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A Local Ring such that the Map between Grothendieck Groups with Rational Coefficients Induced by Completion Is Not Injective

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Dedicated to Professor Melvin Hochster on the occasion of his 65th birthday

1. Introduction

In this paper, we construct a local ring A such that the kernel of the map $G_0(A)_{\mathbb{Q}} \to G_0(\hat{A})_{\mathbb{Q}}$ is not zero, where $\hat{A}$ is the completion of A with respect to the maximal ideal and where $G_0(\cdot)_{\mathbb{Q}}$ is the Grothendieck group of finitely generated modules with rational coefficients. In our example, A is a 2-dimensional local ring that is essentially of finite type over $\mathbb{C}$, but it is not normal.

For a Noetherian ring R, we set

$$G_0(R) = \frac{\bigoplus_{M:\text{ f.g. } R\text{-mod.}} \mathbb{Z}[M]}{\langle [L] + [N] - [M] \mid 0 \to L \to M \to N \to 0 \text{ is exact} \rangle};$$

this is called the *Grothendieck group* of finitely generated R-modules. Here $[M]$ denotes the free basis corresponding to a finitely generated R-module (f.g. R-mod.) M of the free module $\bigoplus \mathbb{Z}[M]$, where $\mathbb{Z}$ is the ring of integers.

For a flat ring homomorphism $R \to A$, we have the induced map $G_0(R) \to G_0(A)$ defined by $[M] \mapsto [M \otimes_R A]$.

We are interested in the following problem (Question 1.4 in [7]).

Problem 1.1. *Let R be a Noetherian local ring. Is the map $G_0(R)_{\mathbb{Q}} \to G_0(\hat{R})_{\mathbb{Q}}$ injective?*

Here $\hat{R}$ denotes the $\mathfrak{m}$-adic completion of R, where $\mathfrak{m}$ is the unique maximal ideal of R. For an abelian group N, $N_{\mathbb{Q}}$ denotes the tensor product with the field of rational numbers $\mathbb{Q}$.

Next we explain motivation and applications.

Assume that S is a regular scheme and that X is a scheme of finite type over S. Then, by the singular Riemann–Roch theorem [3], we obtain an isomorphism

$$\tau_{X/S} \colon G_0(X)_{\mathbb{Q}} \xrightarrow{\sim} A_*(X)_{\mathbb{Q}},$$

where $G_0(X)$ (resp. $A_*(X)$) is the *Grothendieck group* of coherent sheaves on X (resp. *Chow group* of X). We refer the reader to Chapters 1, 18, and 20 in [3] for

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the definitions of $G_0(X)$, $A_*(X)$, and $\tau_{X/S}$. Note that $G_0(X)$ (resp. $\tau_{X/S}$) is denoted by $K_0(X)$ (resp. τ_X) in [3]. The map $\tau_{X/S}$ usually depends on the choice of S. In fact, we have

$$\tau_{\mathbb{P}^1_k/\mathbb{P}^1_k}([\mathcal{O}_{\mathbb{P}^1_k}]) = [\mathbb{P}^1_k] \in A_*(\mathbb{P}^1_k)_{\mathbb{Q}} = \mathbb{Q}[\mathbb{P}^1_k] \oplus \mathbb{Q}[t],$$
$$\tau_{\mathbb{P}^1_k/\operatorname{Spec} k}([\mathcal{O}_{\mathbb{P}^1_k}]) = [\mathbb{P}^1_k] + \chi(\mathcal{O}_{\mathbb{P}^1_k})[t] = [\mathbb{P}^1_k] + [t] \in A_*(\mathbb{P}^1_k)_{\mathbb{Q}},$$

where t is a k-rational closed point of $\mathbb{P}^1_k$ over a field k. Here, for a closed subvariety Y, $[Y]$ denotes the algebraic cycle corresponding to Y. Hence

$$\tau_{\mathbb{P}^1_k/\mathbb{P}^1_k}([\mathcal{O}_{\mathbb{P}^1_k}]) \neq \tau_{\mathbb{P}^1_k/\operatorname{Spec} k}([\mathcal{O}_{\mathbb{P}^1_k}])$$

in this case. However, for a local ring R that is a homomorphic image of a regular local ring T, the map $\tau_{\operatorname{Spec} R/\operatorname{Spec} T}$ is independent of the choice of T in many cases. In fact, if R is a complete local ring or if R is essentially of finite type over either a field or the ring of integers, it is proved in [9, Prop. 1.2] that the map $\tau_{\operatorname{Spec} R/\operatorname{Spec} T}$ is actually independent of T.

From now on, for simplicity we denote $\tau_{\operatorname{Spec} R/\operatorname{Spec} T}$ by $\tau_{R/T}$. It is natural to ask the following question.

PROBLEM 1.2. *Let R be a homomorphic image of a regular local ring T. Is the map $\tau_{R/T}$ independent of T?*

We remark that, by the singular Riemann–Roch theorem, the diagram

$$\begin{array}{ccc} G_0(R)_{\mathbb{Q}} & \xrightarrow{\tau_{R/T}} & A_*(R)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ G_0(\hat{R})_{\mathbb{Q}} & \xrightarrow{\tau_{\hat{R}/\hat{T}}} & A_*(\hat{R})_{\mathbb{Q}} \end{array}$$

is commutative, where the vertical maps are induced by the completion $R \to \hat{R}$. We wish to emphasize that the bottom map as well as the vertical maps are independent of the choice of T because $\hat{R}$ is complete [9, Prop. 1.2]. Hence if the vertical maps are injective, then the top map is also independent of T.

Therefore, if the answer to Problem 1.1 is affirmative then so is the answer to Problem 1.2.

We offer another motivation as follows. Roberts [11] and Gillet–Soulé [4] proved the vanishing theorem of intersection multiplicities for complete intersections. If a local ring R is a complete intersection, then $\tau_{R/T}([R]) = [\operatorname{Spec} R]$ holds, where

$$[\operatorname{Spec} R] = \sum_{\substack{\mathfrak{p} \in \operatorname{Spec} R \\ \dim R/\mathfrak{p} = \dim R}} \ell_{R_\mathfrak{p}}(R_\mathfrak{p})[\operatorname{Spec} R/\mathfrak{p}] \in A_{\dim R}(R)_{\mathbb{Q}}.$$

In [11], Roberts proved the vanishing theorem of intersection multiplicities not only for complete intersections but also for local rings satisfying $\tau_{R/T}([R]) =$

[Spec R]. Inspired by his work, Kurano [9] began a study of local rings satisfying the condition $\tau_{R/T}([R]) = [\operatorname{Spec} R]$ and called them *Roberts* rings. If R is a Roberts ring, then the completion, the henselization, and the localizations of R are also Roberts rings [9]. However, the following problem remained open.

PROBLEM 1.3. *If $\hat{R}$ is a Roberts ring, then must R also be a Roberts ring?*

It is proved in [7, Prop. 6.2] that the answer to Problem 1.3 is affirmative if and only if the answer to Problem 1.1 is.

The following partial result concerning Problem 1.1 was given by Theorem 1.5 in [7].

THEOREM 1.4. *Let R be a homomorphic image of an excellent regular local ring. Assume that R satisfies at least one of the following three conditions*:

(i) *R is henselian*;
(ii) *$R = S_{\mathfrak{n}}$, where S is a standard graded ring over a field and $\mathfrak{n} = \bigoplus_{n>0} S_n$*;
(iii) *R has only isolated singularity.*

Then the induced map $G_0(R) \to G_0(\hat{R})$ is injective.

However, the following example was given by Hochster [6].

EXAMPLE 1.5. Let k be a field. We set

$$T = k[x, y, u, v]_{(x,y,u,v)},$$
$$P = (x, y),$$
$$f = xy - ux^2 - vy^2.$$

Then $\operatorname{Ker}(G_0(T/fT) \to G_0(\widehat{T/fT})) \ni [T/P] \neq 0$. In this case, $2 \cdot [T/P] = 0$.

The ring T/fT is not normal in Example 1.5. Dao [2] has found the following example.

EXAMPLE 1.6. We set

$$R = \mathbb{R}[x, y, z, w]_{(x,y,z,w)}/(x^2 + y^2 - (w + 1)z^2),$$
$$P = (x, y, z),$$

where R is a normal local ring. Then, $\operatorname{Ker}(G_0(R) \to G_0(\hat{R})) \ni [R/P] \neq 0$. In this case, $2 \cdot [R/P] = 0$.

The following result is the main theorem of this paper.

THEOREM 1.7. *There exists a 2-dimensional local ring A, which is essentially of finite type over $\mathbb{C}$, that satisfies*

$$\operatorname{Ker}(G_0(A)_{\mathbb{Q}} \to G_0(\hat{A})_{\mathbb{Q}}) \neq 0.$$

REMARKS 1.8. (1) By Theorem 1.7, we know that the answers to both Problem 1.1 and Problem 1.3 are negative. In other words, there exists a local ring R such that $\hat{R}$ is a Roberts ring but R is not.

(2) Problem 1.2 is still open.

(3) In [10] we defined a notion of numerical equivalence on $G_0(R)$ and $A_*(R)$. Set $\overline{G_0(R)} = G_0(R)/\sim_{\text{num.}}$ and $\overline{A_*(R)} = A_*(R)/\sim_{\text{num.}}$. Then the following statements hold.

(a) $\overline{G_0(R)} \to \overline{G_0(\hat{R})}$ is injective for any local ring R.
(b) The induced map $\overline{\tau_{R/T}} : \overline{G_0(R)}_{\mathbb{Q}} \xrightarrow{\sim} \overline{A_*(R)}_{\mathbb{Q}}$ is independent of T.
(c) R is a numerically Roberts ring if and only if $\hat{R}$ is. (The definition of numerically Roberts rings was given in [10]; the vanishing theorem of intersection multiplicities holds true for numerically Roberts rings.)

(4) The ring A constructed in the main theorem is not normal. We do not know any example of a normal local ring that does not satisfy Problem 1.1.

Theorem 1.7 follows immediately from Lemmas 1.9 and 1.10.

LEMMA 1.9. *Let K be an algebraically closed field, and let $S = \bigoplus_{n \geq 0} S_n$ be a standard graded ring over K—that is, a Noetherian graded ring generated by S_1 over $S_0 = K$. We set $X = \operatorname{Proj} S$ and assume that X is smooth over K with $d = \dim X \geq 1$. Let h be the very ample divisor on X of this embedding. Let $\pi : Y \to \operatorname{Spec} S$ be the blow-up at $\mathfrak{n} = \bigoplus_{n>0} S_n$. We make the following assumptions.*

1. *Set $R = S_{\mathfrak{n}}$ and let $\hat{R}$ be the completion of R. Then the map $A_1(R)_{\mathbb{Q}} \to A_1(\hat{R})_{\mathbb{Q}}$ induced by completion is an isomorphism.*
2. *There exists a smooth connected curve C in Y that satisfies the following two conditions*:
   (i) *C transversally intersects with $\pi^{-1}(\mathfrak{n}) \simeq X$ at two points—namely, P_1 and P_2.*
   (ii) *$[P_1] - [P_2] \neq 0$ in $A_0(X)_{\mathbb{Q}}/h \cdot A_1(X)_{\mathbb{Q}}$.*

Then there exists a local ring A, of dimension $d + 1$, that is essentially of finite type over K and such that

$$\operatorname{Ker}(G_0(A)_{\mathbb{Q}} \to G_0(\hat{A})_{\mathbb{Q}}) \neq 0.$$

LEMMA 1.10. *Set $S = \mathbb{C}[x_0, x_1, x_2]/(f)$, where f is a homogeneous cubic polynomial. Assume that $X = \operatorname{Proj} S$ is smooth over $\mathbb{C}$. Then R satisfies the assumptions in Lemma 1.9 with $d = 1$.*

We shall prove Lemmas 1.9 and 1.10 in Sections 2 and 3, respectively.

2. A Proof of Lemma 1.9

Let $\mathfrak{p}$ be the prime ideal of S satisfying $\operatorname{Spec} S/\mathfrak{p} = \pi(C)$. Set $R = S_{\mathfrak{n}}$ and $\mathfrak{m} = \mathfrak{n}R$. Then C is the normalization of $\operatorname{Spec} S/\mathfrak{p}$. We denote by v_i the normalized valuation of the discrete valuation ring at $P_i \in C$ for $i = 1, 2$.

We begin by showing the following claim.

CLAIM 2.1. *There exists an* $s \in \mathfrak{m}/\mathfrak{p}R$ *such that*

1. $v_1(s) = v_2(s) > 0$, *and*
2. $K[s]_{(s)} \hookrightarrow R/\mathfrak{p}R$ *is finite.*

Proof. Let C' be the smooth projective connected curve over K that contains C as a Zariski open set. We regard P_1 and P_2 as points of C'.

Let $R(C')$ be the field of rational functions on C'. Since P_1 is an ample divisor on C', there exists a $t_1 \in R(C')^\times$ such that

- P_1 is the only pole of t_1, and
- P_2 is neither a zero nor a pole of t_1.

Similarly, one can find a $t_2 \in R(C')^\times$ such that

- P_2 is the only pole of t_2, and
- P_1 is neither a zero nor a pole of t_2.

Replacing t_1 (resp. t_2) with a suitable power of t_1 (resp. t_2), we may assume that $v_1(t_1) = v_2(t_2) < 0$.

Put $t = 1/t_1t_2 \in R(C')^\times$; then $\{P_1, P_2\}$ is the set of zeros of t. Observe that $v_1(t) = v_2(t) > 0$.

Let O_{v_i} be the discrete valuation ring at P_i for $i = 1, 2$. Then $K[t]_{(t)}$ is a subring of

$$O_{v_1} \cap O_{v_2} = \overline{S/\mathfrak{p}} \otimes_{S/\mathfrak{p}} R/\mathfrak{p}R,$$

where the overline denotes normalization of the given ring.

Since $\{P_1, P_2\}$ is just the set of zeros of t, it follows that $O_{v_1} \cap O_{v_2}$ is the integral closure of $K[t]_{(t)}$ in $R(C')$. In particular, $\overline{S/\mathfrak{p}} \otimes_{S/\mathfrak{p}} R/\mathfrak{p}R$ is finite over $K[t]_{(t)}$.

Let I be the conductor ideal of the normalization

$$R/\mathfrak{p}R \subset \overline{S/\mathfrak{p}} \otimes_{S/\mathfrak{p}} R/\mathfrak{p}R.$$

Let $\mathfrak{m}_i$ be the maximal ideal of $\overline{S/\mathfrak{p}} \otimes_{S/\mathfrak{p}} R/\mathfrak{p}R$ corresponding to P_i for $i = 1, 2$. Since I is contained in $\mathfrak{m}/\mathfrak{p}R$, we have

$$I \subset \mathfrak{m}_1 \cap \mathfrak{m}_2$$

and therefore

$$\sqrt{I} = \mathfrak{m}_1 \cap \mathfrak{m}_2 \ni t.$$

Thus, t^n is contained in I for a sufficiently large n. In particular, t^n is in $\mathfrak{m}/\mathfrak{p}R$.

Consider the following commutative diagram:

$$\begin{array}{ccc} K[t^n]_{(t^n)} & \longrightarrow & R/\mathfrak{p}R \\ \downarrow & & \downarrow \\ K[t]_{(t)} & \longrightarrow & \overline{S/\mathfrak{p}} \otimes_{S/\mathfrak{p}} R/\mathfrak{p}R. \end{array}$$

The left and bottom morphisms are both finite. Therefore, all morphisms are finite.

Now put $s = t^n$. Then s satisfies all the requirements. □

Let $R \xrightarrow{\xi} R/\mathfrak{p}R$ be the natural surjective morphism. We set $A = \xi^{-1}(K[s]_{(s)})$. Then

$$\begin{array}{ccc} R & \xrightarrow{\xi} & R/\mathfrak{p}R \\ \uparrow & \square & \uparrow \\ A & \longrightarrow & K[s]_{(s)}. \end{array}$$

In the rest of this section, we shall prove that the ring A satisfies the required condition.

We shall now prove our next claim.

CLAIM 2.2. *The morphism $A \to R$ is finite birational, and A is essentially of finite type over K of dimension $d+1$.*

Proof. Observe that

$$A \supset \mathfrak{p}R \neq 0$$

because the dimension of R is at least 2. Take $0 \neq a \in \mathfrak{p}R$. Since $A[a^{-1}] = R[a^{-1}]$, it follows that $A \to R$ is birational.

One can prove that A is a Noetherian ring by Eakin–Nagata's theorem. However, we shall prove here that A is essentially of finite type over K without using Eakin–Nagata's theorem.

Let B be the integral closure of $K[s]$ in $R/\mathfrak{p}R$, and recall that B is of finite type over K. Since $R/\mathfrak{p}R$ is finite over $K[s]_{(s)}$, we have $B \otimes_{K[s]} K[s]_{(s)} = R/\mathfrak{p}R$. Then

$$\begin{array}{ccccc} R & \xrightarrow{\xi} & R/\mathfrak{p}R & \longleftarrow & B \\ \uparrow & & \uparrow & & \uparrow \\ S & & K[s]_{(s)} & \longleftarrow & K[s]. \end{array}$$

Take an element $s' \in R$ that satisfies $\xi(s') = s$. Suppose $S = K[s_1, \dots, s_n]$. Since $B \otimes_{K[s]} K[s]_{(s)} = R/\mathfrak{p}R$, there must exist $g_i \in B$ and $f_i \in K[s] \setminus (s)$ such that $\xi(s_i) = g_i/f_i$ for $i = 1, \dots, n$. Take an element $f_i' \in K[s']$ such that $\xi(f_i') = f_i$ for $i = 1, \dots, n$. Put

$$S' = K[s', s_1 f_1', \dots, s_n f_n'].$$

Observe that R is a localization of S' and that $\xi(S') \subset B$. Because B is of finite type over K, there exists a ring D that satisfies the following conditions:

- $S' \subset D \subset R$;
- D is of finite type over K;
- R is a localization of D; and
- $\xi(D) = B$.

Put $\phi = \xi|_D$ and $E = \phi^{-1}(K[s])$. Then D is finite over E, and we have

$$\begin{array}{ccc} D & \xrightarrow{\phi} & B \\ \uparrow & \square & \uparrow \\ E & \longrightarrow & K[s]. \end{array}$$

Since $B \otimes_{K[s]} K[s]_{(s)} = R/\mathfrak{p}R$, there is only one prime ideal N of B lying over $(s) \subset K[s]$. Therefore, $\phi^{-1}(N)$ is the only prime ideal lying over the prime ideal $\phi^{-1}((s))$ of E. Localizing all the rings in the previous diagram yields

$$\begin{array}{ccc} D \otimes_E E_{\phi^{-1}((s))} & \longrightarrow & B \otimes_E E_{\phi^{-1}((s))} \\ \uparrow & \square & \uparrow \\ E_{\phi^{-1}((s))} & \longrightarrow & K[s] \otimes_E E_{\phi^{-1}((s))}. \end{array}$$

We remark that

$$D \otimes_E E_{\phi^{-1}((s))} = R, \quad K[s] \otimes_E E_{\phi^{-1}((s))} = K[s]_{(s)}, \quad B \otimes_E E_{\phi^{-1}((s))} = R/\mathfrak{p}R.$$

Therefore, A coincides with $E_{\phi^{-1}((s))}$.

Since D is finite over E and since D is of finite type over K, it follows that E is also of finite type over K. Hence we know that A is essentially of finite type over K and that R is finite over A. It is then easy to see that

$$\dim A = \dim R = \dim S = d + 1. \qquad \square$$

In particular, A is a homomorphic image of a regular local ring T. We thus have the commutative diagram

$$\begin{array}{ccc} G_0(A)_{\mathbb{Q}} & \xrightarrow{\tau_{A/T}} & A_*(A)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ G_0(\hat{A})_{\mathbb{Q}} & \xrightarrow{\tau_{\hat{A}/\hat{T}}} & A_*(\hat{A})_{\mathbb{Q}} \end{array}$$

by the singular Riemann–Roch theorem [3, Chaps. 18 and 20]. Observe that the horizontal maps in the above diagram are isomorphisms. Therefore, in order to prove that $\mathrm{Ker}(G_0(A)_{\mathbb{Q}} \to G_0(\hat{A})_{\mathbb{Q}})$ is not 0, it is sufficient to prove that $\mathrm{Ker}(A_1(A)_{\mathbb{Q}} \to A_1(\hat{A})_{\mathbb{Q}})$ is not 0.

The diagram

$$\begin{array}{ccc} R & \longrightarrow & \hat{R} \\ \uparrow & & \uparrow \\ A & \longrightarrow & \hat{A} \end{array}$$

induces the commutative diagram

$$\begin{array}{ccc} A_1(R)_{\mathbb{Q}} & \longrightarrow & A_1(\hat{R})_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ A_1(A)_{\mathbb{Q}} & \longrightarrow & A_1(\hat{A})_{\mathbb{Q}}, \end{array} \tag{1}$$

where the vertical maps are induced by the finite morphisms $A \to R$ and $\hat{A} \to \hat{R}$ and where the horizontal maps are induced by the completions $A \to \hat{A}$ and $R \to \hat{R}$. By Lemma 1.9(1), the top map in the diagram (1) is an isomorphism.

Here we shall show, for each prime ideal of A, that there exists only one prime ideal of R lying over it. Let $\mathfrak{q}$ be a prime ideal of A, and recall that the conductor ideal $\mathfrak{p}R$ is a prime ideal of both A and R. If $\mathfrak{q}$ does not contain $\mathfrak{p}R$, then $A_{\mathfrak{q}}$ coincides with $R \otimes_A A_{\mathfrak{q}}$. Hence there exists only one prime ideal of R lying over $\mathfrak{q}$ in this case. Next suppose that $\mathfrak{q}$ contains $\mathfrak{p}R$. Then $\mathfrak{q}$ is either $\mathfrak{p}R$ or the unique maximal ideal of A. In any case, there exists only one prime ideal of R lying over $\mathfrak{q}$.

Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Rat}_1(R) & \longrightarrow & Z_1(R) & \longrightarrow & A_1(R) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Rat}_1(A) & \longrightarrow & Z_1(A) & \longrightarrow & A_1(A) & \longrightarrow & 0 \end{array}$$

(see [3, Chap. 1] for the definition of Rat_* and Z_*). Because the morphism $A \to R$ is finite injective, the cokernel of $\mathrm{Rat}_1(R) \to \mathrm{Rat}_1(A)$ is torsion by [3, Chap. 1, Prop. 1.4]. Since, for each prime ideal of A, there is only one prime ideal of R lying over it, it follows that the map $Z_1(R) \to Z_1(A)$ is injective and that the cokernel of this map is a torsion module $\mathbb{Z}/(2v)$, where $v = v_1(s) = v_2(s)$. Therefore, the map on the left-hand side of diagram (1) is also an isomorphism.

By the commutativity of diagram (1) we know that, in order to prove that $\mathrm{Ker}(A_1(A)_{\mathbb{Q}} \to A_1(\hat{A})_{\mathbb{Q}})$ is not 0, it is sufficient to show that

$$\mathrm{Ker}(A_1(\hat{R})_{\mathbb{Q}} \to A_1(\hat{A})_{\mathbb{Q}}) = \mathbb{Q}.$$

We know $\hat{A}/(\mathfrak{p}R)\hat{A} = \widehat{K[s]_{(s)}} = K[[s]]$, so $(\mathfrak{p}R)\hat{A}$ is a prime ideal of $\hat{A}$ of height d. We thus have the following bijective correspondences:

the set of prime ideals of $\hat{R}$ lying over $(\mathfrak{p}R)\hat{A}$

$\longleftrightarrow$ the set of minimal prime ideals of $\widehat{R/\mathfrak{p}R}$

$\longleftrightarrow$ the set of maximal ideals of $\overline{S/\mathfrak{p}} \otimes_{S/\mathfrak{p}} R/\mathfrak{p}R$

$\longleftrightarrow \{P_1, P_2\}$,

where $\overline{S/\mathfrak{p}} \otimes_{S/\mathfrak{p}} R/\mathfrak{p}R$ is the normalization of $R/\mathfrak{p}R$. Hence there are only two prime ideals of $\hat{R}$ lying over $(\mathfrak{p}R)\hat{A}$, which we denote by $\mathfrak{p}_1$ and $\mathfrak{p}_2$.

It is easy to see that $\mathfrak{p}R$ is the conductor ideal of the ring extension $A \subset R$—that is,

$$\mathfrak{p}R = A :_A R.$$

Then $(\mathfrak{p}R)\hat{A} = \hat{A} :_{\hat{A}} \hat{R}$ is satisfied. Therefore, $(\mathfrak{p}R)\hat{A}$ is the conductor ideal of the ring extension $\hat{A} \subset \hat{R}$. Now consider the map

$$\varphi\colon Z_1(\hat{R}) \to Z_1(\hat{A}).$$

Let $\mathfrak{q}$ be a prime ideal of $\hat{A}$ of height d. If $\mathfrak{q}$ does not contain the conductor ideal $(\mathfrak{p}R)\hat{A}$, then there exists only one prime ideal $\mathfrak{q}'$ of $\hat{R}$ lying over $\mathfrak{q}$. Furthermore, $\hat{A}/\mathfrak{q}$ is birational to $\hat{R}/\mathfrak{q}'$. Therefore,

$$\varphi([\operatorname{Spec} \hat{R}/\mathfrak{q}']) = [\operatorname{Spec} \hat{A}/\mathfrak{q}].$$

We will show that

$$\varphi([\operatorname{Spec} \hat{R}/\mathfrak{p}_1]) = \varphi([\operatorname{Spec} \hat{R}/\mathfrak{p}_2]) = v[\operatorname{Spec} \hat{A}/(\mathfrak{p}R)\hat{A}],$$

where $v = v_1(s) = v_2(s)$. Recall that

$$\widehat{O_{v_1}} \times \widehat{O_{v_2}} = (\overline{R/\mathfrak{p}R})^{\wedge} = \overline{\hat{R}/\mathfrak{p}\hat{R}} = \overline{\hat{R}/\mathfrak{p}_1} \times \overline{\hat{R}/\mathfrak{p}_2}.$$

Therefore, we may assume $\widehat{O_{v_i}} \simeq \overline{\hat{R}/\mathfrak{p}_i}$ for $i = 1, 2$. Then

$$\begin{aligned} \operatorname{rank}_{\hat{A}/(\mathfrak{p}R)\hat{A}} \hat{R}/\mathfrak{p}_i &= \operatorname{rank}_{\hat{A}/(\mathfrak{p}R)\hat{A}} \overline{\hat{R}/\mathfrak{p}_i} = \operatorname{rank}_{\hat{A}/(\mathfrak{p}R)\hat{A}} \widehat{O_{v_i}} = \operatorname{rank}_{K[[s]]} \widehat{O_{v_i}} \\ &= \dim_K \widehat{O_{v_i}}/s\widehat{O_{v_i}} = \dim_K O_{v_i}/sO_{v_i} = v \end{aligned}$$

for $i = 1, 2$, where $\dim_K$ denotes the dimension of the given K-vector space. We thus have the exact sequence

$$0 \longrightarrow \mathbb{Z} \cdot ([\operatorname{Spec} \hat{R}/\mathfrak{p}_1] - [\operatorname{Spec} \hat{R}/\mathfrak{p}_2]) \longrightarrow Z_1(\hat{R}) \longrightarrow Z_1(\hat{A}) \longrightarrow \mathbb{Z}/(v) \longrightarrow 0.$$

Now consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Rat}_1(\hat{R}) & \longrightarrow & Z_1(\hat{R}) & \longrightarrow & A_1(\hat{R}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Rat}_1(\hat{A}) & \longrightarrow & Z_1(\hat{A}) & \longrightarrow & A_1(\hat{A}) & \longrightarrow & 0. \end{array}$$

Because the morphism $\hat{A} \to \hat{R}$ is finite injective, the cokernel of $\operatorname{Rat}_1(\hat{R}) \to \operatorname{Rat}_1(\hat{A})$ is torsion (cf. [3, Prop. 1.4]). We therefore have the exact sequence

$$0 \longrightarrow \mathbb{Q} \cdot ([\operatorname{Spec} \hat{R}/\mathfrak{p}_1] - [\operatorname{Spec} \hat{R}/\mathfrak{p}_2]) \longrightarrow A_1(\hat{R})_{\mathbb{Q}} \longrightarrow A_1(\hat{A})_{\mathbb{Q}} \longrightarrow 0.$$

Hence we need only prove that

$$[\operatorname{Spec} \hat{R}/\mathfrak{p}_1] - [\operatorname{Spec} \hat{R}/\mathfrak{p}_2] \neq 0$$

in $A_1(\hat{R})_{\mathbb{Q}}$.

Let $\hat{\pi} : \hat{Y} \to \operatorname{Spec} \hat{R}$ be the blow-up at $\mathfrak{m}\hat{R}$. Since $\hat{\pi}^{-1}(\mathfrak{m}\hat{R}) \simeq X$, it follows that

$$A_1(X)_{\mathbb{Q}} \xrightarrow{i_*} A_1(\hat{Y})_{\mathbb{Q}} \xrightarrow{\hat{\pi}_*} A_1(\hat{R})_{\mathbb{Q}} \longrightarrow 0$$

is exact and that

$$\hat{\pi}_*\big(\big[\mathrm{Spec}\ \overline{\hat{R}/\mathfrak{p}_1}\big] - \big[\mathrm{Spec}\ \overline{\hat{R}/\mathfrak{p}_2}\big]\big) = [\mathrm{Spec}\ \hat{R}/\mathfrak{p}_1] - [\mathrm{Spec}\ \hat{R}/\mathfrak{p}_2],$$

where $i\colon X \to \hat{Y}$ is the inclusion. Consider the following commutative diagram:

$$\begin{array}{ccccc}
P_i & \longrightarrow & \{P_1, P_2\} & \longrightarrow & X \\
\downarrow & \square & \downarrow & \square & \downarrow \\
\mathrm{Spec}\ O_{v_i} & \longrightarrow & \mathrm{Spec}\ \overline{R/\mathfrak{p}} & \longrightarrow & Y \\
 & & \downarrow & & \downarrow \\
 & & \mathrm{Spec}\ R/\mathfrak{p} & \longrightarrow & \mathrm{Spec}\ R.
\end{array}$$

Take the fibre product with Spec $\hat{R}$ over Spec R. We may assume that Spec $\overline{\hat{R}/\mathfrak{p}_i}$ coincides with Spec $\widehat{O_{v_i}}$ for $i = 1, 2$, so the following diagram commutes:

$$\begin{array}{ccccccccc}
 & & P_i & = & P_i & \longrightarrow & \{P_1, P_2\} & \longrightarrow & X \\
 & & \downarrow & \square & \downarrow & \square & \downarrow & \square & \downarrow \\
\mathrm{Spec}\ \overline{\hat{R}/\mathfrak{p}_i} & = & \mathrm{Spec}\ \widehat{O_{v_i}} & \longrightarrow & \mathrm{Spec}\ O_{v_i} \otimes_R \hat{R} & \longrightarrow & \mathrm{Spec}\ \overline{R/\mathfrak{p}} \otimes_R \hat{R} & \longrightarrow & \hat{Y} \\
\downarrow & & & & & & \downarrow & & \downarrow \\
\mathrm{Spec}\ \hat{R}/\mathfrak{p}_i & & & \longrightarrow & & & \mathrm{Spec}\ \hat{R}/\mathfrak{p}\hat{R} & \longrightarrow & \mathrm{Spec}\ \hat{R}.
\end{array}$$

Assume that

$$[\mathrm{Spec}\ \hat{R}/\mathfrak{p}_1] - [\mathrm{Spec}\ \hat{R}/\mathfrak{p}_2] = 0$$

in $A_1(\hat{R})_{\mathbb{Q}}$. Then, there exists a $\delta \in A_1(X)_{\mathbb{Q}}$ such that

$$i_*(\delta) = \big[\mathrm{Spec}\ \overline{\hat{R}/\mathfrak{p}_1}\big] - \big[\mathrm{Spec}\ \overline{\hat{R}/\mathfrak{p}_2}\big].$$

Now consider the map

$$A_1(\hat{Y})_{\mathbb{Q}} \xrightarrow{\ i^!\ } A_0(X)_{\mathbb{Q}};$$

that is, taking the intersection with $\hat{\pi}^{-1}(\mathfrak{m}\hat{R}) = X$. Since $i^! i_*(\delta) = -h \cdot \delta$ and

$$i^!\big(\big[\mathrm{Spec}\ \overline{\hat{R}/\mathfrak{p}_1}\big] - \big[\mathrm{Spec}\ \overline{\hat{R}/\mathfrak{p}_2}\big]\big) = i^!([\mathrm{Spec}\ \widehat{O_{v_1}}] - [\mathrm{Spec}\ \widehat{O_{v_2}}]) = [P_1] - [P_2],$$

we have

$$[P_1] - [P_2] = -h \cdot \delta.$$

This contradicts that

$$[P_1] - [P_2] \neq 0$$

in $A_0(X)_{\mathbb{Q}}/h \cdot A_1(X)_{\mathbb{Q}}$ and thus we have completed the proof of Lemma 1.9.

3. A Proof of Lemma 1.10

Suppose that $S = \mathbb{C}[x_0, x_1, x_2]/(f)$ and $X = \mathrm{Proj}\, S$ satisfy the assumption in Lemma 1.10. Let Z be the projective cone of X: $Z = \mathrm{Proj}\, \mathbb{C}[x_0, x_1, x_2, x_3]/(f)$.

Let $W \xrightarrow{\xi} Z$ be the blow-up at $(0,0,0,1)$. We set $X_\infty = V_+(x_3)$ and $X_0 = \xi^{-1}((0,0,0,1))$. Notice that both X_0 and X_∞ are isomorphic to X. Then, $W \xrightarrow{\eta} X$ is a $\mathbb{P}^1$-bundle.

Take any two closed points $Q_1, Q_2 \in X$. We set $L_i = \eta^{-1}(Q_i)$ for $i = 1,2$. Consider the Weil divisor $L_1 + L_2 + X_\infty$ on W. We shall prove the following claim.

CLAIM 3.1. *The complete linear system* $|L_1 + L_2 + X_\infty|$ *is basepoint-free, and the induced morphism* $W \xrightarrow{f} \mathbb{P}^n$ *satisfies* $\dim f(W) \geq 2$.

Proof. Since the complete linear system $|Q_1 + Q_2|$ on X is basepoint-free, so is $|L_1 + L_2|$. Since the complete linear system $|X_\infty|$ is basepoint-free, so is $|L_1 + L_2 + X_\infty|$.

In order to prove $\dim f(W) \geq 2$, we need only show that the set

$$\{a \in R(W)^\times \mid \mathrm{div}(a) + L_1 + L_2 + X_\infty \geq 0\}$$

contains two algebraically independent elements over $\mathbb{C}$.

Observe that $W \xrightarrow{\eta} X$ is a surjective morphism and so $R(X)$ is contained in $R(W)$. Consider

$$\begin{aligned} &H^0(W, \mathcal{O}_W(L_1 + L_2 + X_\infty)) \\ &\qquad = \{a \in R(W)^\times \mid \mathrm{div}(a) + L_1 + L_2 + X_\infty \geq 0\} \cup \{0\}, \\ &H^0(X, \mathcal{O}_X(Q_1 + Q_2)) = \{a \in R(X)^\times \mid \mathrm{div}(a) + Q_1 + Q_2 \geq 0\} \cup \{0\}. \end{aligned}$$

It is easy to see that

$$H^0(W, \mathcal{O}_W(L_1 + L_2 + X_\infty)) \supset H^0(X, \mathcal{O}_X(Q_1 + Q_2)) \supset \mathbb{C}.$$

The set $H^0(X, \mathcal{O}_X(Q_1 + Q_2))$ contains a transcendental element over $\mathbb{C}$. Since $R(X)$ is algebraically closed in $R(W)$ and since

$$H^0(W, \mathcal{O}_W(L_1 + L_2 + X_\infty)) \neq H^0(X, \mathcal{O}_X(Q_1 + Q_2)),$$

it follows that $H^0(W, \mathcal{O}_W(L_1 + L_2 + X_\infty))$ contains two algebraically independent elements over $\mathbb{C}$. □

Because $|L_1 + L_2 + X_\infty|$ is basepoint-free as in Claim 3.1, we know that

$$\mathrm{div}(a) + L_1 + L_2 + X_\infty$$

is smooth for a general element $a \in H^0(W, \mathcal{O}_W(L_1 + L_2 + X_\infty)) \setminus \{0\}$ (see e.g. [5, III, Cor. 10.9]). Since $\dim f(W) \geq 2$ as in Claim 3.1,

$$\mathrm{div}(a) + L_1 + L_2 + X_\infty$$

is connected for any $a \in H^0(W, \mathcal{O}_W(L_1 + L_2 + X_\infty)) \setminus \{0\}$ [5, III, Exer. 11.3].

Let $\{a_1, \ldots, a_n\}$ be a $\mathbb{C}$-basis of $H^0(W, \mathcal{O}_W(L_1 + L_2 + X_\infty))$. Let α_i be the local equation defining the Cartier divisor $\mathrm{div}(a_i) + L_1 + L_2 + X_\infty$ for $i = 1, \ldots, n$. For $c = (c_1, \ldots, c_n) \in \mathbb{C}^n \setminus \{(0, \ldots, 0)\}$, let D_c denote the Cartier divisor on W defined by $c_1\alpha_1 + \cdots + c_n\alpha_n$.

For a general point $c \in \mathbb{C}^n$, the divisor D_c does not contain X_0 as a component but does intersect with X_0 at two distinct points. Recall that X_0 is isomorphic to X. Set $D_c \cap X_0 = \{Q_{c1}, Q_{c2}\} \subset X$.

Choose an $e \in X$ such that the Weil divisor $3e$ coincides with the very ample divisor corresponding to the embedding $X = \operatorname{Proj} S$. We regard the set of closed points of the elliptic curve X as an abelian group with unit e in the usual way.

Let $\varphi \colon X \to \mathbb{P}^1_{\mathbb{C}}$ be the morphism defined by $|2e|$. For a general point $c \in \mathbb{C}^n$, set

$$\theta(c) = \varphi(Q_{c1} \ominus Q_{c2}) \in \mathbb{P}^1_{\mathbb{C}},$$

where $\ominus$ denotes the difference in the group X. One can prove that there exists a nonempty Zariski open set U of $\mathbb{C}^n$ such that $\theta|_U \colon U \to \mathbb{P}^1_{\mathbb{C}}$ is a nonconstant morphism and D_c is smooth connected for any $c \in U$. Then there exists a nonempty Zariski open set of $\mathbb{P}^1_{\mathbb{C}}$ contained in $\operatorname{Im}(\theta|_U)$. Let F be the set of elements of X of finite order. Then it is well known that F is a countable set. In particular, $\varphi(F)$ does not contain $\operatorname{Im}(\theta|_U)$. Hence there exists a $c \in U$ such that $\theta(c) \notin \varphi(F)$. Then D_c is a smooth connected curve in W, with D_c intersecting $X_0 \simeq X$ at two points $\{P_1, P_2\}$ transversally such that $P_1 \ominus P_2$ has infinite order in X.

Let $\phi \colon X \to A_0(X)$ be a map defined by $\phi(P) = [P] - [e]$. It is well known that ϕ is a group homomorphism. We have the following exact sequence:

$$0 \longrightarrow X \xrightarrow{\ \phi\ } A_0(X) \xrightarrow{\ \deg\ } \mathbb{Z} \longrightarrow 0.$$

Because $\deg(h) = 3$, we have an isomorphism

$$X \otimes_{\mathbb{Z}} \mathbb{Q} \overset{\bar{\phi}}{\simeq} A_0(X)_{\mathbb{Q}}/hA_1(X)_{\mathbb{Q}}.$$

By definition,

$$0 \neq \bar{\phi}(P_1 \ominus P_2) = [P_1] - [P_2]$$

in $A_0(X)_{\mathbb{Q}}/hA_1(X)_{\mathbb{Q}}$.

Let Y be the blow-up of $\operatorname{Spec} S$ at the origin; then Y is an open subvariety of W. We set $C = D_c \cap Y$. Then C satisfies assumption 2 in Lemma 1.9.

Since $H^1(X, \mathcal{O}_X(n)) = 0$ for $n > 0$, we have $\operatorname{Cl}(R) \simeq \operatorname{Cl}(\hat{R})$ by Danilov's theorem (see [1, Prop. 8]). Therefore, R satisfies assumption 1 in Lemma 1.9. This completes the proof of Lemma 1.10.

REMARK 3.2. Let A be a 2-dimensional local ring constructed using Lemmas 1.9 and 1.10. Because A and $\hat{A}$ are 2-dimensional excellent local domains, we have the following isomorphisms:

$$G_0(A) \simeq \mathbb{Z} \oplus A_1(A),$$

$$G_0(\hat{A}) \simeq \mathbb{Z} \oplus A_1(\hat{A}).$$

Therefore,

$$\operatorname{Ker}(G_0(A) \to G_0(\hat{A})) \simeq \operatorname{Ker}(A_1(A) \to A_1(\hat{A})).$$

We can use this expression to prove that

$$\operatorname{Ker}(G_0(A) \to G_0(\hat{A})) \simeq \mathbb{Z}$$

as follows. Consider the diagram

$$
\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
0 & & \mathbb{Z} \\
\downarrow & & \downarrow \\
A_1(R) & \xrightarrow[\sim]{f} & A_1(\hat{R}) \\
\downarrow{\scriptstyle i} & & \downarrow \\
A_1(A) & \xrightarrow{g} & A_1(\hat{A}) \\
\downarrow & & \downarrow \\
\mathbb{Z}/(2v) & \longrightarrow & \mathbb{Z}/(v) \\
\downarrow & & \downarrow \\
0 & & 0.
\end{array}
$$

Let α_i be the element of $A_1(R)$ such that $f(\alpha_i) = [\mathrm{Spec}\ \hat{R}/\mathfrak{p}_i]$ for $i = 1, 2$. Then the kernel of g is generated by

$$i(\alpha_1) - v[\mathrm{Spec}\ A/\mathfrak{p}R].$$

Here, note that

$$2(i(\alpha_1) - v[\mathrm{Spec}\ A/\mathfrak{p}R]) = i(\alpha_1) - i(\alpha_2).$$

Since the kernel of g is not torsion, it must be isomorphic to $\mathbb{Z}$.

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