Michigan Math. J. 57 (2008)

A Property of the Absolute Integral Closure of an Excellent Local Domain in Mixed Characteristic

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Dedicated to Professor Melvin Hochster on the occasion of his sixty-fifth birthday

1. Introduction

Let $(R, \mathfrak{m})$ be a Noetherian local excellent domain and let R^+ be the absolute integral closure of R—that is, the integral closure of R in the algebraic closure of the fraction field of R. The ring R^+ when R is 3-dimensional and of mixed characteristic played an important role in Heitmann's proof of the direct summand conjecture in dimension 3 [3]. In dimension > 3 the direct summand conjecture is still open. This motivates the study of R^+ in mixed characteristic and in dimension > 3.

Hochster and Huneke [4] proved that if R contains a field of characteristic 0 then R^+ is a big Cohen–Macaulay R-algebra; in other words, $H^i_{\mathfrak{m}}(R^+) = 0$ for all $i < \dim R$, and every system of parameters of R is a regular sequence on R^+. Recently, in joint work with Huneke [5], we gave a simpler proof of this result.

This paper is motivated by Huneke's suggestion that perhaps the techniques of our paper [5] could be applied to R^+ in mixed characteristic. Our main result is the following theorem.

Theorem 1.1. *Let $(R, \mathfrak{m})$ be a Noetherian local excellent domain of mixed characteristic, residual characteristic $p > 0$, and dimension ≥ 3. Let $\sqrt{pR}$ (resp. $\sqrt{pR^+}$) be the radical of the principal ideal of R (resp. R^+) generated by p. Set $\bar{R} = R/\sqrt{pR}$ (resp. $\overline{R^+} = R^+/\sqrt{pR^+}$). Then*

(i) $H^1_{\mathfrak{m}}(\overline{R^+}) = 0$, *and*

(ii) *every part of a system of parameters $\{a, b\}$ of $\bar{R}$ of length 2 is a regular sequence on $\overline{R^+}$.*

This theorem suggests the following.

Question. Let $(R, \mathfrak{m})$ be a Noetherian local excellent domain of mixed characteristic. Is $\overline{R^+}$ then a big Cohen–Macaulay $\bar{R}$-algebra? That is:

(i) is $H^i_{\mathfrak{m}}(\overline{R^+}) = 0$ for all $i < \dim \bar{R}$; and

(ii) is every system of parameters of $\bar{R}$ a regular sequence on $\overline{R^+}$?

Received July 5, 2007. Revision received September 6, 2007.

2. Proof of Theorem 1.1

Since $\bar{R}$ is a ring of characteristic p, it follows that, for every $\bar{R}$-algebra $\mathcal{R}$, the standard map $\mathcal{R} \xrightarrow{r \mapsto r^p} \mathcal{R}$ induces a map $f\colon H^i_{\mathfrak{m}}(\mathcal{R}) \to H^i_{\mathfrak{m}}(\mathcal{R})$. This map is called *the action of the Frobenius* on the local cohomology of $\mathcal{R}$.

LEMMA 2.1. *Let R' be a finite normal extension of R contained in R^+, and let $\overline{R'} = R'/\sqrt{pR'}$. The aforementioned action of the Frobenius $f\colon H^1_{\mathfrak{m}}(\overline{R'}) \to H^1_{\mathfrak{m}}(\overline{R'})$ on $H^1_{\mathfrak{m}}(\overline{R'})$ is nilpotent; that is, for some $s \geq 1$, f^s sends $H^1_{\mathfrak{m}}(\overline{R'})$ to zero (here $f^1 = f$ and $f^s = f \circ f^{s-1}$ for $s > 1$).*

Proof. Because R and R' are excellent and normal, their completions with respect to the $\mathfrak{m}$-adic topology also are excellent and normal. Since R' is semilocal, it follows that $\widehat{R'}$ is a product of several complete normal domains $R'_1, R'_2, \ldots$, which are the completions of R' with respect to its maximal ideals. We set $\overline{R'_i} = R'_i/\sqrt{pR'_i}$. Since

$$\widehat{R'}/\sqrt{p\widehat{R'}} \cong \widehat{\overline{R'}} \cong \Pi_i \overline{R'_i} \quad \text{and} \quad H^1_{\mathfrak{m}}(\widehat{\overline{R'}}) \cong H^1_{\mathfrak{m}}(\overline{R'}) \cong \Pi_i H^1_{\mathfrak{m}}(\overline{R'_i})$$

and since the action of the Frobenius is the same on $H^1_{\mathfrak{m}}(\widehat{\overline{R'}})$ as on $H^1_{\mathfrak{m}}(\overline{R'})$ and since the Frobenius acts individually on each $H^1_{\mathfrak{m}}(\overline{R'_i})$, we conclude that it is enough to prove that the action of the Frobenius on each $H^1_{\mathfrak{m}}(\overline{R'_i})$ is nilpotent. Thus, giving $\hat{R}$ and R'_i the names R and R' again, we may assume that R is complete and hence that R' is local. We keep this assumption for the rest of the proof.

At this point we paraphrase a result from [6, 4.1, 4.6b, and the paragraph following the statement of 4.6b]: *Let A be a local ring of characteristic p. Then f is nilpotent on $H^i_{\mathfrak{m}}(A)$ for $i \leq 1$ if and only if* $\dim A \geq 2$ *and the punctured spectrum of the completion of the strict Henselization of A is connected.* Because $\dim \overline{R'} \geq 2$, this implies that f is nilpotent on $H^1_{\mathfrak{m}}(\overline{R'})$ if the punctured spectrum of $B \stackrel{\text{def}}{=} \widehat{(\overline{R'})^{\text{sh}}}$ is connected, where $\widehat{(\overline{R'})^{\text{sh}}}$ is the completion of the strict Henselization of $\overline{R'}$. Hence it is enough to prove that the punctured spectrum of B is connected.

Since R' is excellent, so is its strict Henselization $(R')^{\text{sh}}$ [1, 5.6iii]. Since R' is normal, standard properties of strict Henselization imply that $(R')^{\text{sh}}$ is normal. Because $(R')^{\text{sh}}$ is both excellent and normal, so is its completion $B' \stackrel{\text{def}}{=} \widehat{(R')^{\text{sh}}}$. In particular, B' is a domain.

Since B' is excellent and since $\overline{R'} = R'/\sqrt{pR'}$, standard properties of strict Henselization and completion imply that $B = B'/\sqrt{pB'}$.

Assume that the punctured spectrum of B is disconnected. This is equivalent to the existence of ideals $\tilde{I}_1$ and $\tilde{I}_2$ of B such that $\tilde{I}_1 \cap \tilde{I}_2 = 0$ and $\sqrt{\tilde{I}_1 + \tilde{I}_2} = \mathfrak{m}_B$, where $\mathfrak{m}_B$ is the maximal ideal of B.

Let I_1 and I_2 be the preimages of $\tilde{I}_1$ and $\tilde{I}_2$ (respectively) in B'. Then $\sqrt{pB'} = I_1 \cap I_2$ and $I_1 + I_2$ is $\mathfrak{m}'$-primary, where $\mathfrak{m}'$ is the maximal ideal of B'. Let $\dim B' = \dim R = d$. The Mayer–Vietoris sequence yields

$$H^{d-1}_{(p)}(B') \to H^d_{\mathfrak{m}'}(B') \to H^d_{I_1}(B') \oplus H^d_{I_2}(B'),$$

which is an exact sequence. Then $H^{d-1}_{(p)}(B') = 0$ because (p) is a principal ideal and $d - 1 > 1$, while $H^d_{I_1}(B') = 0$ and $H^d_{I_2}(B') = 0$ by the Hartshorne–Lichtenbaum local vanishing theorem [2, 3.1] (note that B' is a complete local d-dimensional domain). Hence $H^d_{\mathfrak{m}'}(R') = 0$, which is impossible. □

Viewing $\overline{R'}$ as a subring of $\overline{R^+}$ in a natural way, we set

$$\mathcal{R} \stackrel{\text{def}}{=} \{r \in \overline{R^+} \mid r^{p^s} \in \overline{R'}\}.$$

Since every monic polynomial with coefficients in $\overline{R'}$ has a root in $\overline{R^+}$, we know that every element of $\overline{R'}$ has a (p^s)th root in $\overline{R^+}$ and that this (p^s)th root is unique because $\overline{R^+}$ is reduced. Therefore, the $\bar{R}$-algebra homomorphism $\varphi\colon \mathcal{R} \to \overline{R'}$ that sends $r \in \mathcal{R}$ to $r^{p^s} \in \overline{R'}$ is an isomorphism.

Lemma 2.2. *The map $\phi_*\colon H^1_{\mathfrak{m}}(\overline{R'}) \to H^1_{\mathfrak{m}}(\mathcal{R})$ induced by the natural inclusion $\phi\colon \overline{R'} \hookrightarrow \mathcal{R}$ is the zero map.*

Proof. The composition of $\bar{R}$-algebra homomorphisms $\varphi \circ \phi\colon \overline{R'} \to \overline{R'}$ is the standard homomorphism sending $r \in \overline{R'}$ to $r^{p^s} \in \overline{R'}$. Hence the induced map $\varphi_* \circ \phi_*\colon H^1_{\mathfrak{m}}(\overline{R'}) \to H^1_{\mathfrak{m}}(\overline{R'})$ is nothing but f^s, which is the zero map by Lemma 2.1. Because φ is an isomorphism, so is φ_*. Since $\varphi_* \circ \phi_*$ is the zero map and since φ_* is an isomorphism, ϕ_* is the zero map. □

The $\bar{R}$-algebra $\overline{R^+}$ is the direct limit of $\overline{R'}$ as R' ranges over the finite normal extensions of R contained in R^+. Since local cohomology commutes with direct limits, it follows that $H^1_{\mathfrak{m}}(\overline{R^+})$ is the direct limit of $H^1_{\mathfrak{m}}(\overline{R'})$. In other words, $H^1_{\mathfrak{m}}(\overline{R^+})$ is the union of the images of the maps $\phi'_*\colon H^1_{\mathfrak{m}}(\overline{R'}) \to H^1_{\mathfrak{m}}(\overline{R^+})$ induced by the natural inclusion $\phi'\colon \overline{R'} \hookrightarrow \overline{R^+}$. But Lemma 2.2 implies that the image of every ϕ'_* is zero (ϕ'_* factors through ϕ_*). This completes the proof of Theorem 1.1(i).

For Theorem 1.1(ii), let $\{a, b\}$ be a part of a system of parameters of $\bar{R}$. Since $\overline{R^+}$ is a reduced integral extension of $\bar{R}$, a is regular on $\overline{R^+}$. That $H^i_{\mathfrak{m}}(\overline{R^+}) = 0$ for $i = 0, 1$ and the short exact sequence

$$0 \to \overline{R^+} \xrightarrow{\text{mult. by } a} \overline{R^+} \to \overline{R^+}/a\overline{R^+} \to 0$$

together imply that $H^0_{\mathfrak{m}}(\overline{R^+}/a\overline{R^+}) = 0$. Hence $\mathfrak{m}$ is not an associated prime of $\overline{R^+}/a\overline{R^+}$. This implies that the only associated primes of $\overline{R^+}/a\overline{R^+}$ are the minimal primes of $\bar{R}/a\bar{R}$. Indeed, if there is an embedded associated prime, say P, then P is the maximal ideal of the ring $\bar{R}_P$ whose dimension exceeds 1 and P is an associated prime of $(\overline{R^+}/a\overline{R^+})_P = \overline{(R_P)^+}/a\overline{(R_P)^+}$, which is impossible by the foregoing. Since b is not in any minimal prime of $\bar{R}/a\bar{R}$, it must be regular on $\overline{R^+}/a\overline{R^+}$. □

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