

## Syzygies of Multiplier Ideals on Singular Varieties

ROBERT LAZARSFELD, KYUNGYONG LEE,  
& KAREN E. SMITH

*Dedicated to Mel Hochster on the occasion of his sixty-fifth birthday*

### Introduction

It was recently established in [10] that multiplier ideals on a smooth variety satisfy some special syzygetic properties. The purpose of this paper is to show how some of these can be extended to the singular setting.

To set the stage, we review some of the results from [10]. Let  $X$  be a smooth complex variety of dimension  $\dim(X) = d$ , and denote by  $(\mathcal{O}, \mathfrak{m})$  the local ring of  $X$  at a fixed point  $x \in X$ . Let  $\mathcal{J} \subseteq \mathcal{O}$  be any multiplier ideal; that is, assume  $\mathcal{J}$  is the stalk at  $x$  of a multiplier ideal sheaf  $\mathcal{J}(X, \mathfrak{b}^\lambda)$ , where  $\mathfrak{b} \subseteq \mathcal{O}_X$  is an ideal sheaf and  $\lambda$  is a positive rational number. The main result of [10] is that if  $p \geq 1$  then no minimal  $p$ th syzygy of  $\mathcal{J}$  vanishes modulo  $\mathfrak{m}^{d+1-p}$  at  $x$ . In other words, if we consider a minimal free resolution of the ideal  $\mathcal{J}$  over the regular local ring  $\mathcal{O}$ ,

$$\cdots \xrightarrow{u_3} F_2 \xrightarrow{u_2} F_1 \xrightarrow{u_1} F_0 \longrightarrow \mathcal{J} \longrightarrow 0,$$

then no minimal generator of the  $p$ th syzygy module

$$\mathrm{Syz}_p(\mathcal{J}) \stackrel{\mathrm{def}}{=} \mathrm{Im}(u_p) \subseteq F_{p-1}$$

of  $\mathcal{J}$  lies in  $\mathfrak{m}^{d+1-p} \cdot F_{p-1}$ . Although this result places no restriction on the orders of vanishing of the *generators* of  $\mathcal{J}$ , it provides strong constraints on the first and higher *syzygies* of  $\mathcal{J}$ . When  $d = 2$  these conditions hold for any integrally closed ideal, but [10] shows that in dimensions  $d \geq 3$  only rather special integrally closed ideals can arise as multiplier ideals. (In contrast, it was established by Favre–Jonsson [2] and Lipman–Watanabe [12] that any integrally closed ideal on a smooth *surface* is locally a multiplier ideal.)

Multiplier ideals can be defined on any  $\mathbf{Q}$ -Gorenstein variety  $X$  or, more generally, for any pair  $(X, \Delta)$  consisting of an effective Weil  $\mathbf{Q}$ -divisor  $\Delta$  on a normal variety  $X$  such that  $K_X + \Delta$  is  $\mathbf{Q}$ -Cartier. It is natural to wonder whether multiplier ideals in this context satisfy the same sort of algebraic properties as in the smooth case. We will see (Example 3.1) that the result from [10] just quoted does not extend without change. However, we show that at least for *first* syzygies, one obtains a statement by replacing the maximal ideal  $\mathfrak{m}$  by any parameter ideal.

---

Received September 10, 2007. Revision received March 29, 2008.

Research of the first author partially supported by NSF grant DMS-0652845; research of the third author partially supported by NSF grant DMS-0500823.

**THEOREM A.** *Let  $(X, \Delta)$  be a pair with  $\dim X = d$ , let  $(\mathcal{O}, \mathfrak{m})$  be the local ring of  $X$  at a Cohen–Macaulay point  $x \in X$ , and fix a system of parameters*

$$z_1, \dots, z_d \in \mathcal{O}.$$

*Let  $\mathcal{J} \subseteq \mathcal{O}$  be (the germ at  $x$  of) any multiplier ideal on  $(X, \Delta)$ . Then no minimal first syzygy of  $\mathcal{J}$  vanishes modulo  $(z_1, \dots, z_d)^d$ .*

If  $X$  is  $\mathbf{Q}$ -Gorenstein then we can take  $\Delta = 0$ , so that one is dealing with usual multiplier ideals of the form  $\mathcal{J}(X, \mathfrak{b}^\lambda)$ . Of course, the strongest statement is achieved by taking  $z_1, \dots, z_d$  to generate the largest possible ideal—which is to say, by taking the  $z_i$  to generate a reduction of  $\mathfrak{m}$ . In this case, if  $x$  is a smooth point then the  $z_i$  generate the maximal ideal itself, and we recover the original result from [10] in the case  $p = 1$ .

Observe that even though Theorem A doesn't give a uniform bound on the order of vanishing of syzygies of a multiplier ideal, it does uniformly bound the highest power of any ideal generated by a *system of parameters* that can contain a syzygy. It also yields uniform statements provided we bring the multiplier ideal of the trivial line bundle into the picture. For example, we have the following result.

**COROLLARY B.** *Let  $x \in X$  be a Cohen–Macaulay point with maximal ideal  $\mathfrak{m}$ , and set*

$$\tau = \mathcal{J}((X, \Delta); \mathcal{O}_X)_x.$$

*If  $\mathcal{J}$  is the germ at  $x$  of any multiplier ideal, then no first syzygy of  $\mathcal{J}$  vanishes modulo  $\tau \cdot \mathfrak{m}^{2d-1}$ .*

In particular, if  $(X, \Delta)$  is Kawamata log terminal (KLT) then no first syzygy can vanish modulo  $\mathfrak{m}^{2d-1}$ .

Unlike the results for smooth varieties in [10], the statements here deal only with *first* syzygies. This may be more an artifact of our method than a necessary restriction, and it would be interesting to investigate this further.

We begin in Section 1 with a discussion of Skoda's theorem in the singular setting. In Section 2 we modify the arguments from [10] to prove Theorem A. We conclude in Section 3 with some examples and applications.

We are grateful to Craig Huneke for some valuable discussions.

## 1. Skoda Complexes on Singular Varieties

In this section, we discuss the circle of ideas surrounding Skoda's theorem in the singular setting. This appears only briefly in [9], so we thought it would be useful to spell out some of the details. We don't claim any essential novelty for the material in this section.

Let  $(X, \Delta)$  be a pair in the sense of [9, 9.3.55]; this means  $X$  is a normal variety and  $\Delta = \sum d_i D_i$  is an effective Weil  $\mathbf{Q}$ -divisor such that  $K_X + \Delta$  is  $\mathbf{Q}$ -Cartier. Fix ideals  $\mathfrak{b}, \mathfrak{c} \subseteq \mathcal{O}_X$ , and let

$$\mu: X' \rightarrow X$$

be a log resolution of  $(X, \Delta)$ ,  $\mathfrak{b}$ , and  $\mathfrak{c}$ . Then one can attach numbers

$$a(E) \in \mathbf{Q}, \quad b(E), c(E) \in \mathbf{N}$$

to each exceptional divisor of  $\mu$  and also to the proper transforms of the divisors appearing in the support of  $\Delta$  or the zeroes of  $\mathfrak{b}$  and  $\mathfrak{c}$ , characterized by the expressions

$$\begin{aligned} K_{X'} &\equiv_{\text{num}} \mu^*(K_X + \Delta) + \sum a(E)E; \\ \mathfrak{b} \cdot \mathcal{O}_{X'} &= \mathcal{O}_{X'}(-\sum b(E)E), \\ \mathfrak{c} \cdot \mathcal{O}_{X'} &= \mathcal{O}_{X'}(-\sum c(E)E). \end{aligned}$$

Given a rational or real weighting coefficient  $\lambda > 0$ , one then defines the multiplier ideal

$$\mathcal{J}((X, \Delta); \mathfrak{c} \cdot \mathfrak{b}^\lambda) = \mu_* \mathcal{O}_{X'}(\sum (\lceil a(E) - c(E) - \lambda b(E) \rceil) E),$$

this being independent of the resolution. If  $X$  is  $\mathbf{Q}$ -Gorenstein then we can take  $\Delta = 0$ , and if in addition  $X$  is actually Gorenstein then we have the more familiar definition

$$\mathcal{J}(X, \mathfrak{c} \cdot \mathfrak{b}^\lambda) = \mu_* \mathcal{O}_{X'}(K_{X'/X} - [C + \lambda B]),$$

where

$$B = \sum b(E)E, \quad C = \sum c(E)E.$$

The following lemma expresses an elementary but important property of multiplier ideals.

LEMMA 1.1. *For any integer  $m \geq 0$ , there is an inclusion*

$$\mathfrak{c} \cdot \mathcal{J}((X, \Delta); \mathfrak{c}^m \cdot \mathfrak{b}^\lambda) \subseteq \mathcal{J}((X, \Delta); \mathfrak{c}^{m+1} \cdot \mathfrak{b}^\lambda).$$

*Proof.* One has

$$\begin{aligned} \mathfrak{c} \cdot \mathcal{J}((X, \Delta); \mathfrak{c}^m \cdot \mathfrak{b}^\lambda) &\subseteq \mu_* \mathcal{O}_{X'}(-\sum c(E)E) \cdot \mu_* \mathcal{O}_{X'}(\sum (\lceil a(E) - mc(E) - \lambda b(E) \rceil) E) \\ &\subseteq \mu_* \mathcal{O}_{X'}(\sum (\lceil a(E) - (m+1)c(E) - \lambda b(E) \rceil) E) \\ &= \mathcal{J}((X, \Delta); \mathfrak{c}^{m+1} \cdot \mathfrak{b}^\lambda). \end{aligned} \quad \square$$

COROLLARY 1.2. *With  $(X, \Delta)$  as above,*

$$\mathfrak{c} \cdot \mathcal{J}((X, \Delta; \mathcal{O}_X) \subseteq \mathcal{J}((X, \Delta); \mathfrak{c})$$

*for any ideal  $\mathfrak{c}$ . In particular, if  $(X, \Delta)$  is KLT then*

$$\mathfrak{c} \subseteq \mathcal{J}((X, \Delta); \mathfrak{c}).$$

We now turn to Skoda complexes. Because our interest is local, we assume for simplicity of notation that  $X$  is affine. Choose elements

$$f_1, \dots, f_r \in \mathfrak{c},$$

and for compactness write

$$\mathcal{J}(\mathfrak{c}^m \cdot \mathfrak{b}^\lambda) = \mathcal{J}((X, \Delta); \mathfrak{c}^m \cdot \mathfrak{b}^\lambda).$$

It follows from Corollary 1.2 that each  $f_i$  multiplies  $\mathcal{J}(\mathfrak{c}^\ell \cdot \mathfrak{b}^\lambda)$  into  $\mathcal{J}(\mathfrak{c}^{\ell+1} \cdot \mathfrak{b}^\lambda)$ . Hence the  $f_i$  determine a complex  $\text{Skod}_\bullet(m; f)$

$$\dots \longrightarrow \mathcal{O}_X^{\binom{2}{2}} \otimes \mathcal{J}(\mathfrak{c}^{m-2} \cdot \mathfrak{b}^\lambda) \longrightarrow \mathcal{O}_X^r \otimes \mathcal{J}(\mathfrak{c}^{m-1} \cdot \mathfrak{b}^\lambda) \longrightarrow \mathcal{J}(\mathfrak{c}^m \cdot \mathfrak{b}^\lambda) \longrightarrow 0,$$

arising as a subcomplex of the Koszul complex  $K_\bullet(f_1, \dots, f_r) = \text{Kosz}_\bullet(f)$  on the  $f_i$ .

The basic result for our purposes is Theorem 1.3.

**THEOREM 1.3.** *Assume that  $m \geq r$  and that the  $f_i$  generate a reduction of  $\mathfrak{c}$ . Then  $\text{Skod}_\bullet(m; f)$  is exact.*

Recall that the hypothesis on the  $f_i$  is equivalent to requiring that their pull-backs to the log resolution  $X'$  generate the pull-back  $\mathcal{O}_{X'}(-C)$  of  $\mathfrak{c}$ .

*Proof of Theorem 1.3 (sketch).* The pull-backs of the given elements  $f_i \in \mathfrak{c}$  determine an exact Koszul complex of vector bundles on  $X'$ :

$$\dots \longrightarrow \mathcal{O}_{X'}^{\binom{2}{2}} \otimes \mathcal{O}_{X'}(2C) \longrightarrow \mathcal{O}_{X'}^r \otimes \mathcal{O}_{X'}(C) \longrightarrow \mathcal{O}_{X'} \longrightarrow 0.$$

Twisting through by  $\mathcal{O}_{X'}(\sum(\binom{r}{a}E - mc(E) - \lambda b(E)^{\binom{r}{a}})E)$ , we derive an exact sequence all of whose terms have vanishing higher direct images thanks to the local vanishing theorems for multiplier ideals [9, 9.4.17]. The direct image of this twisted Koszul complex, which is the Skoda complex  $\text{Skod}_\bullet(m; f)$ , is therefore exact. □

We conclude this section by recording some consequences of Briançon–Skoda type.

**COROLLARY 1.4.** *Assume as in the theorem that  $m \geq r$  and that  $f_1, \dots, f_r$  generate a reduction of  $\mathfrak{c}$ . Then*

- (i)  $\mathcal{J}((X, \Delta); \mathfrak{c}^m \cdot \mathfrak{b}^\lambda) = (f_1, \dots, f_r) \cdot \mathcal{J}((X, \Delta); \mathfrak{c}^{m-1} \cdot \mathfrak{b}^\lambda)$ ;
- (ii)  $\mathfrak{c}^m \cdot \mathcal{J}((X, \Delta); \mathcal{O}_X) \subseteq (f_1, \dots, f_r)^{m+1-r}$ .

*In particular, if  $(X, \Delta)$  has only log terminal singularities then*

$$\mathfrak{c}^m \subseteq (f_1, \dots, f_r)^{m+1-r}.$$

*Proof.* Part (i) follows from the surjectivity of the last map in the Skoda complex, and it implies inductively that  $\mathcal{J}_m = (f_1, \dots, f_r)^{m+1-r} \mathcal{J}_{r-1}$ . Thus, given (i), for part (ii) one uses Lemma 1.1 to conclude

$$\begin{aligned} \mathfrak{c}^m \cdot \mathcal{J}((X, \Delta); \mathcal{O}_X) &\subseteq \mathcal{J}((X, \Delta); \mathfrak{c}^m) \\ &= (f_1, \dots, f_r)^{m+1-r} \cdot \mathcal{J}((X, \Delta); \mathfrak{c}^{r-1}) \\ &\subseteq (f_1, \dots, f_r)^{m+1-r}. \end{aligned}$$

The last statement in the corollary follows from (ii) because  $\mathcal{J}((X, \Delta); \mathcal{O}_X) = \mathcal{O}_X$  when  $(X, \Delta)$  has log terminal singularities. □

REMARK 1.5. The inclusion  $\mathfrak{c}^m \subseteq (f_1, \dots, f_r)^{m+1-r}$  in the last statement of the corollary holds more generally on any variety with only rational singularities; this follows from Lipman and Tessier’s form of the Briançon–Skoda theorem [11, Thm. 2.1].

### 2. Proof of Theorem A

We now refine the arguments of [10] to prove Theorem A.

As in the statement, let  $(\mathcal{O}, \mathfrak{m})$  be the local ring of  $X$  at the Cohen–Macaulay point  $x \in X$ . Let

$$\mathfrak{c} = (z_1, \dots, z_d) \subseteq \mathcal{O}$$

denote the ideal generated by the given system of parameters, and write

$$\mathcal{J}(\mathfrak{c}^m \cdot \mathfrak{b}^\lambda) = \mathcal{J}((X, \Delta); \mathfrak{c}^m \cdot \mathfrak{b}^\lambda)_x \subseteq \mathcal{O}$$

for the germ at  $x$  of the indicated multiplier ideal.

We claim to begin with that the map

$$\mathrm{Tor}_1(\mathfrak{c}^{d-1} \cdot \mathcal{J}, \mathcal{O}/\mathfrak{c}) \longrightarrow \mathrm{Tor}_1(\mathcal{J}, \mathcal{O}/\mathfrak{c}) \tag{*}$$

vanishes. This follows by observing that [10, Thm. B] remains valid in our setting, but it is more instructive to write out the argument explicitly. In fact, since  $(\mathcal{O}, \mathfrak{m})$  is Cohen–Macaulay and since  $\mathfrak{c}$  is generated by a regular sequence, we may compute each Tor in question via the Koszul complex  $K_*(z_1, \dots, z_d)$  associated to  $z_1, \dots, z_d$ . That being said, consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{O}^{\binom{d}{2}} \otimes \mathfrak{c}^{d-1} \mathcal{J}(\mathfrak{b}^\lambda) & \longrightarrow & \mathcal{O}^d \otimes \mathfrak{c}^{d-1} \mathcal{J}(\mathfrak{b}^\lambda) & \longrightarrow & \mathcal{O} \otimes \mathfrak{c}^d \mathcal{J}(\mathfrak{b}^\lambda) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}^{\binom{d}{2}} \otimes \mathcal{J}(\mathfrak{c}^{d-2} \mathfrak{b}^\lambda) & \longrightarrow & \mathcal{O}^d \otimes \mathcal{J}(\mathfrak{c}^{d-1} \mathfrak{b}^\lambda) & \longrightarrow & \mathcal{O} \otimes \mathcal{J}(\mathfrak{c}^d \mathfrak{b}^\lambda) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}^{\binom{d}{2}} \otimes \mathcal{J}(\mathfrak{b}^\lambda) & \longrightarrow & \mathcal{O}^d \otimes \mathcal{J}(\mathfrak{b}^\lambda) & \longrightarrow & \mathcal{O} \otimes \mathcal{J}(\mathfrak{b}^\lambda). \end{array}$$

The top and bottom rows arise from the Koszul complex (except that we have harmlessly modified the upper term on the right), and the middle row is part of the Skoda complex. The inclusion of the top into the middle row comes from Lemma 1.1.

The groups in (\*) are computed respectively as the homology of the first and third rows in the diagram, with the map arising from the inclusion of one in the other. By Theorem 1.3, the middle row of the diagram is exact. Hence the map in (\*) is zero, as required.

Following the idea of [10, Prop. 2.1], we now deduce Theorem A from (\*). Let  $\mathcal{J} = \mathcal{J}(\mathfrak{b}^\lambda) \subseteq \mathcal{O}$ , and consider a minimal free resolution  $F_\bullet$  of  $\mathcal{J}$ :

$$\dots \xrightarrow{u_3} F_2 \xrightarrow{u_2} F_1 \xrightarrow{u_1} F_0 \xrightarrow{\pi} \mathcal{J} \longrightarrow 0, \tag{1}$$

where  $F_i = \mathcal{O}^{b_i}$ . Assume for a contradiction that the statement of the theorem fails. Then there is a minimal generator  $e \in F_1$  such that  $u_1(e) \in (z_1, \dots, z_d)^d F_0$ . In particular,  $e$  lies in the kernel of the induced map

$$F_1 \otimes \mathcal{O}/\mathfrak{c} \xrightarrow{u_1 \otimes 1} F_0 \otimes \mathcal{O}/\mathfrak{c}$$

and so represents a class

$$\bar{e} \in \text{Tor}_1(\mathcal{J}, \mathcal{O}/\mathfrak{c}) = H_1(F_* \otimes \mathcal{O}/\mathfrak{c}).$$

Furthermore, since  $e$  is *minimal* generator of  $F_1$ , we have  $e \notin \mathfrak{m}F_1$  and hence  $e \notin \text{im}(u_2)$ ; this ensures that the class  $\bar{e}$  represents in Tor is nonzero. To complete the proof of Theorem A, we will show that  $\bar{e}$  lies in the image of the natural map

$$\bar{e} \in \text{im}(\text{Tor}_1(\mathfrak{c}^{d-1}\mathcal{J}, \mathcal{O}/\mathfrak{c}) \longrightarrow \text{Tor}_1(\mathcal{J}, \mathcal{O}/\mathfrak{c})), \tag{**}$$

which contradicts (\*).

For (\*\*), the plan is to explicate the representation of  $e$  as a Koszul cohomology class. Toward this end, let  $h_1, \dots, h_r$  be minimal generators of  $\mathcal{J}$  and let  $g_1, \dots, g_r \in \mathfrak{m}$  be the coefficients of the minimal syzygy represented by  $e \in F_1$ , so that  $\sum g_i h_i = 0$ . By assumption,

$$g_i \in \mathfrak{c}^d = (z_1, \dots, z_d)^d.$$

Now write

$$g_i = z_1 g_{i1} + \dots + z_d g_{id},$$

where each  $g_{ij} \in (z_1, \dots, z_d)^{d-1}$ , and for  $j = 1, \dots, d$  put

$$G_j = h_1 g_{1j} + \dots + h_r g_{rj}.$$

Then  $G_j \in (z_1, \dots, z_d)^{d-1}\mathcal{J}$ . Furthermore, the  $G_j$  give a Koszul relation on the  $z_j$ , that is,

$$z_1 G_1 + \dots + z_d G_d = 0,$$

and so they represent a first cohomology class of the complex  $(z_1, \dots, z_d)^{d-1}\mathcal{J} \otimes K_*(z_1, \dots, z_d)$ ; here, as before,  $K_*(z_1, \dots, z_d)$  denotes the Koszul complex on the  $z_j$ . In other words,  $(G_1, \dots, G_d)$  represents an element

$$\eta \in \text{Tor}_1(\mathfrak{c}^{d-1}\mathcal{J}, \mathcal{O}/\mathfrak{c}).$$

It is not hard to check that the image of  $\eta$  under the natural map to  $\text{Tor}_1(\mathcal{J}, \mathcal{O}/\mathfrak{c})$  is precisely the class  $\bar{e}$ . In other words,  $\bar{e}$  lies in the image of the map in (\*\*), as required.

REMARK 2.1. The “lifting” argument of [10, Prop. 1.1] cannot be carried out in the singular case for  $p$ th syzygies when  $p \geq 2$  because the entries of the matrices defining the maps in the minimal free resolution of  $J$  can be assumed only in the maximal ideal  $\mathfrak{m}$ , not in  $(z_1, \dots, z_d)$ . However, if we happen to know that an ideal  $J$  has a minimal free resolution in which the entries of the matrices describing all

the maps  $\mu_i$  for  $i < p$  lie in the ideal  $(z_1, \dots, z_d)$ , then we can carry out the same “zigzag” argument as in [10, Prop. 1.1] to deduce that no minimal  $p$ th syzygy of  $J$  is in  $(z_1, \dots, z_d)^{d+1-p}$ . This will be the case, for example, when ideals  $J$  are generated by a regular sequence of elements vanishing to high order at  $x$ .

### 3. Corollaries and Examples

We start with an example to show that the results of [10] do not extend without change to the singular case.

EXAMPLE 3.1. Let  $\mathcal{O}$  be the local ring at the origin of the hypersurface in  $\mathbf{C}^3$  defined by the equation

$$x^n + y^n + z^n = 0,$$

where  $n \geq 3$ . Blowing up the singular point yields a log resolution, and it is easy to compute that the multiplier ideal of the trivial ideal is precisely  $\tau = (x, y, z)^{n-2}$ . This multiplier ideal has a minimal syzygy vanishing to order 2: the elements  $x^2, y^2, z^2$  give a minimal syzygy on the (subset of the) minimal generators  $x^{n-2}, y^{n-2}, z^{n-2}$  of  $\tau$ . This shows that, in the singular case, Theorem A of [10] does not hold as stated, which would rule out a minimal syzygy vanishing modulo  $(x, y, z)^2$ . In contrast, as our Theorem A here predicts, this syzygy does not vanish modulo the square of the ideal generated by two of the coordinate functions.  $\square$

We next use results of Briançon–Skoda type to give statements involving the multiplier ideal of  $\mathcal{O}_X$ . For a normal complex variety  $X$  of dimension  $d$ , define an ideal  $\sigma(X) \subseteq \mathcal{O}_X$  by setting

$$\sigma(X) = \sum_{\Delta} \mathcal{J}((X, \Delta); \mathcal{O}_X), \tag{2}$$

where the summation is taken over all effective  $\mathbf{Q}$ -divisors  $\Delta$  such that  $K_X + \Delta$  is  $\mathbf{Q}$ -Cartier.

COROLLARY 3.2. *If  $x \in X$  is a Cohen–Macaulay point and if*

$$\mathcal{J} = \mathcal{J}((X, \Delta); \mathfrak{b}^\lambda)_x \subseteq \mathcal{O} = \mathcal{O}_{x,X}$$

*is the germ at  $x$  of any multiplier ideal, then no minimal first syzygy of  $\mathcal{J}$  can vanish modulo  $\sigma(X) \cdot \mathfrak{m}^{2d-1}$ . In other words, if  $g_1, \dots, g_r$  are the coefficients of a minimal syzygy on minimal generators  $h_1, \dots, h_r \in \mathcal{J}$ , then*

$$g_i \notin \sigma(X) \cdot \mathfrak{m}^{2d-1}$$

*for at least one index  $i$ .*

COROLLARY 3.3. *If  $X$  supports a  $\mathbf{Q}$ -divisor  $\Delta_0$  such that  $(X, \Delta_0)$  is KLT, then no first syzygy of any multiplier ideal  $\mathcal{J}$  can vanish modulo  $\mathfrak{m}^{2d-1}$ .*

*Proof of Corollary 3.2.* Let  $z_1, \dots, z_d$  be a system of parameters at  $x$  generating a reduction of the maximal ideal  $\mathfrak{m} \subseteq \mathcal{O}$ . It follows from Corollary 1.4(ii) that

$$\sigma(X) \cdot \mathfrak{m}^{2d-1} \subseteq (z_1, \dots, z_d)^d.$$

The assertion then follows from Theorem A. □

REMARK 3.4. Using the result of Lipman and Teissier [11] quoted in Remark 1.5, a similar argument shows that the conclusion of Corollary 3.3 holds at any Cohen–Macaulay point of a  $\mathbf{Q}$ -Gorenstein variety with only rational singularities.

EXAMPLE 3.5. Let  $R$  be the local ring at the vertex of the affine cone over a smooth projective hypersurface of degree  $n$  in projective  $d$ -space. Then  $R$  is a  $d$ -dimensional Gorenstein ring with multiplier ideal  $\tau = \mathfrak{m}^{n-d}$ . According to Corollary 3.2, no minimal syzygy of any multiplier ideal can vanish to order  $n - d + (2d - 1) = n + d - 1$ . Note that, since every ideal is contained in the unit ideal, it follows that every multiplier ideal is contained in the multiplier ideal  $\tau = \mathfrak{m}^{n-d}$  of the trivial ideal.

REMARK 3.6. If  $X$  is  $\mathbf{Q}$ -Gorenstein then  $\sigma(X) = \mathcal{J}(X, \mathcal{O}_X)$ , since in this case one can take  $\Delta = 0$  in the sum defining  $\sigma(X)$ . It is known in this setting that  $\mathcal{J}(X, \mathcal{O}_X)$  reduces modulo  $p \gg 0$  to the test ideal  $\tau(X)$  of  $X$  defined using tight closure (see [4; 17]). It would be interesting to know whether there is an analogous interpretation of the ideal  $\sigma(X)$  on an arbitrary normal variety  $X$ . In this connection, observe from Corollary 1.4 that if  $f_1, \dots, f_r \in \mathfrak{m}$  are functions generating a reduction of an ideal  $\mathfrak{c}$ , then

$$\sigma(X) \cdot \mathfrak{c}^m \subseteq (f_1, \dots, f_r)^{m+1-r};$$

in characteristic  $p > 0$ , the analogous formula holds with  $\sigma(X)$  replaced by  $\tau(X)$ .

Our remaining applications make more systematic use of the connection with tight closure just alluded to. Let  $X$  be a  $\mathbf{Q}$ -Gorenstein variety of dimension  $d$ , let  $x \in X$  be a Cohen–Macaulay point, and set

$$\tau = \mathcal{J}(X, \mathcal{O}_X)_x \subseteq \mathcal{O} = \mathcal{O}_{x, X}.$$

In the  $\mathbf{Q}$ -Gorenstein setting, Corollary 3.2 asserts that no minimal first syzygy of a multiplier ideal can vanish modulo  $\tau \cdot \mathfrak{m}^{2d-1}$ . However, according to [17, Thm. 3.1] or [4], the ideal  $\tau$  is a *universal test ideal* for  $\mathcal{O}$  in the sense of tight closure. Roughly speaking, this means that after reducing modulo  $p$  for  $p \gg 0$  the ideal  $\tau$  becomes the test ideal for the corresponding ring  $\mathcal{O}$  modulo  $p$ ; that is, the elements of  $\tau$  multiply the tight closure of any ideal  $I$  back into the ideal  $I$ . (For precise statements we refer to the main theorems of either [17] or [4].) We can now deduce some statements in characteristic 0 by reducing modulo  $p$  and invoking facts from tight closure, as follows.

COROLLARY 3.7. *Assume that  $X$  is  $\mathbf{Q}$ -Gorenstein and let  $J \subset \mathcal{O}$  denote the Jacobian ideal of  $\mathcal{O}$  with respect to some local embedding in a smooth variety. If*



$f = (f_1, \dots, f_r)$  is a minimal (first) syzygy of some multiplier ideal  $\mathcal{J}$ , then some  $f_i$  fails to be in the ideal  $J\mathfrak{m}^{2d-1}$ .

*Proof.* Keeping in mind our remarks from the preceding paragraph, it suffices to show that the Jacobian ideal  $J$  is contained in the multiplier ideal  $\tau$ . Indeed, by [6, Thm. 3.4] we know that, in prime characteristic, the Jacobian ideal is contained in the test ideal; this means that, in characteristic 0, the Jacobian ideal must be contained in the multiplier ideal of the unit ideal by [17, Thm. 3.1]. (Related results comparing multiplier ideals and Jacobian ideals can be found in [1, Sec. 4].)  $\square$

REMARK 3.8. One can replace the multiplier ideal  $\tau = \mathcal{J}(X, \mathcal{O}_X)$  in this discussion by any ideal  $\tau'$  with the property that, after reducing modulo  $p$  for  $p \gg 0$ ,  $\tau'$  is contained in the *parameter test ideal* for the corresponding prime characteristic ring. The point is that the parameter test ideal will multiply the tight closure of any ideal  $I$  generated by monomials in a system of parameters back into  $I$ , so that the equation

$$\tau' \mathfrak{m}^{2d-1} \subseteq (z_1, \dots, z_d)^d$$

will hold for such  $\tau'$ . One could, for example, replace  $\sigma(X)$  in the statement of Corollary 3.2 by a *universal parameter test ideal* if one is known to exist. For instance, when  $\mathcal{O}$  is rationally singular, the universal parameter test ideal exists and is the unit ideal; this is essentially the well-known statement that rationally singular rings correspond, after reduction modulo  $p$  for  $p \gg 0$ , to rings in which all parameter ideals are tightly closed (see the main theorems in [16] and [3] or [13]).

EXAMPLE 3.9. Using Remark 3.8, we can generalize Example 3.5 as follows. Let  $x$  be the vertex of the cone over any rationally singular projective variety  $Y$  with respect to any ample invertible sheaf  $\mathcal{L}$ . In other words, the local ring  $\mathcal{O}$  at  $x$  is obtained by localizing the section ring of  $Y$  with respect to  $\mathcal{L}$  at its unique homogeneous maximal ideal  $\mathfrak{m}$ . Assume that  $\mathcal{O}$  is Cohen–Macaulay and  $\mathbf{Q}$ -Gorenstein (it is always normal), and let  $d$  be its dimension. Let  $a$  be the  $a$ -invariant of  $\mathcal{O}$ ; that is, let  $a$  be the largest integer  $n$  such that the graded module  $\omega_{\mathcal{O}}$  is nonzero in degree  $-n$  (or, alternatively, such that  $\omega_X \otimes \mathcal{L}^{-n}$  has a nonzero global section). Then no minimal first syzygy of  $\mathcal{J}$  can vanish to order  $a + 2d$  at the vertex of the cone.

The point in this case is that, after reducing modulo  $p \gg 0$ , the parameter test ideal includes all elements of degree  $> a$ . In particular,  $\mathfrak{m}^{a+1}$  is contained in the parameter test ideal and so we can apply Remark 3.8.

To see that every element of degree greater than the  $a$ -invariant is contained in the parameter test ideal, recall first that the parameter test ideal of  $\mathcal{O}$  is the annihilator of the tight closure of the zero module in  $H_{\{x\}}^d(\mathcal{O})$  [15, Prop. 4.4]. On the other hand, for a section ring over a rationally singular variety, the tight closure of zero in  $H_{\{x\}}^d(\mathcal{O})$  is precisely the submodule of  $H_{\{x\}}^d(\mathcal{O})$  of nonnegatively graded elements. Since  $H_{\{x\}}^d(\mathcal{O})$  vanishes in degree  $> a$ , it follows that every element of degree  $> a$  annihilates the required tight closure module; see also [8].

EXAMPLE 3.10. The case of a standard graded algebra gives a user-friendly special case of Example 3.9. Let  $R$  be a normal Cohen–Macaulay  $\mathbf{Q}$ -Gorenstein  $\mathbf{N}$ -graded domain generated by its degree-1 elements over its degree-0 part  $\mathbf{C}$ . Assume also that  $R$  has isolated nonrational singularities. Then the first minimal syzygies of every multiplier ideal in  $R$  have degree  $< 2d + a$ , where  $d$  is the dimension of  $R$  and  $a$  is the  $a$ -invariant of  $R$ . (The statement holds even in the nonhomogeneous case, where by “degree” we mean the degree of the smallest degree component of the syzygy.)

EXAMPLE 3.11. In the smooth two-dimensional case, every integrally closed ideal is a multiplier ideal by a theorem of [12] or [2]. As in [10], Theorem A easily implies the existence of integrally closed ideals in dimension  $\geq 3$  that are not multiplier ideals in the singular case also. For example, if  $X$  is a normal Cohen–Macaulay  $\mathbf{Q}$ -Gorenstein variety of dimension  $\geq 3$ , then we can find plenty of regular sequences  $f_1, \dots, f_r$ —where  $r$  is strictly smaller than the dimension—contained in an arbitrarily high power some minimal reduction of  $(z_1, \dots, z_d)$ , of  $\mathfrak{m}$ . For general such  $f_i$ , the ideal  $I$  they generate is radical and hence integrally closed. On the other hand, the Koszul syzygies on the  $f_i$  violate Theorem A. If we want an  $\mathfrak{m}$ -primary example then we can work in the graded case and add a large power of  $\mathfrak{m}$  to  $I$  as in [10, Lemma 2.1]. (The proof of this lemma is given in [10] for polynomial rings, but it works for any graded ring.)

Finally, it would be interesting to know the answer to the following question.

QUESTION 3.12. If  $R$  is a two-dimensional  $\mathbf{Q}$ -Gorenstein rational singular ring essentially of finite type over  $\mathbf{C}$ , then is every integrally closed ideal a multiplier ideal?

The first point to consider would be whether the conclusion of Corollary 3.3 automatically holds in such rings.

## References

- [1] L. Ein, R. Lazarsfeld, K. E. Smith, and D. Varolin, *Jumping coefficients of multiplier ideals*, Duke Math. J. 123 (2004), 469–506.
- [2] C. Favre and M. Jonsson, *Valuations and multiplier ideals*, J. Amer. Math. Soc. 18 (2005), pp. 655–684.
- [3] N. Hara, *A characterization of rational singularities in terms of injectivity of Frobenius maps*, Amer. J. Math. 120 (1998), 981–996.
- [4] ———, *Geometric interpretation of tight closure and test ideals*, Trans. Amer. Math. Soc. 353 (2001), 1885–1906.
- [5] M. Hochster and C. Huneke, *Tight closure, invariant theory, and the Briançon–Skoda theorem*, J. Amer. Math. Soc. 3 (1990), 31–116.
- [6] ———, *Comparison of symbolic and ordinary powers of ideals*, Invent. Math. 147 (2002), 349–369.
- [7] ———, *Tight closure in equal characteristic zero with an introduction to the characteristic  $p$  theory*, Springer-Verlag, New York, 2007.

- [8] C. Huneke and K. E. Smith, *Tight closure and the Kodaira vanishing theorem*, J. Reine Angew. Math. 484 (1997), 127–152.
- [9] R. Lazarsfeld, *Positivity in algebraic geometry, I. & II.* Ergeb. Math. Grenzgeb. (3), 48 & 49, Springer-Verlag, Berlin, 2004.
- [10] R. Lazarsfeld and K. Lee, *Local syzygies of multiplier ideals*, Invent. Math. 167 (2007) 409–418.
- [11] J. Lipman and B. Tesser, *Pseudorational local rings and a theorem of Briançon–Skoda about integral closures of ideals*, Michigan Math. J. 28 (1981), 97–116.
- [12] J. Lipman and K. Watanabe, *Integrally closed ideals in two-dimensional regular local rings are multiplier ideals*, Math. Res. Lett. 10 (2003), 423–434.
- [13] V. B. Mehta and V. Srinivas, *A characterization of rational singularities*, Asian J. Math. 1 (1997), 249–271.
- [14] M. Scott Osborne, *Basic homological algebra*, Grad. Texts in Math., 196, Springer-Verlag, New York, 2000.
- [15] K. E. Smith, *Test ideals in local rings*, Trans. Amer. Math. Soc. 347 (1995), 3453–3472.
- [16] ———, *F-rational rings have rational singularities*, Amer. J. Math. 119 (1997), 159–180.
- [17] ———, *The multiplier ideal is a universal test ideal*, Comm. Algebra 28 (2000), 5915–5929.

R. Lazarsfeld  
Department of Mathematics  
University of Michigan  
Ann Arbor, MI 48109  
rlaz@umich.edu

K. Lee  
Department of Mathematics  
University of Michigan  
Ann Arbor, MI 48109  
kyungl@umich.edu

K. E. Smith  
Department of Mathematics  
University of Michigan  
Ann Arbor, MI 48109  
kesmith@umich.edu