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Rationality of Hilbert–Kunz Multiplicities: A Likely Counterexample

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1. A Conjecture

At a 2004 Banff workshop, I gave a talk to demonstrate that, in many cases of interest, the Hilbert–Kunz multiplicity of a hypersurface is a rational number. (Mel Hochster, in the audience, told me a curious general fact: the set of possible Hilbert–Kunz multiplicities is countable.)

At the time I suspected that Hilbert–Kunz multiplicities must be rational. But soon after the workshop I found reason to change my opinion, and in this paper I suggest that a certain hypersurface defined by a 5-variable polynomial has $\frac{4}{3}+\frac{5}{14\sqrt{7}}$ as its Hilbert–Kunz multiplicity.

Throughout, q will denote a power 2^n of 2 with $n \geq 0$, and H will be the element x^3+y^3+xyz of $\mathbb{Z}/2[x,y,z]$; $e_n(H^j)$ is the colength, $\deg(x^q, y^q, z^q, H^j)$, of the ideal (x^q, y^q, z^q, H^j). It is known [1, Thm. 3] that $e_n(H)$ is $\frac{7q^2-q-3}{3}$ or $\frac{7q^2-q-5}{3}$ according as $q \equiv 1$ or 2 modulo 3. I'll present conjectured formulas of similar type for $e_n(H^j)$, with j arbitrary, that are strongly supported by computer calculation. I show that if these hold then the Hilbert–Kunz multiplicity of $uv + H(x,y,z)$ is $\frac{4}{3}+\frac{5}{14\sqrt{7}}$.

Explicitly, I define numbers u_j and v_j and conjecture that, if $q \geq j$, then $e_n(H^j) = \frac{jq(7q-j)}{3}+u_j$ or $\frac{jq(7q-j)}{3}+v_j$ according as $q \equiv 1$ or 2 modulo 3. The definition of u_j and v_j is complicated and may appear to be unmotivated. In fact, it is related to ideas from [2], and the reader will find a somewhat less mysterious form of our conjecture, connected to these ideas, in Section 3 of this paper.

To define u_j and v_j, I introduce some notation.

Definition 1.1. Γ is the free abelian group on symbols $[0],[1],[2],\dots$ and E. σ_0 and σ_1 are the endomorphisms of Γ that satisfy the following statements.

(1) $\sigma_0([i]) = [i+1]$ for even i and $[i-1]+E$ for odd i; $\sigma_0(E) = 2E$.
(2) $\sigma_1([i]) = [i-1]+E$ for even $i \neq 0$ and $[i+1]$ for odd i; $\sigma_1([0]) = [0]$ and $\sigma_1(E) = 2E$.

Definition 1.2. If $0 \leq j < q$ then we define an element $f(q,j)$ of Γ inductively as follows:

$$f(1,0) = [0], \quad f(2q,2k) = \sigma_0 f(q,k), \quad f(2q,2k+1) = \sigma_1 f(q,k).$$

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Note that, by induction, $f(q,j) = [i] + mE$ for some i and $m \geq 0$.

LEMMA 1.3. $f(4q,j) - f(q,j) = qE$.

Proof. We argue by induction on q. If $q = 1$, then $j = 0$ and we have $f(4,0) = \sigma_0\sigma_0([0]) = f(1,0) + E$. Given that the result holds for a given q, we show that $f(8q,j) - f(2q,j) = 2qE$. Set $k = \lfloor \frac{j}{2} \rfloor$. Then $f(4q,k) - f(q,k) = qE$, and we obtain the result by applying σ_0 or σ_1 according as $j = 2k$ or $2k+1$. $\square$

DEFINITION 1.4. If $f(q,j) = [i] + mE$, then $\langle q,j \rangle$ is the integer $2m + 4i + 3$.

Our conjecture is simply stated using the $\langle q,k \rangle$.

CONJECTURE 1.5. *If* $0 \leq k < q$, *then*

$$e_{n+1}(H^{2k+1}) - \tfrac{1}{2}(e_{n+1}(H^{2k}) + e_{n+1}(H^{2k+2})) = \langle q,k \rangle.$$

Throughout the paper we shall use that $e_{n+1}(H^{2j}) = 8e_n(H^j)$. This follows directly from the observation that $\mathbb{Z}/2[x,y,z]$ is free of rank 8 over $\mathbb{Z}/2[x^2,y^2,z^2]$.

Observe that, if $0 \leq j \leq q$, then

$$\begin{aligned} e_{n+1}(H^{q+j}) &= \deg(x^{2q}, y^{2q}, z^{2q}, (xyz)^q H^j) \\ &= 8q^3 - \deg((x^{2q}, y^{2q}, z^{2q}) : (xyz)^q H^j) \\ &= 8q^3 - \deg((x^q, y^q, z^q) : H^j) = e_n(H^j) + 7q^3. \end{aligned}$$

Hence the left-hand side of Conjecture 1.5 is unchanged when k and n are replaced by $k+q$ and $n+1$. Also, an induction shows that $f(q,k) = f(2q,q+k)$. Thus, if Conjecture 1.5 holds for a given q and k, then it also holds for $2q$ and $q+k$.

REMARKS. When $k = 0$, Conjecture 1.5 states that $e_{n+1}(H) - 4e_n(H) = \frac{2q+7}{3}$ or $\frac{2q+17}{3}$ according as $q \equiv 1$ or 2 modulo 3. This is true because $e_n(H) = \frac{7q^2-q-3}{3}$ (resp. $\frac{7q^2-q-5}{3}$). When $k > 0$, computer calculations using Macaulay 2 strongly support the conjecture. We have verified it for $k < q \leq 256$. This is tedious when $q = 256$, as the program must be exited frequently to avoid computer overflow. To avoid overflow when $128 \leq k < 256$, we made use of the final sentence of the previous paragraph, taking $q = 128$. Teixeira has checked the conjecture in some other cases: when $q = 512$ and $k \leq 101$, when $q = 1024$ and $k \leq 41$, when $q = 2048$ and $k \leq 9$, and when $q = 4096$ and $k \leq 4$.

DEFINITION 1.6. $r_j = \langle q,j \rangle - \frac{2q}{3}$, where q is a power of 2 with $q > j$ and $q \equiv 1 \bmod 3$; s_j is defined similarly but taking $q \equiv 2 \bmod 3$. (By Lemma 1.3, $\langle 4q,j \rangle - \frac{8q}{3} = \langle q,j \rangle - \frac{2q}{3}$; hence r_j and s_j do not depend on q.)

EXAMPLE 1.7. For $0 \leq j \leq 7$, the r_j are $\frac{7}{3}, \frac{25}{3}, \frac{13}{3}, \frac{1}{3}, \frac{7}{3}, \frac{25}{3}, -\frac{5}{3}, \frac{13}{3}$ and the s_j are $\frac{17}{3}, \frac{5}{3}, \frac{29}{3}, \frac{11}{3}, -\frac{1}{3}, \frac{17}{3}, \frac{5}{3}, -\frac{7}{3}$.

We can now introduce the u_j and v_j mentioned previously. We give an alternative version of Conjecture 1.5 expressing $e_n(H^j)$ in terms of u_j or v_j according as $q \equiv 1$ or 2 modulo 3.

DEFINITION 1.8.

(1) $u_1 = -1,\quad v_1 = -\frac{5}{3};$
(2) If $j = 2k,\quad u_j = 8v_k,\quad v_j = 8u_k;$
(3) If $j = 2k+1,\ u_j = 4v_k + 4v_{k+1} + s_k,\ v_j = 4u_k + 4u_{k+1} + r_k.$

For example, $u_2 = -\frac{40}{3}$ and $v_2 = -8$. So $u_3 = 4\big(-\frac{5}{3} - 8\big) + s_1 = -37$, while $v_3 = 4\big(-1 - \frac{40}{3}\big) + r_1 = -49$. Similarly, $u_5 = 4(-8 - 49) + s_2 = -\frac{655}{3}$, while $v_5 = 4\big(-\frac{40}{3} - 37\big) + r_2 = -197$.

THEOREM 1.9. *Suppose Conjecture 1.5 holds, and let $q \geq j$. Then, with u_j and v_j as before, $e_n(H^j) = jq(7q - j)/3 + u_j$ or $jq(7q - j)/3 + v_j$ according as $q \equiv 1$ or 2 modulo 3.*

Proof. Suppose first that $j = 1$. We've seen that Conjecture 1.5 for $k = 0$ implies $e_n(H) = \frac{7q^2-q-3}{3}$ or $\frac{7q^2-q-5}{3}$ according as $q \equiv 1$ or 2 modulo 3. Because u_1 and v_1 are defined to be -1 and $-\frac{5}{3}$, this is the conclusion of Theorem 1.9.

Next suppose that $j > 1$, and let $\delta_{q,j} = e_n(H^j) - \frac{7jq^2}{3} + \frac{j^2q}{3}$. We'll show by induction that $\delta_{2q,j} = u_j$ or v_j according as $2q \equiv 1$ or 2 modulo 3. Set $k = \lfloor \frac{j}{2} \rfloor$. If $j = 2k$ then $\delta_{2q,j} = 8\delta_{q,k}$. If $2q \equiv 1 \bmod 3$ then $q \equiv 2 \bmod 3$, and the induction hypothesis tells us that $\delta_{q,k} = v_k$. So $\delta_{2q,j} = 8v_k = u_j$, and the argument when $2q \equiv 2 \bmod 3$ is the same. If $j = 2k+1$ then $\delta_{2q,j} - \frac{1}{2}(\delta_{2q,2k} + \delta_{2q,2k+2})$ is, by Conjecture 1.5, equal to $\langle q,k\rangle + \frac{2q}{3}((2k+1)^2 - 2k^2 - (2k^2 + 4k + 2))$; this is s_k or r_k according as $2q \equiv 1$ or 2 modulo 3. Also, the induction assumption tells us that $\frac{1}{2}\delta_{2q,2k} = 4\delta_{q,k} = 4v_k$ or $4u_k$; similarly, $\frac{1}{2}\delta_{2q,2k+2}$ is $4v_{k+1}$ or $4u_{k+1}$. In the first case, $\delta_{2q,j} = 4v_k + 4v_{k+1} + s_k = u_j$; in the second case, $\delta_{2q,j} = 4u_k + 4u_{k+1} + r_k = v_j$. □

For example, when $n \geq 2$, Conjecture 1.5 predicts that $e_n(H^3) = q(7q-3) - 37$ or $q(7q-3) - 49$ according as $q \equiv 1$ or 2 modulo 3. As already noted, a Macaulay 2 calculation verifies these formulas for $2 \leq n \leq 13$ as well as the corresponding formulas for $e_n(H^5)$ and $e_n(H^7)$ with $3 \leq n \leq 13$ and for $e_n(H^9)$ with $4 \leq n \leq 13$. One should note that, if the formulas for the $e_n(H^j)$ given in Theorem 1.9 hold, then Conjecture 1.5 is true.

The rest of this section is devoted to the study of the power series $\sum_0^\infty d_n w^n$, where $d_n = \sum_0^{q-1} \langle q,k\rangle$. The result we derive is key to the (conjectural) calculation of the Hilbert–Kunz multiplicity of $H^* = uv + H(x,y,z)$.

DEFINITIONS 1.10.

- σ is the endomorphism $\sigma_0 + \sigma_1$ of Γ.
- δ_n is the element $\sum_0^{q-1} f(q,j)$ of Γ.
- a_n is the coefficient of $[0]$ in δ_n.
- b_n is the image of δ_n under the homomorphism $\rho\colon \Gamma \to \mathbb{Z}$ taking $[i]$ to i and E to 0.
- c_n is the coefficient of E in δ_n.

Definition 1.4 shows that $d_n = 2c_n + 4b_n + 3q$. Observe that $\sigma([0]) = [0] + [1]$, that $\sigma([i]) = [i-1] + [i+1] + E$ if $i \neq 0$, and that $\sigma(E) = 4E$.

LEMMA 1.11. $\delta_{n+1} = \sigma(\delta_n)$.

Proof. We have $\delta_{n+1} = \sum_{k<q} f(2q, 2k) + \sum_{k<q} f(2q, 2k+1)$. The first sum is $\sigma_0(\delta_n)$, and the second is $\sigma_1(\delta_n)$. □

LEMMA 1.12. $2w \sum_0^\infty a_n w^n = (1+2w)(1-4w^2)^{-1/2} - 1$.

Proof. For each n, let $\beta_0 \geq \beta_1 \geq \cdots \geq \beta_n$ be the binomial coefficients $\binom{n}{k}$, $0 \leq k \leq n$, in order of nonincreasing size. An induction on n using Lemma 1.11 shows that $\delta_n = \sum_{i=0}^n \beta_i \cdot [i] +$ a multiple of E. Hence $a_{2n} = \binom{2n}{n}$ and $a_{2n-1} = \binom{2n-1}{n-1} = \frac{1}{2}\binom{2n}{n}$. Then $2w \sum_0^\infty a_{2n} w^{2n} = 2w \sum_0^\infty \binom{2n}{n} w^{2n} = 2w(1-4w^2)^{-1/2}$ and $2w \sum_1^\infty a_{2n-1} w^{2n-1} = \sum_1^\infty \binom{2n}{n} w^{2n} = (1-4w^2)^{-1/2} - 1$. □

LEMMA 1.13. $(1-2w) \sum_0^\infty 2b_n w^n = (1+2w)(1-4w^2)^{-1/2} - 1$.

Proof. If $i \neq 0$ then $\rho(\sigma([i])) = (i-1) + (i+1) = 2\rho([i])$. Furthermore, $\rho(\sigma([0])) = 1$. Since δ_n is $a_n[0] +$ a linear combination of $[1], [2], \dots$ and E, it follows that $\rho\sigma(\delta_n) = 2\rho(\delta_n) + a_n$. So $b_{n+1} - 2b_n = a_n$; moreover, $b_0 = 0$. Then $(1-2w) \sum_0^\infty 2b_n w^n = 2w \sum_0^\infty a_n w^n$, and we can now use Lemma 1.12. □

LEMMA 1.14. $(1-2w)(1-4w) \sum_0^\infty 2c_n w^n = 1 - (1-4w^2)(1-4w^2)^{-1/2}$.

Proof. If $i \neq 0$ then the coefficient of E in $\sigma([i]) - 4[i]$ is 1. Furthermore, $\sigma([0]) - 4[0] = [1] - 3[0]$, while $\sigma(E) - 4E$ is 0. It follows that the coefficient of E in $\sigma(\delta_n) - 4\delta_n$ is the sum of the coefficients of $[1], [2], \dots$ in δ_n. In other words, $c_{n+1} - 4c_n = 2^n - a_n$. Arguing as in the proof of Lemma 1.13, we find that $(1-4w) \sum_0^\infty 2c_n w^n = \left(2w \sum_0^\infty 2^n w^n\right) + 1 - (1+2w)(1-4w^2)^{-1/2}$. Multiplying by $1-2w$ yields $(1-2w)(1-4w) \sum_0^\infty 2c_n w^n = 1 - (1-4w^2)(1-4w^2)^{-1/2}$. □

THEOREM 1.15.

$$(1-2w)(1-4w) \sum_0^\infty d_n w^n = (2-4w) + (1-4w-12w^2)(1-4w^2)^{-1/2}.$$

Proof. We have $d_n = 2c_n + 4b_n + 3 \cdot 2^n$. Lemmas 1.13 and 1.14 then show that

$$\begin{aligned}(1-2w)(1-4w) \sum_0^\infty d_n w^n &= 1 - (1-4w^2)(1-4w^2)^{-1/2} \\ &\quad + (1-4w)((2+4w)(1-4w^2)^{-1/2} - 2) \\ &\quad + 3(1-4w),\end{aligned}$$

giving the desired result. □

COROLLARY 1.16. *In the disc* $|w| < \frac{1}{4}$, $\sum_0^\infty d_n w^n$ *converges to the function* $(1-2w)^{-1}(1-4w)^{-1}((2-4w) + (1-4w-12w^2)(1-4w^2)^{-1/2})$.

Proof. The function is holomorphic in the disc, and Theorem 1.15 tells us that $\sum_0^\infty d_n w^n$ is its Taylor expansion at the origin. □

COROLLARY 1.17. $\sum_0^\infty d_n \left(\frac{1}{16}\right)^n$ *is the irrational number* $\left(\frac{8}{7}\right)\left(\frac{4}{3}\right)\left(\frac{7}{4} + \frac{45}{64} \cdot \frac{8}{3\sqrt{7}}\right) = \frac{8}{3} + \frac{20}{7\sqrt{7}}$.

2. The Conjecture's Consequence

Definition 2.1. H^* is the 5-variable polynomial $uv + H(x, y, z)$.

We begin the study of $e_n(H^*)$ by expressing it in terms of the $e_n(H^j)$, $j \leq q$. (More general results of a similar nature are found in [3] but, because this case is simple, we give a self-contained argument.)

Theorem 2.2. $e_n(H^*) = 2\sum_{j<q} e_n(H^j) + e_n(H^q)$.

The following lemmas lead to the proof of Theorem 2.2. Let L be a field and let U be the vector space $L[u, v]/(u^q, v^q)$. We give U the structure of an $L[T]$-module with T acting by multiplication by uv.

Lemma 2.3. *U is a direct sum of two copies of $L[T]/T$, two copies of $L[T]/T^2$, $\ldots$, two copies of $L[T]/T^{q-1}$, and one copy of $L[T]/T^q$.*

Proof. U is the direct sum of cyclic submodules generated by $u^{q-1}, v^{q-1}, u^{q-2}, v^{q-2}, \ldots, u, v$, and 1. □

If U_1 and U_2 are $L[T]$-modules, and if T_1 and T_2 denote the action of T on U_1 and on U_2 (respectively), then we may give $U_1 \otimes_L U_2$ an $L[T]$-module structure with T acting by $(T_1 \otimes 1) + (1 \otimes T_2)$.

Lemma 2.4. *Let V be an $L[T]$-module, and let $V_s = V \otimes_L L[T]/T^s$ with the $L[T]$-module structure defined previously. Then the vector spaces V_s/TV_s and V/T^sV are isomorphic.*

Proof. An element of V_s has the form $u_s \otimes 1 + u_{s-1} \otimes T + \cdots + u_1 \otimes T^{s-1}$ with the u_i in V, and $v \to v \otimes 1$ imbeds V in V_s. Because $u \otimes T^k \equiv -Tu \otimes T^{k-1} \bmod TV_s$, the map $V \to V_s/TV_s$ is onto. Hence we need only show that $V \cap TV_s = T^sV$. Abbreviate $u_s \otimes 1 + \cdots + u_1T^{s-1}$ as $(u_s, \ldots, u_1)$. Then $T_1 \otimes 1$ takes this element to $(Tu_s, \ldots, Tu_1)$ while $1 \otimes T_2$ takes it to $(0, u_s, \ldots, u_2)$. Thus, if $(v, 0, \ldots, 0) = T(u_s, \ldots, u_1)$, then $v = T(u_s)$, $u_s = -T(u_{s-1})$, $\ldots$, $u_2 = -T(u_1)$, and v lies in $T^s(V)$. Similarly, we see that if $v = T^s(u)$ then $(v, 0, \ldots, 0)$ is in the image of $T = (T_1 \otimes 1) + (1 \otimes T_2)$. □

Lemma 2.5. *Let V be an $L[T]$-module and let U be the $L[T]$-module $L[u, v]/(u^q, v^q)$ of Lemma 2.3. Set $V^* = V \otimes_L U$ and give V^* the product $L[T]$-module structure. Then V^*/TV^* is a direct sum of two copies of each of $V/TV, \ldots, V/T^{q-1}V$ and one copy of V/T^qV.*

Proof. Decompose U as in Lemma 2.3 and then apply Lemma 2.4. □

The proof of Theorem 2.2 is now easy. Let $L = \mathbb{Z}/2$, and let V be $\mathbb{Z}/2[x, y, z]/(x^q, y^q, z^q)$ with T acting by multiplication by H. Then $V^* = V \otimes_L U$ identifies with $\mathbb{Z}/2[x, y, z, u, v]/(x^q, y^q, z^q, u^q, v^q)$ with T acting by multiplication by H^*. By Lemma 2.5, $e_n(H^*) = \dim V^*/TV^* = 2\sum_{j<q} \dim V/T^jV + \dim V/T^qV = 2\sum_{j<q} e_n(H^j) + e_n(H^q)$. □

THEOREM 2.6. *If Conjecture 1.5 holds, then* $e_{n+1}(H^*) - 16e_n(H^*) = 2d_n$.

Proof. Suppose Conjecture 1.5 holds. Then

$$2\sum_{\substack{j \text{ odd} \\ j<2q}} e_{n+1}(H^j) = (e_{n+1}(H^2) + 2\langle q,0\rangle) + (e_{n+1}(H^2) + e_{n+1}(H^4) + 2\langle q,1\rangle) + \cdots + (e_{n+1}(H^{2q-2}) + e_{n+1}(H^{2q}) + 2\langle q,q-1\rangle).$$

This is equal to

$$2\sum_{\substack{j \text{ even} \\ j<2q}} e_{n+1}(H^j) + e_{n+1}(H^{2q}) + 2d_n.$$

Theorem 2.2 then shows that

$$e_{n+1}(H^*) = 4\sum_{\substack{j \text{ even} \\ j<2q}} e_{n+1}(H^j) + 2e_{n+1}(H^{2q}) + 2d_n.$$

This is just $32\sum_{k<q} e_n(H^k) + 16e_n(H^q) + 2d_n$, and we use Theorem 2.2 once again. □

COROLLARY 2.7. *Suppose Conjecture 1.5 holds. Then* μ, *the Hilbert–Kunz multiplicity of* H^*, *is the irrational number* $\frac{4}{3} + \frac{5}{14\sqrt{7}}$.

Proof. Since $e_n(H^*) = \mu \cdot 16^n + O(8^n)$, it follows that $\sum_0^\infty e_n(H^*)w^n$ has a meromorphic extension to the disc $|w| < \frac{1}{8}$ and that the principal part at $w = \frac{1}{16}$ is $\mu/(1-16w)$. Therefore, $\mu = \lim_{w\to 1/16}(1-16w)\sum_0^\infty e_n(H^*)w^n$. By Theorem 2.6 this is $\lim_{w\to 1/16} 1 + 2w\sum_0^\infty d_n w^n$, so we can now apply Corollary 1.17. □

By the Hilbert–Kunz series of H^* we mean the function $\sum_0^\infty e_n(H^*)w^n$ that is holomorphic in the disc $|w| < \frac{1}{16}$. By Corollary 2.7 we (conjecturally) have

$$\begin{aligned}(1-2w)(1-4w)(1-16w)\quad &(\text{Hilbert–Kunz series})\\ &= (1-2w)(1-4w) + (4w-8w^2) + 2w(1-4w-12w^2)(1-4w^2)^{-1/2}\\ &= (1-2w) + 2w(1-4w-12w^2)(1-4w^2)^{-1/2}.\end{aligned}$$

Thus the Hilbert–Kunz series has (conjecturally) a meromorphic extension to $|w| < \frac{1}{2}$ but no meromorphic extension to any region containing $\frac{1}{2}$ or $-\frac{1}{2}$. In all previous cases where one has been able to determine the Hilbert–Kunz series explicitly, it has been shown to be a rational function.

We next give a (conjectural) calculation of a related Hilbert–Kunz function. The alternative formulation of Conjecture 1.5 given in Section 3 will involve this function.

LEMMA 2.8. $\langle q, \lfloor \frac{q}{3} \rfloor \rangle = 4n+3$.

Proof. Induction on n shows that $f(q, \lfloor \frac{q}{3} \rfloor) = [n]$. □

THEOREM 2.9. *Suppose Conjecture 1.5 holds, and let* $H_1 = u^3 + H(x,y,z)$. *Then*

(a) $e_{n+1}(H_1) - 8e_n(H_1) = 8n+6$ *and*
(b) $e_n(H_1) = \frac{99}{49}q^3 - \frac{8n}{7} - \frac{50}{49}$.

Proof. Part (b) follows from (a) by induction, since the case $n = 0$ is trivial. To establish (a), let U be the $\mathbb{Z}/2[T]$-module $\mathbb{Z}/2[u]/u^q$ with T acting by multiplication by u^3. Writing U as a direct sum of submodules generated by u^2, u, and 1, we find that if $q = 3m + 1$ then U is isomorphic to

$$\mathbb{Z}/2[T]/T^m \oplus \mathbb{Z}/2[T]/T^m \oplus \mathbb{Z}/2[T]/T^{m+1},$$

and if $q = 3m + 2$, to

$$\mathbb{Z}/2[T]/T^m \oplus \mathbb{Z}/2[T]/T^{m+1} \oplus \mathbb{Z}/2[T]/T^{m+1}.$$

As in the proof of Theorem 2.2, it follows that $e_n(H_1) = 2e_n(H^m) + e_n(H^{m+1})$ in the first case and is $e_n(H^m) + 2e_n(H^{m+1})$ in the second case.

Suppose now that $q = 3m+1$. Then $e_{n+1}(H_1) = e_{n+1}(H^{2m}) + 2e_{n+1}(H^{2m+1})$. By Conjecture 1.5, this is $2e_{n+1}(H^{2m}) + e_{n+1}(H^{2m+2}) + 2\langle q, m\rangle$. Lemma 2.8 and the result of the previous paragraph show that this is $8e_n(H_1) + 8n + 6$. The argument when $q = 3m + 2$ is similar. It would be of interest to establish (a) unconditionally. □

3. Remarks on the Conjecture, and an Alternative Formulation

Conjecture 1.5 as presented here seems mysterious. Its motivation is contained in ideas from [2] that allow one to calculate all the $e_n(G^j)$ when G is a 2-variable polynomial (or power series) with constant term 0. (The argument leading from Conjecture 1.5 to the calculation of the Hilbert–Kunz series of H^* has its genesis in [3].)

We summarize some results from [2]. Let X be the space of continuous real-valued functions on $[0, 1]$. Call two elements of X *linearly equivalent* if their difference is a linear combination of 1 and the identity function t. Suppose now that k is a finite field of characteristic p and that $G \in k[x, y]$ is separable with constant term 0. Let U_i $(0 \le i < p)$ be the "magnification operators" $f \to p^2 f\left(\frac{t+i}{p}\right)$ on X. Let φ_G be the unique element of X with $\varphi_G\left(\frac{a}{q}\right) = q^{-2}\deg(x^q, y^q, G^a)$ whenever q is a power of p. Teixeira and I prove the following statement.

> *The elements of X obtained from φ_G by applying a finite sequence of operators, each of which is some U_i with $0 \le i < p$, lie in finitely many linear equivalence classes.*

Conjecture 1.5 arose when I attempted to find an analogue to this result for the 3-variable polynomial H. One may define φ_H so that $\varphi_H\left(\frac{a}{q}\right) = q^{-3}\deg(x^q, y^q, z^q, H^a)$. Also, there are operators U_0 and U_1 with $U_0(f) = 8f\left(\frac{t}{2}\right)$ and $U_1(f) = 8f\left(\frac{t+1}{2}\right)$. If we set $\varepsilon = t - t^2$, then $U_0(\varepsilon) = 2\varepsilon + 2t$ and $U_1(\varepsilon) = 2\varepsilon + 2(1-t)$. I investigated functions obtained from φ_H, making use of U_0 and U_1 as before, and found that two functions obtained in this way often appeared to differ by a linear combination of 1, t, and ε.

For example, the relation $e_{n+1}(H^{q+j}) = e_n(H^j) + 7q^3$ of Section 1 tells us that $U_1(\varphi_H) = \varphi_H + 7$.

DEFINITION 3.1.

(a) $\psi_0, \psi_1, \psi_2, \psi_3, \dots$ are the functions $\varphi_H, U_0(\varphi_H), U_1U_0(\varphi_H), U_0U_1U_0(\varphi_H), \dots$ obtained by the alternate application of U_0 and U_1.

(b) If $0 \le l < q$, then $\psi_{q,l}$ is the function $t \to q^3\varphi_H\left(\frac{t+l}{q}\right)$, $[0,1] \to \mathbb{R}$.

REMARKS.

(1) U_0 and U_1 take $\psi_{q,l}$ to $\psi_{2q,2l}$ and $\psi_{2q,2l+1}$, respectively. Hence the $\psi_{q,l}$ are just the functions obtained from φ_H by applying a sequence of operators, each of which is U_0 or U_1.

(2) An induction shows that $\psi_n = \psi_{q,\lfloor q/3 \rfloor}$.

(3) The values of $8\psi_{q,k}$ at 0 and 1 are $8e_n(H^k)$ and $8e_n(H^{k+1})$, respectively, while the value at $\frac{1}{2}$ is $e_{n+1}(H^{2k+1})$.

(4) Let $\mathcal{L}$ be the linear functional $F \to 8F\left(\frac{1}{2}\right) - 4(F(0) + F(1))$. Then $\mathcal{L}$ takes 1 and t to 0 and takes ε to 2. By (3), $\mathcal{L}(\psi_{q,k}) = e_{n+1}(H^{2k+1}) - \frac{1}{2}(e_{n+1}(H^{2k}) + e_{n+1}(H^{2k+2}))$. As a result, Conjecture 1.5 may be rephrased as stating that $\mathcal{L}(\psi_{q,k}) = \langle q, k\rangle$.

(5) Using (2), (4), and calculations made in the proof of Theorem 2.9, we find that $\mathcal{L}(\psi_i) = \frac{1}{2}(e_{i+1}(H_1) - 8e_i(H_1))$ with H_1 as in Theorem 2.9.

We shall use the symbol $\sim$ to denote linear equivalence of functions.

LEMMA 3.2. *If Conjecture 1.5 holds and if $f(q^*, j^*) - f(q, j) = rE$, then we have $\psi_{q^*,j^*} - \psi_{q,j} - r\varepsilon \sim 0$.*

Proof. Let h be the function $\sim$ to $\psi_{q^*,j^*} - \psi_{q,j} - r\varepsilon$ with $h(0) = h(1) = 0$. We shall show that $h = 0$ by proving that h vanishes at all rationals in $[0,1]$ with denominator a power of 2. By Remark (4), $\mathcal{L}(h) = \langle q^*, j^*\rangle - \langle q, j\rangle - 2r = 0$. Hence $h\left(\frac{1}{2}\right) = 0$ and $U_0(h)$ and $U_1(h)$ vanish at 0 and 1, respectively.

Now $f(2q^*, 2j^*) - f(2q, 2j) = 2rE$. Furthermore, $U_0(h) \sim \psi_{2q^*,2j^*} - \psi_{2q,2j} - 2r\varepsilon$. We also have $f(2q^*, 2j^*+1) - f(2q, 2j+1) = 2rE$ and $U_1(h) \sim \psi_{2q^*,2j^*+1} - \psi_{2q,2j+1} - 2r\varepsilon$. The result of the previous paragraph then tells us that $U_0(h)$ and $U_1(h)$ each vanish at $\frac{1}{2}$, so $h\left(\frac{1}{4}\right) = h\left(\frac{3}{4}\right) = 0$. Continuing in this way yields the result. □

THEOREM 3.3. *Suppose Conjecture 1.5 holds. Then*:

(a) $U_1(\psi_i) \sim \psi_{i-1} + \varepsilon$ *when i is even and $i \neq 0$*;

(b) $U_0(\psi_i) \sim \psi_{i-1} + \varepsilon$ *when i is odd*;

(c) $\mathcal{L}(\psi_i) = 4i + 3$ *for all i*.

In view of Remark (5), Theorem 3.3(c) is the same as $e_{i+1}(H_1) - 8e_i(H_1) = 8i + 6$ for all i. An induction shows that this is equivalent to $e_n(H_1) = \frac{99}{49}q^3 - \frac{8n}{7} - \frac{50}{49}$ for all n; see Theorem 2.9.

Proof of Theorem 3.3. If Conjecture 1.5 holds then $\mathcal{L}(\psi_i) = \left\langle 2^i, \left\lfloor \frac{2^i}{3} \right\rfloor \right\rangle$. Since $f\left(2^i, \left\lfloor \frac{2^i}{3} \right\rfloor\right) = [i]$, it follows that $\mathcal{L}(\psi_i) = 4i + 3$, giving (c). Parts (a) and (b) are easy consequences of Lemma 3.2. Rather than writing out the proof, we illustrate by example: $f(32,11) = \sigma_1 f(16,5) = \sigma_1([4]) = [3]+E = f(8,2)+E$. Hence, by Lemma 3.2, $\psi_{32,11} \sim \psi_{8,2} + \varepsilon$. But $\psi_{32,11} = U_1(\psi_4)$ while $\psi_{8,2} = \psi_3$. Similarly, $f(64,20) = \sigma_0 f(32,10) = \sigma_0([5]) = [4] + E = f(16,5) + E$ and so, by Lemma 3.2, $\psi_{64,20} \sim \psi_{16,5} + \varepsilon$. But $\psi_{64,20}$ and $\psi_{16,5}$ are $U_0(\psi_5)$ and ψ_4, respectively. □

We next establish a converse to Theorem 3.3 as follows.

LEMMA 3.4. *Suppose* (a) *and* (b) *of Theorem 3.3 hold and that* $0 \le j < q$. *Then, if* $f(q,j) = [i] + mE$, *we have* $\psi_{q,j} \sim \psi_i + m\varepsilon$.

Proof. We argue by induction on q, since $q = 1$ is easy. Suppose that, for a fixed q and j, we have $f(q,j) = [i] + mE$ and $\psi_{q,j} \sim \psi_i + m\varepsilon$. First assume $i = 0$. Applying σ_0 and σ_1 to the first relation we find that $f(2q,2j)$ and $f(2q,2j+1)$ are $[1] + 2mE$ and $[0] + 2mE$, respectively. Applying U_0 and U_1 to the second relation and noting that $U_0(\psi_0)$ and $U_1(\psi_0)$ are ψ_1 and $\psi_0 + 7$ (respectively), we find that $\psi_{2q,2j} \sim \psi_1 + 2m\varepsilon$ and $\psi_{2q,2j+1} \sim \psi_0 + 2m\varepsilon$, as desired.

Next assume $i \ne 0$ is even. Applying σ_0 and σ_1 to the first relation, we find that $f(2q,2j)$ and $f(2q,2j+1)$ are $[i+1] + 2mE$ and $[i-1] + (2m+1)E$, respectively. Applying U_0 and U_1 to the second relation and using Theorem 3.3(a), we find that $\psi_{2q,2j} \sim \psi_{i+1} + 2m\varepsilon$ and $\psi_{2q,2j+1} \sim \psi_{i-1} + \varepsilon + 2m\varepsilon$, as desired. When i is odd we argue similarly but using Theorem 3.3(b). □

THEOREM 3.5. *With* ψ_i *as in Definition 3.1, Conjecture 1.5 is equivalent to* (a), (b), *and* (c) *of Theorem 3.3 all holding.*

Proof. Suppose (a), (b), and (c) of Theorem 3.3 hold and that $0 \le j < q$. Write $f(q,j) = [i] + mE$. By Lemma 3.4, $\psi_{q,j} \sim \psi_i + m\varepsilon$. Applying $\mathcal{L}$ yields $\mathcal{L}(\psi_{q,j}) = \mathcal{L}(\psi_i) + 2m$, which by (c) is $\langle q,j \rangle$. Now use the last sentence of Remark 4 to obtain Conjecture 1.5. We have already seen that (a), (b), and (c) all follow from Conjecture 1.5.

The new version of Conjecture 1.5, recast as (a), (b), and (c) of Theorem 3.3, has advantages over the old. It avoids the complicated function $\langle q,k \rangle$, displays connections with [2], and emphasizes the importance of establishing the formula for $e_n(H_1)$. □

References

[1] R.-O. Buchweitz and Q. Chen, *Hilbert–Kunz functions of cubic curves and surfaces,* J. Algebra 197 (1997), 246–267.

[2] P. Monsky and P. Teixeira, *p-fractals and power series I,* J. Algebra 280 (2004), 505–536.

[3] ———, *p-fractals and power series II,* J. Algebra 304 (2006), 237–255.

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