phi(a)$ for $a \in Z, b \in \ker\tau$, and $\phi \in \omega_Z$.

Clearly $\alpha\beta(\phi) = \phi$ for all $\phi \in \omega_Z$ and so it suffices to prove that α is injective. Let Z have field of fractions F with algebraic closure $\bar{F}$, and write $\bar{F}\Delta$ as a full matrix ring $M_m(\bar{F})$ for some m. Since $\mathfrak{sl}_m(\bar{F})$ is a simple Lie algebra, it is spanned by commutators and hence is also spanned by $\{[a_i, a_j]\}$ for any given $\bar{F}$-basis $\{a_\ell\}$ of $\mathfrak{sl}_m(\bar{F})$. Now let $\theta \in \ker\tau$; thus, by (2.6), θ kills all commutators in Δ. Because we can pick the $a_\ell \in \Delta$, this implies that θ kills $\mathfrak{sl}_m(\bar{F})$. Since τ is a scalar multiple of the trace map in $\bar{F}\Delta$, it follows that $\theta(\ker\tau) = 0$. Since $\theta|_Z = 0$, this implies that $\theta = 0$. $\square$

In our setting there is the following very precise description of ω_Δ. This is the analogue of the Hurwitz formula in algebraic geometry.

PROPOSITION 2.7. *Assume that R is an essentially affine k-algebra and that Δ is a tame R-order. Then*

$$\omega_\Delta \cong \left(\omega_Z \otimes_Z \prod_{\mathfrak{p}\in\mathfrak{P}_1(Z)} (\Delta \cap \operatorname{rad}(\Delta_\mathfrak{p})) \cdot \mathfrak{p}^{(-1)}\right)^{**}.$$

This result is well known, but we had difficulty locating a reference that was valid in the generality we need. We therefore give the proof here.

Proof of Proposition 2.7. By Lemma 2.2, we can replace R by Z. By Lemma 2.5(2) and (4), we obtain

$$\omega_\Delta \cong \operatorname{Hom}_Z(\Delta, \omega_R) \cong (\omega_Z \otimes \operatorname{Hom}_Z(\Delta, Z))^{**}$$

and so it suffices to prove that

$$\operatorname{Hom}_Z(\Delta, Z) \cong \prod_{\mathfrak{p}\in\mathfrak{P}_1(Z)} (\Delta \cap \operatorname{rad}(\Delta_\mathfrak{p})) \cdot \mathfrak{p}^{(-1)}.$$

As usual, let F denote the field of fractions of Z, set $A = F\Delta$, and write $\tau : A \to F$ for the reduced trace map. By the nondegeneracy of τ, $\operatorname{Hom}_Z(\Delta, Z)$ is isomorphic as a Δ-bimodule to the *inverse different* $\mathcal{I}(\Delta) = \{x \in A : \tau(x\Delta) \subseteq Z\}$ (see e.g. [F, Prop. 1]). Hence we need to show that

$$\mathcal{I}(\Delta) = \prod_{\mathfrak{p}\in\mathfrak{P}_1(Z)} (\Delta \cap \operatorname{rad}(\Delta_\mathfrak{p})) \cdot \mathfrak{p}^{(-1)}.$$

It suffices to prove this result after localizing at a height-1 prime ideal $\mathfrak{p} \in \mathfrak{P}_1(Z)$. In other words, we may assume that $R = Z = Z(\Delta) = (Z, pZ)$ is a discrete valuation ring and that Δ is an hereditary order with Jacobson radical $J = \operatorname{rad}\Delta$. In this case J is invertible and, by [Si, Thm. 2.3], it generates the group $\operatorname{Div}(\Delta) \cong \mathbb{Z}$. Thus we write $p\Delta = J^e$ for some $e \geq 1$ and need to prove that $\mathcal{I}(\Delta) = J^{1-e}$ (in this situation $J^{(a)} = J^a$ for all integers a, so we can omit the parentheses). By Lemma 2.5(4) it follows that $\mathcal{I}(\Delta) = J^a$ for some $a \in \mathbb{Z}$, so we need to show $a = 1 - e$.

If $a < 1 - e$ then $p^{-1} \in J^a$ and so Lemma 2.2 implies that $Z \supseteq \tau(p^{-1} \cdot \Delta) = p^{-1}Z$, which is absurd. Hence $a \geq 1 - e$. To prove equality it is sufficient to prove that $\tau(J^{1-e}) \subset Z$ or, equivalently, that $\tau(J) \subset pZ$.

Pick a finite field extension $\hat{F}/F$ such that $\hat{F} \otimes_F A \cong M_n(\hat{F})$. Let $\hat{Z} \subset \hat{F}$ be a discrete valuation ring lying over Z, write $\hat{\Delta} = \Delta\hat{Z}$, and choose a free $\hat{Z}$-submodule $G = \bigoplus_{i=1}^n \hat{Z}g_i \subset \hat{F}^n$ such that $\hat{\Delta}G \subset G$. We may use the basis $\{g_i\}$ to compute reduced traces. Since $J^e = p\Delta \subset \operatorname{rad}\hat{Z}$, elements of J act nilpotently on $G/\operatorname{rad}\hat{Z}G$. Hence, if $x \in J$ then $\tau(x) \in \operatorname{rad}\hat{Z} \cap F = pZ$. This finishes the proof. $\square$

REMARK 2.8. We emphasize that our definition of ω_R does coincide with the usual commutative notion $\bigwedge^d(\Omega_{R/k})^{**}$ when R is essentially affine with $\operatorname{GKdim} R = d$.

Indeed, if $\omega'_R = \bigwedge^d(\Omega_{R/k})^{**}$, then [Y2, Lemma 5.4] shows that $\omega_S = \omega'_S$ holds for any regular and essentially finite domain S. Because R is normal, it is regular in codimension 1 and so Lemma 2.5(3) implies that $(\omega_R)_{\mathfrak{p}} = (\omega'_R)_{\mathfrak{p}}$ for all height-1 prime ideals $\mathfrak{p} \in \mathfrak{P}_1(R)$. By Lemma 2.5(4), ω_R and ω'_R are reflexive and so $\omega_R = \omega'_R$.

PROPOSITION 2.9. *Assume that Δ is a prime affine k-algebra. Then Δ is homologically homogeneous of dimension d if and only if* $\operatorname{gl\,dim} \Delta < \infty$ *and $D_\Delta = \Omega[d]$ for some invertible Δ-bimodule Ω. If this holds, then $\Omega = \omega_\Delta$.*

REMARK 2.10. In the notation of [V1, Sec. 8], the proposition states that Δ is homologically homogeneous of dimension d if and only if $\operatorname{gl\,dim} \Delta < \infty$ and Δ is AS-Gorenstein. See [SZ, Thms. 1.3 and 1.4] for a closely related result.

Proof of Proposition 2.9. Assume first that Δ is homologically homogeneous of dimension d. The statement of the proposition is independent of the choice of R, so by Noether normalization we may assume that R is a polynomial ring. By Theorem 2.3(1) and (3), Δ is CM and hence free as an R-module. But now $D_R = \omega_R[d] \cong R[d]$ and so $D_\Delta = \operatorname{RHom}_R(\Delta, D_R)$ lives purely in dimension $-d$, whence $D_\Delta = \omega_\Delta[d]$. Lemma 2.5(2) implies that ω_Δ is free and hence is CM as an R-module, so [BH, Cor. 3.1] implies that ω_Δ is a projective Δ-module on either side. On the other hand, since Δ is a free R-module, it follows that Δ is a tame R-order and so Lemma 2.5(4) implies that ω_Δ is invertible in codimension 1. Together with [Si, Prop. 3.1], these observations imply that ω_Δ is invertible, finishing the proof in this direction.

Conversely, assume that $\operatorname{gl\,dim} \Delta < \infty$ and that $D_\Delta = \Omega[d]$ for some invertible bimodule Ω. We must show that every simple Δ-module S has projective dimension $\operatorname{pd} S = d$. As before, we may assume that R is a polynomial ring in (say) d' indeterminates, whence $\omega_R \cong R[d']$. Since $\operatorname{Hom}_R(\Delta, R) \neq 0$ and since D_Δ is concentrated in dimension $-d$, this implies that $d = d' = \operatorname{GKdim} \Delta$. By [YZ1, Cor. 6.9], D_Δ is Auslander and GKdim-Macaulay in the sense of [YZ1, Defs. 2.1 and 2.24]. Because S is finite dimensional, this implies that $\operatorname{Ext}^d_\Delta(S, \Omega) \neq 0$ and so $\operatorname{pd} S \geq d$. If $\operatorname{gl\,dim} \Delta = e > d$ then, by [Ra, Thm. 8], there exists a simple Δ-module S with $\operatorname{Ext}^e_\Delta(S, \Delta) \neq 0$. Since Ω is invertible, this implies that $E = \operatorname{Ext}^e_\Delta(S', \Omega) \neq 0$ for $S' = \Omega \otimes_\Delta S$. By the Auslander property, it follows that $j(E) \geq e - d$ and hence that $\operatorname{Cdim}_{D_\Delta}(E) \leq d - e$ in the sense of [YZ1, Def. 2.9]. By the GKdim-Macaulay property, this implies that $\operatorname{GKdim} S < 0$, which is absurd. $\square$

The following formulas will be useful.

COROLLARY 2.11. *Assume that R is essentially affine with* $\operatorname{GKdim} R = d$, *and let Δ be a tame R-order. Then*

$$\omega_\Delta^{(-1)} = \operatorname{Ext}^d_{\Delta^e}(\Delta, \Delta^e)^{**}. \tag{2.12}$$

If Δ is homologically homogeneous, then

$$\omega_\Delta^{(-1)} = \operatorname{RHom}_{\Delta^e}(\Delta, \Delta^e)[d]. \tag{2.13}$$

Proof. If Δ is homologically homogeneous then it has dimension d by Theorem 2.3. Thus [V1, Prop. 8.4] and Remark 2.10 combine to prove (2.13).

Now suppose that Δ is a tame R-order and set $\Gamma = \Delta_{\mathfrak{p}}$ for some $\mathfrak{p} \in \mathfrak{P}_1(Z)$. Then Γ is an hereditary order and, by [McRo, Thm. 13.10.1], $Z(\Gamma)$ is a local PID. By Lemma 2.5(4), ω_Δ is invertible and hence, just as in the proof of Proposition 2.9, $D_\Delta = \omega_\Delta[d]$. Thus, [V1, Prop. 8.4] can again be applied to show that $\omega_\Gamma^{(-1)} = \operatorname{Ext}^d_{\Gamma^e}(\Gamma, \Gamma^e)^{**}$. In other words, (2.12) holds in codimension 1. Since both sides of that equation are reflexive, it holds everywhere. □

3. Reduction to the Calabi–Yau Case

Let Δ be a homologically homogeneous ring. In Section 4 we will use the structure of ω_Δ to show that Z has rational singularities, but this is awkward to prove when ω_Δ is not cyclic. In this section we show how to use a trick from [NV, Thm. 3.1] to (locally) replace Δ by an order for which ω_Δ is generated by a single central element. This is a noncommutative generalization of a well-known technique in algebraic geometry where one constructs a Gorenstein cover of a $\mathbb{Q}$-Gorenstein singularity.

Given a tame R-order Γ in A and $I \in \operatorname{Div}(\Gamma)$, the *Rees ring* $\Gamma[I]$ of Γ is defined to be the subring $\sum_{n=-\infty}^{\infty} I^{(n)}x^n$ of the Laurent polynomial ring $A[x, x^{-1}]$.

Proposition 3.1. *Assume that Δ is homologically homogeneous. For some $n \geq 1$, suppose that $\omega_\Delta^{\otimes n} \cong \Delta$ as bimodules and choose n minimal with this property. Write*

$$\Lambda = \Delta \oplus \omega_\Delta \oplus \omega_\Delta^{\otimes 2} \oplus \cdots \oplus \omega_\Delta^{\otimes n-1},$$

where the multiplication is defined using the isomorphism $\omega_\Delta^{\otimes n} \cong \Delta$. Then:

(1) *Λ is a prime homologically homogeneous ring*;
(2) *$\omega_\Lambda \cong \Lambda$ as Λ-bimodules.*

Proof. (1) By Theorem 2.3(3) and (4), Z is an affine normal domain and Δ is a tame Z-order in its simple Artinian ring of fractions A. By [YZ1, Cor. 3.6], ω_Δ is Z-central and so Lemma 2.5(4) implies that ω_Δ is isomorphic to a divisorial fractional ideal I. As a result, $I^{(n)} = \Delta a$ for some $a \in L = Z(A)$ and so $\Lambda \cong \Delta[I]/(1 - ax^n)$. The field of fractions of Λ is therefore

$$B = A \otimes_L L[x]/(1 - ax^n).$$

By [Si, Thm. 2.3], $\operatorname{Div}(\Delta)$ is a free abelian group. Therefore, if $a = b^m$ for some $m > 1$ and $b \in L$ then we would have $m \mid n$ and $I^{(n/m)} = \Delta b$, contradicting the minimality of n. It follows that $L[x]/(1 - ax^n)$ is a field and thus B is a central simple algebra. Consequently, Λ is prime.

The ring Λ is strongly graded and hence $\operatorname{gl\,dim} \Lambda = \operatorname{gl\,dim} \Delta$ follows from [McRo, Cor. 7.6.18] together with the equivalence of the categories of Δ-modules and graded Λ-modules. Thus $\operatorname{gl\,dim} \Lambda = \operatorname{gl\,dim} \Delta = \operatorname{GKdim} \Delta = \operatorname{GKdim} \Lambda$ by Theorem 2.3(2) and [McRo, Prop. 8.2.9(ii)]. By Theorem 2.3(1), Δ is CM as a Z-module and hence so is each $\omega_\Delta^{\otimes j}$ and Λ. Thus Λ is homologically homogeneous by Lemma 2.4.

(2) Using the formula $\omega_\Lambda = \mathrm{Hom}_R(\Lambda, \omega_R)$, we compute that

$$\omega_\Lambda = \omega_\Delta \oplus \omega_\Delta^{\otimes 2} \oplus \cdots \oplus \omega_\Delta^{\oplus n-1} \oplus \Delta$$

as $\mathbb{Z}/n\mathbb{Z}$-graded Λ-bimodules. Forgetting the grading gives the result. □

REMARKS 3.2. (1) Assume that Z is an essentially affine k-algebra. Following [Br] or [G], Δ is called *Calabi–Yau of dimension d* if $D_\Delta \cong \Delta[d]$ in $D(\Delta^e)$. (Some authors also require Calabi–Yau algebras to have finite global dimension; see e.g. [IR, Thm. 3.2(iii)].) For a survey on the Calabi–Yau property in an algebraic context, see [G].

By Proposition 2.9, an affine Calabi–Yau algebra of finite global dimension is automatically homologically homogeneous. Conversely, Proposition 3.1 can be regarded as a reduction to the Calabi–Yau case.

(2) Proposition 3.1 can also be regarded as a reduction to the case of orders unramified in codimension 1. In order to explain this, recall that a tame order Δ is unramified in codimension 1 if $\mathfrak{p}\Delta_\mathfrak{p} = \mathrm{rad}\,\Delta_\mathfrak{p}$ for all $p \in \mathfrak{P}_1(Z)$. Given a tame Calabi–Yau order Δ, it follows from Lemma 2.5(5) that $Z \cong Z(\omega_\Delta) = \omega_Z$ and so Proposition 2.7 implies that $\mathfrak{p}\Delta_\mathfrak{p} = \mathrm{rad}\,\Delta_\mathfrak{p}$ for all $\mathfrak{p} \in \mathfrak{P}_1(Z)$.

Even when Δ is homologically homogeneous, there is no reason for ω_Δ to have finite order and so Proposition 3.1 cannot be applied directly. However, ω_Δ has finite order locally, which will be sufficient for our applications. Before stating the result, we prove some elementary facts.

LEMMA 3.3. *If S is a ring with Jacobson radical* $\mathrm{rad}(S)$ *and if P is an invertible S-bimodule, then* $\mathrm{rad}(S)P = P\,\mathrm{rad}(S)$.

Proof. We claim that the image of composition

$$\chi : P^{-1} \otimes_S \mathrm{rad}(S) \otimes_S P \to P^{-1} \otimes_S S \otimes_S P \cong S \tag{3.4}$$

lies in $\mathrm{rad}(S)$. This proves the inclusion $\mathrm{rad}(S)P \subseteq P\,\mathrm{rad}(S)$. To prove the opposite inclusion, interchange P and P^{-1}.

In order to prove the claim, we will show that the image of χ annihilates all simple S-modules. Let M be a simple S-module. We must show that the map

$$P^{-1} \otimes_S \mathrm{rad}(S) \otimes_S P \otimes_S M \to M \tag{3.5}$$

is zero. Tensoring (3.5) on the left by P yields the map

$$\mathrm{rad}(S) \otimes_S P \otimes_S M \to P \otimes_S M. \tag{3.6}$$

Since $P \otimes_S -$ is an autoequivalence of $\mathrm{Mod}(S)$, it follows that $P \otimes_S M$ is a simple module and hence (3.6) is indeed the zero map. □

LEMMA 3.7. *Assume that R is local and that Γ is a tame R-order in A, with $R = Z(\Gamma)$. If P is an R-central invertible Γ-bimodule, then there exists an integer $n > 0$ such that $P^{\otimes n} \cong \Gamma$ as Γ-bimodules.*

Proof. Because P is invertible, tensor powers, symbolic powers, and ordinary powers all coincide. Hence we drop the tensor product sign.

We first prove that $P^n \cong \Gamma$ as left Γ-modules. By Lemma 3.3, $P/\operatorname{rad}(\Gamma)P$ is an invertible bimodule over $\Gamma/\operatorname{rad}(\Gamma)$. Since $\Gamma/\operatorname{rad}(\Gamma)$ is semisimple, it is easy to see that there exists an $n > 0$ such that

$$P^n/\operatorname{rad}(\Gamma)P^n = (P/\operatorname{rad}(\Gamma)P)^n \cong \Gamma/\operatorname{rad}(\Gamma)$$

as left $\Gamma/\operatorname{rad}(\Gamma)$-modules. By Nakayama's lemma it follows that $P^n \cong \Gamma$, again as left Γ-modules.

Let K denote the fraction field of R. Since P is R-central, $K \otimes_R P$ is an invertible A-bimodule. After choosing an isomorphism $K \otimes_R P \cong A$, we may assume that P is a divisorial fractional R-ideal. By [LeVV, Prop. II.4.20], some power $P^{(e)}$ of P lies in the image of $\operatorname{Div}(Z(\Gamma))$ in $\operatorname{Div}(\Gamma)$; that is, $P^{(e)} \cong (\Gamma I)^{**}$ for some reflexive ideal I of R. By the preceding paragraph, we may also assume that $P^e \cong \Gamma$ as left Γ-modules.

Now let $u = \operatorname{rk}_R \Gamma$ and write $\bigwedge^u(M) = \bigwedge^u_R(M)$ for the uth exterior power of an R-module M. Then $\left(\bigwedge^u(\Gamma I)\right)^{**} \cong \bigwedge^u P^e \cong \left(\bigwedge^u(\Gamma)\right)^{**}$ as R-modules. On the other hand,

$$\left(\bigwedge^u(\Gamma I)\right)^{**} = \left(\left(\bigwedge^u \Gamma\right) I^u\right)^{**} = \left(\bigwedge^u \Gamma\right)^{**} (I^u)^{**}$$

and so $\left(\bigwedge^u \Gamma\right)^{**}(I^u)^{**} = \left(\bigwedge^u(\Gamma)\right)^{**}$. Canceling $\left(\bigwedge^u \Gamma\right)^{**}$ gives $(I^u)^{**} \cong R$ as R-modules. Since $P^e \cong (\Gamma I)^{**}$ as Γ-bimodules, we obtain $P^{eu} \cong \Gamma$ as Γ-bimodules. □

Corollary 3.8. *Suppose that Δ is homologically homogeneous k-algebra. Then for every maximal ideal $\mathfrak{m}$ of Z there exist $f \in Z \setminus \mathfrak{m}$ and $n > 0$ with the property that $\omega_{\Delta_f}^{\otimes n} \cong \Delta_f$ as Δ_f-bimodules.*

Proof. By Proposition 2.9, ω_Δ is invertible and (as we have already seen) is Z-central. By Lemma 2.5(3) and Theorem 2.3(4), we can therefore apply Lemma 3.7 to $P = \omega_{\Delta_\mathfrak{m}}$ and conclude that $\omega_{\Delta_\mathfrak{m}}^{\otimes n} \cong \Delta_\mathfrak{m}$ as $\Delta_\mathfrak{m}$-bimodules. As usual, this isomorphism may be "spread out" on a neighborhood of $\mathfrak{m}$ in Spec Z. □

4. The Center of Homologically Homogeneous Rings

In this section we prove Theorem 1.1 from the Introduction. We start with two preparatory lemmas, the first of which gives a useful algebraic criterion for a ring to have rational singularities.

Lemma 4.1. *Let Z be an affine normal CM k-domain with field of fractions K. Then Z has rational singularities if and only if, for all regular affine k-algebras S satisfying $Z \subseteq S \subset K$, we have $\omega_Z \subseteq \omega_S$ inside ω_K.*

Proof. Let $X = \operatorname{Spec} Z$. By Remark 2.8, ω_X in the sense of [KKMS, KoMo] is equal to ω_Z in the sense of this paper and so, by Lemma 2.5(4), ω_X is reflexive. According to [KKMS, p. 50] or [KoMo, Thm. 5.10], X has rational singularities if and only if for one (or for all) resolution(s) of singularities $f: Y \to X$ we have

$f_*\omega_Y = \omega_X$ inside ω_K. Since ω_X and ω_Y are reflexive this is equivalent to $\omega_X \subseteq f_*\omega_Y$, which in turn is equivalent to $(f^*\omega_X)^{**} \subseteq \omega_Y$. This can be checked locally on Y.

So assume that $\omega_Z \subseteq \omega_S$ for all affine regular k-algebras S satisfying $Z \subseteq S \subset K$. Pick Y by the previous paragraph as well as an open affine subset $U \subset Y$. Then $\omega_Z \subseteq \omega_S$ for $S = \mathcal{O}(U)$ and hence $(S \otimes_Z \omega_Z)^{**} \subseteq \omega_S$. Globalizing gives $(f^*\omega_X)^{**} \subseteq \omega_Y$, so Z has rational singularities.

Conversely, assume that Z has rational singularities and let $Z \subseteq S$ be as in the statement of the lemma. Put $U = \operatorname{Spec} S$. We may compactify the map $g\colon U \to X$ to a projective map $\bar{g}\colon Y' \to X$. A priori Y' will not be smooth, but we can resolve it further without touching U (see [KoMo, Thm. 0.2]) to arrive at a resolution of singularities $f\colon Y \to X$. The fact that $(f^*\omega_X)^{**} \subseteq \omega_Y$ then implies that $(S \otimes_Z \omega_Z)^{**} \subseteq \omega_S$ after restricting to U. Therefore, $\omega_Z \subseteq \omega_S$. $\square$

LEMMA 4.2. *Let Λ_1 and Λ_2 be affine k-algebras of finite global dimension that satisfy a polynomial identity. Then $\Lambda_1 \otimes_k \Lambda_2$ has finite global dimension.*

Proof. By the Nullstellensatz [McRo, Thm. 13.10.3], every primitive factor ring of Λ_i is isomorphic to a full matrix ring over k. Hence every primitive factor ring Γ of $\Lambda = \Lambda_1 \otimes_k \Lambda_2$ decomposes as $\Gamma = \Gamma_1 \otimes_k \Gamma_2$ for primitive factor rings Γ_i of Λ_i. Thus, any simple Λ-module M can be written as $M = M_1 \otimes_k M_2$, where each M_i is a simple Λ_i-module. Now use [CE, Prop. IX.2.6]. $\square$

THEOREM 4.3. *If Δ is a homologically homogeneous k-algebra, then $Z = Z(\Delta)$ has rational singularities.*

Proof. It is enough to prove the result locally, so by Corollary 3.8 we can replace Δ by some Δ_f and assume that $\omega_\Delta^{\otimes n} \cong \Delta$ as Δ-bimodules. By Proposition 3.1, the algebra $\Lambda = \Delta \oplus \omega_\Delta \oplus \omega_\Delta^{\otimes 2} \oplus \cdots \oplus \omega_\Delta^{\otimes n-1}$ satisfies $\omega_\Lambda \cong \Lambda$ as Λ-bimodules. Then Λ and hence $Z(\Lambda)$ are $\mathbb{Z}/n\mathbb{Z}$-graded. Moreover, since ω_Δ is Z-central, clearly Z commutes with each $\omega_\Delta^{\otimes j}$ and so $Z \subseteq Z(\Lambda)_0$. The other inclusion is trivial, so $Z = Z(\Lambda)_0$ and Z is a module-theoretic summand of $Z(\Lambda)$. Because a direct summand of a ring with rational singularities has rational singularities [Bou], we may replace Δ by Λ and assume that $\omega_\Delta \cong \Delta$ as bimodules. By Proposition 3.1(1), Δ remains homologically homogeneous.

We will use Lemma 4.1, so fix a ring $Z \subseteq S \subset K$ as in the lemma and let Γ be a maximal (and therefore tame) S-order containing $S\Delta$ inside the simple Artinian ring of fractions A of Δ. Our discussion in Section 2 on dualizing complexes also applies with (R, Δ) replaced by (S, Γ), so $\omega_\Gamma = \operatorname{Hom}_S(\Gamma, \omega_S)$ and $\omega_S = Z(\omega_\Gamma)$ in the notation developed there. We will show that $\omega_\Delta \subseteq \omega_\Gamma$ inside ω_A. Since $S \subset K$, this will yield $Z(\omega_\Delta) \subseteq Z(\omega_\Gamma)$ as subgroups of $Z(\omega_A)$ and so Lemma 2.5(5) will imply that $\omega_Z \subseteq \omega_S$, as required.

In order to prove that $\omega_\Delta \subseteq \omega_\Gamma$ we may as well prove that $\omega_\Delta\Gamma \subseteq \omega_\Gamma$. The bimodule isomorphism $\omega_\Delta \cong \Delta$ means that $\omega_\Delta = c\Delta$ for some central element $c \in \omega_A$. From this we deduce that $\omega_\Delta\Gamma = \Gamma\omega_\Delta$ is an invertible Γ-bimodule with inverse $\omega_\Delta^{(-1)}\Gamma = c^{-1}\Gamma$. By Lemma 2.5(4), ω_Γ is reflexive and so it suffices to prove that $\omega_\Gamma^{(-1)} \subseteq \omega_\Delta^{(-1)}\Gamma$ inside $\omega_A^{(-1)}$.

We claim that

$$\Gamma \overset{L}{\otimes}_\Delta \mathrm{RHom}_{\Delta^e}(M, \Delta^e) \overset{L}{\otimes}_\Delta \Gamma = \mathrm{RHom}_{\Gamma^e}\big(\Gamma \overset{L}{\otimes}_\Delta M \overset{L}{\otimes}_\Delta \Gamma, \Gamma^e\big) \tag{4.4}$$

for any object M in $D^b(\Delta^e)$ with finitely generated cohomology. To prove this, recall that gl dim $\Delta^e < \infty$ by Lemma 4.2. Thus we can replace M by a finite projective resolution of Δ^e-modules, whereafter it suffices prove the claim for $M = \Delta^e$. This case is obvious.

Applying (4.4) with $M = \Delta$ and using the formula $\omega_\Delta^{(-1)} = \mathrm{RHom}_{\Delta^e}(\Delta, \Delta^e)[d]$ from (2.13), we obtain

$$\Gamma \overset{L}{\otimes}_\Delta \omega_\Delta^{(-1)} \overset{L}{\otimes}_\Delta \Gamma = \mathrm{RHom}_{\Gamma^e}\big(\Gamma \overset{L}{\otimes}_\Delta \Gamma, \Gamma^e\big)[d].$$

Because the derived tensor product maps to the ordinary tensor, we can now induce a composed map

$$\begin{aligned}\mathrm{RHom}_{\Gamma^e}(\Gamma, \Gamma^e)[d] \to \mathrm{RHom}_{\Gamma^e}\big(\Gamma \overset{L}{\otimes}_\Delta \Gamma, \Gamma^e\big)[d] \\ = \Gamma \overset{L}{\otimes}_\Delta \omega_\Delta^{(-1)} \overset{L}{\otimes}_\Delta \Gamma \to \omega_\Delta^{(-1)}\Gamma.\end{aligned}$$

Taking cohomology in degree 0 and then biduals gives a map

$$\mathrm{Ext}^d_{\Gamma^e}(\Gamma, \Gamma^e)^{**} \to (\omega_\Delta^{(-1)}\Gamma)^{**} = \omega_\Delta^{(-1)}\Gamma.$$

We can then use (2.12) to induce the map

$$\omega_\Gamma^{(-1)} \to \omega_\Delta^{(-1)}\Gamma. \tag{4.5}$$

We could have performed these computations after tensoring with the field of fractions K of Z. Since $K = KZ = KS$ and $K\Delta = K\Gamma = A$, all morphisms would then have been (canonically) the identity. From this we deduce that (4.5) is an inclusion that takes place inside $\omega_A^{(-1)}$, so we are done. □

REMARKS 4.6. (1) Suppose that Δ is an affine Calabi–Yau k-algebra of finite global dimension. Then Theorem 4.3 and Remark 3.2(1) combine to prove that Z has rational singularities.

(2) Homologically homogeneous rings were defined in [BH] for orders in semisimple rather than simple Artinian rings. However, by [BH, Thm. 5.3], these more general algebras are direct sums of prime homologically homogeneous rings; the more general case also follows from this theorem. Similarly, one can weaken the hypothesis that Δ be finitely generated as a module over its center to the assumption that it be an affine algebra satisfying a polynomial identity—since, by [SZ, Thm. 5.6(iv)], this already forces Δ to be a finitely generated Z-module.

5. Examples

Here we give two examples to illustrate the preceding results. The first shows that [V2, Lemma 4.2] cannot be improved, and the second shows that Theorem 1.1 can fail in finite characteristic.

In addition to our standing hypotheses, suppose that R is an affine Gorenstein k-algebra and that $\Delta = \mathrm{End}_R(M)$ for some finitely generated reflexive R-module

M. Then it follows from [V2, Lemma 4.2] that Δ is homologically homogeneous if and only if $\operatorname{gl}\dim \Delta < \infty$ and Δ is a CM R-module. This is useful for the theory of noncommutative crepant resolutions, so it would be useful if one could weaken the hypotheses in this result.

In our first example, we show that the Gorenstein condition is necessary. Let T be a one-dimensional torus acting on the generators of the polynomial ring $S = k[x_1, x_2, x_3, x_4, x_5]$ with respective weights $1, 1, 1, -1, -1$, and let $R = S^T$. We may also view R as the coordinate ring of the variety of 2×3 matrices of rank ≤ 1.

The T-weights give a grading $S = \bigoplus_{\ell=-\infty}^{\infty} S_\ell$ with $S_0 = R$. According to the proof of [V2, Lemma 8.8], the S_i are isomorphic to reflexive ideals of R with $S_{a+b} = (S_a S_b)^{**}$ for all $a, b \in \mathbb{Z}$. Furthermore, it is easy to see that S_i is not a projective R-module when $i \neq 0$.

It follows from [V2, Lemma 8.1] that S_{-2}, S_{-1}, R, and S_1 are CM R-modules, while R is certainly normal. Therefore, by [V2, Lemma 8.2 and Thm. 8.6],

$$\Delta = \operatorname{End}_R(R \oplus S_1) = \begin{pmatrix} R & S_1 \\ S_{-1} & R \end{pmatrix}$$

has finite global dimension and hence is a tame order over its center R. By [Kn, Kor. 2], the dualizing module ω_R is isomorphic to S_{-1} (where -1 represents the negative of the sum of the weights of the generators of S), from which we deduce that

$$\omega_\Delta = \operatorname{Hom}_R(\Delta, \omega_R) \cong \begin{pmatrix} S_{-1} & R \\ S_{-2} & S_{-1} \end{pmatrix}.$$

Both Δ and ω_Δ are graded for the standard grading on R. For this choice of grading, Δ is graded semilocal and ω_Δ is (as left or right module) not a direct sum of indecomposable graded Δ-projectives. Consequently, ω_Δ is not projective.

By Proposition 2.9, Δ is therefore not homologically homogeneous.

REMARKS 5.1. (1) By the proof of [DV, Prop. 1.2], it follows that ω_Δ defines an element of the *derived* Picard group of Δ.

(2) The methods of [BuLV] allow one to treat this first example in the context of determinantal varieties. It follows from the results given there that one of the simple graded Δ-modules has projective dimension 4 and the other has projective dimension 5.

The example leads naturally to the following question.

QUESTION 5.2. Assume that $Z = Z(\Delta)$ is an affine normal k-domain and that Δ is a finitely generated CM Z-module with finite global dimension. Then, does Z have rational singularities?

We now turn to an example in finite characteristic of a homologically homogeneous ring with CM center but without rational singularities in any reasonable sense.

Assume that F is a field of characteristic 2 and let $C = F[u, v, x, y]/(p, q)$, where

$$p = x + u^2 + x^2 u \quad \text{and} \quad q = y + v^2 + y^2 v.$$

Because the Jacobian matrix of p, q with respect to x, y is invertible, $C/F[x, y]$ is étale and hence C is regular. Consider the action of $G = \mathbb{Z}/(2) = \{1, \sigma\}$ on C by $\sigma(u) = u + x^2$, $\sigma(v) = v + y^2$, $\sigma(x) = x$, and $\sigma(y) = y$. Clearly $B = C^G$ is an affine normal domain of Krull dimension 2 and hence is CM.

Resolutions of singularities are known to exist for surfaces in all characteristics, and there is a corresponding satisfactory theory of rational singularities. We will show that B does not have rational singularities. Let $\mathfrak{m} = (u, v) \subset C$ and observe that $\mathfrak{m} = (u, v, x, y)$ is maximal; thus $\hat{C}_{\mathfrak{m}} = F[[u, v]]$. It suffices to prove that $\hat{B}_{\mathfrak{n}}$ ($\mathfrak{n} = B \cap \mathfrak{m}$) does not have rational singularities. Since $uu^\sigma = u^2 + ux^2 = x \in \hat{C}_{\mathfrak{m}}^G = \hat{B}_{\mathfrak{n}}$ and $vv^\sigma = y \in \hat{B}_{\mathfrak{n}}$, our notation conforms with that of [A, Thm.]. Now $u^2 + x^2u + x = 0 = v^2 + y^2v + y$ implies that $\hat{B}_{\mathfrak{n}}$ does not have rational singularities, by the observation from [A, p. 64].

Finally, let $\Lambda = C[z; \sigma]$ be the twisted polynomial ring; thus $zc = c^\sigma z$ for all $c \in C$. By the Nullstellensatz, every simple Λ-module is finite dimensional and so, by [McRo, Thm. 7.9.16], Λ is homologically homogeneous of dimension 3. Since $\sigma^2 = 1$, the element z^2 is central. It follows routinely that $Z(\Lambda) = B[z^2]$. Therefore, $Z(\Lambda)$ also does not have rational singularities.

The basic reason why such counterexamples exist in bad characteristic is that a fixed ring S^G need not be a summand of the ring S. The example of a homologically homogeneous ring with a non-CM center [BHMa, Ex. 7.3] occurs for a similar reason. So it is natural to pose the following question.

Question 5.3. Suppose that Λ is a homologically homogeneous ring whose center $Z(\Lambda)$ is an affine F-algebra for field F of characteristic $p > 0$. If $Z(\Lambda)$ is a $Z(\Lambda)$-module summand of Λ, then does $Z(\Lambda)$ have rational singularities?

It was conjectured in [V3] and proved in [V2, Thm. 6.6.3] that a three-dimensional k-variety with terminal singularities has a noncommutative crepant resolution if and only if it has a commutative one (see also [IR, Cor. 8.8]). We end by remarking that this is not true in higher dimensions. One way to produce counterexamples is with the fixed ring $R = \mathbb{C}[V]^G$ of a finite group $G \subset \mathrm{SL}(V)$, where $V = \mathbb{C}^n$. In this case, the twisted group ring $\mathbb{C}[V] * G \cong \mathrm{End}_R(\mathbb{C}[V])$ is a noncommutative crepant resolution of R [V2, Ex. 1.1], but it is well known that such a ring R need not have a commutative crepant resolution (see e.g. [Ka1, Thm. 1.7]).

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