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Oriented Cohomology, Borel–Moore Homology, and Algebraic Cobordism

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Warmly dedicated to Mel Hochster, who,
among other important lessons, taught me that
sometimes my pencil is smarter than my brain

Introduction

The notion of *oriented cohomology* has been introduced, in various forms and in various settings, in the work of Panin [10], Levine and Morel [6], and others. A related notion, that of *oriented Borel–Moore homology,* appears in [6]. Mocanasu [7] has examined the relation of these two notions and, with a somewhat different axiomatic than appears in either [6] or [10], has given an equivalence of the two theories; the relation is that the cohomology with supports in a closed subset X of a smooth scheme M becomes the Borel–Moore homology of X.

Our main goal in this paper is to tie all these theories together. Our first step is to extend results of [10] in order to show that an orientation on a ring cohomology theory gives rise to a good theory of projective push-forwards on the cohomology with supports. This extension of Panin's results allows us to use the ideas and results of Mocanasu, which in essence show that many of the properties and structures associated with the cohomology of a smooth scheme M with supports in a closed subset X depend only on X; we require resolution of singularities for this step. We axiomatize this into the notion of an *oriented duality theory,* which one can view as a version of the classical notion of a Bloch–Ogus twisted duality theory. The main difference between a general oriented duality theory (H, A) and a Bloch–Ogus theory is that we do not assume that the Chern class map $L \mapsto c_1(L)$ satisfies the usual additivity with respect to tensor product of line bundles:

$$c_1(L \otimes M) = c_1(L) + c_1(M).$$

This relation is replaced by the *formal group law* $F_A(u, v) \in A(\operatorname{Spec} k)[[u, v]]$ of the underlying oriented cohomology theory A, defined by the relation

$$c_1(L \otimes M) = F_A(c_1(L), c_1(M)).$$

In fact, the Chern classes $c_1(L)$ and formal group law F_A are not explicitly given as part of the axioms but instead follow from the more basic structures—namely, the pull-back, the projective push-forward, and the projective bundle formula.

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We conclude with a discussion of the two theories that form our primary interest: the theory of algebraic cobordism Ω_* of [6]; and the bi-graded theory $\mathrm{MGL}^{*,*}$, also known as algebraic cobordism but defined via the algebraic Thom complex MGL in the Morel–Voevodsky motivic stable homotopy category $\mathcal{SH}(k)$ (see [12]). Assuming that k admits resolution of singularities, we show how one may apply our general theory to $\mathrm{MGL}^{*,*}$ and so give rise to an associated oriented Borel–Moore homology theory $\mathrm{MGL}'_{*,*}$, which together form an oriented duality theory $(\mathrm{MGL}'_{*,*}, \mathrm{MGL}^{*,*})$. Concerning Ω_*, we show how this theory comes with a canonical "classifying map"

$$\vartheta_H : \Omega_* \to H_{2*,*}$$

for each bi-graded oriented duality theory (H, A). Taking the case $(\mathrm{MGL}'_{*,*}, \mathrm{MGL}^{*,*})$, we achieve an extension of the natural transformation $\vartheta^{\mathrm{MGL}} : \Omega^* \to \mathrm{MGL}^{2*,*}$, discussed in [6], to a natural transformation

$$\vartheta_{\mathrm{MGL}} : \Omega_* \to \mathrm{MGL}'_{2*,*}.$$

We conjecture that ϑ is an isomorphism, extending the conjecture of [6] that ϑ^{MGL} is an isomorphism, and we outline a program for proving this conjecture. In fact, the extension of the conjecture of [6]—and how this extension to the setting of Borel–Moore homology could lead to a proof of the conjecture—is the main motivation behind this paper.

In Section 1 we review Panin's theory of oriented ring cohomology and show how his method of defining projective push-forwards for such cohomologies extends to give projective push-forwards for cohomology with supports. In Section 2 we recall Mocanasu's theory of algebraic oriented cohomology, give a modified version of this theory, and show that the projective push-forward with supports defined in Section 1 endows an oriented ring cohomology theory with the structure of an algebraic oriented cohomology theory. In Section 3 we introduce the notion of an oriented duality theory and show that an oriented ring cohomology theory extends uniquely to an oriented duality theory. In Section 4 we apply our results to $\mathrm{MGL}^{*,*}$, construct the classifying map $\vartheta_H : \Omega_* \to H_{2*,*}$, and discuss the conjecture that ϑ_{MGL} is an isomorphism.

1. Integration with Support

Panin has made a study of properties of *oriented ring cohomology theories,* showing how a good theory of Chern classes of line bundles gives rise to push-forward maps for projective morphisms (he calls this latter structure an *integration*). For our purposes we will need push-forward maps for projective morphisms of pairs, so we need to extend Panin's theory a bit. Fortunately, the extension is mainly a matter of making a few changes in the definitions and then noting that most of Panin's arguments extend without major change to the more general setting. In this section we give the necessary extension of Panin's theory.

We fix a base-field k and let $\mathbf{Sm}/k$ denote the category of smooth quasi-projective schemes over k. We denote the base-scheme Spec k by pt. Panin uses the category $\mathbf{SmOp}$ of *smooth open pairs* over k, which is the category of pairs (M, U), $M, U \in \mathbf{Sm}/k$, with $U \subset M$ an open subscheme. A morphism $f : (M, U) \to (N, V)$ is a morphism $f : M \to N$ in $\mathbf{Sm}/k$ with $f(U) \subset V$.

Let $\mathbf{SP}$ denote the category with objects (M, X), with $M \in \mathbf{Sm}/k$ and $X \subset M$ a closed subset. A morphism $f : (M, X) \to (M, Y)$ in $\mathbf{SP}$ is a morphism $f : M \to N$ in $\mathbf{Sm}/k$ such that $f^{-1}(Y) \subset X$. We call $\mathbf{SP}$ the category of *smooth pairs.*

REMARK 1.1. The reader should take care that a morphism $(M, X) \to (N, Y)$ in $\mathbf{SP}$ is *not* the same as a morphism of pairs. The latter notion is (as in topology) just a morphism $f : M \to N$ with $f(X) \subset Y$.

The categories $\mathbf{SmOp}$ and $\mathbf{SP}$ are seen to be isomorphic by sending $(M, U) \in \mathbf{SmOp}$ to $(M, M \setminus U) \in \mathbf{SP}$. Throughout we will use $\mathbf{SP}$ instead of $\mathbf{SmOp}$. We have the inclusion functor $\iota : \mathbf{Sm}/k \to \mathbf{SP}$ sending M to (M, M) and $f : M \to N$ to the induced map $f : (M, M) \to (N, N)$.

Because our intention here is to add support conditions to Panin's theory and this requires some additional commutativity conditions not imposed in [10], we will add the simplifying assumption that a ring cohomology theory will always be $\mathbb{Z}/2$-graded. We likewise require that the boundary maps in the underlying cohomology theory be of odd degree and that the pull-back maps preserve degree. After these modifications, we have the following version of Panin's notion of a cohomology theory, and a ring cohomology theory, on $\mathbf{SP}$.

DEFINITION 1.2. A *cohomology theory* A on $\mathbf{SP}$ is a functor $A : \mathbf{SP}^{\mathrm{op}} \to \mathbf{Gr}_{\mathbb{Z}/2}\mathbf{Ab}$ together with a collection of degree-1 operators

$$\partial_{M,X} : A(M, X) \to A(M \setminus X, M \setminus X)$$

satisfying the axioms of [10, Def. 2.0.1].

For a smooth pair (M, X), we write $A_X(M)$ for $A(M, X)$, we write $A(M)$ for $A(M, M) = A_M(M)$, and for $f : (M, X) \to (N, Y)$ a morphism in $\mathbf{SP}$ we write $f^* : A_Y(N) \to A_X(M)$ for the map $A(f)$. For a smooth pair (M, X), the identity map on M induces the "forget the support map" $\mathrm{id}_M^* : A_X(M) \to A(M)$. With this notation, the axioms are as follows.

(1) *Localization.* For each $(M, X) \in \mathbf{SP}$, let $U = M \setminus X$ and let $j : U \to M$ be the inclusion. Then the sequence

$$A(M) \xrightarrow{j^*} A(U) \xrightarrow{\partial_{M,X}} A_X(M) \xrightarrow{\mathrm{id}_M^*} A(M) \xrightarrow{j^*} A(U)$$

is exact.

In addition, the maps $\partial_{M,X}$ are natural with respect to morphisms in $\mathbf{SP}$: Given a morphism $f : (M, X) \to (N, Y)$ in $\mathbf{SP}$, the diagram

$$\begin{array}{ccc} A(N \setminus Y) & \xrightarrow{\partial_{N,Y}} & A_Y(N) \\ {\scriptstyle f^*|_{N\setminus Y}}\downarrow & & \downarrow{\scriptstyle f^*} \\ A(M \setminus X) & \xrightarrow[\partial_{M,X}]{} & A_X(M) \end{array}$$

commutes.

(2) *Excision.* Let $f\colon M' \to M$ be an étale morphism in $\mathbf{Sm}/k$, let $X \subset M$ be a closed subset, and suppose that $f\colon f^{-1}(X) \to X$ is an isomorphism (giving X and $f^{-1}(X)$ the reduced scheme structure). Then the map

$$f\colon (M', f^{-1}(X)) \to (M, X)$$

induces an isomorphism $f^*\colon A_X(M) \to A_{f^{-1}(X)}(M')$.

(3) *Homotopy.* For $M \in \mathbf{Sm}/k$, the map $p^*\colon A(M) \to A(M \times \mathbb{A}^1)$ induced by the projection $p\colon M \times \mathbb{A}^1 \to M$ is an isomorphism.

REMARK 1.3. One should think of $A_X(M)$ as "A-cohomology of M with support in the closed subset X". For example, if $k = \mathbb{C}$, let

$$A_X(M) := H^{\text{even}}_{X^{\text{an}}}(M^{\text{an}}, \mathbb{Z}) \oplus H^{\text{odd}}_{X^{\text{an}}}(M^{\text{an}}, \mathbb{Z})$$

where $M^{\text{an}}, X^{\text{an}}$ are the $\mathbb{C}$-points of M, X with the classical topology and where

$$H^*_{X^{\text{an}}}(M^{\text{an}}, \mathbb{Z}) := H^*(M^{\text{an}}, M^{\text{an}} - X^{\text{an}}, \mathbb{Z})$$

is the singular cohomology with support. Then $(M, X) \mapsto A_X(M)$ defines a cohomology theory on $\mathbf{SP}$.

REMARKS 1.4. (1) The localization and excision axioms yield a long exact Mayer–Vietoris sequence. Similarly, if $X \subset X' \subset M$ are closed subsets of $M \in \mathbf{Sm}/k$, then putting together the localization sequences for $X \subset M$, $X' \subset M$, and $X' \setminus X \subset M \setminus X$ gives the exact sequence of the triple (M, X', X):

$$A_{X'}(M) \xrightarrow{j^*} A_{X'\setminus X}(M \setminus X) \xrightarrow{\partial_{M,X',X}} A_X(M) \xrightarrow{\text{id}^*_M} A_{X'}(M) \xrightarrow{j^*} A_{X'\setminus X}(M \setminus X).$$

See [10, 2.2.3] for details.

(2) Let $p\colon V \to M$ be an affine space bundle and $X \subset M$ a closed subset. Together with localization and Mayer–Vietoris, the homotopy axiom implies that

$$p^*\colon A_X(M) \to A_{p^{-1}(X)}(V)$$

is an isomorphism. Indeed, $V \to M$ is Zariski locally isomorphic to the projection $M \times \mathbb{A}^n \to M$.

DEFINITION 1.5. A *ring cohomology theory* on $\mathbf{SP}$ is a cohomology theory A on $\mathbf{SP}$ together with graded maps for each pair of smooth pairs (M, X), (N, Y),

$$\times\colon A_X(M) \otimes A_Y(N) \to A_{X\times Y}(M \times N),$$

and an element $1 \in A^{\text{ev}}(pt)$ that satisfies the following axioms of [10, Def. 2.4.2].

(1) *Associativity.* $(a \times b) \times c = a \times (b \times c)$.
(2) *Unit.* $a \times 1 = 1 \times a = a$.
(3) *Partial Leibniz rule.* Given smooth pairs (M, X), (M, X'), (N, Y) with $X \subset X'$, we have the exact sequence of the triple $(M \times N,\ X' \times Y,\ X \times Y)$ (Remark 1.4(1)) with boundary map

$$\partial_{M\times N,\,X'\times N,\,X\times N}\colon A_{(X'\setminus X)\times Y}((M \setminus Z) \times N) \to A_{X\times Y}(M \times N).$$

We also have the triple (M, X', X), with boundary map

$$\partial_{M,X',X}\colon A_{X'\setminus X}(M \setminus X) \to A_X(M).$$

Then

$$\partial_{M\times N,\,X'\times N,\,X\times N}(a \times b) = \partial_{M,X',X}(a) \times b$$

for $a \in A_{X'\setminus X}(M \setminus X)$, $b \in A_Y(N)$.

We add a fourth axiom.

(4) *Graded commutativity.* For $a \in A_X(M)$ of degree p and $b \in A_Y(N)$ of degree q and with $\tau\colon N \times M \to M \times N$ denoting the symmetry isomorphism,

$$\tau^*(a \times b) = (-1)^{pq}(b \times a).$$

For smooth pairs (M, X) and (M, Y), pull-back by the diagonal gives the *cup product with supports*

$$\cup\colon A_X(M) \otimes A_Y(M) \to A_{X\cap Y}(M), \quad \cup := \delta_X^* \circ \times.$$

In particular, $A(M)$ is a $\mathbb{Z}/2$-graded, graded-commutative ring with unit for each $M \in \mathbf{Sm}/k$, and $A_X(M)$ is an $A(M)$-module for each smooth pair (M, X); $A_X(M)$ is itself a $\mathbb{Z}/2$-graded, graded-commutative ring without unit. The partial Leibniz rule implies that, for a triple (M, X', X), the boundary map

$$\partial_{M,X',X}\colon A_{X'\setminus X}(M \setminus X) \to A_X(M)$$

is a right $A_{X''}(M)$-module map for all closed subsets $X' \subset X'' \subset M$. More generally, for any closed $X'' \subset M$ we have

$$\partial_{M,\,X'\cap X'',\,X\cap X''}(a \cup b) = \partial_{M,X',X}(a) \cup b \in A_{X\cap X''}(M)$$

for $b \in A_{X''}(M)$ and $a \in A_{X'\setminus X}(M \setminus X)$.

Remark 1.6. Instead of a $\mathbb{Z}/2$ grading, one can work in the $\mathbb{Z}$-graded or bi-graded setting. One requires that the pull-back maps f^* preserve the (bi-)grading and that ∂ be of degree $+1$ or bi-degree $(+1, 0)$.

At various places in the theory, Panin requires various elements to have certain commutativity properties (see e.g. [10, Def. 2.4.4]); we will replace these conditions with the condition that these elements have even degree. With these modifications, Panin defines four structures on a ring cohomology theory A.

(1) An *orientation* on A is an assignment of a graded $A(M)$-module isomorphism $th_X^E\colon A_X(M) \to A_X(E)$, for each smooth pair (M, X) and each vector bundle E on M, that satisfies the properties listed in [10, Def. 3.1.1].

(2) A *Chern structure* on A is an assignment of an even-degree element $c_1(L) \in A^{\mathrm{ev}}(M)$, for each line bundle L on $M \in \mathbf{Sm}/k$, that satisfies the properties of functoriality and nondegeneracy: a $\mathbb{P}^1$-bundle formula and vanishing for $L = O_M$ the trivial line bundle on M (see [10, Def. 3.2.1]).
(3) A *Thom structure* on A is the assignment of an even-degree element $th(L) \in A^{\mathrm{ev}}_M(L)$, for each line bundle L on $M \in \mathbf{Sm}/k$, that satisfies the properties of functoriality and nondegeneracy: cup product with $th(O_M)$ is an isomorphism $\bigcup th(O_M)\colon A(M) \to A_M(M \times \mathbb{A}^1)$ (see [10, Def. 3.2.2]).
(4) An *integration* on A is an assignment $(f\colon N \to M) \mapsto f_*\colon A(N) \to A(M)$, for each projective morphism $f\colon N \to M$ in $\mathbf{Sm}/k$, that satisfies the properties of [10, Def. 4.1.2] (since we are in the $\mathbb{Z}/2$-graded setting, we require that f_* preserve the grading).

The main results of [10] are that each of these structures gives rise in a uniquely determined manner to all the other structures and that each "loop" in this process induces the identity transformation. Our goal in this section is to extend these results to a more widely defined integration structure.

REMARK 1.7. In the $\mathbb{Z}$-graded or bi-graded situation one requires that the Chern class $c_1(L)$ or the Thom class $th(L)$ be of degree 2 (in the graded case) or bi-degree $(2,1)$ (in the bi-graded case) and that the push-forward f_* shifts (bi-)degrees

$$f_*\colon A^n(N) \to A^{n+2d}(M), \qquad f_*\colon A^{p,q}(N) \to A^{p+2d,q+d}(M),$$

where d is the codimension of f, $d = \dim_k M - \dim_k N$. With these modifications, one recovers Panin's main results in the (bi-)graded case.

Let $\mathbf{SP}'$ be the category with objects the smooth pairs (M, X), $M \in \mathbf{Sm}/k$, $X \subset M$, where a morphism $f\colon (M, X) \to (N, Y)$ is a projective morphisms of pairs (i.e., a projective morphism $f\colon M \to N$ in $\mathbf{Sm}/k$ such that $f(X) \subset Y$). A morphism $f\colon N \to M$ in $\mathbf{Sm}/k$ and closed subsets $Z \subset M$ and $Y \subset N$ give rise to the map

$$f^*(\cdot)\cup\colon A_Z(M) \otimes A_Y(N) \to A_{Y\cap f^{-1}(Z)}(N)$$

sending $a \otimes b$ to $f^*(a) \cup b$, where $f^*\colon A_Z(M) \to A_{f^{-1}(Z)}(N)$ is the pull-back. For $Y \subset f^{-1}(Z)$, the map $f^*(\cdot)\cup$ makes $A_Y(N)$ an $A_Z(M)$-module.

DEFINITION 1.8. Let A be a $\mathbb{Z}/2$-graded ring cohomology theory on $\mathbf{SP}$. An *integration with supports* on A is an assignment of a graded push-forward map

$$f_*\colon A_Y(N) \to A_X(M),$$

for each morphism $f\colon (N, Y) \to (M, X)$ in $\mathbf{SP}'$, that satisfies the following six conditions.

(1) $(f \circ g)_* = f_* \circ g_*$ for composable morphisms.
(2) For $f\colon (N, Y) \to (M, X)$ in $\mathbf{SP}'$ and Z a closed subset of M, f_* is an $A_Z(M)$-module map; that is, the diagram

$$\begin{tikzcd} A_Z(M) \otimes A_Y(N) \arrow[r, "f^*(\cdot)\cup"] \arrow[d, "\mathrm{id}\otimes f_*"'] & A_{Y\cap f^{-1}(Z)}(N) \arrow[d, "f_*"] \\ A_Z(M) \otimes A_X(M) \arrow[r, "\cup"'] & A_{X\cap Z}(M) \end{tikzcd}$$

commutes.

(3) Let $i\colon (N,Y) \to (M,X)$ be morphism in $\mathbf{SP}'$ such that $i\colon N \to M$ is a closed embedding, and let $g\colon (\tilde{M},\tilde{X}) \to (M,X)$ be a morphism in $\mathbf{SP}$. Let $\tilde{N} := N \times_M \tilde{M}$, let $\tilde{g}\colon \tilde{N} \to N$ and $\tilde{i}\colon \tilde{N} \to \tilde{M}$ be the projections, and let $\tilde{Y} := \tilde{i}^{-1}(Y)$. Suppose in addition that $\tilde{N}$ is in $\mathbf{Sm}/k$ and that the square

$$\begin{tikzcd} \tilde{N} \arrow[r, "\tilde{i}"] \arrow[d, "\tilde{g}"'] & \tilde{M} \arrow[d, "g"] \\ N \arrow[r, "i"'] & M \end{tikzcd}$$

is transverse. Then the diagram

$$\begin{tikzcd} A_{\tilde{Y}}(\tilde{N}) \arrow[r, "\tilde{i}_*"] & A_{\tilde{X}}(\tilde{M}) \\ A_Y(N) \arrow[r, "i_*"'] \arrow[u, "\tilde{g}^*"] & A_X(M) \arrow[u, "g^*"'] \end{tikzcd}$$

commutes.

(4) Let $f\colon (N,Y) \to (M,X)$ be a morphism in $\mathbf{SP}$, and let $p_N\colon \mathbb{P}^n \times N \to N$ and $p_M\colon \mathbb{P}^n \times M \to M$ be the projections. Then the diagram

$$\begin{tikzcd} A_{\mathbb{P}^n\times Y}(\mathbb{P}^n \times N) \arrow[d, "p_{N*}"'] & A_{\mathbb{P}^n\times X}(\mathbb{P}^n \times M) \arrow[l, "(\mathrm{id}\times f)^*"'] \arrow[d, "p_{M*}"] \\ A_Y(N) & A_X(M) \arrow[l, "f^*"] \end{tikzcd}$$

commutes.

(5) Given smooth pairs (M,X) and (M,Y) with $X \subset Y$, the maps

$$\mathrm{id}_{M*}\colon A_X(M) \to A_Y(M)$$

and

$$\mathrm{id}_M^*\colon A_X(M) \to A_Y(M)$$

are equal.

(6) Let $f\colon N \to M$ be a projective morphism in $\mathbf{Sm}/k$, let $Y \subset Y' \subset N$ and $X \subset X' \subset M$ be closed subsets, and suppose that $f^{-1}(X) \cap Y' = Y$ and $f(Y') \subset X'$. Then the diagram

$$\begin{array}{ccc} A_{Y'\setminus Y}(N\setminus Y) & \xrightarrow{\partial_{N,Y',Y}} & A_Y(N) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ A_{X'\setminus X}(M\setminus X) & \xrightarrow[\partial_{M,X',X}]{} & A_X(M) \end{array}$$

commutes. Here $\partial_{N,Y',Y}$ and $\partial_{M,X',X}$ are the boundary maps in the respective long exact sequence for the triples (N, Y', Y) and (M, X', X), and the push-forward map $f_*\colon A_{Y'\setminus Y}(N\setminus Y) \to A_{X'\setminus X}(M)$ is the composition

$$A_{Y'\setminus Y}(N\setminus Y) \xrightarrow{j^*} A_{Y'\setminus Y}(N\setminus f^{-1}(X)) \xrightarrow{f|_{N\setminus f^{-1}(X)*}} A_{X'\setminus X}(M\setminus X).$$

Note that an integration with supports on A determines an integration on A by restricting f_* to $f_*\colon A(N) \to A(M)$. One also has a $\mathbb{Z}$-graded or bi-graded version.

For later use, we give an extension of the properties (3) and (4) of Definition 1.8.

LEMMA 1.9. *Let A be a $\mathbb{Z}/2$-graded ring cohomology theory on* **SP**, *with an integration with supports. Let $f\colon (N, Y) \to (M, X)$ be a morphism in* $\mathbf{SP}'$ *and $g\colon (\tilde{M}, \tilde{X}) \to (M, X)$ a morphism in* **SP**. *Let $\tilde{N} := N \times_M \tilde{M}$, let $\tilde{g}\colon \tilde{N} \to N$ and $\tilde{f}\colon \tilde{N} \to \tilde{M}$ be the projections, and let $\tilde{Y} := \tilde{f}^{-1}(X)$. Suppose in addition that there is an open neighborhood U of Y in N such that $U \times_M \tilde{M}$ is in* $\mathbf{Sm}/k$, *the diagram*

$$\begin{array}{ccc} U \times_M \tilde{M} & \xrightarrow{\tilde{f}} & \tilde{M} \\ {\scriptstyle \tilde{g}}\downarrow & & \downarrow{\scriptstyle g} \\ U & \xrightarrow[f]{} & M \end{array}$$

is transverse, and the closure $\hat{U}$ of $U \times_M \tilde{M}$ in $\tilde{N}$ is smooth. Then the diagram

$$\begin{array}{ccc} A_{\tilde{Y}}(\hat{U}) & \xrightarrow{\hat{f}_*} & A_{\tilde{X}}(\tilde{M}) \\ {\scriptstyle \hat{g}^*}\uparrow & & \uparrow{\scriptstyle g^*} \\ A_Y(N) & \xrightarrow[f_*]{} & A_X(M) \end{array}$$

commutes, where $\hat{f}\colon \hat{U} \to \tilde{M}$ is the restriction of $\tilde{f}$ and $\hat{g}\colon \hat{U} \to N$ is the restriction of $\tilde{g}$.

In particular, if $\tilde{N}$ is in $\mathbf{Sm}/k$ *and the Cartesian diagram*

$$\begin{array}{ccc} \tilde{N} & \xrightarrow{\tilde{f}} & \tilde{M} \\ {\scriptstyle \tilde{g}}\downarrow & & \downarrow{\scriptstyle g} \\ N & \xrightarrow[f]{} & M \end{array}$$

is transverse, then the diagram

$$\begin{array}{ccc} A_{\tilde{Y}}(\tilde{N}) & \xrightarrow{\tilde{f}_*} & A_{\tilde{X}}(\tilde{M}) \\ \uparrow{\scriptstyle \hat{g}^*} & & \uparrow{\scriptstyle g^*} \\ A_Y(N) & \xrightarrow[f_*]{} & A_X(M) \end{array}$$

commutes.

Proof. Factor $f: N \to M$ as $p \circ i$, with $i: N \to \mathbb{P}^n \times M$ a closed immersion and $p: \mathbb{P}^n \to M$ the projection. The statement for $f = p$ is just Definition 1.8(3), so we need only handle the case of $f = i$ a closed immersion. Also, it suffices to handle the case $\tilde{X} = g^{-1}(X)$.

Set $F := N \setminus U$, and let $V := M \setminus F$ and $\tilde{V} := g^{-1}(V)$. Then V is a neighborhood of X in M and $g^{-1}(V)$ is a neighborhood of $\tilde{X} = g^{-1}(X)$ in $\tilde{M}$. Letting $\tilde{U} = U \times_M \tilde{M}$, we have the transverse Cartesian diagram

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\tilde{i}_U} & \tilde{V} \\ {\scriptstyle \tilde{g}_U}\downarrow & & {\scriptstyle U}\downarrow{\scriptstyle g} \\ U & \xrightarrow[i_U]{} & V \end{array}$$

in $\mathbf{Sm}/k$ with i_U a closed immersion. Definition 1.8(2) gives us the commutative diagram

$$\begin{array}{ccc} A_{\tilde{Y}}(\tilde{U}) & \xrightarrow{\tilde{i}_{U*}} & A_{\tilde{X}}(\tilde{V}) \\ \uparrow{\scriptstyle \hat{g}_U^*} & & \uparrow{\scriptstyle g_U^*} \\ A_Y(U) & \xrightarrow[i_{U*}]{} & A_X(V). \end{array}$$

Now we just use the excision isomorphisms

$$A_{\tilde{Y}}(\hat{U}) \to A_{\tilde{Y}}(\tilde{U}), \qquad A_Y(N) \to A_Y(U),$$
$$A_{\tilde{X}}(\tilde{M}) \to A_{\tilde{X}}(\tilde{V}), \qquad A_X(M) \to A_X(V)$$

and Definition 1.8(2) to give the commutativity of

$$\begin{array}{ccc} A_{\tilde{Y}}(\hat{U}) & \xrightarrow{\hat{i}_*} & A_{\tilde{X}}(\tilde{M}) \\ \uparrow{\scriptstyle \hat{g}^*} & & \uparrow{\scriptstyle g^*} \\ A_Y(N) & \xrightarrow[i_*]{} & A_X(M). \end{array}$$

□

Remark 1.10 (Projection formula). Condition (2) of Definition 1.8 is just the projection formula "with supports"—in other words, that

$$f_*(f^*(a) \cup b) = a \cup f_*(b) \in A_{X \cap Z}(M)$$

for $a \in A_Z(M)$, $b \in A_Y(N)$, and $f : (N, Y) \to (M, X)$ a morphism in $\mathbf{SP}'$. This axiom may also be stated using the external product instead of the cup product: Let $f : (N, Y) \to (M, X)$ and $g : (N', Y') \to (M', X')$ be morphisms in $\mathbf{SP}'$; then

$$(f \times g)_*(a \times b) = f_*(a) \times g_*(b) \in A_{X \times X'}(M \times M') \tag{1.1}$$

for all $a \in A_Y(N)$ and $b \in A_{Y'}(N')$. Indeed, to recover (2), take g to be id: $(M, Z) \to (M, Z)$ and then apply (3) to the transverse Cartesian diagram

$$\begin{tikzcd} N \arrow[r, "(\mathrm{id}, f)"] \arrow[d, "f"'] & N \times M \arrow[d, "f \times \mathrm{id}"] \\ M \arrow[r, "\delta"'] & M \times M \end{tikzcd}$$

and the morphism $\delta : (M, X \cap Z) \to (M \times M, X \times Z)$.

To show that (2) implies (1.1), since $f \times g = (f \times \mathrm{id}) \circ (\mathrm{id} \times g)$ it suffices to handle the case $g = \mathrm{id}$. From the commutative diagram

$$\begin{tikzcd} N \times M' \arrow[r, "p'_2"] \arrow[d, "f \times \mathrm{id}"'] & M' \arrow[d, equal] \\ M \times M' \arrow[r, "p_2"'] & M' \end{tikzcd}$$

we have $p_2'^*(b) = (f \times \mathrm{id})^*(p_2^*(b))$. Applying (3) to the Cartesian transverse diagram

$$\begin{tikzcd} N \times M' \arrow[r, "p'_1"] \arrow[d, "f \times \mathrm{id}"'] & N \arrow[d, "f"] \\ M \times M' \arrow[r, "p_1"'] & M \end{tikzcd}$$

and using (2) yields

$$\begin{aligned} (f \times \mathrm{id})_*(a \times b) &= (f \times \mathrm{id})_*(p_1'^*(a) \cup p_2'^*(b)) \\ &= (f \times \mathrm{id})_*(p_1'^*(a) \cup (f \times \mathrm{id})^*(p_2^*(b))) \\ &= (f \times \mathrm{id})_*(p_1'^*(a)) \cup p_2^*(b) \\ &= p_1^*(f_*(a)) \cup p_2^*(b) \\ &= f_*(a) \times b. \end{aligned}$$

To set up a one-to-one correspondence between integrations with support and the other structures, we rephrase the compatibility condition [10, Def. 4.1.3].

DEFINITION 1.11. Let ω be an orientation of A and let $L \mapsto c_1(L)$ be the corresponding Chern structure on A (given by [10, 3.7.5]). We say that an integration with supports $f \mapsto f_*$ on A is *subjected to* the orientation ω if, for each smooth pair (M, X) and each line bundle $p\colon L \to M$ with zero section $s\colon M \to L$, the endomorphism

$$A_X(M) \xrightarrow{s_*} A_{p^{-1}(X)}(L) \xrightarrow{s^*} A_X(M)$$

of $A_X(M)$ is given by cup product with $c_1(L)$.

In case $X = M$, this condition is just saying that $c_1(L) = s^*(s_*(1))$, which the reader can easily check is equivalent to the condition given in Panin's definition [10, Def. 4.1.3].

Our main result is the following theorem (cf. [10, Thm. 4.1.4]).

THEOREM 1.12. *Let A be a $\mathbb{Z}/2$-graded ring cohomology theory. Given an orientation ω on A, there is a unique integration with supports on A subjected to ω.*

COROLLARY 1.13. *Let A be a $\mathbb{Z}/2$-graded ring cohomology theory. Given an integration $f \mapsto f_*$ on A, there is a unique integration with supports on A extending f.*

Proof. Let ω be the orientation on A corresponding to f by [10, Thm. 4.1.4]. By Theorem 1.12, there is a unique integration with supports ι on A subjected to ω. Since the restriction of ι to an integration on A (without supports) is subjected to ω, it follows from the uniqueness in [10, Thm. 4.1.4] that the restriction of ι to an integration on A is the given one $f \mapsto f_*$. Thus an extension of $f \mapsto f_*$ to an integration with supports on A exists.

If now ι' is another extension, write the push-forward map for f as f'_*; note that $f'_* = f_*$ if we omit supports. Take $a \in A_X(M)$ and let $p\colon L \to M$ be a line bundle with zero section s. Then

$$\begin{aligned} s^*(s'_*(a)) &= s^*(s'_*(s^*p^*(a) \cup 1)) \\ &= s^*(p^*(a) \cup s_*(1)) \\ &= a \cup s^*(s_*(1)) \\ &= a \cup c_1(L). \end{aligned}$$

Thus ι' is subjected to ω and hence $\iota = \iota'$ by the uniqueness in Theorem 1.12. □

Theorem 1.12 is proven by copying the construction in [10] of an integration subjected to a given orientation ω, making at each stage the extension to an integration with supports.

Step 1. *The case of a closed immersion*
Let $i\colon N \to M$ be a closed immersion in $\mathbf{Sm}/k$, let $Y \subset N$ be a closed subset, and let $\nu \to N$ be the normal bundle of N in M. The deformation to the normal bundle [10, Sec. 2.2.7] gives the diagram

$$\begin{tikzcd}
\nu \arrow[r, "i_0"] & M_t & M \arrow[l, "i_1"'] \\
N \arrow[u, "s"] \arrow[r, "i_0"'] & N \times \mathbb{A}^1 \arrow[u, "\tilde{i}"'] & N \arrow[l, "i_1"] \arrow[u, "i"'] \\
Y \arrow[u, hook] \arrow[r, "i_0"'] & Y \times \mathbb{A}^1 \arrow[u, hook] & Y. \arrow[l, "i_1"] \arrow[u, hook]
\end{tikzcd}$$

From this deformation diagram, we arrive at the maps

$$A_Y(\nu) \xleftarrow{i_0^*} A_{Y\times\mathbb{A}^1}(M_t) \xrightarrow{i_1^*} A_Y(M).$$

LEMMA 1.14. *The maps i_0^* and i_1^* are isomorphisms.*

Proof. When $Y = N$ this is [10, Thm. 2.2.8]. In general, let $U = M \setminus i(Y)$ and $V = N \setminus Y$, let ν' be the normal bundle of V in U, and let U_t be the deformation space constructed from the closed immersion $i' \colon V \to U$.

Let $j \colon U_t \to M_t \setminus Y \times \mathbb{A}^1$ and $\bar{j} \colon \nu' \to \nu \setminus Y$ be the inclusions. We then have the commutative diagram

$$\begin{tikzcd}
A_V(\nu \setminus Y) \arrow[d, "\bar{j}^*"'] & A_{V\times\mathbb{A}^1}(M_t \setminus Y \times \mathbb{A}^1) \arrow[l, "i_0^*"'] \arrow[r, "i_1^*"] \arrow[d, "j^*"'] & A_V(U) \arrow[d, equal] \\
A_V(\nu') & A_{V\times\mathbb{A}^1}(U_t) \arrow[l, "i_0^*"] \arrow[r, "i_1^*"'] & A_V(U);
\end{tikzcd}$$

the maps j^* and $\bar{j}^*$ are isomorphisms by excision. By [10, Thm. 2.2.8], the horizontal maps in the bottom row are isomorphisms and so the horizontal maps in the top row are isomorphisms as well.

We have the commutative diagram

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & {} \arrow[d] \\
A_V(\nu \setminus Y) \arrow[d, "\partial"'] & A_{V\times\mathbb{A}^1}(M_t \setminus Y \times \mathbb{A}^1) \arrow[l, "i_0^*"'] \arrow[r, "i_1^*"] \arrow[d, "\partial"] & A_V(U) \arrow[d, "\partial"] \\
A_Y(\nu) \arrow[d] & A_{Y\times\mathbb{A}^1}(M_t) \arrow[l, "i_0^*"'] \arrow[r, "i_1^*"] \arrow[d] & A_Y(M) \arrow[d] \\
A_N(\nu) \arrow[d] & A_{N\times\mathbb{A}^1}(M_t) \arrow[l, "i_0^*"] \arrow[r, "i_1^*"'] \arrow[d] & A_N(M), \arrow[d] \\
{} & {} & {}
\end{tikzcd}$$

where the columns are the long exact sequences of triples (ν, N, Y), $(M_t, N \times \mathbb{A}^1, Y \times \mathbb{A}^1)$, and (M, N, Y). Hence the case $Y = N$, our foregoing remarks, and the five lemma together show that the horizontal maps in the middle row are isomorphisms, as desired. $\square$

Now let $X \subset M$ be a closed subset containing $i(Y)$. We have the diagram

$$A_Y(\nu) \xleftarrow{i_0^*} A_{Y\times\mathbb{A}^1}(M_t) \xrightarrow{i_1^*} A_X(M)$$

with i_0^* an isomorphism. Let

$$i_*\colon A_Y(N) \to A_X(M)$$

be given by the composition

$$A_Y(N) \xrightarrow{th_Y^\nu} A_Y(\nu) \xrightarrow{i_1^*\circ(i_0^*)^{-1}} A_X(M). \tag{1.2}$$

Proposition 1.15. *Let A be a $\mathbb{Z}/2$-graded oriented ring cohomology theory.*

(1) *For $i\colon N \to M$ a closed immersion in* **Sm**$/k$, *the map $i_*\colon A(N) \to A(M)$ defined previously agrees with the map i_{gys} defined in* [10, Sec. 4.2].

(2) *Let $i\colon N \to M$ be a closed immersion in* **Sm**$/k$, *let Y be a closed subset of N, and let X be a closed subset of M such that $i(Y) \subset X$. Then, for $Z \subset M$ a closed subset, $i_*\colon A_Y(N) \to A_X(M)$ is an $A_Z(M)$-module homomorphism* (*in the sense of Definition 1.8*(2)).

(3) *Let $i_1\colon N \to M$ and $i_2\colon P \to N$ be closed immersions in* **Sm**$/k$, *and let $X \subset M$, $Y \subset N$, and $Z \subset P$ be closed subsets with $i_1(X) \subset Y$ and $i_2(Y) \subset Z$. Then*

$$(i_1 \circ i_2)_* = i_{1*} \circ i_{2*}\colon A_Z(P) \to A_X(M).$$

(4) *Let N_1, N_2 be in* **Sm**$/k$, *and let $j_i\colon N_i \to N := N_1 \amalg N_2$ be the canonical inclusions, $i = 1, 2$. Let $i\colon N \to M$ be a closed immersion in* **Sm**$/k$, *let $Y_i \subset N_i$ $(i = 1, 2)$ be a closed subset, and let $X \subset M$ be a closed subset containing $i(Y_1 \amalg Y_2)$. Let i_j be the restriction of i to N_j, $j = 1, 2$. Then*

$$i_* = i_{1*} \circ j_1^* + i_2 \circ j_2^*\colon A_{Y_1\amalg Y_2}(N_1 \amalg N_2) \to A_X(M).$$

(5) *Let $i\colon (N, Y) \to (M, X)$ be a morphism in* **SP**$'$ *such that $i\colon N \to M$ is a closed immersion, and let $g\colon (\tilde{M}, \tilde{X}) \to (M, X)$ be a morphism in* **SP**. *Let $\tilde{N} := N \times_M \tilde{M}$, let $\tilde{g}\colon \tilde{N} \to N$ and $\tilde{i}\colon \tilde{N} \to \tilde{M}$ be the projections, and let $\tilde{Y} := \tilde{f}-1(Y)$. Suppose in addition that $\tilde{N}$ is in* **Sm**$/k$ *and that the square*

$$\begin{tikzcd} \tilde{N} \arrow[r, "\tilde{i}"] \arrow[d, "\tilde{g}"'] & \tilde{M} \arrow[d, "g"] \\ N \arrow[r, "i"'] & M \end{tikzcd}$$

is transverse. Then the diagram

$$\begin{tikzcd} A_{\tilde{Y}}(\tilde{N}) \arrow[r, "\tilde{i}_*"] & A_{\tilde{X}}(\tilde{M}) \\ A_Y(N) \arrow[u, "\tilde{g}^*"] \arrow[r, "i_*"'] & A_X(M) \arrow[u, "g^*"'] \end{tikzcd}$$

commutes.

(6) *For $M \in \mathbf{Sm}/k$ with closed subsets $Y \subset X$, we have*

$$\mathrm{id}_{M*} = \mathrm{id}_M^* \colon A_Y(M) \to A_X(M).$$

(7) *Let $i \colon (N, Y) \to (M, X)$ be a morphism in $\mathbf{SP}'$ such that i is a closed immersion. Let $M \setminus Y \to M$ be the inclusion. Then the sequence*

$$A_Y(N) \xrightarrow{i_*} A_X(M) \xrightarrow{j^*} A_{X \setminus Y}(M \setminus Y)$$

is exact.

(8) *Let $i \colon N \to M$ be a closed immersion in $\mathbf{Sm}/k$, let $Y \subset Y'$ be closed subsets of N, and let $X \subset X'$ be closed subsets of M such that $i^{-1}(X) \cap Y' = Y$ and $i(Y') \subset X'$. Then the diagram*

$$\begin{tikzcd} A_{Y' \setminus Y}(N \setminus Y) \arrow[r, "\partial_{N,Y',Y}"] \arrow[d, "i_*"'] & A_Y(N) \arrow[d, "i_*"] \\ A_{X' \setminus X}(M) \arrow[r, "\partial_{M,X',X}"'] & A_X(M) \end{tikzcd}$$

commutes.

Proof. Part (1) follows from the definitions. The proofs of parts (2)–(7) are exactly as the proofs given in [10, Sec. 4.4] of the analogous statements without support, altered by adding the supports to the notation.

For (8), we may replace Y' with $Y' \cup i^{-1}(X)$ and Y with $i^{-1}(X)$; changing notation, we may assume that $i^{-1}(X) = Y$. Use (1) and (6) to factor $i_* \colon A_Y(N) \to A_X(M)$ as the composition

$$A_Y(N) \xrightarrow{i_*} A_Y(M) \xrightarrow{\mathrm{id}_M^*} A_X(M),$$

and similarly factor $i_* \colon A_{Y' \setminus Y}(N \setminus Y) \to A_{X' \setminus X}(M \setminus X)$ as

$$A_{Y' \setminus Y}(N \setminus Y) \xrightarrow{i_*} A_{Y' \setminus Y}(M \setminus Y) \xrightarrow{j^*} A_{X' \setminus X}(M \setminus X).$$

Because the long exact sequence of a triple is natural with respect to pull-back, this reduces us to the case of $X = Y$ and $X' = Y'$.

Panin [10, Lemma 3.7.2] shows that there is a "Thom classes theory" on A; that is, for each vector bundle $p \colon E \to M$ ($M \in \mathbf{Sm}/k$), an even-degree element $th(E) \in A_M(E)$ such that the orientation isomorphism $th_X^E \colon A_X(M) \to A_X(E)$ is given by the composition

$$A_X(M) \xrightarrow{p^*} A_{p^{-1}(X)}(E) \xrightarrow{\cup th(E)} A_X(E).$$

The classes $th(E)$ satisfy additional properties (see [10, Def. 3.7.1]); in particular, for $f\colon N \to M$ we have $f^*(th(E)) = th(f^*E)$. Let $p\colon \nu \to N$ be the normal bundle of N in M, and let $j\colon N \setminus Y \to N$ be the inclusion. Since the boundary map in the long exact sequence of a triple (M, X', X) is natural with respect to pull-backs and is an $A_{X'}(M)$-module map, this shows that the diagram

$$\begin{tikzcd}
A_{Y'\setminus Y}(N \setminus Y) \arrow[r, "th^{j^*\nu}_{Y'\setminus Y}"] \arrow[d, "\partial_{N,Y',Y}"'] & A_{Y'\setminus Y}(j^*\nu) \arrow[d, "\partial_{\nu,Y',Y}"] \\
A_Y(N) \arrow[r, "th^{\nu}_{Y}"'] & A_Y(\nu)
\end{tikzcd}$$

commutes. Looking at the definition (1.2) of the Gysin map, this commutativity—together with the naturality of the long exact sequence of a triple with respect to pull-back—finishes the proof of (8). $\square$

REMARK 1.16. Let $i\colon N \to M$ be a closed immersion in $\mathbf{Sm}/k$, let $Y \subset N$ be a closed subset, and let $X = i(Y)$. Then

$$i_*\colon A_Y(N) \to A_X(M)$$

is an isomorphism. Indeed,

$$i_*\colon = i_1^* \circ (i_0^*)^{-1} \circ th^{\nu}_{Y};$$

th^{ν}_{Y} is isomorphism by the definition of an orientation, and i_0^*, i_1^* are isomorphisms by Lemma 1.14.

Step 2. *The case of a projection*

This step relies on the formal group law associated to an oriented theory. We sketch the main points here, following [10, Sec. 3.9].

We recall that an oriented theory A satisfies the *projective bundle formula* [10, Thm. 3.3.1]: For $M \in \mathbf{Sm}/k$,

$$A(\mathbb{P}^n \times M) \cong A(M)[t]/(t^{n+1})$$

(with t in even degree), where the isomorphism sends t to $c_1(\mathcal{O}(1))$. Here $L \mapsto c_1(L)$ is the Chern structure associated to the given orientation.

REMARK 1.17 [10, Cor. 3.3.8]. Using the exact sequences of the pairs (M, X) and $(\mathbb{P}^n \times M, \mathbb{P}^n \times X)$, the projective bundle formula extends to give an isomorphism of $A_{X'}(M)$-modules (for any closed subset X' of M containing X)

$$A_{\mathbb{P}^n \times X}(\mathbb{P}^n \times M) \cong A_X(M) \otimes_{A(M)} A(M)[t]/(t^{n+1}),$$

with $a \otimes t^i$ mapping to $p_2^*(a) \cup c_1(\mathcal{O}(1))^i$.

We set $pt := \operatorname{Spec} k$. Defining

$$A(\mathbb{P}^\infty \times M) := \varprojlim_{\mathbb{N}} A(\mathbb{P}^N \times M)$$

(using the system of inclusions $\mathbb{P}^N \to \mathbb{P}^{N+1}$ as the hyperplane $X_{N+1} = 0$), we have

$$A(\mathbb{P}^\infty \times M) \cong A(M)[[t]].$$

Similarly,

$$A(\mathbb{P}^\infty \times \mathbb{P}^\infty \times M) \cong A(M)[[u, v]],$$

the latter isomorphism sending u to $c_1(p_1^*\mathcal{O}(1))$ and v to $c_1(p_2^*\mathcal{O}(1))$. Hence there is a well-defined element $F_A(u, v) \in A(pt)[[u, v]]$ with

$$F_A(c_1(p_1^*\mathcal{O}(1)), c_1(p_2^*(\mathcal{O}(1))) = c_1(p_1^*\mathcal{O}(1) \otimes p_2^*(\mathcal{O}(1)).$$

By Jouanolou's trick [4] and functoriality, this gives

$$F_A(c_1(L), c_1(L')) = c_1(L \otimes L')$$

for each pair of line bundles L, L' on some $M \in \mathbf{Sm}/k$. That the set of isomorphism class of line bundles on $M \in \mathbf{Sm}/k$ is a group under tensor product directly implies that $F_A(u, v)$ defines a (commutative, rank-1) formal group law over $A(pt)$:

(1) $F_A(u, 0) = F_A(0, u) = u$;
(2) $F_A(u, v) = F_A(v, u)$;
(3) $F_A(F_A(u, v), w) = F_A(u, F_A(v, w))$.

The inverse is given by the power series $I_A(t) \in A(pt)[[t]]$ corresponding to $c_1(\mathcal{O}(-1))$ under the isomorphism $A(\mathbb{P}^\infty) \cong A(pt)[[t]]$.

REMARK 1.18. Since $c_1(L)$ has even degree, it follows that all the coefficients of $F_A(u, v)$ and $I_A(t)$ have even degree and so we actually have a formal group law over the commutative ring $A^{\mathrm{ev}}(pt)$.

For a commutative ring R, let $\Omega^{ctn}_{R[[t]]/R} := \Omega_{R[t]/R} \otimes_R R[[t]]$. Given a commutative formal group law $F(u, v) \in R[[u, v]]$ over R, there is a unique normalized invariant differential form $\omega_F \in \Omega^{ctn}_{R[[t]]/R}$. We write ω_A for $\omega_{F_A} \in \Omega^{ctn}_{A^{\mathrm{ev}}(pt)[[t]]/A^{\mathrm{ev}}(pt)}$. Using the canonical generator dt for $\Omega^{ctn}_{A^{\mathrm{ev}}(pt)[[t]]/A^{\mathrm{ev}}(pt)}$, we have

$$\omega_A = \left(1 + \sum_{n \geq 1} a_n t^n\right) dt = dt + a_1 t\, dt + \cdots$$

with $a_n \in A^{\mathrm{ev}}(pt)$ (here "normalized" means the first term is dt; i.e., $a_0 = 1$).

DEFINITION 1.19. We denote the projection $\mathbb{P}^n \times M \to M$ by p^n. For a smooth pair (M, X), define the map

$$p^n_*\colon A_{\mathbb{P}^n \times X}(\mathbb{P}^n \times M) \to A_X(M)$$

by

$$p^n_*(a \otimes t^i) := a_{n-i} \cdot a.$$

Here we use the isomorphism $A_{\mathbb{P}^n \times X}(\mathbb{P}^n \times M) \cong A_X(M) \otimes_{A(M)} A(M)[t]/(t^{n+1})$ given by the projective bundle formula as well as the canonical $A^{\mathrm{ev}}(pt)$-module structure on $A_X(M)$.

REMARK 1.20. If we forget supports, the map p^n_* agrees with the map p^n_{quil} defined in [10, Sec. 4.3].

REMARK 1.21. Since the coefficients a_n of ω_A are in A^{ev}, it follows that p^n_* is an $A_Z(M)$-module map for all closed subsets $Z \subset M$ (in the sense of Definition 1.8(2)). In particular, p^n_* is an $A(M)$-module map.

REMARK 1.22. Using the obvious modification of the push-forward map p^n_*, we have maps

$$p^n_*\colon A_{X\times\mathbb{P}^n\times Y}(M\times\mathbb{P}^n\times N)\to A_{X\times Y}(M\times N)$$

for smooth pairs (M,X) and (N,Y). Because the basis elements in the projective bundle formula are of even degree, we need not worry about the order of the factors.

PROPOSITION 1.23. *Let A be a $\mathbb{Z}/2$-graded oriented ring cohomology theory.*

(1) *For (M,X) a smooth pair, the following diagram commutes*:

$$\begin{tikzcd}
A_{\mathbb{P}^n\times\mathbb{P}^m\times X}(\mathbb{P}^n\times\mathbb{P}^m\times M) \arrow[r, "p^m_*"] \arrow[d, "p^n_*"'] & A_{\mathbb{P}^n\times X}(\mathbb{P}^n\times M) \arrow[d, "p^n_*"] \\
A_{\mathbb{P}^m\times X}(\mathbb{P}^m\times M) \arrow[r, "p^m_*"'] & A_X(M).
\end{tikzcd}$$

(2) *Let $f\colon (N,Y)\to(M,X)$ be a morphism in* **SP**. *Then the diagram*

$$\begin{tikzcd}
A_{\mathbb{P}^n\times X}(\mathbb{P}^n\times M) \arrow[r, "f^*"] \arrow[d, "p^n_*"'] & A_{\mathbb{P}^n\times Y}(\mathbb{P}^n\times N) \arrow[d, "p^n_*"] \\
A_X(M) \arrow[r, "f^*"'] & A_Y(N)
\end{tikzcd}$$

commutes.

(3) *Let $i\colon\mathbb{P}^n\to\mathbb{P}^m$ be a linear closed immersion and (M,X) a smooth pair. Then the diagram*

$$\begin{tikzcd}
A_{\mathbb{P}^n\times X}(\mathbb{P}^n\times M) \arrow[r, "(i\times\mathrm{id})_*"] \arrow[d, "p^n_*"'] & A_{\mathbb{P}^m\times X}(\mathbb{P}^m\times M) \arrow[d, "p^m_*"] \\
A_X(M) \arrow[r, equal] & A_X(M)
\end{tikzcd}$$

commutes.

(4) *Let $i\colon N\to M$ be a closed immersion, and let $X\subset M$ and $Y\subset N$ be closed subsets with $i(Y)\subset X$. Then the diagram*

$$\begin{tikzcd}
A_{\mathbb{P}^n\times Y}(\mathbb{P}^n\times N) \arrow[r, "(\mathrm{id}\times i)_*"] \arrow[d, "p^n_*"'] & A_{\mathbb{P}^n\times X}(\mathbb{P}^n\times M) \arrow[d, "p^n_*"] \\
A_Y(N) \arrow[r, "i_*"'] & A_X(M)
\end{tikzcd}$$

commutes.

(5) *Let $s\colon M \to \mathbb{P}^n \times M$ be a section to the projection and let $X \subset M$ be a closed subset. Then $p^n_* \circ s_* = \mathrm{id}_{A_X(M)}$.*
(6) *Let $X \subset X'$ be closed subsets of $M \in \mathbf{Sm}/k$. Then the diagram*

$$\begin{array}{ccc} A_{\mathbb{P}^n\times(X'\setminus X)}(\mathbb{P}^n \times (M\setminus X)) & \xrightarrow{\partial_{\mathbb{P}^n\times M,\mathbb{P}^n\times X',\mathbb{P}^n\times X}} & A_{\mathbb{P}^n\times X}(\mathbb{P}^n\times M) \\ {\scriptstyle p^n_*}\downarrow & & \downarrow{\scriptstyle p^n_*} \\ A_{X'\setminus X}(M\setminus X) & \xrightarrow[\partial_{M,X',X}]{} & A_X(M) \end{array}$$

commutes.

Proof. The proofs of (1)–(3) are exactly as the proofs of the corresponding properties in [10, Sec. 4.5], adding the supports throughout. We give a proof of part (4) that is different from the approach used in [10].

By the projective bundle formula, it suffices to check the commutativity on elements of $A_{\mathbb{P}^n\times Y}(\mathbb{P}^n \times N)$ of the form $t^m \times a = p^{n*}(t^m) \cup p_Y^*(a)$, $t = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$. Since the Gysin map $(\mathrm{id}\times i)_*$ is a $A(\mathbb{P}^n \times M)$-module map, we have

$$\begin{aligned}(\mathrm{id}\times i)_*(t^m \times a) &= p^{n*}(t^m) \cup (\mathrm{id}\times i)_*(p_N^*(a))\\ &= p^{n*}(t^m) \cup p_M^*(i_*(a)))\\ &= t^m \times i_*(a),\end{aligned}$$

where the second equality follows from Proposition 1.15(5). Thus

$$\begin{aligned}p^n_*((\mathrm{id}\times i)_*(t^m\times a)) &= p^n_*(t^m \times i_*(a))\\ &= a_{n-m}\cdot i_*(a).\end{aligned}$$

On the other hand,

$$\begin{aligned}i_*(p^n_*(t^m\times a)) &= i_*(a_{n-m}\cdot a)\\ &= a_{n-m}\cdot i_*(a),\end{aligned}$$

where the second equality follows because i_* is an $A(M)$-module map and hence an $A(pt)$-module map. This proves (4).

For (5), the case without supports (proven in [10, Sec. 4.6]) gives in particular the identity

$$p^n_*(s_*(1)) = 1 \in A(M),$$

where $1 \in A(M)$ is the identity. Now take an arbitrary element $a \in A_X(M)$ and write p for p^n. Using that both s_* and p_* satisfy the projection formula (for i_*, this is just Proposition 1.15(2); for p^n_*, this is Remark 1.21), we have

$$\begin{aligned}p_*(s_*(a)) &= p_*(s_*(s^*p^*(a)\cup 1))\\ &= p_*(p^*(a)\cup s_*(1))\\ &= a \cup p_*(s_*(1))\\ &= a\cup 1 = a.\end{aligned}$$

Finally, (6) follows from the partial Leibniz rule for ∂, which implies that $\partial_{X,Z',Z}$ and $\partial_{\mathbb{P}^n\times X,\mathbb{P}^n\times Z',\mathbb{P}^n\times Z}$ are $A(pt)$-module maps, together with the naturality of ∂ with respect to pull-back. Thus,

$$\begin{aligned} p^n_*(\partial_{\mathbb{P}^n\times M,\mathbb{P}^n\times X',\mathbb{P}^n\times X}(t^m\times a)) &= p^n_*(t^m\cup\partial_{\mathbb{P}^n\times M,\mathbb{P}^n\times X',\mathbb{P}^n\times X}(p^{n*}(a)))\\ &= p^n_*(t^m\cup p^{n*}(\partial_{M,X',X}(a)))\\ &= a_{n-m}\cdot\partial_{M,X',X}(a)\\ &= \partial_{M,X',X}(a_{n-m}\cdot a)\\ &= \partial_{M,X',X}(p^n_*(t^m\times a)). \end{aligned}\qquad\square$$

Step 3. *The general case*
Let $f\colon (N,Y)\to(M,X)$ be a morphism in $\mathbf{SP}'$. Factor $f\colon M\to N$ as $f=p\circ i$, with $i\colon N\to\mathbb{P}^n\times M$ a closed immersion and $p=p^n\colon\mathbb{P}^n\times M\to M$ the projection. Define $f_*\colon A_Y(N)\to A_X(M)$ as the composition

$$A_Y(N)\xrightarrow{i_*}A_{\mathbb{P}^n\times X}(\mathbb{P}^n\times M)\xrightarrow{p_*}A_X(M).$$

THEOREM 1.24. *Let A be a $\mathbb{Z}/2$-graded oriented ring cohomology theory on* $\mathbf{SP}$.

(1) *For a morphism $f\colon(N,Y)\to(M,X)$ in $\mathbf{SP}'$, the morphism $f_*\colon A_Y(N)\to A_X(M)$ does not depend on the choice of factorization $f=p\circ i$.*
(2) *For a morphism $f=i\colon(N,Y)\to(M,X)$ in $\mathbf{SP}'$ with $i\colon N\to M$ a closed immersion, f_* agrees with the Gysin morphism defined in Step 1. For $f=p^n\colon(\mathbb{P}^n\times M,\mathbb{P}^n\times X)\to(M,X)$ the projection, f_* agrees with the map p^n_* defined in Step 2.*
(3) *For a projective morphism $f\colon N\to M$, the map $f_*\colon A(N)\to A(M)$ agrees with the map f_* defined in* [10, Sec. 4.7].
(4) *The assignment $[f\colon(N,Y)\to(M,X)]\mapsto f_*\colon A_Y(N)\to A_X(M)$ defines an integration with supports on A (Definition 1.8) subjected to the given orientation on A.*

Proof. The proof of (1) is exactly as in the proof of the analogous result [10, Thm. 4.7.1], adding the supports where needed. Part (2) follows directly from (1), since we may take $n=0$ if f is a closed immersion and i the identity (and $i_*=\mathrm{id}$ as well) if $f=p^n$. Part (3) follows from Proposition 1.15(1) and Remark 1.20.

For (4), the proofs of (1), (3) and (4) in Definition 1.8 are exactly as in the proof of [10, Thm. 4.7.1], adding the supports. Definition 1.8(2) follows from Proposition 1.15(2) and Remark 1.21.

Definition 1.8(5) follows from Proposition 1.15(6), while Definition 1.8(6) follows from Proposition 1.15(8) and Proposition 1.23(6). Therefore,

$$[f\colon(N,Y)\to(M,X)]\mapsto f_*\colon A_Y(N)\to A_X(M)$$

defines an integration with supports on A.

To complete the proof, we need only check that the integration with supports is subjected to the given orientation—in other words, that for a line bundle $p\colon L\to M$ with zero section s, the composition

$$A_X(M)\xrightarrow{s_*}A_{p^{-1}(X)}(L)\xrightarrow{s^*}A_X(M)$$

is cup product with $c_1(L)$. By (3) and [10, Thm. 4.1.4], this is the case for $X=M$; in particular, we have

$$s^*(s_*(1)) = c_1(L) \in A(M).$$

In general, take $a \in A_X(M)$. Then

$$\begin{aligned} s^*(s_*(a)) &= s^*(s_*(s^*p^*(a) \cup 1)) \\ &= s^*(p^*(a) \cup s_*(1)) \\ &= a \cup s^*(s_*(1)) \\ &= a \cup c_1(L), \end{aligned}$$

as desired. □

Proof of Theorem 1.12
The existence of an integration with supports subjected to a given orientation ω on A follows from Theorem 1.24. For the uniqueness, suppose we have two integrations,

$$(f: (N, Y) \to (M, X)) \mapsto f^1_*, f^2_*: A_Y(N) \to A_X(M),$$

both subjected to the same orientation ω. Let $\omega_A = \left(1 + \sum_{n\geq 1} a_n t^n\right) dt$ be the normalized invariant 1-form for the formal group law F_A.

By the uniqueness part of [10, Thm. 4.1.4], $f^1_* = f^2_*: A(N) \to A(M)$. In particular, taking $q: \mathbb{P}^n \to \operatorname{Spec} k$ to be the structure map and letting $t = c_1(\mathcal{O}_{\mathbb{P}^n}(1)) \in A(\mathbb{P}^n)$, we have

$$q^1_*(t^m) = q^2_*(t^m) = a_{n-m}.$$

Now let (M, X) be a smooth pair, take $a \in A_X(M)$, and let $p: \mathbb{P}^n \times M \to M$ be the projection. Then, for $i = 1, 2$,

$$\begin{aligned} p^i_*(t^m \times a) &= q^i_*(t^m) \cdot a \\ &= a_{n-m} \cdot a \end{aligned}$$

and so $p^1_* = p^2_*$.

Next, consider a closed immersion $i: N \to M$ in $\mathbf{Sm}/k$ and let $Y \subset N$ and $X \subset M$ be closed subsets with $i(Y) \subset X$. Suppose $i = \mathrm{id}_M$. Then, by Definition 1.8(5), $i^1_* = \mathrm{id}^*_M = i^2_*$; this reduces us to the case $X = i(Y)$.

Because M is quasi-projective, we can find a sequence of smooth closed subschemes

$$N = N_0 \subset N_1 \subset \cdots \subset N_r = M$$

such that N_{i-1} is a smooth codimension-1 subscheme of N_i for $i = 1, \ldots, r$. This reduces us to the case of a codimension-1 closed immersion.

Consider the deformation to the normal bundle

$$\begin{tikzcd} \nu \arrow[r, "i_0"] & M_t & M \arrow[l, "i_1"'] \\ N \arrow[u, "s"] \arrow[r, "i_0"'] & N \times \mathbb{A}^1 \arrow[u, "\tilde{i}"'] & N \arrow[u, "i"'] \arrow[l, "i_1"] \\ Y \arrow[u, hook] \arrow[r, "i_0"'] & Y \times \mathbb{A}^1 \arrow[u, hook] & Y. \arrow[u, hook] \arrow[l, "i_1"] \end{tikzcd}$$

This gives us the commutative diagram (for $j = 1, 2$)

$$\begin{tikzcd}
A_Y(N) \arrow[d, "s_*^j"'] & A_{Y\times\mathbb{A}^1}(N\times\mathbb{A}^1) \arrow[l, "i_0^*"'] \arrow[r, "i_1^*"] \arrow[d, "\tilde{i}_*^j"'] & A_Y(N) \arrow[d, "i_*^j"] \\
A_Y(\nu) & A_{Y\times\mathbb{A}^1}(M_t) \arrow[l, "i_0^*"] \arrow[r, "i_1^*"'] & A_Y(M).
\end{tikzcd}$$

It follows easily from the homotopy property for A that the maps i_0^*, i_1^* in the upper row are isomorphisms; the maps i_0^*, i_1^* in the lower row are isomorphisms by Lemma 1.14. It therefore suffices to show that $s_*^1 = s_*^2$.

Using excision, we can now replace ν with the $\mathbb{P}^1$-bundle $\mathbb{P}(\nu \oplus O_N)$. Let p: $\mathbb{P}(\nu \oplus O_N) \to N$ be the projection. Clearly $p_*^j : A_Y(\mathbb{P}(\nu \oplus O_N)) \to A_Y(N)$ is inverse to s_*^j ($j = 1, 2$), so it suffices to show that

$$p_*^1 = p_*^2 \colon A_Y(\mathbb{P}(\nu \oplus O_N)) \to A_Y(N).$$

The map p_*^j factors through the "enlarge the support map"

$$\mathrm{id}^* \colon A_Y(\mathbb{P}(\nu \oplus O_N)) \to A_{p^{-1}(Y)}(\mathbb{P}(\nu \oplus O_N));$$

therefore, id^* is injective. Hence we need only show that the maps

$$p_*^j \colon A_{p^{-1}(Y)}(\mathbb{P}(\nu \oplus O_N)) \to A_Y(N)$$

are equal.

We have the extended projective bundle formula [10, Cor. 3.3.8]: Let

$$\alpha \colon A_Y(N) \oplus A_Y(N) \to A_{p^{-1}(Y)}(\mathbb{P}(\nu \oplus O_N))$$

be the map sending (a, b) to $p^*(a) + p^*(b) \cup c_1(\mathcal{O}(1))$; then α is an isomorphism.

The projection formula implies

$$p_*^j(\alpha(a, b)) = a \cup p_*^j(1_{\mathbb{P}(\nu\oplus O_N)}) + b \cup p_*^j(c_1(\mathcal{O}(1))).$$

By the uniqueness part of [10, Thm. 4.1.4],

$$p_*^1 = p_*^2 \colon A(\mathbb{P}(\nu \oplus O_N)) \to A(N);$$

hence $p_*^1(1_{\mathbb{P}(\nu\oplus O_N)}) = p_*^2(1_{\mathbb{P}(\nu\oplus O_N)})$ and $p_*^1(c_1(\mathcal{O}(1))) = p_*^2(c_1(\mathcal{O}(1)))$. As a result, $p_*^1 = p_*^2 \colon A_{p^{-1}(Y)}(\mathbb{P}(\nu \oplus O_N)) \to A_Y(N)$.

Since each projective morphism f factors as $p \circ i$, the two cases of a projection and a closed immersion imply that $f_*^1 = f_*^2$ for all f, completing the proof. □

2. Algebraic Oriented Cohomology

Mocanasu [7] has considered a version of oriented cohomology, with somewhat different axioms from what we have discussed so far, and has shown that such a theory gives rise to a Borel–Moore homology theory on quasi-projective schemes (over a fixed base-field k). In short, the Borel–Moore homology theory H corresponding to an oriented ring cohomology theory A is given by

$$H(X) := A_X(M)$$

for any smooth pair (M, X). The main point is to show that this is independent of the choices: both in the smooth "envelope" M for a given X as well as for morphisms $F\colon M \to M'$ extending a given projective morphism $f\colon X \to X'$. In this section, we give a modified version of Mocanasu's notion of an algebraic oriented theory and show that the integration with supports defined on an oriented ring cohomology theory satisfies the axioms (assuming that the base-field admits resolution of singularities). We fix a base-field k and an oriented $\mathbb{Z}/2$-graded ring cohomology theory A on **SP**. We let $\mathbf{Sch}_k$ denote the category of quasi-projective k-schemes and $\mathbf{Sch}'_k$ the subcategory with the same objects but with only the projective morphisms. We will assume throughout this section that k admits resolution of singularities.

Let (M, X) and (N, Y) be smooth pairs, and let $F\colon M \to N$ be a morphism such that $F(X) \subset Y$ and the restriction $f\colon X \to Y$ of F is projective. Let $\mathcal{C}_F$ be the category of all dense open immersions $j\colon M \to \bar{M}$ with $\bar{M} \in \mathbf{Sm}/k$ and extensions $\bar{F}\colon \bar{M} \to N$ such that $\bar{F}$ is projective; a morphism $\mu\colon (j\colon M \to \bar{M}, \bar{F}) \to (j'\colon M \to \bar{M}', \bar{F}')$ is a morphism $\mu\colon \bar{M} \to \bar{M}'$ with $j' = \mu \circ j$ and $\bar{F}' \circ \mu = \bar{F}$. Note that μ is necessarily projective and birational. Also, since f is projective, $j(X)$ is closed in $\bar{M}$.

LEMMA 2.1. *The category $\mathcal{C}_F$ is left-filtering, and there is is at most one morphism between any two objects.*

Proof. This follows easily from resolution of singularities. The category $\mathcal{C}_F$ is nonempty: since M is quasi-projective, the map F factors through a locally closed immersion $M \to \mathbb{P}^n \times N$. We can close up M in $\mathbb{P}^n \times N$ and resolve singularities to construct $\bar{M}$, j, and F.

Given two objects $\alpha_1 := (j_1\colon M \to \bar{M}_1, \bar{F}_1\colon \bar{M}_1 \to N)$ and $\alpha_2 := (j_2\colon M \to \bar{M}_2, \bar{F}_2\colon \bar{M}_2 \to N)$ in $\mathcal{C}_F$, resolve the singularities of the closure of $(j_1, j_2)(M)$ in $M_1 \times_k M_2$ to construct $j_3\colon M \to \bar{M}_3$ and $\bar{F}_3\colon \bar{M}_3 \to N$ dominating α_1 and α_2. Because M is assumed dense in $\bar{M}$, there is at most one morphism between any two objects of $\mathcal{C}_F$, completing the proof. $\square$

DEFINITION 2.2. Let (M, X) and (N, Y) be smooth pairs, and let $F\colon M \to N$ be a morphism such that $F(X) \subset Y$ and the restriction $f\colon X \to Y$ of F is projective. Define the push-forward morphism

$$F_*\colon A_X(M) \to A_Y(N)$$

by taking $(j\colon M \to \bar{M}, \bar{F}\colon \bar{M} \to N)$ in $\mathcal{C}_F$ and setting

$$F_*\colon = \bar{F}_* \circ (j^*)^{-1},$$

where $j^*\colon A_{j(X)}(\bar{M}) \to A_X(M)$ is the excision isomorphism.

We note that F_* is well-defined by Lemma 2.1.

LEMMA 2.3. *Given composable morphisms* $F: M \to N$ *and* $G: N \to P$ *as well as smooth pairs* $(M, X), (N, Y), (U, Z)$, *suppose that* $F(X) \subset Y$ *and* $G(Y) \subset Z$ *and that the restrictions of* F *and* G, $f: X \to Y$ *and* $g: Y \to Z$, *are projective. Then*

$$G_* \circ F_* = (G \circ F)_*: A_X(M) \to A_Z(P).$$

Proof. Take $j_1: N \to \bar{N}$ and $\bar{G}: \bar{N} \to P$ in $\mathcal{C}_G$, $j_2: M \to \bar{M}$ and $\bar{F}: \bar{M} \to N$ in $\mathcal{C}_F$, and $j_3: \bar{M} \to \bar{M}'$ and $\bar{F}': \bar{M}' \to \bar{N}$ in $\mathcal{C}_{j \circ \bar{F}}$. Then $j_3 \circ j_2: M \to \bar{M}'$ and $\bar{G} \circ \bar{F}': \bar{M}' \to P$ are in $\mathcal{C}_{G \circ F}$, so

$$\begin{aligned}(G \circ F)_* &= \bar{G}_* \circ \bar{F}'_* \circ (j_2^* \circ j_3^*)^{-1};\\ F_* &= \bar{F}_* \circ (j_2^*)^{-1},\\ G_* &= \bar{G}_* \circ (j_1^*)^{-1}.\end{aligned}$$

Since the diagram

$$\begin{tikzcd} \bar{M} \arrow[r, "j_3"] \arrow[d, "\bar{F}"'] & \bar{M}' \arrow[d, "\bar{F}'"] \\ N \arrow[r, "j_1"'] & \bar{N} \end{tikzcd}$$

is transverse Cartesian, we have (by Lemma 1.9)

$$j_1^* \circ \bar{F}'_* = \bar{F}_* \circ j_3^*.$$

Thus

$$\begin{aligned}G_* \circ F_* &= \bar{G}_* \circ (j_1^*)^{-1} \circ \bar{F}_* \circ (j_2^*)^{-1}\\ &= \bar{G}_* \circ \bar{F}'_* \circ (j_3^*)^{-1} \circ (j_2^*)^{-1}\\ &= (G \circ F)_*.\end{aligned}$$

□

PROPOSITION 2.4. *Let* (M, X) *and* (M', X') *be smooth pairs, and let* $F, G: M \to M'$ *be two morphisms such that* F *and* G *restrict to the same projective morphism* $f: X \to X'$. *Then*

$$F_* = G_*: A_X(M) \to A_{X'}(M').$$

Proof. We first reduce to the case of affine M'. Indeed, Jouanolou [4] tells us that there is an affine space bundle $q: E \to M'$ with E affine. Because $E \to M'$ is smooth, the reduction is achieved by replacing X' with $q^{-1}(X')$, M with $M \times_{M'} E$, and X with $X \times_{M'} E$ and then using the extended homotopy property (Remark 1.4(2)).

Next we reduce to the case in which $M' = \mathbb{A}^n$ for some n. Since M' is affine, there is a closed immersion $i: M' \to \mathbb{A}^n$. By Remark 1.16, the push-forward $i_*: A_{X'}(M') \to A_{i(X')}(\mathbb{A}^n)$ is an isomorphism, so we may replace (M', X') with $(\mathbb{A}^n, i(X'))$ and then change notation.

Consider the product map

$$(F,G)\colon M \to \mathbb{A}^n \times_k \mathbb{A}^n.$$

Since F and G are both equal to f when restricted to X, we have the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dd, hook] & X' \arrow[d, hook] \\
 & \mathbb{A}^n \arrow[d, "\delta"] \\
M \arrow[r, "{(F,G)}"'] & \mathbb{A}^n \times \mathbb{A}^n,
\end{tikzcd}$$

where δ is the diagonal.

Consider the map

$$\varphi\colon \mathbb{A}^1 \times \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^n \times \mathbb{A}^n,$$
$$\varphi(t,x,y) := (t,x,ty+(1-t)x).$$

For $a \in k$, let $\varphi_a\colon \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^n \times \mathbb{A}^n$ be the fiber of φ over a. Observe that

$$\varphi \circ (\mathrm{id} \times \delta) = \mathrm{id} \times \delta\colon \mathbb{A}^1 \times \mathbb{A}^n \to \mathbb{A}^1 \times \mathbb{A}^n \times \mathbb{A}^n.$$

Thus, we may form the following commutative diagram of schemes over $\mathbb{A}^1$:

$$\begin{tikzcd}
\mathbb{A}^1 \times X \arrow[r, "\mathrm{id}\times f"] \arrow[dd, hook] & \mathbb{A}^1 \times X' \arrow[d, hook] \\
 & \mathbb{A}^1 \times \mathbb{A}^n \arrow[d, "\mathrm{id}\times\delta"] \\
\mathbb{A}^1 \times M \arrow[r, "{\varphi\circ[\mathrm{id}\times(F,G)]}"'] & \mathbb{A}^1 \times \mathbb{A}^n \times \mathbb{A}^n.
\end{tikzcd}$$

Let $j\colon \mathbb{A}^n \to \mathbb{P}^n$ be the standard open immersion. Because M is quasi-projective, there is an open immersion $g\colon U \hookrightarrow \mathbb{P}^N$ for some N and also a closed immersion $i\colon M \to U$. Thus, we may factor $\varphi \circ [\mathrm{id} \times (F,G)]$ as a composition of maps over $\mathbb{A}^1$,

$$\mathbb{A}^1 \times M \xrightarrow{\iota} \mathbb{A}^1 \times U \times \mathbb{A}^n \times \mathbb{A}^n \xrightarrow{\gamma} \mathbb{A}^1 \times \mathbb{P}^N \times \mathbb{A}^n \times \mathbb{A}^n \xrightarrow{q} \mathbb{A}^1 \times \mathbb{A}^n \times \mathbb{A}^n,$$

with ι a closed immersion, $\gamma = \mathrm{id} \times g \times \mathrm{id}$, and q the projection. Let $\mathcal{M}^*$ be the closure of $\gamma \circ \iota(\mathbb{A}^1 \times M)$ in $\mathbb{A}^1 \times \mathbb{P}^N \times \mathbb{P}^n \times \mathbb{A}^n$ and let $\mu\colon \mathcal{M} \to \mathcal{M}^*$ be a resolution of singularities of $\mathcal{M}^*$. We note that $\mathcal{M}^*$, and hence $\mathcal{M}$, is naturally a scheme over $\mathbb{A}^1$; similarly, the map $\varphi \circ [\mathrm{id} \times (F,G)]$ extends to a map

$$\pi\colon \mathcal{M} \to \mathbb{A}^1 \times \mathbb{P}^n \times \mathbb{A}^n$$

over $\mathbb{A}^1$. Since $\gamma \circ \iota(\mathbb{A}^1 \times M)$ is a smooth dense open subscheme of $\mathcal{M}^*$, we may take $\mathcal{M}$ such that $\mu\colon \mathcal{M} \to \mathcal{M}^*$ is an isomorphism over $\gamma \circ \iota(\mathbb{A}^1 \times M)$. Let $h\colon \mathbb{A}^1 \times M \to \mathcal{M}$ be the resulting open dense immersion.

This gives us the following commutative diagram of schemes over $\mathbb{A}^1$:

$$\begin{tikzcd}
\mathbb{A}^1 \times X \arrow[r, "\mathrm{id}\times f"] \arrow[dd, hook] & \mathbb{A}^1 \times X' \arrow[d, hook] \\
& \mathbb{A}^1 \times \mathbb{A}^n \arrow[d, "\mathrm{id}\times\delta"] \\
\mathbb{A}^1 \times M \arrow[r, "\varphi\circ[\mathrm{id}\times(F,G)]"'] \arrow[d, hook, "h"'] & \mathbb{A}^1 \times \mathbb{A}^n \times \mathbb{A}^n \arrow[d, "\gamma"] \\
\mathcal{M} \arrow[r, "\pi"'] \arrow[d, "p"'] & \mathbb{A}^1 \times \mathbb{P}^n \times \mathbb{A}^n \arrow[d, "p_1"] \\
\mathbb{A}^1 \arrow[r, equal] & \mathbb{A}^1.
\end{tikzcd}$$

For $a \in k$, let $\pi_a\colon \mathcal{M}_a \to \mathbb{P}^n \times \mathbb{A}^n$ be the fiber of π over a. We remark that $\mathcal{M}_a$ contains M as an open subscheme and that π_a extends $\varphi_a \circ [\mathrm{id} \times (F, G)]$. We let $\bar{M}_a \subset \mathcal{M}_a$ be the closure of M in $\mathcal{M}_a$. Blowing up $\mathcal{M}$ further and changing notation if necessary, we may assume that $\bar{M}_0$ and $\bar{M}_1$ are smooth. Let $\iota_0\colon \bar{M}_0 \to \mathcal{M}$ and $\iota_1\colon \bar{M}_1 \to \mathcal{M}$ denote the inclusions.

Noting that $\delta(\mathbb{A}^n)$ is closed in $\mathbb{P}^n \times \mathbb{A}^n$, we see that $\mathbb{A}^1 \times \delta(X')$ is closed in $\mathbb{A}^1 \times \mathbb{P}^n \times \mathbb{A}^n$. Since $\mathrm{id} \times f\colon \mathbb{A}^1 \times X \to \mathbb{A}^1 \times X'$ is projective, this implies that $h(\mathbb{A}^1 \times X)$ is closed in $\mathcal{M}$; hence both $h(0 \times X)$ and $h(1 \times M)$ are closed and are contained in $\bar{M}_0 \subset p^{-1}(0)$ and $\bar{M}_1 \subset p^{-1}(1)$, respectively. We have the commutative diagram

$$\begin{tikzcd}
A_{h(0\times X)}(\bar{M}_0) \arrow[d, "h_0^*"'] & A_{h(\mathbb{A}^1\times X)}(\mathcal{M}) \arrow[l, "\iota_0^*"'] \arrow[r, "\iota_1^*"] \arrow[d, "h^*"] & A_{h(1\times X)}(\bar{M}_1) \arrow[d, "h_1^*"] \\
A_X(M) & A_{\mathbb{A}^1\times X}(\mathbb{A}^1 \times M) \arrow[l, "i_0^*"] \arrow[r, "i_1^*"'] & A_X(M),
\end{tikzcd}$$

where $i_0, i_1\colon M \to \mathbb{A}^1 \times M$ are the 0-, 1-sections and h_0, h_1 are the restrictions of h.

By the homotopy property for A, the maps i_0^*, i_1^* are isomorphisms and $i_0^* = i_1^*$. The maps h, h_0^*, and h_1^* are isomorphisms by excision.

Because $h(\mathbb{A}^1 \times M) \subset \mathcal{M}$ is an open neighborhood of $h(\mathbb{A}^1 \times X)$ in $\mathcal{M}$ that is smooth over $\mathbb{A}^1$, we may apply Lemma 1.9 to obtain the commutative diagram

$$\begin{tikzcd}
A_X(M) & A_{\mathbb{A}^1\times X}(\mathbb{A}^1\times M) \arrow[l, "i_0^*"'] \arrow[r, "i_1^*"] & A_X(M) \\
A_{h(0\times X)}(\bar{M}_0) \arrow[u, "h_0^*"] \arrow[d, "\pi_{0*}"'] & A_{h(\mathbb{A}^1\times X)}(\mathcal{M}) \arrow[l, "\iota_0^*"'] \arrow[r, "\iota_1^*"] \arrow[u, "h^*"'] \arrow[d, "\pi_*"] & A_{h(1\times X)}(\bar{M}_1) \arrow[u, "h_1^*"'] \arrow[d, "\pi_{1*}"] \\
A_{\delta(X')}(\mathbb{P}^n\times\mathbb{A}^n) & A_{\mathbb{A}^1\times\delta(X')}(\mathbb{A}^1\times\mathbb{P}^n\times\mathbb{A}^n) \arrow[l, "i_0^*"] \arrow[r, "i_1^*"'] & A_{\delta(X')}(\mathbb{P}^n\times\mathbb{A}^n).
\end{tikzcd}$$

Since $i_1^*\circ(i_0^*)^{-1}=\mathrm{id}$ (for both the top row and the bottom row), this yields

$$\pi_{0*}\circ(h_0^*)^{-1}=\pi_{1*}\circ(h_1^*)^{-1}\colon A_X(M)\to A_{\delta(X')}(\mathbb{P}^n\times\mathbb{A}^n).$$

Composing with the push-forward for the projection $p_2\colon\mathbb{P}^n\times\mathbb{A}^n\to\mathbb{A}^n$, we have

$$p_{2*}\circ\pi_{0*}\circ(h_0^*)^{-1}=p_{2*}\circ\pi_{1*}\circ(h_1^*)^{-1}\colon A_X(M)\to A_{X'}(\mathbb{A}^n). \tag{2.1}$$

Since

$$p_2\circ\varphi\circ[\mathrm{id}\times(F,G)]\circ i_0=F,\qquad p_2\circ\varphi\circ[\mathrm{id}\times(F,G)]\circ i_1=G,$$

we have commutative diagrams

$$\begin{tikzcd}
M \arrow[r, "F"] \arrow[d, "h_0"'] & \mathbb{A}^n \\
\bar{M}_0, \arrow[ur, "p_2\circ\pi_0"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "G"] \arrow[d, "h_1"'] & \mathbb{A}^n \\
\bar{M}_1. \arrow[ur, "p_2\circ\pi_1"'] &
\end{tikzcd}$$

Thus, $(h_0\colon M\to\bar{M}_0,\,p_2\circ\pi_0)$ is in $\mathcal{C}_F$ and $(h_1\colon M\to\bar{M}_0,\,p_2\circ\pi_1)$ is in $\mathcal{C}_G$; hence

$$F_*=(p_2\circ\pi_0)_*\circ(h_0^*)^{-1},$$
$$G_*=(p_2\circ\pi_1)_*\circ(h_0^*)^{-1}.$$

Together with (2.1), this gives $F_*=G_*$. □

LEMMA 2.5. *Let $F\colon M\to N$ be a morphism in $\mathbf{Sm}/k$ and let $(M,X),(N,Y)$ be smooth pairs. Suppose that $F(X)=Y$ and that the restriction of F to $f\colon X\to Y$ is an isomorphism* (*using the reduced scheme structures*). *Then $F_*\colon A_X(M)\to A_Y(N)$ is an isomorphism.*

Proof. Extending F to $\bar{F}\colon\bar{M}\to N$ for some $(M\hookrightarrow\bar{M},\bar{F})$ in $\mathcal{C}_F$ and then changing notation, we may assume that F is projective. Factoring F as $F=p\circ i$, with $p\colon\mathbb{P}^n\times N\to N$ the projection and $i\colon M\to\mathbb{P}^n\times M$ a closed immersion, it suffices to handle the two cases $F=i$ and $F=p$.

For $F=i$, this is Remark 1.16. In the case of a projection, let $s\colon Y\to\mathbb{P}^n\times N$ be the section induced by the isomorphism $p\colon X\to Y$. Suppose we have an extension of s to a section $t\colon N\to\mathbb{P}^n\times N$. Letting $M:=t(N)$, with closed immersion $i\colon M\to\mathbb{P}^n\times N$, we have the commutative diagram

$$\begin{tikzcd} A_X(M) \arrow[r, "i_*"] \arrow[dr, "(p\circ i)_*"'] & A_X(\mathbb{P}^n \times N) \arrow[d, "p_*"] \\ & A_Y(N). \end{tikzcd}$$

Since $p \circ i\colon (M, X) \to (N, Y)$ is an isomorphism of smooth pairs, it follows that the map $(p \circ i)_*\colon A_X(M) \to A_Y(N)$ is an isomorphism. From the case of a closed immersion, $i_*\colon A_X(M) \to A_X(\mathbb{P}^n \times N)$ is also an isomorphism; hence $p_*\colon A_X(\mathbb{P}^n \times N) \to A_Y(N)$ is an isomorphism as well.

We claim that N admits a Zariski open cover

$$N = \bigcup_{i=1}^{s} U_i$$

such that the restriction of s to $U_i \cap Y$ extends to a section $t_i\colon U_i \to \mathbb{P}^n \times U_i$. Using Mayer–Vietoris and the case (handled previously) in which a section extends, this will prove the result in general. To prove our claim, let y be a point of Y. Shrinking N to some affine neighborhood U of y, we may assume that $s(Y)$ is contained in a product $\mathbb{A}^n \times N$, where $\mathbb{A}^n$ is some standard affine subset of $\mathbb{P}^n$. The map s is then given by a morphism $\bar{s}\colon Y \to \mathbb{A}^n$, that is, by n regular functions $\bar{s}_1, \dots, \bar{s}_n$ on Y. Because U is affine, each $\bar{s}_i$ lifts to a regular function $\bar{t}_i$ on U and thus gives the desired section $t\colon U \to \mathbb{A}^n \times U \subset \mathbb{P}^n \times U$ extending s. $\square$

We can also extend the compatibility of push-forward with the boundary in the long exact sequence of a pair (Definition 1.8(5)).

Lemma 2.6. *Let $F\colon M \to N$ be a morphism in $\mathbf{Sm}/k$, and let $X \subset X' \subset M$ and $Y \subset Y' \subset N$ be closed subsets. Suppose that $F(X') \subset Y'$, that the restriction of F to $f\colon X' \to Y'$ is projective, and that $f^{-1}(Y) \cap X' = X$. Then the diagram*

$$\begin{tikzcd} A_{X'\setminus X}(M \setminus X) \arrow[r, "\partial_{M,X',X}"] \arrow[d, "F_*"'] & A_X(M) \arrow[d, "F_*"] \\ A_{Y'\setminus Y}(N \setminus Y) \arrow[r, "\partial_{N,Y',Y}"'] & A_Y(N) \end{tikzcd}$$

commutes.

Proof. Just take $M \to \bar{M}$ and $\bar{F}\colon \bar{M} \to N$ in $\mathcal{C}_f$ and then apply Definition 1.8(5). $\square$

We give a modified version of Mocanasu's notion of an algebraic oriented theory on $\mathbf{SP}$. In what follows, for (M, X) a smooth pair we consider X as a scheme by giving it the reduced structure.

Definition 2.7. An *algebraic oriented theory* on $\mathbf{SP}$ consists of the following data.

(D1) A functor $A\colon \mathbf{SP}^{\mathrm{op}} \to \mathbf{Ab}$. For a morphism $G\colon (M,X) \to (N,Y)$ in $\mathbf{SP}$, we write $G^*\colon A_Y(N) \to A_X(M)$ for $A(G)$.

(D2) Let $(M,X),(N,Y)$ be smooth pairs, and let $F\colon M \to N$ be a morphism such that $F(X) \subset Y$ and the restriction of F to $f\colon X \to Y$ is projective. Then there is a "push-forward" $F_*\colon A_X(M) \to A_Y(N)$.

These data satisfy the following axioms.

(A1) A is additive: For $X, Y \in \mathbf{Sm}/k$, the canonical map $A(X \amalg Y) \to A(X) \times A(Y)$ is an isomorphism.

(A2) (i) Let $(M,X),(N,Y)$ be smooth pairs, and let $F, G\colon M \to N$ be morphisms in $\mathbf{Sm}/k$ such that $F(X) \subset Y$ and $G(X) \subset Y$ and such that F and G restrict to the same projective morphism $f\colon X \to Y$. Then $F_* = G_*\colon A_X(M) \to A_Y(N)$.

(ii) Let $(M,X),(N,Y)$ be smooth pairs, and let $F\colon M \to N$ be a morphism such that $F(X) \subset Y$ and the restriction of F to $f\colon X \to Y$ is an isomorphism. Then $F_*\colon A_X(M) \to A_Y(N)$ is an isomorphism.

(A3) For smooth pairs $(M_1,X_1),(M_2,X_2),(M_3,X_3)$ and a commutative diagram

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, hook] & X_2 \arrow[r, "f_2"] \arrow[d, hook] & X_3 \arrow[d, hook] \\
M_1 \arrow[r, "F_1"'] & M_2 \arrow[r, "F_2"'] & M_3
\end{tikzcd}$$

such that f_1 and f_2 are projective, $F_{2*} \circ F_{1*} = (F_2 \circ F_1)_*$.

(A4) For smooth pairs (M,X), (M',X'), (N,Y), and (N',Y'), suppose we have the commutative diagram

$$\begin{tikzcd}
& X' \arrow[rr, "f'"] \arrow[dl, "g'"'] \arrow[dd, hook] & & Y' \arrow[dl, "g"] \arrow[dd, hook] \\
X \arrow[rr, "f"] \arrow[dd, hook] & & Y \arrow[dd, hook] & \\
& M' \arrow[rr, "F'"] \arrow[dl, "G'"'] & & N' \arrow[dl, "G"] \\
M \arrow[rr, "F"'] & & N &
\end{tikzcd}$$

such that the top, bottom, left, and right squares are Cartesian and the bottom square is transverse. Suppose further that f and f' are projective. Finally, suppose that G and G' either are closed immersions or are smooth and equi-dimensional. Then

$$F'_* \circ G'^* = G^* \circ F'_*\colon A_X(M) \to A_{Y'}(N').$$

(A5) Let (M, X), (N, X), (M', X), and (N', X') be smooth pairs. Suppose we have a Cartesian diagram

$$\begin{tikzcd} M \arrow[r, "G'"] \arrow[d, "F'"'] & M' \arrow[d, "F"] \\ N \arrow[r, "G"'] & N' \end{tikzcd}$$

with $X = G^{-1}(X')$, $X = F^{-1}(X')$, $G'(X) = X$, and $F'(X) = X$ and such that the restrictions $G'\colon X \to X$ and $F'\colon X \to X$ are the identity. Suppose that F and G are open immersions. We have the diagram

$$\begin{tikzcd} A_X(M) \arrow[r, "G'_*"] \arrow[d, "F'_*"'] & A_X(M') \\ A_X(N) & A_{X'}(N'); \arrow[l, "G^*"] \arrow[u, "F^*"'] \end{tikzcd}$$

observe that G'_* and F'_* are isomorphisms by (A2)(ii). Then

$$(G'_*)^{-1} \circ F^* = (F'_*)^{-1} \circ G^*.$$

(A6) Let

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] & V \arrow[d, "p"] \\ Z' \arrow[r, hook] & X \end{tikzcd}$$

be a Cartesian diagram, where the horizontal arrows are inclusions of reduced closed subschemes and $p\colon V \to X$ is an affine space bundle. Then $p^*\colon A_{Z'}(X) \to A_Z(V)$ is an isomorphism.

(A7) Let (X, Z) be a smooth pair. Then $\mathrm{id}_{X*}\colon A_Z(X) \to A_Z(X)$ is the identity map.

(A8) Let $X \subset Y \subset M$ be closed subsets of $M \in \mathbf{Sm}/k$. Then

$$\mathrm{id}_{M*} = \mathrm{id}_M^*\colon A_X(M) \to A_Y(M).$$

REMARK 2.8. Other than notational or organizational changes, our axioms for an oriented algebraic theory differ from Mocanasu's notion [7, Def. 1.15] of an oriented algebraic theory at the following points.

1. Mocanasu's axiom (A4) differs from ours in that she does not assume that the bottom square is Cartesian and does not require the bottom square to be transverse if G and G' are closed immersions. However, in all uses of (A4) in [7], the bottom square *is* transverse Cartesian, so this does not lead to any difference in the applications.

2. Mocanasu's axiom (A5) differs from ours in that she allows the morphisms F and G to be smooth and equi-dimensional rather than requiring them to be

open immersions. This causes a difference in the associated Borel–Moore homology theories: our versions have only functorial pull-back morphisms for open immersions whereas Mocanasu's versions have functorial pull-back morphisms for smooth equi-dimensional morphisms that are "embeddable".

3. We have strengthened the homotopy axiom (A6) from that of [7] by allowing V to be an affine space bundle rather than a vector bundle.

4. We have added the axiom (A8), which appears as an additional condition on an algebraic oriented theory in the statement of [7, Prop. 4.2].

THEOREM 2.9. *Suppose that k admits resolution of singularities. Let A be an oriented $\mathbb{Z}/2$-graded ring cohomology theory on* **SP**. *Then the functor $A\colon \mathbf{SP}^{\mathrm{op}} \to$* **Ab** *(forget the $\mathbb{Z}/2$-grading) and the push-forward maps of Definition 2.2 define an algebraic oriented theory on* **SP**.

Proof. We are using the integration with supports on A given by Theorem 1.12. Axiom (A1) of Definition 2.7 follows from Mayer–Vietoris. Axiom (A2)(i) is Proposition 2.4, (A2)(ii) is Lemma 2.5, and (A3) is Lemma 2.3. Axiom (A4) follows from Lemma 1.9, and (A6) follows from the homotopy property for A together with Mayer–Vietoris; (A7) follows from (A8), and (A8) is Definition 1.8(4). Axiom (A5) follows from the functoriality of pull-back together with the identities

$$(G'_*)^{-1} = G'^*, \qquad (F'_*)^{-1} = F'^*. \qquad \square$$

3. Oriented Duality Theories

We describe an analogue of Bloch–Ogus twisted duality theory [1] for oriented cohomology. As in the previous section, we will assume that the base-field k admits resolution of singularities, although this assumption is not needed for Definition 3.1.

DEFINITION 3.1. An *oriented duality theory* (H, A) on $\mathbf{Sch}_k$ consists of the following data.

(D1) A functor $H\colon \mathbf{Sch}'_k \to \mathbf{Gr}_{\mathbb{Z}/2}\mathbf{Ab}$.

(D2) A $\mathbb{Z}/2$-graded oriented ring cohomology theory A on **SP**.

(D3) For each open immersion $j\colon Y \to X$ in $\mathbf{Sch}_k$, a map $j^*\colon H(X) \to H(Y)$.

(D4) (i) For each smooth pair (M, X) and each morphism $f\colon Y \to M$ in $\mathbf{Sch}_k$, a graded cap product map

$$f^*(\cdot)\cap\colon A_X(M) \otimes H(Y) \to H(Y \cap f^{-1}(X)).$$

(ii) For $X, Y \in \mathbf{Sch}_k$, a graded external product

$$\times\colon H(X) \otimes H(Y) \to H(X \times Y).$$

(D5) For each smooth pair (M, X), an isomorphism

$$\alpha_{M,X}\colon H(X) \to A_X(M).$$

(D6) For $X \in \mathbf{Sch}_k$ and for $Y \subset X$ a closed subset, a degree-1 map

$$\partial_{X,Y}\colon H(X \setminus Y) \to H(Y).$$

We let $[F\colon (M,X)\to (N,Y)]$ in $\mathbf{SP}'\mapsto F_*\colon A_X(M)\to A_Y(N)$ be the integration with supports on A subjected to the given orientation. The data (D1)–(D6) satisfy the following axioms.

(A1) Let $(M,X),(N,Y)$ be smooth pairs, and let $j\colon M\to N$ be an open immersion with $j^{-1}(Y)=X$. Let $j_Y\colon X\to Y$ be the restriction of j. Then the diagram

$$\begin{tikzcd} H(Y) \arrow[r, "\alpha_{N,Y}"] \arrow[d, "j_Y^*"'] & A_Y(N) \arrow[d, "j^*"] \\ H(X) \arrow[r, "\alpha_{M,X}"'] & A_X(M) \end{tikzcd}$$

commutes.

(A2) Let $(M,X),(N,Y)$ be smooth pairs, let $f\colon X\to Y$ be a projective morphism in $\mathbf{Sch}_k$, and suppose f extends to a projective morphism $F\colon M\to N$. Then the diagram

$$\begin{tikzcd} H(X) \arrow[r, "\alpha_{M,X}"] \arrow[d, "f_*"'] & A_X(M) \arrow[d, "F_*"] \\ H(Y) \arrow[r, "\alpha_{N,Y}"'] & A_Y(N) \end{tikzcd}$$

commutes.

(A3) Let (M,X) and (N,Y) be smooth pairs.

(i) Let $F\colon N\to M$ be a morphism in $\mathbf{Sm}/k$, and let $f\colon Y\to M$ be the restriction of F. Let

$$F^*(\cdot)\cup\colon A_X(M)\otimes A_Y(N)\to A_{Y\cap f^{-1}(X)}(N)$$

be the map $a\otimes b\mapsto F^*(x)\cup y$, where $F^*\colon A_X(M)\to A_{f^{-1}(X)}(N)$ is the pull-back. Then the diagram

$$\begin{tikzcd} A_X(M)\otimes H(Y) \arrow[r, "\mathrm{id}\otimes\alpha_{N,Y}"] \arrow[d, "f^*(\cdot)\cap"'] & A_X(M)\otimes A_Y(N) \arrow[d, "F^*(\cdot)\cup"] \\ H(Y\cap f^{-1}(X)) \arrow[r, "\alpha_{N,Y\cap f^{-1}(X)}"'] & A_{Y\cap f^{-1}(X)}(N) \end{tikzcd}$$

commutes.

(ii) The diagram

$$\begin{tikzcd} H(X)\otimes H(Y) \arrow[r, "\times"] \arrow[d, "\alpha_{M,X}\otimes\alpha_{N,Y}"'] & H(X\times Y) \arrow[d, "\alpha_{M\times N,X\times Y}"] \\ A_X(M)\otimes A_Y(N) \arrow[r, "\times"'] & A_{X\times Y}(M\times N) \end{tikzcd}$$

commutes.

(A4) Let (M,X) be a smooth pair and let $Y\subset X$ be a closed subset. Then the diagram

$$\begin{array}{ccc} H(X\setminus Y) & \xrightarrow{\alpha_{M\setminus Y, X\setminus Y}} & A_{X\setminus Y}(M\setminus Y) \\ {\scriptstyle \partial_{X,Y}}\downarrow & & \downarrow{\scriptstyle \partial_{M,X,Y}} \\ H(Y) & \xrightarrow[\alpha_{M,Y}]{} & A_Y(M) \end{array}$$

commutes.

The functor H together with the additional structures (D2)–(D6) is the *oriented Borel–Moore homology theory* underlying the oriented duality theory.

REMARK 3.2. Oriented duality theories on $\mathbf{Sch}_k$ form a category in the obvious manner. Given a $\mathbb{Z}/2$-graded oriented ring cohomology theory on **SP**, an *extension* of A to an oriented duality theory on $\mathbf{Sch}_k$ is an oriented duality theory (H, A') together with an isomorphism $A \cong A'$ of $\mathbb{Z}/2$-graded oriented ring cohomology theories on **SP**. Clearly, two extensions (H_1, A_1) and (H_2, A_2) of A are uniquely isomorphic as extensions of A: the only possible choice of isomorphism $H_1 \cong H_2$ compatible with the given isomorphisms $A_1 \xrightarrow{\beta} A \xleftarrow{\gamma} A_2$ is given by the isomorphisms

$$H_1(X) \xrightarrow{\beta\circ\alpha^1_{M,X}} A_X(M) \xleftarrow{\gamma\circ\alpha^2_{M,X}} H_2(X)$$

for any choice of smooth pair (M, X).

REMARK 3.3. One also has the $\mathbb{Z}$-graded or bi-graded versions of oriented duality theories. For the graded version, one typically uses homological grading on H so that the comparison isomorphisms α are of the form

$$\alpha_{M,X}\colon H_n(X) \to A_X^{2d-n}(M),$$

where $d = \dim_k M$ (by additivity, we may assume that M is equi-dimensional over k). By Remark 1.7, the projective push-forward map f_* preserve the grading, as do the pull-back maps for open immersions. The cap products become

$$A_X^m(M) \otimes H_n(Y) \xrightarrow{f^*(\cdot)\cap} H_{n-m}(Y \cap f^{-1}(X)).$$

In the bi-graded case, we index H to give comparison isomorphisms

$$\alpha_{M,X}\colon H_{p,q}(X) \to A_X^{2d-p,d-q}(M).$$

The second index in the bi-grading plays the role of the “weight” in the classical Bloch–Ogus theory. The projective push-forward and open pull-back preserve the bi-grading, and the cap products are

$$A_X^{m,n}(M) \otimes H_{p,q}(Y) \xrightarrow{f^*(\cdot)\cap} H_{p-m,q-n}(Y \cap f^{-1}(X)).$$

THEOREM 3.4. *Suppose that k admits resolution of singularities. Let A be an oriented $\mathbb{Z}/2$-graded ring cohomology theory on* **SP**. *Then there is a unique extension of A to an oriented duality theory (H, A) on $\mathbf{Sch}_k$.*

Proof. We have already discussed the uniqueness. Existence follows from the results of [7, Sec. 2.1] with some minor modifications. We give a sketch of the construction for the reader's convenience, referring to [7] for details. We will use throughout that an oriented $\mathbb{Z}/2$-graded ring cohomology theory defines an oriented algebraic cohomology theory (Theorem 2.9).

Call morphisms $F, G\colon (M, X) \to (N, Y)$ in $\mathbf{SP}'$ *equivalent* if F and G induce the same morphism $X \to Y$, and let $\overline{\mathbf{SP}'}$ be the quotient of $\mathbf{SP}'$ by this equivalence relation.

We have the restriction functor $\mathrm{res}\colon \overline{\mathbf{SP}'} \to \mathbf{Sch}'_k$ sending (M, X) to X and $[F]\colon (M, X) \to (N, Y)$ to the restriction $F|_X\colon X \to Y$. We let $\mathcal{H}\mathbf{SP}'$ be the category formed from $\overline{\mathbf{SP}'}$ by inverting all morphisms over an isomorphism in $\mathbf{Sch}'_k$. For each X in $\mathbf{Sch}'_k$, the fiber of res over X is a left-filtering category with at most one morphism between any two objects, so the induced map $\mathrm{res}\colon \mathcal{H}\mathbf{SP}' \to \mathbf{Sch}'_k$ is an equivalence of categories.

By Axioms (A2), (A3), and (A7) of Definition 2.7, sending (M, X) to $A_X(M)$ and $F\colon (M, X) \to (N, Y)$ in $\mathbf{SP}'$ to $F_*\colon A_X(M) \to A_Y(N)$ descends to a well-defined functor

$$A_-(\cdot)\colon \mathcal{H}\mathbf{SP}' \to \mathbf{Gr}_{\mathbb{Z}/2}\mathbf{Ab}.$$

Since $\mathrm{res}\colon \mathcal{H}\mathbf{SP}' \to \mathbf{Sch}'_k$ is an equivalence, this gives us the functor $H\colon \mathbf{Sch}'_k \to \mathbf{Gr}_{\mathbb{Z}/2}\mathbf{Ab}$ and the natural isomorphisms

$$\alpha_{M,X}\colon H(X) \to A_X(M)$$

satisfying Axiom (A2).

To define the pull-back map $j^*\colon H(X) \to H(Y)$ associated to an open immersion $j\colon Y \to X$, choose a smooth pair (M, X). It is easy to see that there exist a smooth pair (N, Y) and an open immersion $\tilde{j}\colon N \to M$ extending j. Let $j^*\colon H(X) \to H(Y)$ be the unique map making the diagram

$$\begin{tikzcd}
H(X) \arrow[r, "\alpha_{M,X}"] \arrow[d, "j^*"'] & A_X(M) \arrow[d, "\tilde{j}^*"] \\
H(Y) \arrow[r, "\alpha_{N,Y}"'] & A_Y(N)
\end{tikzcd}$$

commute.

To verify (A1), let $(M', X), (N', Y)$ be smooth pairs and let $g\colon N' \to M'$ be an open immersion extending j. We have the commutative diagram

$$\begin{tikzcd}
Y \arrow[rr, equal] \arrow[dd, equal] \arrow[dr, hook] & & Y \arrow[d, hook] \arrow[dddr, bend left, "j"] & \\
& N' \times N \arrow[r, "\mathrm{id}\times\tilde{j}"] \arrow[d, "g\times\mathrm{id}"'] & N' \times M \arrow[d, "g\times\mathrm{id}"] & \\
Y \arrow[r, hook] \arrow[drrr, bend right, "j"'] & M' \times N \arrow[r, "\mathrm{id}\times\tilde{j}"'] & M' \times M & \\
& & & X. \arrow[ul, hook]
\end{tikzcd}$$

By Definition 2.7(A5),

$$((g \times \mathrm{id})_*)^{-1} \circ (\mathrm{id} \times \tilde{j})^* = ((\mathrm{id} \times \tilde{j})_*)^{-1} \circ (g \times \mathrm{id})^*.$$

From the commutative diagram

$$\begin{tikzcd}
Y \arrow[r, hook] \arrow[d, "j"'] & N' \times M \arrow[r, "p_1"] \arrow[d, "g \times \mathrm{id}"'] & N' \arrow[d, "g"] \\
X \arrow[r, hook] & M' \times M \arrow[r, "p_1"] & M'
\end{tikzcd}$$

and Definition 2.7(A4), it follows that

$$p_{1*} \circ (g \times \mathrm{id})^* = g^* \circ p_{1*}.$$

Similarly, the commutative diagram

$$\begin{tikzcd}
Y \arrow[r, hook] \arrow[d, "j"'] & M' \times N \arrow[r, "p_2"] \arrow[d, "g \times \mathrm{id}"'] & N \arrow[d, "g"] \\
X \arrow[r, hook] & M' \times M \arrow[r, "p_2"] & M
\end{tikzcd}$$

yields

$$p_{2*} \circ (\mathrm{id} \times \tilde{j})^* = \tilde{j}^* \circ p_{2*}.$$

This gives us the commutative diagrams

$$\begin{tikzcd}
H(Y) \arrow[dr, "\alpha_{M' \times N, Y}"'] \arrow[drr, bend left, "\alpha_{N,Y}"] & & \\
& A_Y(M' \times N) \arrow[r, "p_{2*}"'] & A_Y(N) \\
& A_X(M' \times M) \arrow[u, "(\mathrm{id} \times \tilde{j})^*"] \arrow[r, "p_{2*}"] & A_X(M), \arrow[u, "\tilde{j}^*"'] \\
H(X) \arrow[uuu, "j^*"] \arrow[ur, "\alpha_{M' \times M, X}"] \arrow[urr, bend right, "\alpha_{M,X}"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
H(Y) \arrow[rr, equal] \arrow[dd, equal] \arrow[dr, "\alpha_{N' \times N, Y}"] & & H(Y) \arrow[d, "\alpha_{N' \times M, Y}"'] & \\
& A_Y(N' \times N) \arrow[r, "(\mathrm{id} \times \tilde{j})_*"] \arrow[d, "(g \times \mathrm{id})_*"'] & A_Y(N' \times M) & \\
H(Y) \arrow[r, "\alpha_{M' \times N, Y}"'] & A_Y(M' \times N) & A_Y(M' \times M) \arrow[l, "(\mathrm{id} \times \tilde{j})^*"] \arrow[u, "(g \times \mathrm{id})^*"'] & \\
& & & H(X), \arrow[ul, "\alpha_{M' \times M, X}"'] \arrow[uuul, bend right, "j^*"'] \arrow[ulll, bend left, "j^*"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
H(Y) \arrow[dr, "\alpha_{N'\times M,Y}"'] \arrow[drr, bend left, "\alpha_{N',Y}"] & & \\
& A_Y(M'\times N) \arrow[r, "p_{1*}"'] & A_Y(N') \\
& A_X(M'\times M) \arrow[u, "(g\times \mathrm{id})^*"] \arrow[r, "p_{1*}"] & A_X(M'). \arrow[u, "g^*"'] \\
H(X) \arrow[ur, "\alpha_{M'\times M,X}"] \arrow[urr, bend right, "\alpha_{M',X}"'] & &
\end{tikzcd}$$

Hence, the diagram

$$\begin{tikzcd}
H(Y) \arrow[r, "\alpha_{N',Y}"] & A_Y(N') \\
H(X) \arrow[u, "j^*"] \arrow[r, "\alpha_{M',X}"'] & A_X(M') \arrow[u, "g^*"']
\end{tikzcd}$$

commutes, as desired.

To define the cap product pairing (D4)(i) for a smooth pair (M, X) and a morphism $f: Y \to M$ with $f(Y) \subset X$, choose a smooth pair $(N, i: Y \to N)$ and embed Y in $N \times M$ by (i, f). Let $f^*(\cdot)\cap$ be the unique morphism making

$$\begin{tikzcd}
A_X(M)\otimes H(Y) \arrow[r, "\mathrm{id}\otimes\alpha_{N\times M,Y}"] \arrow[d, "f^*(\cdot)\cap"'] & A_X(M)\otimes A_Y(N\times M) \arrow[d, "p_2^*(\cdot)\cup"] \\
H(Y\cap f^{-1}(X)) \arrow[r, "\alpha_{N\times M, Y\cap f^{-1}(X)}"'] & A_{Y\cap p_2^{-1}(X)}(N\times M)
\end{tikzcd}$$

commute. For another smooth pair $(N', i': Y \to N')$ and morphism $G: N' \to M$ extending f, consider the commutative diagram

$$\begin{tikzcd}
N' \arrow[r, "(i',G)"] \arrow[dr, "G"'] & N'\times M \arrow[d, "G\circ p_2"] \\
& M.
\end{tikzcd}$$

We embed Y in $N' \times M$ by (i', f). The projection formula gives, for $b \in A_Y(N')$ and $a \in A_X(M)$,

$$(i', G)_*(G^*(a) \cup b) = (i', G)_*((i', G)^* p_2^*(a) \cup b) = p_2^*(a) \cup (i', G)_*(b),$$

so we can replace $(i': Y \to N', G: N' \to M)$ with $((i', f): Y \to N' \times M, p_2)$. Similarly, we have the embedding $(i', i, f): Y \to N' \times N \times M$ and so, for $a \in A_X(M)$ and $b \in A_Y(N' \times N \times M)$, we have

$$p^{N'NM}_{N'M*}(p^{N'NM*}_M(a)\cup b) = p^{N'M*}_M(a)\cup p^{N'NM}_{N'M*}(b),$$
$$p^{N'NM}_{NM*}(p^{N'NM*}_M(a)\cup b) = p^{NM*}_M(a)\cup p^{N'NM}_{NM*}(b).$$

Here $p^{N'NM}_{NM}$ is the projection $N'\times N\times M\to N\times M$, et cetera. The commutativity in (A3) follows from these identities.

For the external product (D4)(ii), we fix as before smooth pairs (M, X), (N, Y) and define $\times\colon H(X)\otimes H(Y)\to H(X\times Y)$ as the unique map making

$$\begin{tikzcd}
H(X)\otimes H(Y) \arrow[r, "\times"] \arrow[d, "\alpha_{M,X}\otimes\alpha_{N,Y}"'] & H(X\times Y) \arrow[d, "\alpha_{M\times N,\, X\times Y}"] \\
A_X(M)\otimes A_Y(N) \arrow[r, "\times"'] & A_{X\times Y}(M\times N)
\end{tikzcd}$$

commute. For other smooth pairs (M', X), (N', Y), consider the diagram

$$\begin{tikzcd}
A_X(M\times M')\otimes A_Y(N\times N') \arrow[r, "\times"] \arrow[d, "p_{1*}\otimes p_{1*}"'] & A_{X\times Y}(M\times M'\times N\times N') \arrow[d, "p_{13*}"] \\
A_X(M)\otimes A_Y(N) \arrow[r, "\times"'] & A_{X\times Y}(M\times N).
\end{tikzcd}$$

By Remark 1.10, this diagram commutes. Using a similar diagram, but with M', N' replacing M, N in the bottom row, verifies (A3)(ii).

Finally, for (A4), choose a smooth pair (M, X) and let $\partial_{X,Y}$ be the unique map making

$$\begin{tikzcd}
H(X\setminus Y) \arrow[r, "\alpha_{M\setminus Y,\, X\setminus Y}"] \arrow[d, "\partial_{X,Y}"'] & A_{X\setminus Y}(M\setminus Y) \arrow[d, "\partial_{M,X,Y}"] \\
H(Y) \arrow[r, "\alpha_{M,Y}"'] & A_Y(M)
\end{tikzcd}$$

commute. If we have another smooth pair (M', X) then we also have the smooth pair $(M\times M', X)$ and the commutative diagram

$$\begin{tikzcd}
A_{X\setminus Y}(M\setminus Y) \arrow[d, "\partial_{M,X,Y}"'] & A_{X\setminus Y}(M\times M'\setminus Y) \arrow[l, "p_{1*}"'] \arrow[r, "p_{2*}"] \arrow[d, "\partial_{M\times M',X,Y}"] & A_{X\setminus Y}(M'\setminus Y) \arrow[d, "\partial_{M',X,Y}"] \\
A_Y(M) & A_Y(M\times M') \arrow[l, "p_{1*}"'] \arrow[r, "p_{2*}"] & A_Y(M')
\end{tikzcd}$$

(see Lemma 2.6), from which (A4) follows directly. □

Of course, the role of the Borel–Moore homology theory H in an oriented duality theory is just to say that certain properties of cohomology with supports $A_X(M)$ depend only on X, not on the choice of smooth pair (M, X). In addition to the properties (projective push-forward, open pull-back, cup product, and boundary map) given by the axioms, one has the following properties and structures.

Functoriality of open pull-back, cap products, and external products. For $j: U \to V$ and $g: V \to X$, open immersions in $\mathbf{Sch}_k$, we have

$$j^* \circ g^* = (g \circ j)^*: H(X) \to H(U)$$

and $\mathrm{id}_X^* = \mathrm{id}_{H(X)}$. This follows from the functoriality of open pull-back for the oriented ring cohomology theory A, using (A1) to compare.

For the cap products, we have three functorialities as follows.

(1) Take $X, Y \in \mathbf{Sch}_k$ and $M \in \mathbf{Sm}/k$, a smooth pair (N, X), and morphisms $f: Y \to M$ and $g: M \to N$. Then

$$(g \circ f)^*(a) \cap b = f^*(g^*(a)) \cap b$$

for $a \in A_X(N)$ and $b \in H(Y)$, where $g^*: A_X(N) \to A_{g^{-1}(X)}(M)$ is the pull-back.

(2) Let $h: Y \to Z$ be a projective morphism in $\mathbf{Sch}_k$, let (M, X) be a smooth pair, and let $f: Z \to M$ be a morphism. Then

$$h_*((f \circ h)^*(a) \cap b) = f^*(a) \cap h_*(b).$$

(3) Let $j: U \to Y$ be an open immersion in $\mathbf{Sch}_k$, let (M, X) be a smooth pair, and let $f: Y \to M$ be a morphism. Then

$$j^*(f^*(a) \cap b) = (f \circ j)^*(a) \cap j^*(b)$$

for $a \in A_X(M)$ and $b \in H(Y)$.

The first and third identities follow from the naturality of cup product with respect to pull-back, and the second follows from the projection formula. Finally, the fact that pull-back is a ring homomorphism yields the identity

$$f^*(a \cup b) \cap c = f^*(a) \cap (f^*(b) \cap c)$$

for $a, b \in A_X(M)$ and $c \in H(Y)$.

The external products are functorial for push-forward: For projective morphisms $f: X \to X'$ and $g: Y \to Y'$ we have

$$(f \times g)_*(a \times b) = f_*(a) \times g_*(b) \in H(X' \times Y'), \quad a \in H(X),\ b \in H(Y).$$

This follows from Remark 1.10.

Long exact sequence of a pair and Mayer–Vietoris. Let $i: Y \to X$ be a closed subset of $X \in \mathbf{Sch}_k$ and let $j: U \to X$ be the open complement. Then the sequence

$$\cdots \to H(U) \xrightarrow{\partial_{X,Y}} H(Y) \xrightarrow{i_*} H(X) \xrightarrow{j^*} H(U) \to \cdots$$

is exact. Indeed, after choosing a smooth pair (M, X) we use (A1), (A2), and (A4) to compare with the long exact sequence of the triple (M, X, Y).

If we have an open cover of some $X \in \mathbf{Sch}_k$ ($X = U \cup V$), then the exact sequence of a pair gives formally the long exact Mayer–Vietoris sequence

$$\cdots \to H(X) \xrightarrow{(j_U^*, j_V^*)} H(U) \oplus H(V)$$
$$\xrightarrow{j_{UV}^{U*} - j_{UV}^{V*}} H(U \cap V) \xrightarrow{\partial_{X,U,V}} H(U \cap V) \to \cdots .$$

The localization sequence is natural with respect to pull-back by open immersions and by push-forward with respect to projective morphisms.

PROPOSITION 3.5. *Let (H, A) be an oriented duality theory.*

(1) *Let $Y \subset Y' \subset X$ be closed subsets of $X \in \mathbf{Sch}_k$. Let $U = X \setminus Y$ and $U' = X \setminus Y'$, with inclusions $j : U' \to U$ and $i : Y \to Y'$. Then the diagram*

$$\begin{array}{ccc} H_{a,b}(U) & \xrightarrow{\partial_{X,Y}} & H_{a-1,b}(Y) \\ {\scriptstyle j^*}\downarrow & & \downarrow{\scriptstyle i_*} \\ H_{a,b}(U') & \xrightarrow[\partial_{X,Y'}]{} & H_{a-1,b}(Y') \end{array}$$

commutes.

(2) *Let $f : X' \to X$ be a projective morphism in $\mathbf{Sch}_k$, let $Y \subset X$ be a closed subset, let $Y' = f^{-1}(Y)$, $U = X \setminus Y$, and $U' = X' \setminus Y'$, and let $f_U : U' \to U$ and $f_Y : Y' \to Y$ be the respective restrictions of f. Then f_U is projective and the diagram*

$$\begin{array}{ccc} H_{a,b}(U') & \xrightarrow{\partial_{X',Y'}} & H_{a-1,b}(Y') \\ {\scriptstyle f_{U*}}\downarrow & & \downarrow{\scriptstyle f_{Y*}} \\ H_{a,b}(U) & \xrightarrow[\partial_{X,Y}]{} & H_{a-1,b}(Y) \end{array}$$

commutes.

Proof. For (1), take a closed immersion $X \to M$ with $M \in \mathbf{Sm}/k$. Let $N = M \setminus Y$ and $N' = M \setminus Y'$. The identity map on M gives the map in $\mathbf{SP}$, $(M, Y') \to (M, Y)$, and the inclusion $N' \to N$ gives the map $(N, U) \to (N', U')$; these arise from the map of triples $(M, X, Y) \to (M, X, Y')$. Via the comparison isomorphisms α_{**}, the diagram in (1) is isomorphic to

$$\begin{array}{ccc} E^{p,q}_U(N) & \xrightarrow{\partial_{M,X,Y}} & E^{p+1,q}_Y(M) \\ {\scriptstyle j^*}\downarrow & & \downarrow{\scriptstyle \mathrm{id}^*} \\ E^{p,q}_{U'}(N') & \xrightarrow[\partial_{X,Y'}]{} & E^{p+1,q}_{Y'}(M). \end{array}$$

The commutativity of this diagram follows directly from the naturality of ∂_{**} (Definition 1.2(1)) and the construction of the long exact sequence of a triple.

A similar argument proves (2). Indeed, take a closed immersion $X \to M$ with $M \in \mathbf{Sm}/k$. Because $f : X' \to X$ is projective, we can factor f as a closed immersion $i : X' \to X \times \mathbb{P}^n$ followed by the projection $X \times \mathbb{P}^n \to X$. This gives us the closed immersion $X' \to M \times \mathbb{P}^n$, and the projection $M \times \mathbb{P}^n \to M$ extends f to give the map $(M \times \mathbb{P}^n, X') \to (M, X)$ in $\mathbf{SP}'$. Using the naturality of ∂_{**} described in Definition 1.8(6) completes the proof. $\square$

Pull-back by a smooth projection. Although it appears that smooth pull-back depends on the choice of smooth pair, one does have a well-defined pull-back

$$p^*\colon H(X) \to H(X \times F)$$

for $F \in \mathbf{Sm}/k$.

LEMMA 3.6. *For $F \in \mathbf{Sm}/k$ and $X \in \mathbf{Sch}_k$, let $p\colon X \times F \to F$ be the projection. Then there is a pull-back map $p^*\colon H(X) \to H(X \times F)$ such that, for each smooth pair (M, X), the diagram*

$$\begin{tikzcd} H(X) \arrow[r, "\alpha_{M,X}"] \arrow[d, "p^*"'] & A_X(M) \arrow[d, "p_*"] \\ H(X \times F) \arrow[r, "\alpha_{M\times F, X\times F}"'] & A_{X\times F}(M \times F) \end{tikzcd}$$

commutes.

Proof. Of course, we define $p^*\colon H(X) \to H(X \times F)$ to be the unique map making the lemma's diagram commute for one fixed choice (M, X) of a smooth pair.

Let (N, X) be another smooth pair. We have the Cartesian transverse diagram

$$\begin{tikzcd} N \times F \arrow[d, "q"'] & M \times N \times F \arrow[l, "\pi_{23}"'] \arrow[r, "\pi_{13}"] \arrow[d, "\pi_{12}"] & M \times F \arrow[d, "p"] \\ N & M \times N \arrow[l, "p_2"] \arrow[r, "p_1"'] & M, \end{tikzcd}$$

where the maps are the respective projections. This gives us the commutative diagram

$$\begin{tikzcd} A_{X\times F}(N \times F) & A_{X\times F}(M \times N \times F) \arrow[l, "\pi_{23*}"'] \arrow[r, "\pi_{13*}"] & A_{X\times F}(M \times F) \\ A_X(N) \arrow[u, "q^*"] & A_X(M \times N) \arrow[l, "p_{2*}"] \arrow[r, "p_{1*}"'] \arrow[u, "\pi_{12}^*"'] & A_X(M), \arrow[u, "p^*"'] \end{tikzcd}$$

which gives the desired commutativity. □

The cap product is also natural with respect to this pull-back, and we have

$$p_U^* \circ j^* = (j \times \mathrm{id})^* \circ p^*$$

for an open immersion $j\colon U \to X$, where $p_U\colon U \times F \to U$ is the projection. Finally, for $g\colon V \to F$ an open immersion in $\mathbf{Sm}/k$, let $p_V\colon X \times V \to X$ be the projection. Then

$$p_V^* = (\mathrm{id} \times g)^* \circ p^*.$$

Homotopy invariance. Let $p\colon \mathbb{A}^n \times X \to X$ be the projection. Then

$$p^*\colon H(X) \to H(\mathbb{A}^n \times X)$$

is an isomorphism. This follows directly from the homotopy invariance of A together with the existence of the well-defined pull-back p^*.

Chern class operators. Let $E \to X$ be a vector bundle of rank r on some $X \in \mathbf{Sch}_k$. Since X is quasi-projective, we choose a closed immersion $i\colon X \to U$ with $U \subset \mathbb{P}^n$ an open subscheme. This gives us the very ample line bundle $O_X(1)$ on X. For $m \gg 0$, the vector bundle $E(m)$ is generated by global sections; a choice of generating sections $s_0, \ldots, s_M$ then gives a morphism $f\colon X \to \mathrm{Grass}(M, r)$ with $f^*(E_{M,r}) \cong E(m)$, where $E_{M,r} \to \mathrm{Grass}(M, r)$ is the universal bundle. Thus, we have the locally closed immersion $(i, f)\colon X \to \mathbb{P}^n \times \mathrm{Grass}(M, r)$ with $(i, f)^*(\mathcal{O}(-m) \boxtimes E_{M,r}) \cong E$; choosing an open subscheme $V \subset \mathbb{P}^n \times \mathrm{Grass}(M, r)$ such that $(i, f)\colon X \to V$ is a closed immersion, we have a smooth pair (V, X) and a vector bundle $\mathcal{E}$ on V that restricts to E on X. Define the *Chern class operator*

$$\tilde{c}_p(L)\colon H(X) \to H(X)$$

by setting $\tilde{c}_p(L)(b) := (i, f)^*(c_p(\mathcal{E})) \cap b$.

One needs to check that $\tilde{c}_p(E)$ is independent of the choices we have made. This follows from our next result.

PROPOSITION 3.7 [3, Sec. 3.2, Lemma]. *For $X \in \mathbf{Sch}_k$, the pull-back of locally free sheaves induces an isomorphism*

$$K_0(X) \to \varinjlim_{f\colon X \to V \in \mathbf{Sm}/k} K_0(V).$$

Now suppose that we have two smooth pairs (M, X) and (N, X), with (respective) vector bundles E_M on M and E_N on N, restricting to E on X. By the proposition, there exist a $V \in \mathbf{Sm}/k$, a vector bundle E_V on V, and a commutative diagram

$$\begin{tikzcd} & N \\ X \arrow[ur, hook, "i_N"] \arrow[r, hook, "i_V"] \arrow[dr, hook, "i_M"'] & V \arrow[u, "f"'] \arrow[d, "g"] \\ & M \end{tikzcd}$$

such that $[E_V] = [f^* E_N] = [g^* E_M] \in K_0(V)$. In particular, this implies that $c_p(E_V) = f^*(c_p(E_N)) = g^*(c_p(E_M))$ in $A(V)$ and thus

$$i_M^*(c_p(E_M)) \cap (\cdot) = i_N^*(c_p(E_N)) \cap (\cdot)\colon H(Y) \to H(Y).$$

Proposition 3.7 yields the following statement.

LEMMA 3.8. *Let E, E' be vector bundles on $X \in \mathbf{Sch}_k$. Then, for all p, q, the Chern class operators $\tilde{c}_p(E), \tilde{c}_q(E')$ commute. If $p \geq 1$ then $\tilde{c}_p(E)$ is nilpotent, $\tilde{c}_0(E)$ is the identity operator, and $\tilde{c}_p(E) = 0$ for $p >$* rank E.

Indeed, for the Chern classes $c_p(E) \in A(V)$ and $V \in \mathbf{Sm}/k$, these properties follow from [10, Thm. 3.6.2].

Similarly, one has the Whitney product formula for the total Chern class operator. Let $\tilde{c}(E) = \sum_{p=0}^{\operatorname{rank} E} \tilde{c}_p(E)$.

LEMMA 3.9. *Let* $0 \to E' \to E \to E'' \to 0$ *be an exact sequence of vector bundles on* $X \in \mathbf{Sch}_k$*. Then*

$$\tilde{c}(E) = \tilde{c}(E') \circ \tilde{c}(E'') = \tilde{c}(E'') \circ \tilde{c}(E').$$

Indeed, this follows from Proposition 3.7 together with the Whitney product formula for the total Chern class $c(E) := \sum_{p=0}^{\operatorname{rank} E} c_p(E) \in A^{\mathrm{ev}}(V)$ for $E \to V$ a vector bundle and $V \in \mathbf{Sm}/k$ (see [10, Thm. 3.6.2]).

The same reasoning shows that the formal group law for A extends to H, as follows.

LEMMA 3.10. *Let* L *and* M *be line bundles on* $X \in \mathbf{Sch}_k$*. Then*

$$F_A(\tilde{c}_1(L), \tilde{c}_1(M)) = \tilde{c}_1(L \otimes M).$$

The properties of the cap product with respect to pull-back and push-forward give the following.

(1) Let $f: Y \to X$ be a projective morphism in $\mathbf{Sch}_k$ and let $E \to X$ be a vector bundle. Then
$$f_* \circ \tilde{c}_p(f^*E) = \tilde{c}_p(E) \circ f_*.$$

(2) Let $j: U \to X$ be an open immersion and let $p: X \times F \to X$ be a projection with $F \in \mathbf{Sm}/k$. Then, for $E \to X$ a vector bundle,
$$j^* \circ \tilde{c}(E) = \tilde{c}(j^*E) \circ j^*, \qquad p^* \circ \tilde{c}(E) = \tilde{c}(p^*E) \circ p^*.$$

Finally, the projective bundle formula with supports (Remark 1.17) and the cap products give the projective bundle formula for H: For $X \in \mathbf{Sch}_k$, let $p: \mathbb{P}^n \times X \to X$ be the projection and let

$$\alpha_i: H(X) \to H(\mathbb{P}^n \times X)$$

be the composition $\tilde{c}_1(O(1))^i \circ p^*$; then

$$\sum_{i=0}^{n} \alpha_i: H(X)^{n+1} \to H(\mathbb{P}^n \times X)$$

is an isomorphism.

4. Algebraic Cobordism

We want to consider the two varieties of algebraic cobordism: the bi-graded theory $\mathrm{MGL}^{*,*}$, represented by the algebraic Thom complex $\mathrm{MGL} \in \mathcal{SH}(k)$; and the theory Ω_*, the universal oriented Borel–Moore homology theory on $\mathbf{Sch}_k$ (in the sense of [6, Def. 5.1.2]). As before, we will assume that k admits resolution of singularities. For the basic definitions and notions of motivic homotopy theory used here, see [2; 8; 9; 12].

The Thom complex MGL is constructed from the Thom spaces of the universal bundles $E_n \to \mathrm{BGL}_n$, where $\mathrm{MGL}_n := \mathrm{Th}(E_n) := E_n/(E_n \setminus 0_{\mathrm{BGL}_n})$ and

$$\mathrm{MGL} := (pt, \mathrm{MGL}_1, \mathrm{MGL}_2, \dots).$$

The bonding maps are given via the inclusions $i_n\colon \mathrm{BGL}_n \to \mathrm{BGL}_{n+1}$, noting that $i_n^*(E_{n+1}) \cong E_n \oplus O_{\mathrm{BGL}_n}$; thus we have

$$\Sigma_t \,\mathrm{Th}(E_n) \cong \mathrm{Th}(E_n \oplus O_{\mathrm{BGL}_n}) \cong \mathrm{Th}(i_n^* E_{n+1}) \xrightarrow{\tilde{i}_n} \mathrm{Th}(E_{n+1}).$$

We recall from [10, 3.8.7] the orientation on $\mathrm{MGL}^{*,*}$. First of all, $\mathrm{MGL}^{*,*}$ is a bi-graded ring cohomology theory on **SP** with

$$\mathrm{MGL}^{p,q}_X(M) := \mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma^\infty_t M/(M \setminus X), \Sigma^{p,q}\,\mathrm{MGL}).$$

The ring structure is given by the canonical lifting of MGL to a ring object in the category of symmetric T-spectra (see e.g. [11]). The orientation is given by a Thom structure and Panin's theorem [10, Thm. 3.7.4], which associates an orientation to a ring cohomology theory with a Thom structure. The Thom structure is induced by choosing a Thom class on the universal Thom space $\mathrm{Th}(O_{\mathbb{P}^\infty}(1))$, which we now describe. Since $\mathbb{P}^\infty = \mathrm{BGL}_1$ and since $O_{\mathbb{P}^\infty}(1)$ is the universal bundle on BGL_1, the Thom space

$$\mathrm{Th}(O_{\mathbb{P}^\infty}(1)) := O_{\mathbb{P}^\infty}(1)/(O_{\mathbb{P}^\infty}(1) \setminus \mathbb{P}^\infty)$$

is by definition equal to MGL_1. The identity map on $\mathrm{Th}(O_{\mathbb{P}^\infty}(1))$ thus extends canonically to a map

$$\iota\colon \Sigma^\infty_t \,\mathrm{Th}(O_{\mathbb{P}^\infty}(1)) \to \Sigma_t\,\mathrm{MGL} = \Sigma^{2,1}\,\mathrm{MGL},$$

giving the universal Thom class $[\iota] \in \mathrm{MGL}^{2,1}_{\mathbb{P}^\infty}(O_{\mathbb{P}^\infty}(1))$. If now $L \to M$ is a line bundle on some $M \in \mathbf{Sm}/k$, then Jouanolou's trick gives us an affine space bundle $p\colon M' \to M$ with M' affine. We replace $L \to M$ with $L' \to M'$, giving the $\mathbb{A}^1$ weak equivalence $\mathrm{Th}(\tilde{p})\colon \mathrm{Th}(L') \to \mathrm{Th}(L)$ and thus the isomorphism

$$\mathrm{Th}(\tilde{p})^*\colon \mathrm{MGL}^{*,*}_M(L) \to \mathrm{MGL}^{*,*}_{M'}(L').$$

Since M' is affine, it follows that L' is generated by global sections; hence there is a morphism $f\colon M' \to \mathbb{P}^\infty$ with $L' \cong f^*(O_{\mathbb{P}^\infty}(1))$. Define

$$th(L) \in \mathrm{MGL}^{2,1}_M(L)$$

as $th(L) = (\mathrm{Th}(\tilde{p})^*)^{-1} \circ f^*([\iota])$.

PROPOSITION 4.1. *Let k be a field admitting resolution of singularities. Then there is a unique bi-graded oriented duality theory* $(\mathrm{MGL}'_{*,*}, \mathrm{MGL}^{*,*})$ *such that the orientation on* $\mathrm{MGL}^{*,*}$ *is the one with associated Thom structure given by the universal Thom class* $[\iota] \in \mathrm{MGL}^{2,1}_{\mathbb{P}^\infty}(O_{\mathbb{P}^\infty}(1))$.

We use the notation $\mathrm{MGL}'_{*,*}$ to distinguish the Borel–Moore homology theory from the homology theory

$$\mathrm{MGL}_{p,q}(X) := \mathrm{Hom}_{\mathcal{SH}(k)}(S^{p,q}_k, \mathrm{MGL} \wedge \Sigma^\infty_t X_+).$$

Proof of Proposition 4.1. Indeed, the Thom class assignment $L \mapsto th(L) \in \mathrm{MGL}^{*,*}_M(L)$ just described has been shown (see e.g. [11]) to give a Thom structure on $\mathrm{MGL}^{*,*}$. Panin's theorem [10, Thm. 3.7.4] gives the associated orientation for $\mathrm{MGL}^{*,*}$, and we may apply the bi-graded version of Theorem 3.4 to complete the proof. □

We now turn to the "geometric" theory Ω_*. In spite of the terminology, Ω_* does not satisfy all the properties of the underlying Borel–Moore homology theory of a $\mathbb{Z}$-graded oriented duality theory: instead of the long exact sequence of a pair $i: Y \to X$, one has a right-exact sequence for each $n \geq 0$

$$\Omega_n(Y) \xrightarrow{i_*} \Omega_n(X) \xrightarrow{j^*} \Omega_n(X \setminus Y) \to 0.$$

In any case, Ω_* does act as if it were at least part of a universal theory. Given a functor

$$H_{*,*}: \mathbf{Sch}'_k \to \mathbf{bi\text{-}GrAb}$$

and an $X \in \mathbf{Sch}_k$, we let $H_{2*,*}(X) = \bigoplus_n H_{2n,n}(X)$; this gives the functor

$$H_{2*,*}: \mathbf{Sch}'_k \to \mathbf{GrAb}.$$

PROPOSITION 4.2. *Let k be a field admitting resolution of singularities, and let (H, A) be a bi-graded oriented duality theory. Then there is a unique natural transformation*

$$\vartheta_H: \Omega_* \to H_{2*,*}$$

of functors $\mathbf{Sch}'_k \to \mathbf{GrAb}$ that satisfies the following conditions.

(1) *Let $j: U \to X$ be an open immersion in $\mathbf{Sch}_k$. Then the diagram*

$$\begin{array}{ccc} \Omega_*(X) & \xrightarrow{\vartheta_H(X)} & H_{2*,*}(X) \\ {\scriptstyle j^*}\downarrow & & \downarrow{\scriptstyle j^*} \\ \Omega_*(U) & \xrightarrow[\vartheta_H(X)]{} & H_{2*,*}(U) \end{array}$$

commutes.

(2) *Let $f: M \to N$ be a morphism in $\mathbf{Sm}/k$, and let $d_N = \dim_k N$, $d_M = \dim_k M$, and $d = \mathrm{codim}\, f := d_N - d_M$. Then the diagram*

$$\begin{array}{ccc} \Omega_*(N) & \xrightarrow{\alpha_{N,N} \circ \vartheta_H(N)} & A^{2d_N - 2*, -*}(N) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \Omega_{*-d}(M) & \xrightarrow[\alpha_{M,M} \circ \vartheta_H(M)]{} & A^{2d_N - 2*, -*}(M) \end{array}$$

commutes.

(3) *Let $L \to X$ be a line bundle on some $X \in \mathbf{Sch}_k$. Then the diagram*

$$\begin{array}{ccc} \Omega_*(X) & \xrightarrow{\vartheta_H(X)} & H_{2*,*}(X) \\ {\scriptstyle \tilde{c}_1(L)}\downarrow & & \downarrow{\scriptstyle \tilde{c}_1(L)} \\ \Omega_{*-1}(X) & \xrightarrow[\vartheta_H(X)]{} & H_{2*-2,*-1}(X) \end{array}$$

commutes.

(4) *For $M \in \mathbf{Sm}/k$ of dimension d_M over k, set $\Omega^n(M) := \Omega_{d_M-n}(M)$. Then the map $\alpha_{M,M} \circ \vartheta_H : \Omega^*(M) \to A^{2*,*}(M)$ is a homomorphism of graded rings.*

Proof. We let $\mathbb{L}_*$ denote the Lazard ring—that is, the coefficient ring of the universal rank-1 commutative formal group law,

$$F_{\mathbb{L}}(u, v) := u + v + \sum_{i,j\geq 1} a_{ij} u^i v^j.$$

Here $\mathbb{L}$ is generated as a commutative $\mathbb{Z}$-algebra by the coefficients a_{ij}, and we give $\mathbb{L}$ the grading with $\deg(a_{ij}) = i + j - 1$. We use the construction of Ω_* as the universal "oriented Borel–Moore functor of geometric type" on $\mathbf{Sch}_k$ (see [6, Defs. 2.1.1, 2.1.12, 2.2.1, and Thm. 2.3.13]). Since $H_{2*,*}$ does not have all the properties of an oriented Borel–Moore functor of geometric type, we are forced to go through the actual construction of Ω_*; we give a sketch of this three-step process, referring the reader to [6, Sec. 2] for the details.

Step 1. For $Y \in \mathbf{Sm}/k$ of dimension d_Y over k, let $p_Y : Y \to pt$ be the structure morphism and define the *fundamental class* $[Y]_H \in H_{2d_Y,d_Y}(Y)$ by

$$[Y]_H = \alpha_{Y,Y}^{-1}(p_Y^*(1)),$$

where $1 \in A^{0,0}(pt)$ is the unit.

For $X \in \mathbf{Sch}_k$, let $\mathcal{Z}_n(X)$ denote the group of *dimension-n cobordism cycles* on X. This is the group generated by tuples $(f : Y \to X;\ L_1, \dots, L_r)$ with $Y \in \mathbf{Sm}/k$ irreducible of dimension $n + r$ over k, f a projective morphism, and $L_1, \dots, L_r$ line bundles on Y (we allow $r = 0$). We identify two cobordism cycles by isomorphism over X (see [6, Def. 2.1.6]; note that this includes reordering the L_i). Now $\mathcal{Z}_*(X)$ has the following operations.

(i) *Projective push-forward.* For $f : X \to X'$ a projective map in $\mathbf{Sch}_k$, set

$$g_*((f : Y \to X;\ L_1, \dots, L_r)) := (g \circ f : Y \to X';\ L_1, \dots, L_r).$$

(ii) *Smooth pull-back.* Let $h : X' \to X$ be a smooth quasi-projective morphism of relative dimension d. Set

$$h^*((f : Y \to X;\ L_1, \dots, L_r)) := (p_2 : Y \times_X X' \to X';\ p_1^* L_1, \dots, p_1^* L_r) \in \mathcal{Z}_{d+n}(X').$$

(iii) *Chern class operator.* Let $L \to X$ be a line bundle. Set

$$\tilde{c}_1(L)((f : Y \to X;\ L_1, \dots, L_r)) := (f : Y \to X;\ L_1, \dots, L_r, f^* L) \in \mathcal{Z}_{n-1}(X).$$

(iv) *External products.* Define

$$\times\colon \mathcal{Z}_n(X) \times \mathcal{Z}_m(X') \to \mathcal{Z}_{n+m}(X \times X')$$

by

$$\begin{aligned}&(f\colon Y \to X;\ L_1,\dots,L_r) \times (f'\colon Y' \to X';\ M_1,\dots,M_s)\\ &\quad = (f \times f'\colon Y \times Y' \to X \times X';\ p_1^*L_1,\dots,p_1^*L_r, p_2^*M_1,\dots,p_2^*M_s).\end{aligned}$$

Define $\vartheta_H^1(X)\colon \mathcal{Z}_*(X) \to H_{2*,*}(X)$ by

$$\begin{aligned}&\vartheta_H^1(X)((f\colon Y \to X;\ L_1,\dots,L_r))\\ &\qquad := f_*(\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_r)([Y]_H)) \in H_{2d_Y-2r,d_Y-r}(X).\end{aligned}$$

The properties of projective push-forward, pull-back for open immersions, and Chern class operators for H that were discussed in Section 3 imply that (a) the $\vartheta_H^1(X)$ define a natural transformation of functors

$$[\vartheta_H^1\colon \mathcal{Z}_* \to H_{2*,*}]\colon \mathbf{Sch}'_k \to \mathbf{GrAb}$$

and (b) ϑ_H^1 is compatible with pull-back for open immersions and with the respective Chern class operators for line bundles.

For $M \in \mathbf{Sm}/k$ of dimension d_M over k, let $\mathcal{Z}^*(M) := \mathcal{Z}_{d_M-*}(M)$. We have the map

$$\vartheta_1^A(M) := \alpha_{M,M} \circ \vartheta_H^1(M)\colon \mathcal{Z}^*(M) \to A^{2*,*}(M).$$

It is easy to see that ϑ_1^A has the same compatibilities as ϑ_H and, in addition, that ϑ_1^A is compatible with smooth pull-back and external products.

Step 2. The formal group law $F_A(u,v) \in A^{2*,*}(pt)[[u,v]]$ gives rise to the classifying map

$$\varphi_A\colon \mathbb{L}_* \to A^{-2*,-*}(pt),$$

a homomorphism of graded rings. Via the structure morphism $p_X\colon X \to pt$, $H_{2*,*}(X)$ becomes a graded module over $A^{-2*,-*}(pt)$; then the projective push-forward, open pull-back, and Chern class operators are all $A^{-2*,-*}(pt)$-module maps. Via φ_A, $H_{2*,*}(X)$ becomes a graded module over $\mathbb{L}_*$, and the projective push-forward, open pull-back, and Chern class operators are all $\mathbb{L}_*$-module maps. Thus, ϑ_H^1 gives rise to the natural transformation

$$[\vartheta_H^2\colon \mathbb{L}_* \otimes \mathcal{Z}_* \to H_{2*,*}]\colon \mathbf{Sch}'_k \to \mathbf{Gr}_{\mathbb{L}_*}\,\mathrm{Mod},$$

which is compatible with open pull-back and Chern class operators.

Similarly, we have the maps

$$\vartheta_2^A(M)\colon \mathbb{L}^* \otimes \mathcal{Z}^*(M) \to A^{2*,*}(M),$$

which are compatible with projective push-forward, smooth pull-back, Chern class operators, and external products. Here $\mathbb{L}^n := \mathbb{L}_{-n}$, giving the graded ring $\mathbb{L}^*$ and the graded ring homomorphism $\varphi_A\colon \mathbb{L}^* \to A^{2*,*}(pt)$. The graded group $A^{2*,*}(M)$ is thereby a graded $\mathbb{L}^*$-module, and the projective push-forward, smooth pull-back, and Chern class operators are all $\mathbb{L}^*$-linear. The external products are $\mathbb{L}^*$-bilinear.

Step 3. The group $\Omega_*(X)$ is defined as a quotient of $\mathbb{L}_* \otimes \mathcal{Z}_*(X)$ by imposing relations on the Borel–Moore functor $\mathbb{L}_* \otimes \mathcal{Z}_*$ (see [6, Def. 2.2.1]) as follows.

(a) *The dimension axiom.* For each $X \in \mathbf{Sch}_k$, let $\underline{\mathcal{Z}}_*(X)$ be the quotient of $\mathcal{Z}_*(X)$ by the subgroup generated by elements of the form

$$(f : Y \to X, \pi^*(L_1), \dots, \pi^*(L_r), M_1, \dots, M_s),$$

where $\pi : Y \to Z$ is a smooth morphism in $\mathbf{Sm}/k$, the $L_1, \dots, L_r$ are line bundles on Z, and $r > \dim_k Z$.

(b) *The Gysin axiom.* For each $X \in \mathbf{Sch}_k$, let $\underline{\Omega}_*(X)$ be the quotient of $\underline{\mathcal{Z}}_*(X)$ by the subgroup generated by elements of the form

$$(f : Y \to X;\ L_1, \dots, L_r) - (f \circ i : Z \to X;\ i^*L_1, \dots, i^*L_{r-1}),$$

where $i : Z \to Y$ is the inclusion of a smooth codimension-1 closed subscheme Z such that $O_Y(Z) \cong L_r$.

(c) *The formal group law.* $\Omega_*(X)$ is the quotient of $\mathbb{L}_* \otimes \underline{\Omega}_*$ by the $\mathbb{L}_*$-submodule generated by elements of the form

$$f_*([F_A(\tilde{c}_1(L), \tilde{c}_1(M)) - \tilde{c}_1(L \otimes M)](\eta)),$$

as $f : Y \to X$ runs over projective morphisms with $Y \in \mathbf{Sm}/k$ irreducible, L, M run over line bundles on Y, and η runs over elements of $\mathcal{Z}_*(Y)$ of the form $\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_r)(\mathrm{id}_Y : Y \to Y)$ for line bundles $L_1, \dots, L_r$ on Y.

It follows from the results of [6, Sec. 2.4] that the constructions (a)–(c) are well-defined and $\underline{\mathcal{Z}}_*$, $\underline{\Omega}_*$, and Ω_* inherit the operations of projective push-forward, smooth pull-back, Chern class operators, and external products from $\mathcal{Z}_*$. Finally, by [6, Thm. 2.4.13], Ω_* is the universal oriented Borel–Moore $\mathbb{L}_*$-functor of geometric type.

To extend ϑ_H^2 to the desired natural transformation ϑ_H, we need only show that ϑ_H^2 sends to zero the elements described in (a)–(c) above. In fact, we have the following lemma.

LEMMA 4.3. *Let A be a bi-graded oriented ring cohomology theory on* **SP**. *Then the restriction of $A^{2*,*}$* (*with the integration on A subjected to the given orientation*) *to* $\mathbf{Sm}/k$ *defines an oriented cohomology theory on* $\mathbf{Sm}/k$ *in the sense of* [6, Def. 1.1.2].

Proof. Indeed, an oriented cohomology theory on $\mathbf{Sm}/k$ (following [6]) is a contravariant functor A^* from $\mathbf{Sm}/k$ to graded, commutative rings with unit, plus push-forward maps $f_* : A^*(Y) \to A^{*+d}(X)$ for each projective morphism $f : Y \to X$, $d = \operatorname{codim} f$, satisfying the functoriality of projective push-forward, commutativity of pull-back and push-forward in transverse Cartesian squares, the projective bundle formula (with $c_1(L) := s^*s_*(1_X)$ for $L \to X$ a line bundle with zero section s) and an extended homotopy property:

$$p^* : A^*(X) \to A^*(E)$$

is an isomorphism for each affine space bundle $p : E \to X$. These properties for an oriented ring cohomology theory are all verified in [10]. $\square$

By [6, Thm. 7.1.1], the structures we have defined on Ω^* admit a unique extension that makes Ω^* an oriented cohomology theory on $\mathbf{Sm}/k$ in the sense of [6]. By [6, Thm. 7.1.3], Ω^* is the universal oriented cohomology theory on $\mathbf{Sm}/k$. Thus, given a bi-graded oriented ring cohomology theory A on $\mathbf{SP}$, there is a unique natural transformation of oriented cohomology theories on $\mathbf{Sm}/k$:

$$\vartheta^A\colon \Omega^* \to A^{2*,*}.$$

By [6, Prop. 5.2.1], the Chern class operators in Ω^* are given by cup product with the Chern classes $c_1(L)$. Because the image $[f\colon Y \to M;\ L_1,\dots,L_r]$ in $\Omega^*(M)$ of a cobordism cycle $(f\colon Y \to M;\ L_1,\dots,L_r)$ is equal to $f_*(\tilde{c}_1(L)\circ\cdots\circ\tilde{c}_1(L_r)(p_Y^*(1)))$, it follows that

$$\vartheta^A([f\colon Y \to M;\ L_1,\dots,L_r]) = \vartheta_1^A((f\colon Y \to M;\ L_1,\dots,L_r)).$$

Also, $\vartheta^A\colon \Omega^*(pt) \to A^{2*,*}(pt)$ is a graded ring homomorphism and ϑ^A is a $\Omega^*(pt)$-module homomorphism, so

$$\vartheta^A(a\cdot[f\colon Y \to M;\ L_1,\dots,L_r]) = \vartheta_2^A(a\otimes(f\colon Y \to M;\ L_1,\dots,L_r))$$

for all $a \in \mathbb{L}_*$. In other words, ϑ_2^A descends to the natural transformation ϑ^A: $\Omega^* \to A^{2*,*}$.

This immediately implies that ϑ_H^2 descends to a natural transformation

$$\vartheta_H\colon \Omega_* \to H_{2*,*}.$$

Indeed, the elements described in (a)–(c) above are all of the form $f_*(\tau)$ for τ an element of $\mathcal{Z}_*(Y)$, $\underline{\mathcal{Z}}_*(Y)$, or $\mathbb{L}_*\otimes\underline{\Omega}_*(Y)$ with $Y \in \mathbf{Sm}/k$, $f\colon Y \to X$ a projective morphism, and τ going to zero in $\Omega_*(Y)$. Since

$$0 = \vartheta_2^A(Y)(\tau) = \alpha_{Y,Y}(\vartheta_H^2(Y)(\tau)),$$

it follows that $\vartheta_H^2(Y)(\tau) = 0$ and thus

$$0 = f_*(\vartheta_H^2(Y)(\tau)) = \vartheta_H^2(X)(f_*(\tau)).$$

Therefore, ϑ_H^2 descends uniquely to

$$\vartheta_H\colon \Omega_* \to H_{2*,*},$$

completing the proof of Proposition 4.2(1)–(4). □

There is still the question of the behavior of ϑ_H with respect to cap products and external products. We recall [6, Thm. 7.1.1], which states that Ω_* admits functorial pull-back maps for all local complete intersection morphisms in $\mathbf{Sch}_k$, extending the pull-back maps for smooth morphisms, and satisfying the axioms of an oriented Borel–Moore homology theory on $\mathbf{Sch}_k$ (in the sense of [6, Def. 5.1.2]). This enables us to define a cap product map

$$f^*(\cdot)\cap\colon \Omega^q(M)\otimes\Omega_p(Y) \to \Omega_{p-q}(Y)$$

for each morphism $f\colon Y \to M$ with $M \in \mathbf{Sm}/k$. Indeed, we have the external product

$$\times\colon \Omega^q(M)\otimes\Omega_p(Y)\to\Omega_{p-q+d_M}(M\times Y).$$

Because M is smooth, the graph embedding $(f,\mathrm{id}_Y)\colon Y\to M\times Y$ is a regular embedding of codimension d_M and so we have a well-defined pull-back

$$(f,\mathrm{id}_Y)^*\colon \Omega_{p-q+d_M}(M\times Y)\to\Omega_{p-q}(Y).$$

We set $f^*(a)\cap b := (f,\mathrm{id}_Y)^*(a\times b)$ for $a\in\Omega^q(M)$ and $b\in\Omega_p(Y)$.

PROPOSITION 4.4. *Let $f\colon Y\to M$ be a morphism in $\mathbf{Sch}_k$ with $M\in\mathbf{Sm}/k$. Then the diagram*

$$\begin{tikzcd}
\Omega^q(M)\otimes\Omega_p(Y) \arrow[r, "f^*(\cdot)\cap"] \arrow[d, "\alpha_{M,M}\circ\vartheta^A(M)\otimes\vartheta_H(Y)"'] & \Omega_{p-q}(Y) \arrow[d, "\vartheta_H(Y)"] \\
A^{2q,q}(M)\otimes H_{2p,p}(Y) \arrow[r, "f^*(\cdot)\cap"'] & H_{2(p-q),p-q}(Y)
\end{tikzcd}$$

commutes.

Proof. Suppose first that Y is in $\mathbf{Sm}/k$ and of dimension d_Y over k. It is easy to see that the map

$$f^*(\cdot)\cap\colon \Omega^q(M)\otimes\Omega_p(Y)\to\Omega_{p-q}(Y)$$

is given by

$$f^*(a)\cap b = f^*(a)\cup b$$

after making the identification $\Omega_n(Y)=\Omega^{d_Y-n}(Y)$, where the f^* on the right-hand side is the pull-back map $f^*\colon\Omega^*(Y)\to\Omega^*(M)$ and $\cup$ is the product on $\Omega^*(Y)$. The analogous formula on the (H,A) side follows from Definition 3.1(A4). Thus, the proposition is true for $Y\in\mathbf{Sm}/k$.

For the general case, we recall from [6, Lemma 2.5.11] that $\Omega_*(Y)$ is generated (as an abelian group) by the classes of the form $[g\colon W\to Y]$ with $W\in\mathbf{Sm}/k$ and g projective. For both Ω_* and $H_{*,*}$, we have the identity

$$g_*((f\circ g)^*(a)\cap b) = f^*(a)\cap g_*(b).$$

Since ϑ_H commutes with projective push-forward and since ϑ^A commutes with pull-back by arbitrary morphisms in $\mathbf{Sm}/k$, the case of smooth Y implies the general case. □

PROPOSITION 4.5. *The natural transformation ϑ_H is compatible with external products: For $a\in\Omega_p(X)$ and $b\in\Omega_q(Y)$,*

$$\vartheta_H(a\times b)=\vartheta_H(a)\times\vartheta_H(b)\in H_{2(p+q),p+q}(X\times Y).$$

Proof. The proof is similar to that of Proposition 4.4. Because the external products are compatible with push-forward (as in Definition 3.1(A3)(ii)), it suffices to handle the case of smooth X and Y. The statement is then a consequence of the fact that $\vartheta^A(M)\colon\Omega^*(M)\to A^{2*,*}(M)$ is a ring homomorphism. □

Comparing Ω^* *and* $\mathrm{MGL}'_{2*,*}$. Putting Proposition 4.1 and Proposition 4.2 together yields the natural transformation

$$[\vartheta_{\mathrm{MGL}'}\colon \Omega_* \to \mathrm{MGL}'_{2*,*}]\colon \mathbf{Sch}'_k \to \mathbf{GrAb},$$

which extends the natural transformation of oriented cohomology theories on $\mathbf{Sm}/k$,

$$\vartheta^{\mathrm{MGL}}\colon \Omega^* \to \mathrm{MGL}^{2*,*},$$

discussed in [6].

CONJECTURE 4.6. *Let k be a field of characteristic 0. Then $\vartheta_{\mathrm{MGL}'}\colon \Omega_* \to \mathrm{MGL}'_{2*,*}$ is an isomorphism.*

The analogous conjecture for ϑ^{MGL} was stated in [6]. In fact, the extension of ϑ^{MGL} to $\vartheta_{\mathrm{MGL}'}$ should allow one to use localization to prove Conjecture 4.6. We give a sketch of the argument here; details may be found in the preprint [5].

It follows from unpublished work of Hopkins and Morel, in which a spectral sequence from $\mathbb{L}^* \otimes H^*(\cdot, \mathbb{Z}(*))$ converging to $\mathrm{MGL}^{*,*}$ is constructed, that the map $\vartheta^{\mathrm{MGL}}(\mathrm{Spec}\, F)$ is an isomorphism for any field F finitely generated over the base-field k (in characteristic 0). Now that we have the extension to $\vartheta_{\mathrm{MGL}'}$, we can use the right-exact localization sequence and induction on the Krull dimension to prove the result in general.

Indeed, for a given $X \in \mathbf{Sch}_k$, let

$$\Omega_*^{(1)}(X) = \varinjlim_{W \subset X} \Omega_*(W),$$

where the limit is over all closed subsets of W not containing any generic point of X. Define $\mathrm{MGL}'^{(1)}_{2*,*}(X)$ similarly. We have the commutative diagram

$$\begin{tikzcd}
& \Omega_*^{(1)}(X) \arrow[r, "i_*"] \arrow[d, "\vartheta^{(1)}(X)"'] & \Omega_*(X) \arrow[r, "j^*"] \arrow[d, "\vartheta(X)"'] & \Omega_*(k(X)) \arrow[r] \arrow[d, "\vartheta(k(X))"] & 0 \\
\mathrm{MGL}'_{2*+1,*}(k(X)) \arrow[r, "\partial"'] & \mathrm{MGL}'^{(1)}_{2*,*}(X) \arrow[r, "i_*"'] & \mathrm{MGL}'_{2*,*}(X) \arrow[r, "j^*"'] & \mathrm{MGL}_{2*,*}(k(X)) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Assuming $\vartheta^{(1)}(X)$ is an isomorphism and noting that $\vartheta(k(X))$ is an isomorphism, we already find that $\vartheta(X)$ is surjective. To show that $\vartheta(X)$ is injective, we need only lift the map ∂ to a commutative diagram

$$\begin{tikzcd}
\widetilde{\mathrm{MGL}}'_{2*+1,*}(k(X)) \arrow[r, "\partial'"] \arrow[d, "\vartheta'"'] & \Omega_*^{(1)}(X) \arrow[d, "\vartheta^{(1)}(X)"] \\
\mathrm{MGL}'_{2*+1,*}(k(X)) \arrow[r, "\partial"'] & \mathrm{MGL}'^{(1)}_{2*,*}(X)
\end{tikzcd}$$

such that $i_* \circ \partial' = 0$ and ϑ' is surjective. For this, one uses the Hopkins–Morel spectral sequence to get a handle on elements generating $\mathrm{MGL}'_{2*+1,*}(k(X))$ and then the formal group law to understand the boundary map ∂ on the generators.

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