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Frobenius Splitting of Certain Rings of Invariants

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Dedicated to Professor Melvin Hochster on the occasion of his sixty-fifth birthday

1. Introduction

The concept of F-purity was introduced by Hochster and Roberts [6]; the F-purity for a Noetherian ring of prime characteristic is equivalent to the splitting of the Frobenius map when the ring is finitely generated over its subring of pth powers. It is closely related to the Frobenius splitting property à la Mehta and Ramanathan [11] for algebraic varieties; more precisely, the F-split property for an irreducible projective variety X over an algebraically closed field of positive characteristic is equivalent to the F-purity of the ring $\bigoplus_{n\geq 0} H^0(X; L^n)$ for any ample line bundle L over X (cf. [3; 13; 14]). We feel that it is only appropriate to dedicate this paper to Professor Hochster on the occasion of his sixty-fifth birthday and thus make a modest contribution to this birthday volume.

Let k be an algebraically closed field of characteristic $p > 0$ and let X be a k-scheme. One has the Frobenius morphism (which is only an $\mathbb{F}_p$-morphism) $F: X \to X$ defined as the identity map of the underlying topological space of X, where the morphism of structure sheaves $F^{\#}: \mathcal{O}_X \to \mathcal{O}_X$ is the pth power map. The morphism F induces a morphism of $\mathcal{O}_X$-modules $\mathcal{O}_X \to F_*\mathcal{O}_X$. The variety X is called *Frobenius split* (or *F-split,* or simply *split*) if there exists a splitting $\varphi: F_*\mathcal{O}_X \to \mathcal{O}_X$ of the morphism $\mathcal{O}_X \to F_*\mathcal{O}_X$. Equivalently, X is Frobenius split if there exists a morphism of sheaves of abelian groups $\varphi: \mathcal{O}_X \to \mathcal{O}_X$ such that (i) $\varphi(f^p g) = f\varphi(g)$ with $f, g \in \mathcal{O}_X$ and (ii) $\varphi(1) = 1$. Basic examples of varieties that are Frobenius split are smooth affine varieties, toric varieties (cf. [1]), generalized flag varieties, and Schubert varieties [11]. Smooth projective curves of genus > 1 are examples of varieties that are *not* Frobenius split.

Frobenius splitting is an interesting property to study. If X is Frobenius split, then it is weakly normal [1, Prop. 1.2.5] and reduced [1, Prop. 1.2.1]. Indeed, projective varieties that are Frobenius split are very special. We refer the reader to [1] for further details.

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If $X = \mathrm{Spec}(R)$, then X is Frobenius split if and only if the Frobenius homomorphism $R \to R$ defined as $a \mapsto a^p$ admits a splitting $\varphi\colon R \to R$ such that $\varphi(a^p b) = a\varphi(b)$ and $\varphi(1) = 1$.

If a linearly reductive group G acts on a k-algebra R that is Frobenius split, then the invariant ring R^G is Frobenius split (see [1, Exer. 1.1.E(5)]). To quote Smith [13, p. 571], "The story of F-splitting and global F-regularity for quotients by reductive groups in characteristics p that are not linearly reductive is much more subtle and complicated." We intend to show that, although the groups $\mathrm{SO}(n)$, $n \geq 3$, and $\mathrm{SL}(n)$, $n \geq 2$, are not linearly reductive, certain rings of invariants for these groups *are* Frobenius split.

We next state the main results of this paper.

Let k be an algebraically closed field of characteristic $p > 2$, and let V be an n-dimensional vector space over k with a symmetric nondegenerate bilinear form. Denote by A_m the the coordinate ring of $V_m := V^{\oplus m}$, $m \geq 1$, and consider the action of $\mathrm{SO}(V)$ on A_m induced by the diagonal action of $\mathrm{SO}(V)$ on $V^{\oplus m}$.

THEOREM 1.1. *The invariant ring $A_m^{\mathrm{SO}(V)}$ is Frobenius split for all $m \geq 1$.*

The group $\mathrm{SL}(V)$ acts on V as well as on the dual vector space $V^* = \mathrm{Hom}_k(V, k)$. Now consider the diagonal action of $\mathrm{SL}(V)$ on the vector space

$$V_{m,q} := V^{\oplus m} \oplus V^{*\oplus q} \quad \text{for } m, q \geq 1.$$

This leads to an action of $\mathrm{SL}(V)$ on the coordinate ring $A_{m,q}$ of $V_{m,q}$.

THEOREM 1.2. *The invariant ring $A_{m,q}^{\mathrm{SL}(V)}$ is Frobenius split for any $m, q \geq n$.*

We shall now sketch the proofs of main results (assuming $m, q > n$). Let S be the invariant ring in question, and let R be the ring of invariants under the larger group $\tilde{G} = \mathrm{GL}(V)$ (resp. $\tilde{G} = \mathrm{O}(V)$). Then (cf. [2; 8]) R is the coordinate ring of a certain determinantal variety in $M_{m,q}$, the space of $m \times q$ matrices (resp. $\mathrm{Sym}\, M_m$, the space of symmetric $m \times m$ matrices) with entries in k. Now a determinantal variety in $M_{m,q}$ (resp. $\mathrm{Sym}\, M_m$) can be canonically identified (cf. [8]) with an open subset in a certain Schubert variety in $G_{q,m+q}$, the Grassmannian variety of q-dimensional subspaces of k^{m+q} (resp. the symplectic Grassmannian variety, the variety of all maximal isotropic subspaces of a $2m$-dimensional vector space over k endowed with a nondegenerate skew-symmetric bilinear form). Hence we obtain that R is Frobenius split (since Schubert varieties are Frobenius split).

Let $X = \mathrm{Spec}(S)$ and $Y = \mathrm{Spec}(R)$, and let $\pi : X \to Y$ be the morphism induced by the inclusion $R \subset S$. When $G = \mathrm{SO}(V)$, we show that π is a double cover that is étale over a "large" open subvariety—that is, a subvariety whose complement has codimension ≥ 2. For $G = \mathrm{SL}(n)$ we show that, when restricted to a large open subvariety, π is a $\mathbb{G}_m$ bundle. The main theorems are then deduced using normality of S.

Hashimoto [4] has shown that, if a connected reductive group G acts on a polynomial ring A over k (of positive characteristic) with good filtration, then the ring

A^G of invariants is strongly F-regular, a property that is closely related to Frobenius splitting. Granting the results of [9] and [7]—we don't need all the results of these papers, only some of the relatively easier ones—the arguments used in our proofs are straightforward and quite elementary; the techniques used in [4] are representation theoretic.

Theorem 1.1 will be proved in Section 2 and Theorem 1.2 in Section 3.

2. Splitting SO(n)-Invariants

Let k be an algebraically closed field of characteristic $p > 0$. Suppose S is an affine k-algebra that is Frobenius split, and suppose a finite group Γ acts on S as k-algebra automorphisms. Then the invariant ring $R = S^\Gamma$ is Frobenius split provided the order of Γ is not divisible by p [1, Exer. 1.1.E(5)]. We first obtain a partial converse to this statement when Γ is of order 2.

Assume that $\mathrm{char}(k) > 2$. Let S be an affine k-domain and let $\Gamma = \{1, \gamma\} \cong \mathbb{Z}/2\mathbb{Z}$ act effectively on S. Denote by R the invariant subalgebra S^Γ. Then R is an affine k-algebra and S is quadratic and integral over R. Indeed, any $b \in S$ can be expressed as $b = b_0 + b_1$, where $b_0 = (1/2)(b + \gamma(b)) \in R$ and $b_1 = (1/2)(b - \gamma(b))$ satisfies $\gamma(b_1) = -b_1$. Thus, we can choose generators $u_1, \ldots, u_s$ for the R-algebra S to be in the -1 eigenspace of γ. Clearly $u_i^2 = -u_i\gamma(u_i) =: p_i \in R$ for all $i \leq s$. Furthermore,

$$\gamma(u_iu_j) = u_iu_j =: p_{i,j} \in R \quad \text{for all } \; i, j \leq s \quad (\text{with } p_{i,i} = p_i).$$

Observe that $p_{i,j}^2 = p_ip_j$.

We shall assume that S is reduced so that $p_i \neq 0$ for all i. Now let $R_i = R[1/p_i]$ for $1 \leq i \leq n$ and let $S_i = S[1/u_i]$. We claim that $S_i = R_i[u_i]$. To see this, first observe that $R_i[u_i] \subset S[1/u_i]$, since $1/p_i = (1/u_i)^2 \in S[1/u_i]$. To show that $S[1/u_i] \subset R_i[u_i]$, it suffices to show that $u_j \in R_i[u_i]$ for all j and that $(1/u_i) \in R_i[u_i]$. Indeed, $1/u_i = u_i/u_i^2 = u_i/p_i \in R_i[u_i]$ and so $u_j = p_{i,j}/u_i \in R_i[u_i]$.

Write $X = \mathrm{Spec}(S)$ and $Y = \mathrm{Spec}(R)$, and let $\pi : X \to Y$ be the morphism (induced by the inclusion $R \subset S$). As before, let $S_i = S[1/u_i]$ and let $U_i = \mathrm{Spec}(S_i) \subset X$. Let $U := \bigcup_{1 \leq i \leq s} U_i$, which is the full inverse image under π of $\bigcup_{1 \leq i \leq s} \mathrm{Spec}(R_i)$. It is readily verified that $\pi|_U : U \to \pi(U)$ is étale. Indeed, S_i is a free R_i module with basis $\{1, u_i\}$ and with $\det(u_i) = -p_i \neq 0$, so $\pi|_{U_i}$ is étale and hence $\pi|_U$ is étale.

On the other hand, if $y \in Y$ is a closed point such that $p_i(y) = 0$ for all $i \leq s$, then the fibre $f^{-1}(y) = \mathrm{Spec}(S_y \otimes_{R_y} k)$ is the scheme whose coordinate ring is $S_y \otimes_{R_y} k = k[u_1, \ldots, u_s]/\langle u_i^2, 1 \leq i \leq s\rangle$. Here R_y is the local ring at y. Thus $f^{-1}(y)$ is nonreduced. It follows that the ramification locus of π is *equal* to $Y \setminus \pi(U)$ (see [12, Sec. III.10, Thm. 3]).

PROPOSITION 2.1. *Let k be an algebraically closed field of characteristic $p > 2$. Let S be an affine normal domain over k acted on by $\Gamma \cong \mathbb{Z}/2\mathbb{Z}$, and let $R := S^\Gamma$ be Frobenius split. Suppose that the ramification locus of the double cover*

$\pi : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ *has codimension at least* 2 *in* $\mathrm{Spec}(R)$. *Then, any splitting* $\varphi\colon R \to R$ *extends uniquely to a splitting* $\psi : S \to S$.

Proof. We use the notation introduced previously.

Since X is normal and since the codimension of the ramification locus of π is at least 2, it suffices to show that $U = \bigcup_{1\le i\le s} U_i$ is Frobenius split (cf. [1, Lemma 1.1.7(iii)]).

Let $\varphi\colon R \to R$ be a splitting of $Y = \mathrm{Spec}(R)$. First, we shall extend φ to a splitting $\psi_i : S_i \to S_i$ of $U_i = \mathrm{Spec}(S_i)$ ($= \mathrm{Spec}(R_i[u_i])$) for each i and verify that these splittings agree on the overlaps $U_i \cap U_j$ for $1 \le i, j \le s$. Thus we obtain a splitting of $U = \bigcup_{1\le i\le s} U_i$. By normality of X and the hypothesis on the codimension of the ramification locus, we will conclude that this splitting extends to a splitting of X. Next, we shall establish the uniqueness of the extension.

Recall that $\{1, u_i\}$ is an R_i-basis for S_i. Since $u_i = u_i^{-p} p_i^{(1+p)/2}$ on U_i it follows that, if $\psi_i : S_i \to S_i$ is *any* splitting of U_i that extends the splitting φ_i of R_i defined by φ, then we must have $\psi_i(au_i) = \psi_i((1/u_i)^p a p_i^{(p+1)/2}) = (1/u_i)\varphi_i(a p_i^{(p+1)/2})$. By additivity, then,

$$\psi_i(au_i + b) = (1/u_i)\varphi_i(a p_i^{(p+1)/2}) + \varphi_i(b) = (u_i/p_i)\varphi_i(a p_i^{(p+1)/2}) + \varphi_i(b),$$

where $a, b \in R_i$. Thus the extension ψ_i, if it exists, is unique.

We now *define* ψ_i by the preceding equation and claim that ψ_i is indeed a splitting of S_i. First, observe that $\psi_i(1) = 1$ by the very definition of ψ_i. Now, for any $x, y, a \in R_i$,

$$\begin{aligned}\psi_i((xu_i + y)^p a u_i) &= \psi_i(x^p p_i^{(p+1)/2} a + y^p a u_i)\\ &= x\varphi(p_i^{(p+1)/2} a) + y\varphi(au_i)\\ &= x(p_i/u_i)\psi_i(au_i) + y\varphi(au_i)\\ &= xu_i\psi_i(au_i) + y\psi(au_i)\\ &= (xu_i + y)\varphi(au_i).\end{aligned}$$

An entirely similar (and easier) computation shows that

$$\psi_i((xu_i + y)^p b) = (xu_i + y)\psi_i(b),$$

completing the verification that ψ_i is a splitting.

We verify by another straightforward computation that these ψ_i agree on the overlaps $U_i \cap U_j$. Indeed, writing $u_j = u_i p_{i,j}/p_i$, we have

$$\begin{aligned}\psi_i(u_j) = \psi_i(u_i p_{i,j}/p_i) &= (u_i/p_i)\varphi((p_{i,j}/p_i) p_i^{(p+1)/2})\\ &= (u_i/p_i)\varphi(p_{i,j} p_i^{(p-1)/2}).\end{aligned}$$

Since $p_i = p_{i,j}^2/p_j$ on $U_i \cap U_j$, it follows that

$$\begin{aligned}\varphi(p_{i,j} p_i^{(p-1)/2}) &= \varphi(p_{i,j}^p (p_i/p_{i,j}^2)^{(p-1)/2})\\ &= p_{i,j}\varphi(p_j^{(1-p)/2}) = (p_{i,j}/p_j)\varphi(p_j^{(p+1)/2}).\end{aligned}$$

Substituting in the previous expression for $\psi_i(u_j)$ yields

$$\psi_i(u_j) = (u_i p_{i,j}/(p_i p_j))\varphi(p_j^{(p+1)/2}) = (u_j/p_j)\varphi(p_j^{(p+1)/2}) = \psi_j(u_j).$$

This implies that the extensions $\{\psi_i\}$ patch to yield a well-defined splitting on U as claimed. As observed before, the normality of X and the hypothesis on the codimension of the ramification locus implies that this splitting extends to a *unique* splitting $\psi : S \to S$.

Finally, if ψ' is another splitting of X that also extends φ, then ψ' and ψ agree on U_i (for any i) as already observed. Because X is irreducible, U_i is dense in X and we conclude that $\psi' = \psi$. □

As a corollary, we obtain the following result.

THEOREM 2.2. *Let $\pi : X \to Y$ be a double cover of a Noetherian scheme whose ramification locus has codimension at least 2. Suppose that X is reduced, irreducible, and normal and that Y is Frobenius split. Then X is Frobenius split.*

Proof. Cover X by finitely many affine patches X_α. Let $Y_\alpha := \pi X_\alpha$. Then each $\pi|_{X_\alpha}$ satisfies the hypotheses of Proposition 2.1. Let φ be a splitting of Y, and let ψ_α be the unique splitting of X_α that extends the splitting $\varphi|_{Y_\alpha}$. The ψ_α agree on overlaps and hence define a unique splitting of X that "extends" φ. □

Proof of Theorem 1.1. Denote by S the ring of $\mathrm{SO}(V)$-invariants of A_m, where A_m is the coordinate ring of V_m. Let R be the ring of $\mathrm{O}(V)$-invariants.

We shall assume that $m > n$. By [2; 8], $Y := \mathrm{Spec}(R)$ is the determinantal variety $D_n(\mathrm{Sym}\, M_m)$ consisting of all matrices in $\mathrm{Sym}\, M_m$ (the space of symmetric $m \times m$ matrices with entries in k) of rank at most n; we also have (cf. [8]) an identification of $D_n(\mathrm{Sym}\, M_m)$ with an open subset of a certain Schubert variety in the Lagrangian Grassmannian variety (of all maximal isotropic subspaces of a $2m$-dimensional vector space over k endowed with a nondegenerate skew-symmetric bilinear form). Hence we obtain that Y is Frobenius split (since, by [11], Schubert varieties are Frobenius split).

Observe that $\Gamma := \mathrm{O}(n)/\mathrm{SO}(n) \cong \mathbb{Z}/2\mathbb{Z}$ acts on S (the subring of $\mathrm{SO}(V)$-invariants of A_m) and that S^Γ equals R. As before, let $X := \mathrm{Spec}(S)$ and let $\pi : X \to Y$ be the morphism induced by the inclusion $R \subset S$. We need only verify the hypotheses of Theorem 2.2. It is well known that S is an affine normal domain. It remains to verify that the codimension of the branch locus is at least 2. This was proved in [7]. In fact, the ramification locus of Y *equals* the singular locus of Y, but this more refined assertion is not relevant here. Since Y is normal it follows that the codimension of the ramification locus is at least 2. Thus, by Theorem 2.2, X is Frobenius split.

The case $m = n$ is treated separately as Lemma 2.3. When $m < n$, it is easy to see that $S = R$. Again, R is a polynomial algebra over k and hence S is Frobenius split. □

Assume that $m = n$. In this case $R = k[y_{i,j} : 1 \le i \le j \le n]$ is a polynomial ring, since R is the ring of polynomial functions on the space of $n \times n$ symmetric matrices. As an R-algebra, $S = R[u]/\langle u^2 - f \rangle$, where f denotes the determinant function of the symmetric $n \times n$ matrix whose entry in position (i, j) for $1 \leq i \leq j \leq n$ is $y_{i,j}$.

LEMMA 2.3. *Let $m = n$. The ring S of* SO(V)*-invariants is Frobenius split in this case, too.*

Proof. There is a natural identification of $\mathrm{Spec}(R)$ with an affine patch of the symplectic Grassmannian, and the vanishing locus of f under this identification becomes an open part of a Schubert variety [7; 8]. Thus, by [11] (see also [1]), there exists a splitting of $\mathrm{Spec}(R)$ that compatibly splits $\mathrm{Spec}(R/(f))$. Let φ be such a splitting. Continue to denote by φ the restriction of φ to the open part $\mathrm{Spec}(R[1/f])$. Arguing as in the proof of Proposition 2.1, we may "lift" the restriction φ to a splitting (also denoted φ) of $\mathrm{Spec}(S[1/f])$. We claim that φ maps S to S and hence extends to a splitting of $\mathrm{Spec}(S)$. Indeed, a general element of S is of the form $au + b$ with a, b in R, so

$$\varphi(au + b) = \varphi\left(\frac{au^{p+1}}{u^p} + b\right) = \frac{\varphi(af^{(p+1)/2})}{u} + \varphi(b).$$

Since φ compatibly splits the vanishing locus of f, it follows that $\varphi(af^{(p+1)/2})$ belongs to the ideal (f). Writing $\varphi(af^{(p+1)/2}) = cf$, we have

$$\varphi(au + b) = \frac{cf}{u} + \varphi(b) = cu + \varphi(b) \in S. \qquad \square$$

We conclude this section with the following remarks.

REMARKS 2.4. (i) The condition on codimension of U in Proposition 2.1 will be satisfied if S is generated over R by two or more elements u_i such that there exist u_i, u_j such that the supports D_i and D_j of the reduced scheme defined by $u_i = 0$ and $u_j = 0$ have no component in common.

(ii) Theorem 2.2 is not valid when the hypothesis on the codimension of the ramification locus is not satisfied. For example, if $\Gamma \cong \mathbb{Z}/2\mathbb{Z}$ is generated by the involution of a hyperelliptic curve X of genus $g \geq 2$, then the quotient is a smooth projective curve that is Frobenius split. However, X is not split because ω_X is ample but $H^1(X; \omega) \cong k$, whereas higher cohomologies for ample line bundles over Frobenius split projective varieties vanish.

(iii) We do not know if Theorem 2.2 remains valid if Γ is *any* finite group whose order is prime to the characteristic p of k, even in the case when Γ is cyclic.

(iv) One has a surjection of SL(2) onto SO(3) such that the SO(3) action on $V = k^3$ corresponds to the conjugation action of SL(2) on trace-0 2×2 matrices. For this case the Frobenius splitting property of A_m was proved by Mehta and Ramadas [10, Thm. 6]. It should be noted that, when $\dim V = 3$, the completion of the stalk at the origin in A_m is isomorphic to the completion of the stalk at the point corresponding to the class of the trivial rank-2 vector bundle in the moduli space of equivalence classes of semistable rank-2 vector bundles with trivial determinant on a smooth projective curve of genus $m > 2$ (see [10]).

3. Splitting SL(n)-Invariants

In this section we shall establish Theorem 1.2. Let V be an n-dimensional vector space over an algebraically closed field k of characteristic $p \geq 2$ and let V^* denote its dual. Let $V_{m,q} := V^{\oplus m} \oplus V^{*\oplus q}$, and let A denote the ring of regular functions on $V_{m,q}$. By fixing dual bases for V and V^*, we shall view elements of V and V^* as row and column vectors, respectively, so that $V^{\oplus m}$ (resp. $V^{*\oplus q}$) is identified with the space $M_{m,n}$ of $m \times n$ matrices (resp. the space $M_{n,q}$ of $n \times q$ matrices) over k; moreover, we shall identify $\mathrm{GL}(V)$ with $\mathrm{GL}_n(k)$, the group of invertible $n \times n$ matrices over k. In the sequel, we denote $\mathrm{GL}_n(k)$ simply by $\mathrm{GL}(n)$. Then the action of $\mathrm{GL}(V)$ on $V^{\oplus m}$ is identified with the multiplication on the right of $M_{m,n}$ by $\mathrm{GL}(n)$. Similarly, the action of $g \in \mathrm{GL}(V)$ on $V^{*\oplus q}$ is identified with the multiplication on the left of $M_{n,q}$ by g^{-1}. The diagonal action of $\mathrm{GL}(V)$ on $V^{\oplus m} \oplus V^{*\oplus}$ is therefore defined as $(u,\xi).g = (ug, g^{-1}\xi)$, where $g \in \mathrm{GL}(n)$ and $(u,\xi) \in \mathcal{V}_{m,q} := M_{m,n} \times M_{n,q}$. (Note that $\mathcal{V}_{m,q}$ is just the same as the vector space $V_{m,q}$ in matrix notation.) We identify A with the coordinate ring of $\mathcal{V}_{m,q}$.

We denote by R and S the rings of invariants $A^{\mathrm{GL}(n)}$ and $A^{\mathrm{SL}(n)}$, respectively. Let $Y = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(S)$. Note that Y and X are the GIT quotients $\mathcal{V}_{m,q}/\!\!/\mathrm{GL}(n)$ and $\mathcal{V}_{m,q}/\!\!/\mathrm{SL}(n)$, respectively.

Let $m, q \geq n$. By [2; 8] we have that Y is the determinantal variety $D_n(M_{m,q})$ consisting of all matrices in $M_{m,q}$ (the space of $m \times q$ matrices with entries in k) of rank at most n; further, we have (cf. [8]) an identification of $D_n(M_{m,q})$ with an open subset of a certain Schubert variety in the Grassmannian variety (of q-dimensional subspaces of k^{m+q}). Hence Y is Frobenius split (since Schubert varieties are Frobenius split). The multiplication map $\mu\colon \mathcal{V}_{m,q} \to M_{m,q}$ factors through Y; furthermore, under $\pi : X \to Y$ (induced by the inclusion $R \subset S$), we have $\pi([u,\xi]) = u\xi \in M_{m,q}$, where $[u,\xi] \in X$ is the image of $(u,\xi) \in \mathcal{V}_{m,q}$ under the GIT quotient $\mathcal{V}_{m,q} \to X$.

Let $I(n,m)$ denote the set of all n-element subsets I of $\{1,2,\ldots,m\}$. Any such I determines a regular function $u_I : \mathcal{V}_{m,q} \to k$ that maps (u,ξ) to the determinant of the $n \times n$ submatrix $u(I)$ of $u \in M_{m,n}$ with column entries given by I. Clearly u_I is invariant under the action of $\mathrm{SL}(n)$ on $\mathcal{V}_{m,q}$ and hence yields a regular function u_I on X.

We define $\xi(J)$ and ξ_J for $J \in I(n,q)$ analogously; ξ_J is also an $\mathrm{SL}(n)$-invariant. Then $u_I\xi_J =: p_{I,J} \in R$ for all $I \in I(n,m)$ and $J \in I(n,q)$; indeed, $p_{I,J}([u,\xi])$ is just the determinant of the $n \times n$ submatrix of $u\xi \in M_{m,q}$ with row and column indices given by I and J, respectively. It is shown in [9] that S is generated as an R-algebra by u_I, ξ_J with $I \in I(n,m)$ and $J \in I(n,q)$ and that the ideal of relations is generated by $u_I\xi_J - p_{I,J}$ with $I \in I(n,m)$ and $J \in I(n,q)$ together with certain quadratic relations among the u_I and certain quadratic relations among the ξ_J. Moreover, a standard monomial basis is constructed in [9] for S; as a particular consequence, we have that each u_I (resp. ξ_J) is algebraically independent over R for $I \in I(n,m)$ (resp. $J \in I(n,q)$).

For $K \in I(n,m)$ and $L \in I(n,q)$, let

$$R_{K,L} = R[1/p_{K,L}], \qquad Y_{K,L} = \mathrm{Spec}(R_{K,L}).$$

For a given $I \in I(n,m)$ and $J \in I(n,q)$, let

$$Y_I = \bigcup_{J' \in I(n,q)} Y_{I,J'}, \qquad Y_J = \bigcup_{I' \in I(n,m)} Y_{I',J}.$$

Observe that, for $I \in I(n,m)$, any $Y_{I,J'}$ is contained in Y_I; similarly, for $J \in I(n,q)$, any $Y_{I',J}$ is contained in Y_J.

Set $X_I = \pi^{-1}(Y_I) \subset X$ and $X_J = \pi^{-1}(Y_J) \subset X$. Note that u_I (resp. ξ_J) is nonzero on X_I (resp. X_J). Denote by $f_I \colon X_I \to Y_I \times k^*$ the morphism $f_I = (\pi|_{X_I}, u_I|_{X_I})$ and by $f_J \colon X_J \to Y_J \times k^*$ the morphism $f_J = (\pi|_{X_J}, \xi_J|_{X_J})$.

Lemma 3.1. *The morphisms $f_I \colon X_I \to Y_I \times k^*$ and $f_J \colon X_J \to Y_J \times k^*$ are isomorphisms for any $I \in I(n,m)$ and $J \in I(n,q)$.*

Proof. We shall prove that f_I is an isomorphism; the proof in the case of f_J is the same.

Let $X_{I,J} = \pi^{-1}(Y_{I,J})$; then $X_{I,J}$ equals $\operatorname{Spec}(S_{I,J})$ (where $S_{I,J} = S[1/p_{I,J}]$) and $X_{I,J}$ is contained in X_I. The morphism $f_{I,J} \colon X_{I,J} \to Y_{I,J}$ defined by the restriction of f_I is induced by the $R_{I,J}$-algebra map $f_{I,J}^* \colon R_{I,J}[t,t^{-1}] \to S_{I,J}$ that maps t to u_I. Note that $p_{I,J} = u_I \xi_J$ implies that u_I is invertible in $S_{I,J}$ $(= S[1/p_{I,J}])$.

We must show that:

(1) $f_{I,J}^*$ is an isomorphism of k-algebras; and
(2) $f_{I,J}$ and $f_{I,J'}$ agree on the overlap $X_{I,J} \cap X_{I,J'}$ for any two $J, J' \in I(n,q)$.

(1) Observe that $u_{I'} = u_I u_{I'} \xi_J / p_{I,J} = u_I p_{I',J}/p_{I,J} = f_{I,J}^*(p_{I',J}/p_{I,J} t)$. Hence $u_{I'}$ is in the image of $f_{I,J}^*$ for any $I' \in I(n,m)$. Similarly, $\xi_{J'}$ is also in the image of $f_{I,J}^*$ for any $J' \in I(n,q)$. Hence $f_{I,J}^*$ is surjective. Now suppose that $P(t) \in R_{I,J}[t,t^{-1}]$ is in the kernel of $f_{I,J}^*$. We may assume that $P(t)$ is a polynomial in t and that the coefficients of $P(t)$ are actually in R. Then $0 = f_{I,J}^*(P(t)) = P(u_I)$. Since $X_{I,J}$ is open in X, which is irreducible, it follows that $P(u_I) = 0$ must hold in S. This contradicts the algebraic independence of u_I over R (cf. [9, Thm. 6.06(3)]). Consequently, $f_{I,J}^*$ is an isomorphism.

(2) It is evident that $f_{I,J}^*(t) = u_I \in S_{I,J}$ and $f_{I,J'}^*(t) = u_I \in S_{I,J'}$ both restrict to the same regular function (namely, $u_I|_{X_{I,J} \cap X_{I,J'}}$) on the overlap $X_{I,J} \cap X_{I,J'} = \operatorname{Spec}(S[1/p_{I,J}, 1/p_{I,J'}])$. It follows that $f_{I,J}$ and $f_{I,J'}$ agree on $X_{I,J} \cap X_{I,J'}$. This completes the proof that f_I is an isomorphism. □

We remark that if $J, J' \in I(n,q)$ then $\xi_J/\xi_{J'} \in S[1/\xi_{J'}]$ defines a regular function on $Y_{J'}$. This is because $\xi_J/\xi_{J'} = (u_I\xi_J)/(u_I\xi_{J'}) = p_{I,J}/p_{I,J'}$ on $Y_{I,J'}$. It is immediately seen that, on $Y_{I,J'} \cap Y_{I',J'}$, the two regular functions $p_{I,J}/p_{I,J'}$ and $p_{I',J}/p_{I',J'}$ agree. Therefore we conclude that $\xi_J/\xi_{J'}$ is a well-defined regular function on $Y_{J'}$. Clearly it is invertible on $Y_J \cap Y_{J'}$. Similar statements concerning $u_I/u_{I'}$ hold for any $I, I' \in I(n,m)$.

Notation. Let $m, q \geq n$. Denote by $\mathcal{I}$ the disjoint union $I(n,m) \coprod I(n,q)$. We set

$$\lambda_{\beta,\alpha} = \begin{cases} u_\alpha/u_\beta & \text{if } \alpha,\beta \in I(n,m), \\ \xi_\beta/\xi_\alpha & \text{if } \alpha,\beta \in I(n,q), \\ p_{\alpha,\beta} & \text{if } \beta \in I(n,q),\ \alpha \in I(n,m), \\ 1/p_{\beta,\alpha} & \text{if } \beta \in I(n,m),\ \alpha \in I(n,q). \end{cases}$$

Consider the covering $\{Y_\alpha\}_{\alpha\in\mathcal{I}}$ of the open subvariety $Y_0 := \bigcup_{\alpha\in\mathcal{I}} Y_\alpha \subset Y$. The cocycle condition $\lambda_{\alpha,\beta}\lambda_{\beta,\gamma} = \lambda_{\alpha,\gamma}$ is readily verified for any $\alpha,\beta,\gamma \in \mathcal{I}$. Thus we obtain a $\mathbb{G}_m$-bundle over Y_0; call it $\mathcal{E}$. Let $X_0 := \bigcup_{\alpha\in\mathcal{I}} X_\alpha$.

LEMMA 3.2. *Assume that $m,q \geq n$. Then the total space of the $\mathbb{G}_m$-bundle $\mathcal{E}$ over Y_0 is isomorphic to the open subvariety $X_0 := \bigcup_{\alpha\in\mathcal{I}} X_\alpha \subset X$.*

Proof. The total space of the $\mathbb{G}_m$-bundle corresponding to $\mathcal{D}$ is $\coprod_{\alpha\in\mathcal{I}} Y_\alpha \times k^*/\!\sim$, where $(\pi([u,\xi]),t) \in Y_\alpha \times k^*$ is identified with $(\pi([u;\xi]), \lambda_{\beta,\alpha}(\pi([u,\xi]).t)) \in Y_\beta \times k^*$ whenever $\pi([u,\xi]) \in Y_\alpha \cap Y_\beta$. One has the following commuting diagram for any $\alpha,\beta \in \mathcal{I}$:

$$\begin{array}{ccccccc} Y_\alpha \times k^* & \supset & (Y_\alpha \cap Y_\beta) \times k^* & \xrightarrow{\lambda_{\beta,\alpha}} & (Y_\beta \cap Y_\alpha) \times k^* & \subset & Y_\beta \times k^* \\ {\scriptstyle f_\alpha}\uparrow & & {\scriptstyle f'_\alpha}\uparrow & & \uparrow{\scriptstyle f'_\beta} & & \uparrow{\scriptstyle f_\beta} \\ X_\alpha & \supset & X_\alpha \cap X_\beta & = & X_\beta \cap X_\alpha & \subset & X_\beta, \end{array}$$

where f'_α is the restriction of f_α. By Lemma 3.1, the f_α are isomorphisms of varieties and so it follows that the total space of the $\mathbb{G}_m$-bundle over Y_0 is isomorphic to the union $X_0 := \bigcup_\alpha X_\alpha \subset X$. □

We shall now compute the codimension of $Z := X - X_0$. We give the reduced scheme structure on Z. It is evident that Z is defined by the equations $p_{I,J} = 0$ for all $I \in I(n,m)$ and $J \in I(n,q)$. We claim $Z = Z_u \cup Z_\xi$, where Z_u is the closed subvariety with reduced scheme structure defined by the equations $u_I = 0$ for all $I \in I(n,m)$ and Z_ξ is defined by the equations $\xi_J = 0$ for all $J \in I(n,q)$. It is clear that $Z_u \cup Z_\xi \subset Z$. On the other hand, if $[u,\xi]$ is not in $Z_u \cup Z_\xi$, then $u_I([u,\xi]) \neq 0$ for some I and $\xi_J([u,\xi]) \neq 0$ for some J. This implies that $p_{I,J}([u,\xi]) \neq 0$. Hence $[u,\xi] \in X_0$ and so $Z_u \cup Z_\xi = Z$.

LEMMA 3.3. *Let $m > n$ (resp. $q > n$). Then the codimension of Z_u (resp. Z_ξ) in X is at least* 2.

Proof. Consider the closed subvariety $M_u := D_{n-1}(M_{m,n}) \times M_{n,q} \subset \mathcal{V}_{m,q}$ (with reduced scheme structure). Then $\dim M_u = (n-1)(m+1) + nq$. (Note the dimension of the determinantal variety $D_t(M_{r,s})$, consisting of $r \times s$ matrices of rank at most t, equals $t(r+s-t)$; cf. [8].) Clearly, M_u is stable under the $\mathrm{SL}(n)$-action and $M_u /\!/ \mathrm{SL}(n) = Z_u$. We shall find an open subset $Z_{u,0}$ of Z_u such that (i) $\mathrm{SL}(n)$ acts *freely* on the inverse image of $Z_{u,0}$ under the quotient morphism $\eta\colon M_u \to Z_u$ and (ii) $\eta^{-1}(Z_{u,0}) /\!/ \mathrm{SL}(n) = Z_{u,0}$. It would then follow that

$$\begin{aligned}\dim(Z_u) &= \dim(\eta^{-1}(Z_u)) - \dim(\mathrm{SL}(n))\\ &= (n-1)(m+1) + nq - (n^2-1)\\ &= (m+n)q - (n^2-1) - (m-n+1)\\ &= \dim(X) - (m-n+1) \le \dim(X) - 2.\end{aligned}$$

Here $\dim(X) = (m+n)q - (n^2-1)$ (cf. [9]).

Define

$$W_u = D_n(M_{m,n}) \times M^0_{n,q},$$

where $M^0_{n,q} := \{\xi \in M_{n,q} \mid \xi_J(\xi) \neq 0 \text{ for some } J \in I(n,q)\}$. Then W_u is the inverse image of

$$Z_{u,0} := \{[u,\xi] \mid \xi_J(\xi) \neq 0\}$$

under the quotient morphism $\eta\colon M_u \to Z_u$. The assertion that the $\mathrm{SL}(n)$-action is free on W_u follows because the $\mathrm{SL}(n)$-action on $M^0_{n,q}$ is free.

An entirely similar argument shows that Z_ξ has codimension at least 2; consequently, the codimension of Z in X is at least 2. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $m,q > n$. As already observed, we have that $Y = D_n(M_{m,q})$ can be identified with an open subset of a certain Schubert variety in the Grassmann variety $\mathrm{SL}(m+q)/P_q$ of q-dimensional vector subspaces in k^{m+q}. Since Schubert varieties in the Grassmann variety are Frobenius split, it follows that Y is Frobenius split. Since Y_0 is open in Y, it follows that it is also Frobenius split. The variety X_0, as the total space of a $\mathbb{G}_m$-bundle over Y_0, is Frobenius split by [1, Lemma 1.1.11]. Now, since X is normal and since the codimension of X_0 in X is at least 2, it follows that X is Frobenius split (cf. [1, Lemma 1.1.7(iii)]).

If $m,q < n$ then $X = Y = M_{m,q}$ and hence X is Frobenius split. The case $m = n$ is treated separately in Lemma 3.4. □

Assume that $q = n = m$. In this case $Y = M_{n,n}$. Denote the (i,j)th coordinate function on Y by $y_{i,j}$. The sets $I(n,m)$ and $I(n,q)$ are singletons and so $S = R[u,\xi]/(u\xi - f)$, where f is the determinant function on $Y = M_{m,q}$.

Lemma 3.4. *Let $q = n = m$. The ring S of* $\mathrm{SL}(V)$*-invariants is Frobenius split in this case, too.*

Proof. Let φ be a splitting of $\mathrm{Spec}(R)$. Continue to denote by φ the restriction of φ to the open part $\mathrm{Spec}(R[1/f])$. We can "lift" φ to the $\mathbb{G}_m$-bundle $\mathrm{Spec}(R[1/f][u,u^{-1}])$ (over $\mathrm{Spec}(R[1/f])$) as follows. Define

$$\varphi\Big(a + \sum b_i u^i + \sum c_j u^{-j}\Big) := \varphi(a) + \sum \varphi(b_i) u^{i/p} + \sum \varphi(c_j) u^{-j/p},$$

where the summations are over positive integers and where $u^{i/p}$ (resp. $u^{-j/p}$) is interpreted to be 0 unless i (resp. $-j$) is an integral multiple of p. Observe that

$R[1/f][u, u^{-1}] = S[1/f]$ and so we have a splitting of $\mathrm{Spec}(S[1/f])$, which we still denote φ. We claim that φ maps S to S and hence extends to a splitting of $\mathrm{Spec}(S)$. Indeed, a general element s of S is of the form $a + \sum b_i u^i + \sum c_j \xi^j$ with a, b_i, and c_j in R, so that

$$\begin{aligned}\varphi(s) &= \varphi\Big(a + \sum b_i u^i + \sum c_j f^j u^{-j}\Big) \\ &= \varphi(a) + \sum \varphi(b_i) u^{i/p} + \sum \varphi(c_j f^j) u^{-j/p}.\end{aligned}$$

Rewriting $\varphi(c_j f^j) u^{-j/p}$ as $\varphi(c_j) f^{j/p} u^{-j/p} = \varphi(c_j) \xi^{j/p}$, we see that $\varphi(s)$ belongs to S. □

Remark 3.5. If one of $\{m, q\}$ is less than n and the other is not less than n, we expect the ring of invariants to be Frobenius split. At the moment, however, we do not have a proof of this assertion.

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