Michigan Math. J. 57 (2008)

Local Cohomology on Diagrams of Schemes

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Dedicated to Professor Melvin Hochster
on the occasion of his sixty-fifth birthday

1. Introduction

Let S be a scheme, G a flat S-group scheme, and X a G-scheme (i.e., an S-scheme on which G acts). In [18], a G-linearization of an invertible sheaf on X is defined. As quasi-coherent sheaves are important in studying a scheme, G-linearized quasi-coherent sheaves are important in studying a scheme with a group action. If S, G, and $X = \operatorname{Spec} A$ are all affine, then the category $\operatorname{Lin}(G, X)$ of G-linearized quasi-coherent sheaves on X is equivalent to the category of (G, A)-modules (see [8]). In particular, if $S = \operatorname{Spec} k = X$ with k a field, then $\operatorname{Lin}(G, X)$ is equivalent to the category of G-modules. However, the definition of a G-linearization in [18] is complicated, and probably it is difficult to study the homological algebra of $\operatorname{Lin}(G, X)$ only from the definition. In [9], the diagram $B_G^M(X)$ of schemes is defined and the category of quasi-coherent sheaves $\operatorname{Qch}(G, X) = \operatorname{Qch}(B_G^M(X))$ is studied. Note that $\operatorname{Lin}(G, X)$ and $\operatorname{Qch}(G, X)$ are equivalent. The category $\operatorname{Qch}(X)$ of quasi-coherent sheaves on X is embedded in the category of $\mathcal{O}_X$-modules $\operatorname{Mod}(X)$, and this embedding gives some flexibility to the homological algebra of $\operatorname{Qch}(X)$. Similarly, $\operatorname{Qch}(G, X)$ is embedded in $\operatorname{Mod}(G, X) := \operatorname{Mod}(B_G^M(X))$, and the homological algebra of $\operatorname{Qch}(G, X)$ is considered in $\operatorname{Mod}(G, X)$. Note that $B_G^M(X)$ is a diagram of schemes of the form

$$G \times_S G \times_S X \;\substack{\xrightarrow{1_G \times a} \\ \xrightarrow{\mu \times 1_X} \\ \xrightarrow{p_{23}}}\; G \times_S X \;\substack{\xrightarrow{a} \\ \xrightarrow[p_2]{}}\; X,$$

where $a \colon G \times_S X \to X$ is the action, $\mu \colon G \times_S G \to G$ is the product, and p_2 and p_{23} are appropriate projections. Thus, in the study of sheaves on diagrams of schemes, it is important to consider $\operatorname{Lin}(G, X)$.

Local cohomology is a powerful tool in commutative ring theory. The local cohomology $H_{\mathfrak{m}}^i$ on a local ring $(A, \mathfrak{m})$ is especially important. However, when we consider a group action, "local phenomena" sometimes occur on nonaffine schemes; see Example 8.19. Thus, to construct a theory of equivariant local

Received July 31, 2007. Revision received February 17, 2008.

cohomology, it seems that we need to discuss local cohomology on diagrams of not necessarily affine schemes.

The objectives of this paper are to give foundations of local cohomology on diagrams of schemes and give an application to invariant theory. We also introduce the notion of G-localness of a G-scheme.

The local cohomology is a derived functor of the local section functor $\underline{\Gamma}_{U,V}$ for a pair of open subdiagrams of schemes U and V of a diagram of schemes X such that $U \supset V$. As in the usual single-scheme case, $\underline{\Gamma}_{U,V}$ depends only on $U \setminus V$. However, it is interesting to point out that $U \setminus V$ may not be a subdiagram of schemes. Moreover, not all families of locally closed subsets (Z_i) of X_i can be expressed as $Z_i = U_i \setminus V_i$ for a pair of open subdiagrams U and V such that $U \supset V$.

Because unbounded homological algebra is getting more and more important, we discuss unbounded derived functor of $\underline{\Gamma}_{U,V}$. We introduce the notion of K-flabby property over a diagram of schemes.

Section 2 consists of preliminaries. Problems of commutativity of diagrams is inevitable in studying sheaves over diagrams of schemes. In Section 3, we prove some basic commutativity of diagrams. We also prove that, for a locally quasi-coherent sheaf over a locally Noetherian diagram of schemes, the local section functor can be expressed in terms of some inductive limit of hom functors. In Section 4, we discuss local cohomology and the K-flabby property; in Section 5, we slightly modify and discuss Kempf's quasi-flabby property. In Section 6, we state and prove the flat base change. In the theory of local cohomology for the usual single schemes, the flat base change and the independence theorem (see Corollary 4.17) are important. We generalize and prove these theorems. In Section 7, we consider the group action. We study the local cohomology with a group action, the *equivariant local cohomology*. This is realized as a cohomology on the diagram of schemes $B_G^M(X)$. We prove that the local section functor $\underline{\Gamma}_{U,V}$ is compatible with the G-invariance functor. In Section 8 we define an equivariant version of a local scheme, a G-local G-scheme (Definition 8.13); give some examples; and prove some basic properties. It seems that this notion has some importance in invariant theory, since if G is a (strongly) geometrically reductive k-group scheme (see Section 8.22 for the definition), A is a G-algebra, and $\mathfrak{p} \in \operatorname{Spec} A^G$, then $A_{\mathfrak{p}} := A \otimes_{A^G} A_{\mathfrak{p}}^G$ is G-local (Proposition 8.27).

In Section 9 we apply equivariant local cohomology on a G-local G-scheme to prove the following result.

THEOREM 9.5. *Let k be a field, G a linearly reductive k-group scheme, and X a Cohen–Macaulay Noetherian G-scheme. Let $\pi : X \to Y$ be a geometric quotient under the action of G in the sense of* [18]. *Assume that π is an affine morphism. Then Y is Noetherian and Cohen–Macaulay.*

As we will show, this is a generalization of the special case (that the ring in question contains a field) of the theorem of Hochster and Eagon [11, Prop. 13] on the Cohen–Macaulay property of the invariant subrings under the action of finite groups.

The authors are grateful to Professor Melvin Hochster for kindly communicating Theorem 9.10 and Corollary 9.11.

2. Preliminaries

2.1. We use the notation, terminology, and results from [9] freely (however, see Section 2.11).

2.2. Let $f\colon \mathbb{Y} \to \mathbb{X}$ be a ringed continuous functor between ringed sites as defined in [9, (2.3), (2.4), (2.19)]. As in [9], let $\mathrm{PM}(\mathbb{X})$ and $\mathrm{Mod}(\mathbb{X})$ denote the category of presheaves and sheaves, respectively, of $\mathcal{O}_{\mathbb{X}}$-modules. Let $f^{\#}_{\heartsuit}\colon \heartsuit(\mathbb{X}) \to \heartsuit(\mathbb{Y})$ denote the canonical pull-back and let $f_{\#}^{\heartsuit}\colon \heartsuit(\mathbb{Y}) \to \heartsuit(\mathbb{X})$ denote its left adjoint, where $\heartsuit$ denotes either PM or Mod.

For $b, c \in \mathrm{Mod}(\mathbb{Y})$, $\Delta_{\mathrm{Mod}}\colon f_{\#}^{\mathrm{Mod}}(b \otimes c) \to f_{\#}^{\mathrm{Mod}} b \otimes f_{\#}^{\mathrm{Mod}} c$ is the composite of the following (see [9, (1.40), (2.19), (2.20), (2.52)]):

$$\begin{aligned}
f_{\#}(b \otimes c) = af_{\#}qa(qb \otimes^p qc) &\xrightarrow{\varepsilon^{-1}\otimes^p \varepsilon^{-1}} af_{\#}qa(qaqb \otimes^p qaqc) \\
&\xrightarrow{u\otimes^p u} af_{\#}qa(qaf^{\#}f_{\#}qb \otimes^p qaf^{\#}f_{\#}qc) \\
&\xrightarrow{\theta\otimes^p \theta} af_{\#}qa(qf^{\#}af_{\#}qb \otimes qf^{\#}af_{\#}qc) \\
&\xrightarrow{c\otimes^p c} af^{\#}qa(f^{\#}qaf_{\#}qb \otimes^p f^{\#}qaf_{\#}qc) \\
&\xrightarrow{m_{\mathrm{PM}}} af^{\#}qaf^{\#}(qaf_{\#}qb \otimes^p qaf_{\#}qc) \\
&\xrightarrow{\theta} af_{\#}qf^{\#}a(qaf_{\#}qb \otimes^p qaf_{\#}qc) \\
&\xrightarrow{c} af_{\#}f^{\#}qa(qaf_{\#}qb \otimes^p qaf_{\#}qc) \\
&\xrightarrow{\varepsilon} aqa(qaf_{\#}qb \otimes^p qaf_{\#}qc) \\
&\xrightarrow{\varepsilon} a(qaf_{\#}qb \otimes^p qaf_{\#}qc) = f_{\#}b \otimes f_{\#}c.
\end{aligned}$$

See [9, Sec. 2] for the notation. It is easy to see that this composite map agrees with

$$\begin{aligned}
f_{\#}(b \otimes c) = af_{\#}qa(qb \otimes^p qc) \xrightarrow{u^{-1}} af_{\#}(qb \otimes^p qc) &\xrightarrow{\Delta_{\mathrm{PM}}} a(f_{\#}qb \otimes^p f_{\#}qc) \\
&\xrightarrow{u\otimes^p u} a(qaf_{\#}qb \otimes^p qaf_{\#}qc) = f_{\#}b \otimes f_{\#}c
\end{aligned}$$

(see [9, Lemma 2.34]).

2.3. Lemma. *Let notation be as before. If $\Delta_{\mathrm{PM}}\colon f_{\#}(qb \otimes^p qc) \to f_{\#}qb \otimes^p f_{\#}qc$ is an isomorphism, then so is $\Delta_{\mathrm{Mod}}\colon f_{\#}(b \otimes c) \to f_{\#}b \otimes f_{\#}c$.*

Proof. This follows immediately from the previous discussion and [9, Lemma 2.18]. $\square$

2.4. The map $\Gamma(x, \Delta_{\mathrm{PM}})$, which is the map Δ_{PM} at the section at $x \in \mathbb{X}$, is given as follows. It is the map

$$\begin{aligned}
&\Gamma(x, f_\#(b \otimes^p c)) \\
&\quad = \varinjlim \Gamma(x, \mathcal{O}_{\mathbb{X}}) \otimes_{\Gamma(y,\mathcal{O}_{\mathbb{Y}})} (\Gamma(y,b) \otimes_{\Gamma(y,\mathcal{O}_{\mathbb{Y}})} \Gamma(y,c)) \to \Gamma(x, f_\# b \otimes^p f_\# c) \\
&\quad = (\varinjlim \Gamma(x, \mathcal{O}_{\mathbb{X}}) \otimes_{\Gamma(y',\mathcal{O}_{\mathbb{Y}})} \Gamma(y', b)) \\
&\qquad \otimes_{\Gamma(x,\mathcal{O}_{\mathbb{X}})} (\varinjlim \Gamma(x, \mathcal{O}_{\mathbb{X}}) \otimes_{\Gamma(y'',\mathcal{O}_{\mathbb{Y}})} \Gamma(y'', c))
\end{aligned}$$

given by $\alpha \otimes (\beta \otimes \gamma) \mapsto (\alpha \otimes \beta) \otimes (1 \otimes \gamma)$, where the colimits are taken over $y, y', y'' \in (I_x^f)^{\mathrm{op}}$, respectively. This description is obtained from the definition of Δ [9, (1.40)] and the explicit descriptions of u, m, and ε in [9, (2.20), (2.50)]. If the category $(I_x^f)^{\mathrm{op}}$ (cf. [9, (2.6)]) is filtered, then the inverse of $\Gamma(x, \Delta_{\mathrm{PM}})$ is given explicitly by mapping $(\alpha \otimes \beta) \otimes (\alpha' \otimes \gamma)$ to $\alpha\alpha' \otimes (\beta \otimes \gamma)$. Thus we have our next lemma.

2.5. LEMMA. *Assume $(I_x^f)^{\mathrm{op}}$ is filtered for every $x \in \mathbb{X}$. Then $\Delta_{\mathrm{PM}} \colon f_\#(b \otimes^p c) \to f_\# b \otimes^p f_\# c$ is an isomorphism for $b, c \in \mathrm{PM}(\mathbb{Y})$. Hence $\Delta_{\mathrm{Mod}} \colon f_\#(b' \otimes c') \to f_\# b' \otimes f_\# c'$ is an isomorphism for $b', c' \in \mathrm{Mod}(\mathbb{Y})$.*

We remark that $C \colon f^\# \mathcal{O}_{\mathbb{Y}} \to \mathcal{O}_{\mathbb{X}}$ is also an isomorphism if $(I_x^f)^{\mathrm{op}}$ is filtered for every $x \in \mathbb{X}$.

2.6. For $b, c \in \mathrm{Mod}(\mathbb{X})$, the evaluation map $\mathrm{ev} \colon [b,c] \otimes b \to c$ is the composite

$$[b,c] \otimes b = a(q[b,c] \otimes^p qc) \xrightarrow{\bar{H}} a([qb,qc] \otimes^p qc) \xrightarrow{\mathrm{ev}_{\mathrm{PM}}} aqc \xrightarrow{\varepsilon} c,$$

where $[b,c]$ denotes $\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathbb{X}}}(b,c)$ and so on.

2.7. Let the notation be as in Section 2.2. Assume that, for any $x \in \mathbb{X}$, the category $(I_x^f)^{\mathrm{op}}$ is filtered. Then, by Lemma 2.5, Δ_{PM} and Δ_{Mod} are isomorphisms. Thus $P \colon f_\#[b,c] \to [f_\# b, f_\# c]$ is defined for $b, c \in \mathrm{Mod}(\mathbb{Y})$ (see [9, (1.50)]; it is the composite

$$\begin{aligned}
f_\#[b,c] = af_\# q[b,c] &\xrightarrow{\varepsilon^{-1}} aqaf_\# q[b,c] \xrightarrow{\mathrm{tr}} a[qaf_\# qb, qaf_\# q[b,c] \otimes^p qaf_\# qb] \\
&\xrightarrow{\bar{P}} [aqaf_\# qb, a(qaf_\# q[b,c] \otimes^p qaf_\# qb)] \\
&\xrightarrow{\varepsilon^{-1}} [af_\# qb, a(qaf_\# q[b,c] \otimes^p qaf_\# qb)] \\
&\xrightarrow{(u \otimes^p u)^{-1}} [af_\# qb, a(f_\# q[b,c] \otimes^p f_\# qb)] \\
&\xrightarrow{\Delta^{-1}} [af_\# qb, af_\#(q[b,c] \otimes qb)] \\
&\xrightarrow{u} [af_\# qb, af_\# qa(q[b,c] \otimes qb)] \\
&\xrightarrow{H} [af_\# qb, af_\# qa([qb,qc] \otimes qb)] \\
&\xrightarrow{\mathrm{ev}} [af_\# qb, af_\# qaqc] \xrightarrow{\varepsilon} [af_\# qb, af_\# qc] = [f_\# b, f_\# c]
\end{aligned}$$

by [9, (2.48)] and Section 2.2. It is straightforward to check that this composite map agrees with

$$
\begin{aligned}
f_\#[b,c] = af_\# q[b,c] \xrightarrow{\bar{H}} af_\#[qb,qc] \xrightarrow{P} a[f_\# qb, f_\# qc] \\
\xrightarrow{\bar{P}} [af_\# qb, af_\# qc] = [f_\# b, f_\# c]. \qquad (1)
\end{aligned}
$$

2.8. Let $\mathbb{X}$ be as in Section 2.7. By the definition of P [9, (1.50)] and the explicit descriptions of tr, Δ, and ev in [9, (2.42)], Section 2.4, and [9, (2.41)], the map P_{PM} is described as follows:

$$
\begin{aligned}
\Gamma(x, f_\#[b,c]) = \varinjlim \Gamma(x, \mathcal{O}_{\mathbb{X}}) \otimes_{\Gamma(y,\mathcal{O}_{\mathbb{Y}})} \operatorname{Hom}_{\mathcal{O}_{\mathbb{Y}/y}}(b|_y, c|_y) \\
\to \operatorname{Hom}_{\mathcal{O}_{\mathbb{X}/x}}((f_\# b)|_x, (f_\# c)|_x) = \Gamma(x, [f_\# b, f_\# c]),
\end{aligned}
$$

which sends $\beta \otimes \varphi$ to the map that sends $\beta' \otimes \alpha$ to $\beta\beta' \otimes \varphi(\alpha)$ for $\beta \in \Gamma(x, \mathcal{O}_{\mathbb{X}})$, $\varphi\colon b|_y \to c|_y$, $\beta' \in \Gamma(x', \mathcal{O}_{\mathbb{X}})$, and $\alpha \in \Gamma(y', b)$ for some commutative diagram

$$
\begin{tikzcd}
x' \arrow[r] \arrow[d] & fy' \arrow[d, "f\rho"] & y' \arrow[d, "\rho"] \\
x \arrow[r] & fy & y,
\end{tikzcd}
$$

where the colimit is taken over $y \in (I_x^f)^{\mathrm{op}}$.

Thus we have the following statement.

2.9. Lemma. *Let $j\colon U \to X$ be an open immersion of ringed spaces. Then $P\colon j^*[b,c] \to [j^*b, j^*c]$ is an isomorphism for $b, c \in \heartsuit(X)$ for $\heartsuit = \mathrm{PM}, \mathrm{Mod}$.*

Proof. First consider the case $\heartsuit = \mathrm{PM}$. Then, for $V \subset U$,

$$\Gamma(V, P)\colon \Gamma(V, j^*[b,c]) \to \Gamma(V, [j^*b, j^*c])$$

is the identity map of $\operatorname{Hom}_{\mathcal{O}_V}(b|_V, c|_V)$. Thus it is an isomorphism.

Now consider the case of $\heartsuit = \mathrm{Mod}$. Then P_{Mod} is the composite

$$j^*[b,c] = aj^*q[b,c] \xrightarrow{\bar{H}} aj^*[qb,qc] \xrightarrow{P_{\mathrm{PM}}} a[j^*qb, j^*qc] \xrightarrow{\bar{P}} [aj^*qb, aj^*qc]$$

as described in Section 2.7. Note that $\bar{H}$ is an isomorphism by definition [9, Lemma 2.38]. The natural map P_{PM} is also an isomorphism, as we have just seen, and $\bar{P}$ is an isomorphism because j^*qc is a sheaf (see [9, (2.39)]). Thus P_{Mod} is also an isomorphism. □

2.10. Proposition. *Let $f\colon X \to Y$ be a morphism of schemes and let $b, c \in \mathrm{Mod}(Y)$. If one of the following conditions holds, then $P\colon f^*[b,c] \to [f^*b, f^*c]$ is an isomorphism.*

(i) *f is locally an open immersion—that is, there exists an open covering (U_λ) of X such that $f|_{U_\lambda}$ is an open immersion for every λ;*
(ii) *f is flat and b is coherent.*

Proof. (i) By [9, Lemma 1.54] we may assume that f is an open immersion, and this case is Lemma 2.9.

(ii) By [9, Lemma 1.54] and Lemma 2.9, we may assume that both X and Y are affine. Hence there is a presentation of the form $\mathcal{O}_Y^n \to \mathcal{O}_Y^m \to b \to 0$. By the five lemma, we may assume that $b = \mathcal{O}_Y$, and this case is easy. $\square$

2.11. Let I be a small category. For a category $\mathcal{C}$, the functor category $\mathrm{Func}(I^{\mathrm{op}}, \mathcal{C})$ is denoted by $\mathcal{P}(I, \mathcal{C})$. We denote the category of schemes by $\underline{\mathrm{Sch}}$. An object of $\mathcal{P}(I, \underline{\mathrm{Sch}})$ is called an I^{op}*-diagram* of schemes. Although in [9], whose notation we mainly use, diagrams of schemes are denoted by $X_\bullet, Y_\bullet, Z_\bullet, \dots$, we write $X, Y, Z, \dots$ for simplicity of notation. Similarly, morphisms in $\mathcal{P}(I, \underline{\mathrm{Sch}})$ are denoted by $f_\bullet, g_\bullet, h_\bullet, \dots$ in [9], but we use $f, g, h, \dots$.

Let $X \in \mathcal{P}(I, \underline{\mathrm{Sch}})$. For $i \in I$, $X(i)$ is denoted by X_i; for $\phi \in \mathrm{Mor}(I)$, $X(\phi)$ is denoted by X_ϕ. For a property of schemes $\mathbb{P}$, we say that X *satisfies* $\mathbb{P}$ if X_i satisfies $\mathbb{P}$ for every $i \in I$. Let $\mathbb{Q}$ be a property of morphisms. We say that X *has* $\mathbb{Q}$ *arrows* if X_ϕ satisfies $\mathbb{Q}$ for each $\phi \in \mathrm{Mor}(I)$. Let S be a scheme and consider the case $X \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. We say that X is $\mathbb{Q}$ *over* S if the structure morphism $X_i \to S$ satisfies $\mathbb{Q}$ for each i. Let $f : X \to Y$ be a morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}})$. We say that f is $\mathbb{Q}$ if f_i is $\mathbb{Q}$ for each i. We say that f is *Cartesian* if the commutative diagram $Y_\phi f_j = f_i X_\phi$ is a Cartesian square for each $(\phi : i \to j) \in \mathrm{Mor}(I)$.

For a subcategory J of I, the restriction of $X_\bullet$ to J was written $X_\bullet|_J$ in [9]. In this paper, $X_\bullet$ is written as X and $X|_J$ is likewise simplified to X_J. Similarly, for a morphism f of $\mathcal{P}(I, \underline{\mathrm{Sch}})$, the restriction of f to J is denoted by f_J rather than $f|_J$.

2.12. Let $X \in \mathcal{P}(I, \underline{\mathrm{Sch}})$. As in [9], we denote the category of $\mathcal{O}_X$-modules by $\mathrm{Mod}(X)$. Let $\mathcal{M} \in \mathrm{Mod}(X)$. The restriction of $\mathcal{M}$ to X_i is denoted by $\mathcal{M}_i$ for $i \in I$. We say that $\mathcal{M}$ is locally quasi-coherent (resp. locally coherent) if $\mathcal{M}_i$ is quasi-coherent (resp. coherent) for each $i \in I$. We say that $\mathcal{M}$ is quasi-coherent (resp. coherent) if $\mathcal{M}$ is locally quasi-coherent (resp. locally coherent) and equivariant [9, (4.14)]. We denote the full subcategory of $\mathrm{Mod}(X)$ consisting of equivariant (resp. locally quasi-coherent, locally coherent, quasi-coherent, coherent) sheaves by $\mathrm{EM}(X)$ (resp. $\mathrm{Lqc}(X)$, $\mathrm{Lch}(X)$, $\mathrm{Qch}(X)$, $\mathrm{Coh}(X)$). The derived categories such as $D(\mathrm{Mod}(X))$, $D^+_{\mathrm{Lqc}(X)}(\mathrm{Mod}(X))$, and $D^b_{\mathrm{Qch}(X)}(\mathrm{Lqc}(X))$ are denoted (respectively) by $D(X)$, $D^+_{\mathrm{Lqc}}(X)$, and $D^b_{\mathrm{Qch}}(\mathrm{Lqc}(X))$ for short.

2.13. LEMMA. *Let $f : X \to Y$ be a morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}})$ and let $b, c \in \mathrm{Mod}(Y)$. Then*

$$P : f^* \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(b, c) \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(f^* b, f^* c)$$

is an isomorphism if one of the following holds:

(i) *f is locally an open immersion and b is equivariant*;
(ii) *f is flat and b is coherent.*

Proof. By [9, Lemma 1.59], the diagram

$$\begin{tikzcd}
(?)_i f^*[b,c] \arrow[r, "\theta^{-1}"] \arrow[d, "P"] & f_i^*(?)_i[b,c] \arrow[r, "H"] & f_i^*[b_i, c_i] \arrow[d, "P"] \\
(?)_i[f^*b, f^*c] \arrow[r, "H"] & {[(?)_i f^*b, (?)_i f^*c]} \arrow[r, "{[\theta, \theta^{-1}]}"] & {[f_i^*(?)_i b, f_i^*(?)_i c]}
\end{tikzcd}$$

is commutative for every $i \in I$, where $[\cdot,\cdot]$ denotes the Hom sheaf. Since b is assumed to be equivariant in both cases, the horizontal arrows are isomorphisms by [9, Lemma 7.22] and [9, Lemma 6.33]. By Proposition 2.10, the rightmost vertical P is an isomorphism. Thus, the leftmost P is also an isomorphism. Since i is arbitrary, we are done. □

A morphism of schemes is said to be *concentrated* if it is quasi-compact and quasi-separated. A scheme X is said to be concentrated if the unique morphism $X \to \operatorname{Spec} \mathbb{Z}$ is concentrated.

2.14. LEMMA. *Let*

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "h"] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

be a Cartesian square in $\mathcal{P}(I, \underline{\mathrm{Sch}})$, and let $\mathcal{M} \in \mathrm{Mod}(Y')$. Then $\theta\colon f^*g_*\mathcal{M} \to h_*(f')^*\mathcal{M}$ is an isomorphism if one of the following holds:

(i) *f is locally an open immersion*;
(ii) *g is concentrated, f is flat, and $\mathcal{M} \in \mathrm{Lqc}(Y')$.*

Proof. Using [9, Lemma 1.22] twice, it is easy to see that the diagram

$$\begin{tikzcd}
f_i^*(g_i)_*(?)_i \arrow[r, "c"] \arrow[d, "\theta"] & f_i^*(?)_i g_* \arrow[r, "\theta"] & (?)_i f^* g_* \arrow[d, "\theta"] \\
(h_i)_*(f_i')^*(?)_i \arrow[r, "\theta"] & (h_i)_*(?)_i(f')^* \arrow[r, "c"] & (?)_i h_*(f')^*
\end{tikzcd}$$

is commutative. Because the horizontal arrows are isomorphisms, we may assume that the all diagrams of schemes are single schemes.

For (i), the case of f an open immersion is proved in the first paragraph of the proof of [9, Lemma 7.12], and the general case follows from [9, Lemma 1.23]. Part (ii) is nothing but [9, Lemma 7.12]. □

3. Local Section Functor for Diagrams

3.1. Let X be an I^{op}-diagram of schemes. As in [9], the category of presheaves and sheaves of abelian groups are denoted by $\mathrm{PA}(X)$ and $\mathrm{AB}(X)$, respectively. Let U be an open subdiagram of schemes of X, and let V be an open subdiagram of schemes of U. Let both $f: U \to X$ and $g: V \to U$ be the inclusion.

For $\mathcal{M} \in \mathrm{Mod}(X)$ or $\mathcal{M} \in \mathrm{AB}(X)$, we denote the kernel of the unit of adjunction

$$u\colon f_* f^* \mathcal{M} \to f_* g_* g^* f^* \mathcal{M}$$

by $\underline{\Gamma}_{U,V}\mathcal{M}$. We denote the canonical inclusion $\underline{\Gamma}_{U,V}\mathcal{M} \hookrightarrow f_* f^* \mathcal{M}$ by ι. Note that the formation of $\underline{\Gamma}_{U,V}$ is compatible with the forgetful functor $\mathrm{Mod}(X) \to \mathrm{AB}(X)$.

If $U = X$ then there is an exact sequence

$$0 \to \underline{\Gamma}_{X,V}\mathcal{M} \xrightarrow{\iota'} \mathcal{M} \xrightarrow{u} g_* g^* \mathcal{M},$$

where ι' is the composite

$$\underline{\Gamma}_{X,V}\mathcal{M} \xrightarrow{\iota} (\mathrm{id}_X)_*(\mathrm{id}_X)^*\mathcal{M} \xrightarrow{u^{-1}} \mathcal{M}.$$

3.2. LEMMA. $\underline{\Gamma}_{U,V} : \mathrm{Mod}(X) \to \mathrm{Mod}(X)$ *is a left exact functor.*

Proof. Let $0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N}$ be an exact sequence in $\mathrm{Mod}(X)$. Since f^* and g^* are exact and since f_* and g_* are left-exact, it follows that the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \underline{\Gamma}_{U,V}\mathcal{L} \arrow[r, "\iota"] \arrow[d] & f_* f^* \mathcal{L} \arrow[r, "u"] \arrow[d] & f_* g_* g^* f^* \mathcal{L} \arrow[d] \\
0 \arrow[r] & \underline{\Gamma}_{U,V}\mathcal{M} \arrow[r, "\iota"] \arrow[d] & f_* f^* \mathcal{M} \arrow[r, "u"] \arrow[d] & f_* g_* g^* f^* \mathcal{M} \arrow[d] \\
0 \arrow[r] & \underline{\Gamma}_{U,V}\mathcal{N} \arrow[r, "\iota"] & f_* f^* \mathcal{N} \arrow[r, "u"] & f_* g_* g^* f^* \mathcal{N}
\end{tikzcd}$$

has exact rows and that the second and the third columns are exact. Hence the first column is exact, and the assertion follows. $\square$

3.3. Let X, U, V, f, and g be as in Section 3.1. Let J be a subcategory of I, and let $\mathcal{M} \in \mathrm{Mod}(X)$. Then we have the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \underline{\Gamma}_{U_J,V_J}\mathcal{M}_J \arrow[r, "\iota"] & (f_J)_* f_J^* \mathcal{M}_J \arrow[r, "u"] \arrow[d, "c^{-1}\theta"] & (f_J)_*(g_J)_* g_J^* f_J^* \mathcal{M}_J \arrow[d, "c^{-1}c^{-1}\theta\theta"] \\
0 \arrow[r] & (\underline{\Gamma}_{U,V}\mathcal{M})_J \arrow[r, "\iota"] & (f_* f^* \mathcal{M})_J \arrow[r, "u"] & (f_* g_* g^* f^* \mathcal{M})_J,
\end{tikzcd}$$

where $c^{-1}\theta$ is the composite isomorphism

$$(f_J)_* f_J^*(?)_J \xrightarrow{\theta} (f_J)_*(?)_J f^* \xrightarrow{c^{-1}} (?)_J f_* f^*;$$

see [9, Example 5.6, **2**], [9, Lemma 6.25], and [9, Lemma 1.24]. Similarly, $c^{-1}c^{-1}\theta\theta$ is the composite

$$\begin{aligned}(f_J)_*(g_J)_* g_J^* f_J^*(?)_J \xrightarrow{\theta} (f_J)_*(g_J)_* g_J^*(?)_J f^* \xrightarrow{\theta} (f_J)_*(g_J)_*(?)_J g^* f^* \\ \xrightarrow{c^{-1}} (f_J)_*(?)_J g_* g^* f^* \xrightarrow{c^{-1}} (?)_J f_* g_* g^* f^*.\end{aligned}$$

Thus we obtain a unique natural map

$$\hat{\gamma} = \hat{\gamma}_{U,V,J} \colon \underline{\Gamma}_{U_J,V_J}(?)_J \to (?)_J \underline{\Gamma}_{U,V}$$

such that $\iota\hat{\gamma} = c^{-1}\theta\iota$. Note that $\hat{\gamma}$ is an isomorphism by the five lemma.

3.4. Lemma. *With notation as before, let K be a subcategory of J. Then the composite*

$$\begin{aligned}\underline{\Gamma}_{U_K,V_K}(?)_K \xrightarrow{c} \underline{\Gamma}_{U_K,V_K}(?)_{K,J}(?)_J \xrightarrow{\hat{\gamma}} (?)_{K,J}\underline{\Gamma}_{U_J,V_J}(?)_J \\ \xrightarrow{\hat{\gamma}} (?)_{K,J}(?)_J\underline{\Gamma}_{U,V} \xrightarrow{c^{-1}} (?)_K\underline{\Gamma}_{U,V}\end{aligned}$$

is $\hat{\gamma}_{U,V,K}$.

Proof. Consider the diagram

$$\begin{tikzcd}
\underline{\Gamma}_{U_K,V_K}(?)_K \arrow[r, "\iota"] \arrow[d, "c"] \arrow[dr, phantom, "\text{(a)}"] & (f_K)_* f_K^*(?)_K \arrow[d, "c"] \arrow[dddd, bend left=80, "c^{-1}\theta"] \\
\underline{\Gamma}_{U_K,V_K}(?)_{K,J}(?)_J \arrow[r, "\iota"] \arrow[d, "\hat{\gamma}"] \arrow[dr, phantom, "\text{(b)}"] & (f_K)_* f_K^*(?)_{K,J}(?)_J \arrow[d, "c^{-1}\theta"] \arrow[d, phantom, shift left=10ex, "\text{(e)}"] \\
(?)_{K,J}\underline{\Gamma}_{U_J,V_J}(?)_J \arrow[r, "\iota"] \arrow[d, "\hat{\gamma}"] \arrow[dr, phantom, "\text{(c)}"] & (?)_{K,J}(f_J)_* f_J^*(?)_J \arrow[d, "c^{-1}\theta"] \\
(?)_{K,J}(?)_J\underline{\Gamma}_{U,V} \arrow[r, "\iota"] \arrow[d, "c^{-1}"] \arrow[dr, phantom, "\text{(d)}"] & (?)_{K,J}(?)_J f_* f^* \arrow[d, "c^{-1}"] \\
(?)_K\underline{\Gamma}_{U,V} \arrow[r, "\iota"] & (?)_K f_* f^*.
\end{tikzcd}$$

The commutativity of (a) and (d) is trivial. The commutativity of (b) and (c) follows from the definition of $\hat{\gamma}$. The commutativity of (e) is a consequence of [9, Lemma 1.4] and [9, Lemma 1.22]. So the whole diagram is commutative, and the assertion then follows from the definition of $\hat{\gamma}$. $\square$

3.5. Let

$$\begin{tikzcd} V' \arrow[r, "g'"] \arrow[d, "h_V"'] \arrow[dr, phantom, "\text{(a)}"] & U' \arrow[r, "f'"] \arrow[d, "h_U"] \arrow[dr, phantom, "\text{(b)}"] & X' \arrow[d, "h"] \\ V \arrow[r, "g"] & U \arrow[r, "f"] & X \end{tikzcd} \tag{2}$$

be a commutative diagram in $\mathcal{P}(I, \underline{\mathrm{Sch}})$ such that the horizontal arrows are inclusion maps of open subdiagrams.

By [9, Lemma 1.24], we have the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \underline{\Gamma}_{U,V} h_* \arrow[r, "\iota"] & f_* f^* h_* \arrow[r, "u"] \arrow[d, "c\theta"] & f_* g_* g^* f^* h_* \arrow[d, "cc\theta\theta"] \\ 0 \arrow[r] & h_* \underline{\Gamma}_{U',V'} \arrow[r, "\iota"] & h_* f'_*(f')^* \arrow[r, "u"] & h_* f'_* g'_*(g')^*(f')^*, \end{tikzcd} \tag{3}$$

where $c\theta$ is the composite

$$f_* f^* h_* \xrightarrow{\theta} f_*(h_U)_*(f')^* \xrightarrow{c} h_* f'_*(f')^*$$

and $cc\theta\theta$ is the composite

$$\begin{aligned} f_* g_* g^* f^* h_* &\xrightarrow{\theta} f_* g_* g^*(h_U)_*(f')^* \xrightarrow{\theta} f_* g_*(h_V)_*(g')^*(f')^* \\ &\xrightarrow{c} f_*(h_U)_* g'_*(g')^*(f')^* \xrightarrow{c} h_* f'_* g'_*(g')^*(f')^*. \end{aligned}$$

In this way we induce the unique natural map

$$\bar{\gamma} = \bar{\gamma}_{U,V,U',V',h} \colon \underline{\Gamma}_{U,V} h_* \to h_* \underline{\Gamma}_{U',V'}$$

such that $\iota\bar{\gamma} = c\theta\iota$. In particular, considering the case where $X' = X$ and h is the identity,

$$\bar{\gamma} = \bar{\gamma}_{U,V,U',V'} \colon \underline{\Gamma}_{U,V} \to \underline{\Gamma}_{U',V'}$$

is defined.

3.6. LEMMA. *Assume that* (a) *and* (b) *in diagram* (2) *are Cartesian. Then* $\bar{\gamma}_{U,V,U',V',h}$ *is an isomorphism.*

Proof. By the five lemma, it suffices to show that $c\theta$ and $cc\theta\theta$ in (3) are isomorphisms. Yet this is an immediate consequence of Lemma 2.14. $\square$

3.7. Lemma. *Let*

$$\begin{tikzcd}
V'' \arrow[r, "g''"] \arrow[d, "k_V"] & U'' \arrow[r, "f''"] \arrow[d, "k_U"] & X'' \arrow[d, "k"] \\
V' \arrow[r, "g'"] \arrow[d, "h_V"] & U' \arrow[r, "f'"] \arrow[d, "h_U"] & X' \arrow[d, "h"] \\
V \arrow[r, "g"] & U \arrow[r, "f"] & X
\end{tikzcd}$$

be a commutative diagram in $\mathcal{P}(I, \underline{\mathrm{Sch}})$ such that the horizontal maps are inclusions of open subdiagrams. Then the composite

$$\underline{\Gamma}_{U,V}(hk)_* \xrightarrow{c} \underline{\Gamma}_{U,V} h_* k_* \xrightarrow{\bar{\gamma}} h_* \underline{\Gamma}_{U',V'} k_* \xrightarrow{\bar{\gamma}} h_* k_* \underline{\Gamma}_{U'',V''} \xrightarrow{c^{-1}} (hk)_* \underline{\Gamma}_{U'',V''}$$

equals $\bar{\gamma}$.

Proof. Consider the diagram

$$\begin{tikzcd}
\underline{\Gamma}_{U,V}(hk)_* \arrow[r, "\iota"] \arrow[d, "c"] \arrow[dr, phantom, "\text{(a)}"] & f_* f^*(hk)_* \arrow[d, "c"] \arrow[dddd, bend left=60, "c\theta"] \\
\underline{\Gamma}_{U,V} h_* k_* \arrow[r, "\iota"] \arrow[d, "\bar{\gamma}"] \arrow[dr, phantom, "\text{(b)}"] & f_* f^* h_* k_* \arrow[d, "c\theta"] \arrow[r, phantom, "\text{(e)}"] & {} \\
h_* \underline{\Gamma}_{U',V'} k_* \arrow[r, "\iota"] \arrow[d, "\bar{\gamma}"] \arrow[dr, phantom, "\text{(c)}"] & h_* f'_*(f')^* k_* \arrow[d, "c\theta"] \\
h_* k_* \underline{\Gamma}_{U'',V''} \arrow[r, "\iota"] \arrow[d, "c^{-1}"] \arrow[dr, phantom, "\text{(d)}"] & h_* k_* f''_*(f'')^* \arrow[d, "c^{-1}"] \\
(hk)_* \underline{\Gamma}_{U'',V''} \arrow[r, "\iota"] & (hk)_* f''_*(f'')^*.
\end{tikzcd}$$

The commutativity of (a) and (d) is trivial. The commutativity of (b) and (c) follows from the definition of $\bar{\gamma}$. The commutativity of (e) is a consequence of [9, Lemma 1.4] and [9, Lemma 1.22]. So the whole diagram is commutative, and the assertion then follows from the definition of $\bar{\gamma}$. □

3.8. Lemma. *Let* (2) *be as in Section 3.5 and let J be a subcategory of I. Then the diagram*

$$\begin{tikzcd}
\underline{\Gamma}_{U_J,V_J}(?)_J h_* \arrow[r, "\hat{\gamma}"] \arrow[d, "c"] & (?)_J \underline{\Gamma}_{U,V} h_* \arrow[r, "\bar{\gamma}"] & (?)_J h_* \underline{\Gamma}_{U',V'} \arrow[d, "c"] \\
\underline{\Gamma}_{U_J,V_J}(h_J)_*(?)_J \arrow[r, "\bar{\gamma}"] & (h_J)_* \underline{\Gamma}_{U'_J,V'_J}(?)_J \arrow[r, "\hat{\gamma}"] & (h_J)_*(?)_J \underline{\Gamma}_{U',V'}
\end{tikzcd}$$

is commutative.

Proof. Consider the diagram

$$\begin{tikzcd}
\underline{\Gamma}_{U_J,V_J}(?)_J h_* \arrow[r, "\hat{\gamma}"] \arrow[d, "\iota"] \arrow[ddd, bend right=60, "c"'] \arrow[dr, phantom, "\text{(a)}"] & (?)_J \underline{\Gamma}_{U,V} h_* \arrow[r, "\bar{\gamma}"] \arrow[d, "\iota"] \arrow[dr, phantom, "\text{(b)}"] & (?)_J h_* \underline{\Gamma}_{U',V'} \arrow[d, "\iota"] \arrow[ddd, bend left=60, "c"] \\
(f_J)_* f_J^*(?)_J h_* \arrow[r, "c^{-1}\theta"] \arrow[d, "c"] \arrow[d, phantom, shift right=12ex, "\text{(c)}"] & (?)_J f_* f^* h_* \arrow[r, "c\theta"] \arrow[d, phantom, "\text{(d)}"] & (?)_J h_* f'_*(f')^* \arrow[d, "c"] \arrow[d, phantom, shift left=12ex, "\text{(e)}"] \\
(f_J)_* f_J^*(h_J)_*(?)_J \arrow[r, "c\theta"] & (h_J)_*(f'_J)_*(f'_J)^*(?)_J \arrow[r, "c^{-1}\theta"] & (h_J)_*(?)_J f'_*(f')^* \\
\underline{\Gamma}_{U_J,V_J}(h_J)_*(?)_J \arrow[r, "\bar{\gamma}"] \arrow[u, "\iota"] \arrow[ur, phantom, "\text{(f)}"] & (h_J)_* \underline{\Gamma}_{U'_J,V'_J}(?)_J \arrow[r, "\hat{\gamma}"] \arrow[u, "\iota"] \arrow[ur, phantom, "\text{(g)}"] & (h_J)_*(?)_J \underline{\Gamma}_{U',V'}. \arrow[u, "\iota"]
\end{tikzcd}$$

By the definition of $\hat{\gamma}$, (a) and (g) are commutative; by the definition of $\bar{\gamma}$, (b) and (f) are commutative. The commutativity of (c) and (e) is trivial, and the commutativity of (d) follows from [9, Lemma 1.4] and [9, Lemma 1.22]. Since ι is a monomorphism, the assertion follows. $\square$

3.9. LEMMA. *Given* (2) *as before, assume $X = X'$ and $h = \mathrm{id}$. If $U_i \setminus V_i = U'_i \setminus V'_i$ for any $i \in I$, then $\bar{\gamma}_{U,V,U',V'} \colon \underline{\Gamma}_{U,V} \to \underline{\Gamma}_{U',V'}$ is an isomorphism.*

Proof. In view of Lemma 3.8, we may assume that X is a single scheme. Let $\mathcal{M} \in \mathrm{AB}(X)$ or $\mathcal{M} \in \mathrm{Mod}(X)$. For any open set $W \subset X$, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(W, \underline{\Gamma}_{U,V}\mathcal{M}) \arrow[r, "\iota"] \arrow[d, "\bar{\gamma}"] & \Gamma(W \cap U, \mathcal{M}) \arrow[r, "\mathrm{res}"] \arrow[d, "\mathrm{res}"] & \Gamma(W \cap V, \mathcal{M}) \arrow[d, "\mathrm{res}"] \\
0 \arrow[r] & \Gamma(W, \underline{\Gamma}_{U',V'}\mathcal{M}) \arrow[r, "\iota"] & \Gamma(W \cap U', \mathcal{M}) \arrow[r, "\mathrm{res}"] & \Gamma(W \cap V', \mathcal{M}).
\end{tikzcd}$$

By assumption, $U = V \cup U'$ and $V' = V \cap U'$. Hence

$$0 \to \Gamma(W \cap U, \mathcal{M}) \to \Gamma(W \cap V, \mathcal{M}) \oplus \Gamma(W \cap U', \mathcal{M}) \to \Gamma(W \cap V', \mathcal{M})$$

is exact. Thus $\bar{\gamma}$ is bijective, as can be seen easily. $\square$

3.10. Let $X \in \mathcal{P}(I, \underline{\mathrm{Sch}})$. Let Y be a Cartesian closed subdiagram of schemes of X; that is, Y is a subdiagram of schemes such that the inclusion $Y \hookrightarrow X$ is a Cartesian closed immersion. Let Z be a Cartesian closed subdiagram of schemes of Y. If we now let $U_i = X_i \setminus Z_i$ then $U = U(Z)$ is a Cartesian open subdiagram of schemes of X, and if we let $V_i = X_i \setminus Y_i$ then $V = U(Y)$ is a Cartesian open subdiagram of schemes of U. Thus $\underline{\Gamma}_{Y;Z} := \underline{\Gamma}_{U(Z),U(Y)}$ is defined.

If Z is empty, then $\underline{\Gamma}_{Y;\emptyset}$ is denoted by $\underline{\Gamma}_Y$. There is an exact sequence

$$0 \to \underline{\Gamma}_Y \xrightarrow{\iota'} \mathrm{Id} \xrightarrow{u} g_* g^*,$$

where $g \colon U(Y) \to X$ is the inclusion.

For a subcategory J of I, we have $U(Y)_J = U(Y_J)$ and $U(Z)_J = U(Z_J)$. Thus the isomorphism

$$\hat{\gamma}_{Y;Z;J} := \hat{\gamma}_{U(Z),U(Y),J} \colon \underline{\Gamma}_{Y_J;Z_J}(?)_J \to (?)_J \underline{\Gamma}_{Y;Z}$$

is defined (see Section 3.3). We denote $\hat{\gamma}_{Y;\emptyset;J}$ by $\hat{\gamma}_{Y;J}$.

3.11. Let notation be as before, and let $h : X' \to X$ be a morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}})$. Then $Y' := h^{-1}(Y)$ is a Cartesian closed subdiagram of X' and $Z' := h^{-1}(Z)$ is a Cartesian closed subdiagram of Y'. Thus

$$\bar{\gamma}_{Y;Z;h} := \bar{\gamma}_{U(Z),U(Y),U(Z'),U(Y'),h}$$

is defined (see Section 3.5), and we denote $\bar{\gamma}_{Y;\emptyset;h}$ by $\bar{\gamma}_{Y;h}$. By Lemma 3.6, we immediately have the following result.

3.12. Lemma. *Let notation be as before. Then $\bar{\gamma}_{Y;Z;h}$ is an isomorphism.*

3.13. Let $X \in \mathcal{P}(I, \underline{\mathrm{Sch}})$. A collection $Z = (Z_i)_{i \in I}$ is called a *locally closed system* of X if there exist some open subdiagram of schemes U of X and an open subdiagram of schemes V of U such that $Z_i = U_i \setminus V_i$. Such a pair (U, V) is called a *UV-pair* of Z. If Z is a locally closed system of X, then Z_i is a locally closed subset of X_i for any i. If $((U_\lambda, V_\lambda))$ is a family of UV-pairs of Z, then $\left(\bigcup U_\lambda, \bigcup V_\lambda\right)$ is also a UV-pair of Z. So if Z is a locally closed system of X, then there is a largest UV-pair $(U(Z), V(Z))$ of Z.

We define $\underline{\Gamma}_Z := \underline{\Gamma}_{U(Z),V(Z)}$ for a locally closed system Z of X. If (U, V) is a UV-pair of Z, then $\bar{\gamma} \colon \underline{\Gamma}_Z \to \underline{\Gamma}_{U,V}$ is an isomorphism by Lemma 3.9. If Z is a Cartesian closed subdiagram of schemes of X, then Z can be viewed as a locally closed system of X and thus $\underline{\Gamma}_Z$ is defined. This definition of $\underline{\Gamma}_Z$ agrees with the one in Section 3.10, so there is no conflict.

3.14. Let the commutative diagram (2) be as in Section 3.5. Assume that h is flat. Then there is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & h^*\underline{\Gamma}_{U,V} \arrow[r, "\iota"] & h^* f_* f^* \arrow[r, "u"] \arrow[d, "d\theta"] & h^* f_* g_* g^* f^* \arrow[d, "dd\theta\theta"] \\
0 \arrow[r] & \underline{\Gamma}_{U',V'} h^* \arrow[r, "\iota"] & f'_*(f')^* h^* \arrow[r, "u"] & f'_* g'_* (g')^* (f')^* h^*,
\end{tikzcd}$$

where $d\theta$ is the composite

$$h^* f_* f^* \xrightarrow{\theta} f'_* h_U^* f^* \xrightarrow{d} f'_*(f')^* h^*$$

and $dd\theta\theta$ is the composite

$$h^* f_* g_* g^* f^* \xrightarrow{\theta} f'_* h_U^* g_* g^* f^* \xrightarrow{\theta} f'_* g'_* h_V^* g^* f^*$$
$$\xrightarrow{d} f'_* g'_* (g')^* h_U^* f^* \xrightarrow{d} f'_* g'_* (g')^* (f')^* h^*.$$

Hence there is a unique natural map $\bar{\delta} = \bar{\delta}_{U,V,U',V',h} \colon h^*\underline{\Gamma}_{U,V} \to \underline{\Gamma}_{U',V'} h^*$ such that $\iota\bar{\delta} = d\theta\iota$.

3.15. LEMMA. *Assume that the squares* (a) *and* (b) *in* (2) *are Cartesian and that h is flat. Let $\mathcal{M} \in \mathrm{Mod}(X)$. Then $\bar{\delta}\colon h^*\underline{\Gamma}_{U,V}\mathcal{M} \to \underline{\Gamma}_{U',V'}h^*\mathcal{M}$ is an isomorphism if one of the following conditions holds*:

(i) *h is locally an open immersion*;

(ii) *f and g are quasi-compact and $\mathcal{M}$ is locally quasi-coherent.*

Proof. In both cases, $\theta\colon h^*f_* \to f'_*h_U^*$ and $\theta\colon h_U^*g_* \to g'_*h_V^*$ are isomorphisms by Lemma 2.14. The assertion then follows from the five lemma. $\square$

3.16. Let notation be as described in Section 3.11, and assume that h is flat. Then we define $\bar{\delta}_{Y;Z;h} := \bar{\delta}_{U(Z),U(Y),h^{-1}(U(Z)),h^{-1}(U(Y)),h}$. By Lemma 3.15, if h is locally an open immersion or if X is locally Noetherian and $\mathcal{M}$ is locally quasi-coherent, then $\bar{\delta}_{Y;Z;h}$ is an isomorphism.

3.17. LEMMA. *Given the notation of Lemma 3.7, assume that h and k are flat. Then the composite*

$$(hk)^*\underline{\Gamma}_{U,V} \xrightarrow{d^{-1}} k^*h^*\underline{\Gamma}_{U,V} \xrightarrow{\bar{\delta}} k^*\underline{\Gamma}_{U',V'}h^* \xrightarrow{\bar{\delta}} \underline{\Gamma}_{U'',V''}k^*h^* \xrightarrow{d} \underline{\Gamma}_{U'',V''}(hk)^*$$

is $\bar{\delta}$.

Proof. Consider the diagram

$$\begin{tikzcd}
(hk)^*\underline{\Gamma}_{U,V} \arrow[r, "\iota"] \arrow[d, "d^{-1}"] \arrow[dr, phantom, "\text{(a)}"] & (hk)^*f_*f^* \arrow[d, "d^{-1}"] \arrow[dddd, bend left=70, "d\theta"] \\
k^*h^*\underline{\Gamma}_{U,V} \arrow[r, "\iota"] \arrow[d, "\bar{\delta}"] \arrow[dr, phantom, "\text{(b)}"] & k^*h^*f_*f^* \arrow[d, "d\theta"] \arrow[d, phantom, shift left=8ex, "\text{(e)}"] \\
k^*\underline{\Gamma}_{U',V'}h^* \arrow[r, "\iota"] \arrow[d, "\bar{\delta}"] \arrow[dr, phantom, "\text{(c)}"] & k^*f'_*(f')^*h^* \arrow[d, "d\theta"] \\
\underline{\Gamma}_{U'',V''}k^*h^* \arrow[r, "\iota"] \arrow[d, "d"] \arrow[dr, phantom, "\text{(d)}"] & f''_*(f'')^*k^*h^* \arrow[d, "d"] \\
\underline{\Gamma}_{U'',V''}(hk)^* \arrow[r, "\iota"] & f''_*(f'')^*(hk)^*.
\end{tikzcd}$$

The commutativity of (a) and (d) is trivial. The commutativity of (b) and (c) follows from the definition of $\bar{\delta}$. The commutativity of (e) is a consequence of the opposite assertion of [9, Lemma 1.4] and [9, Lemma 1.23]. So the whole diagram is commutative, and the assertion then follows from the definition of $\bar{\delta}$. $\square$

3.18. LEMMA. *With notation as in Lemma 3.8, assume that h is flat. Then the diagram*

$$\begin{tikzcd}
h_J^*(?)_J\underline{\Gamma}_{U,V} \arrow[r, "\hat{\gamma}^{-1}"] \arrow[d, "\theta"] & h_J^*\underline{\Gamma}_{U_J,V_J}(?)_J \arrow[r, "\bar{\delta}"] & \underline{\Gamma}_{U'_J,V'_J}h_J^*(?)_J \arrow[d, "\theta"] \\
(?)_J h^*\underline{\Gamma}_{U,V} \arrow[r, "\bar{\delta}"] & (?)_J\underline{\Gamma}_{U',V'}h^* \arrow[r, "\hat{\gamma}^{-1}"] & \underline{\Gamma}_{U'_J,V'_J}(?)_J h^*
\end{tikzcd}$$

is commutative.

Proof. Consider the diagram

$$\begin{tikzcd}
h_J^*(?)_J\underline{\Gamma}_{U,V} \arrow[d, "\iota"] \arrow[ddd, bend right=60, "\theta"'] \arrow[dr, phantom, "\text{(a)}"] & h_J^*\underline{\Gamma}_{U_J,V_J}(?)_J \arrow[l, "\hat{\gamma}"'] \arrow[r, "\bar{\delta}"] \arrow[d, "\iota"] \arrow[dr, phantom, "\text{(b)}"] & \underline{\Gamma}_{U'_J,V'_J}h_J^*(?)_J \arrow[d, "\iota"] \arrow[ddd, bend left=60, "\theta"] \\
h_J^*(?)_J f_* f^* \arrow[d, "\theta"] \arrow[d, phantom, "\text{(c)}"', shift right=6ex] \arrow[drr, phantom, "\text{(d)}"] & h_J^*(f_J)_* f_J^*(?)_J \arrow[l, "c^{-1}\theta"'] \arrow[r, "d\theta"] & (f'_J)_*(f'_J)^*h_J^*(?)_J \arrow[d, "\theta"] \arrow[d, phantom, "\text{(e)}", shift left=6ex] \\
(?)_J h^* f_* f^* \arrow[r, "d\theta"] & (?)_J f'_*(f')^*h^* & (f'_J)_*(f'_J)^*(?)_J h^* \arrow[l, "c^{-1}\theta"'] \\
(?)_J h^*\underline{\Gamma}_{U,V} \arrow[u, "\iota"] \arrow[r, "\bar{\delta}"] \arrow[ur, phantom, "\text{(f)}"] & (?)_J\underline{\Gamma}_{U',V'}h^* \arrow[u, "\iota"] \arrow[ur, phantom, "\text{(g)}"] & \underline{\Gamma}_{U'_J,V'_J}(?)_J h^*. \arrow[l, "\hat{\gamma}"'] \arrow[u, "\iota"]
\end{tikzcd}$$

By the definition of $\hat{\gamma}$, (a) and (g) are commutative; by the definition of $\bar{\delta}$, (b) and (f) are commutative. The commutativity of (c) and (e) is trivial, and the commutativity of (d) follows from [9, Lemma 1.22] and [9, Lemma 1.23]. Since ι is a monomorphism, the assertion follows. □

3.19. LEMMA. *Let*

$$\begin{tikzcd}
 & V_{X'} \arrow[rr, "g_{X'}"] \arrow[dd, "h_V"] & & U_{X'} \arrow[rr, "f_{X'}"] \arrow[dd, "h_U"] & & X' \arrow[dd, "h"] \\
V_{Z'} \arrow[ur, "k'_V"] \arrow[rr, "g_{Z'}"] \arrow[dd, "\bar{h}_V"] & & U_{Z'} \arrow[ur, "k'_U"] \arrow[rr, "f_{Z'}"] \arrow[dd, "\bar{h}_U"] & & Z' \arrow[ur, "k'"] \arrow[dd, "\bar{h}"] & \\
 & V \arrow[rr, "g"] & & U \arrow[rr, "f"] & & X \\
V_Z \arrow[ur, "k_V"] \arrow[rr, "g_Z"] & & U_Z \arrow[ur, "k_U"] \arrow[rr, "f_Z"] & & Z \arrow[ur, "k"] &
\end{tikzcd}$$

be a commutative diagram in $\mathcal{P}(I, \underline{\mathrm{Sch}})$. Assume that f, f_Z, $f_{X'}$, $f_{Z'}$, g, g_Z, $g_{X'}$, and $g_{Z'}$ are inclusions of open subdiagrams, and assume that h and $\bar{h}$ are flat. Then the diagram

$$
\begin{tikzcd}
h^*k_*\underline{\Gamma}_{U_Z,V_Z} \arrow[d, "\theta"] & h^*\underline{\Gamma}_{U,V}k_* \arrow[l, "\bar{\gamma}"'] \arrow[r, "\bar{\delta}"] & \underline{\Gamma}_{U_{X'},V_{X'}}h^*k_* \arrow[d, "\theta"] \\
k'_*\bar{h}^*\underline{\Gamma}_{U_Z,V_Z} \arrow[r, "\bar{\delta}"] & k'_*\underline{\Gamma}_{U_{Z'},V_{Z'}}\bar{h}^* & \underline{\Gamma}_{U_{X'},V_{X'}}k'_*\bar{h}^* \arrow[l, "\bar{\gamma}"']
\end{tikzcd}
$$

is commutative.

Proof. Consider the diagram

$$
\begin{tikzcd}
h^*k_*\underline{\Gamma}_{U_Z,V_Z} \arrow[d, "\iota"] \arrow[dr, phantom, "\text{(a)}"] \arrow[ddd, bend right=60, "\theta"'] & h^*\underline{\Gamma}_{U,V}k_* \arrow[l, "\bar{\gamma}"'] \arrow[r, "\bar{\delta}"] \arrow[d, "\iota"] \arrow[dr, phantom, "\text{(b)}"] & \underline{\Gamma}_{U_{X'},V_{X'}}h^*k_* \arrow[d, "\iota"] \arrow[ddd, bend left=60, "\theta"] \\
h^*k_*(f_Z)_*f_Z^* \arrow[d, "\theta"] \arrow[d, phantom, shift right=10ex, "\text{(c)}"] \arrow[drr, phantom, "\text{(d)}"] & h^*f_*f^*k_* \arrow[l, "c\theta"'] \arrow[r, "d\theta"] & (f_{X'})_*f_{X'}^*h^*k_* \arrow[d, "\theta"] \arrow[d, phantom, shift left=10ex, "\text{(e)}"] \\
k'_*\bar{h}^*(f_Z)_*f_Z^* \arrow[r, "d\theta"] & k'_*(f_{Z'})_*f_{Z'}^*\bar{h}^* & (f_{X'})_*f_{X'}^*k'_*\bar{h}^* \arrow[l, "c\theta"'] \\
k'_*\bar{h}^*\underline{\Gamma}_{U_Z,V_Z} \arrow[u, "\iota"] \arrow[r, "\bar{\delta}"] \arrow[ur, phantom, "\text{(f)}"] & k'_*\underline{\Gamma}_{U_{Z'},V_{Z'}}\bar{h}^* \arrow[u, "\iota"] \arrow[ur, phantom, "\text{(g)}"] & \underline{\Gamma}_{U_{X'},V_{X'}}k'_*\bar{h}^*. \arrow[u, "\iota"] \arrow[l, "\bar{\gamma}"']
\end{tikzcd}
$$

By the definition of $\bar{\gamma}$, (a) and (g) are commutative; by the definition of $\bar{\delta}$, (b) and (f) are commutative. The commutativity of (c) and (e) is trivial, and the commutativity of (d) follows from [9, Lemma 1.22] and [9, Lemma 1.23]. Since ι is a monomorphism, the assertion follows. $\square$

3.20. Let $X \in \mathcal{P}(I, \underline{\mathrm{Sch}})$, and assume that X has flat arrows. Let Y be a Cartesian closed subdiagram of schemes of X (i.e., a closed subdiagram such that the inclusion $j \colon Y \hookrightarrow X$ is Cartesian) so that the defining ideal $\mathcal{I}$ of Y is quasi-coherent. Set $U := X \setminus Y$. Then U is an open subdiagram of schemes of X. Note that $f \colon U \to X$ is also Cartesian.

Because the sequence

$$\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{I} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}^n \to 0$$

is exact, $\mathcal{O}_X/\mathcal{I}^n$ is coherent for $n \geq 1$, since coherent sheaves are closed under tensor products and cokernels. Applying $(?)_i$ to the exact sequence, we obtain $(\mathcal{O}_X/\mathcal{I}^n)_i \cong \mathcal{O}_{X_i}/\mathcal{I}_i^n$.

For $\mathcal{M} \in \mathrm{Mod}(X)$, there is a canonical monomorphism

$$\Phi_Y \colon \varinjlim \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathcal{M}) \to \mathcal{M}$$

induced by the obvious maps

$$\Phi_n \colon \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathcal{M}) \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M}) \cong \mathcal{M}.$$

The composite

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathcal{M}) \xrightarrow{\Phi_Y} \mathcal{M} \xrightarrow{u} f_*f^*\mathcal{M} \tag{4}$$

factors through

$$f_*f^*\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathcal{M}) \cong f_*\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(f^*(\mathcal{O}_X/\mathcal{I}^n), f^*\mathcal{M}).$$

Since $f^*(\mathcal{O}_X/\mathcal{I}^n) = 0$, it follows that (4) is zero; thus we induce the monomorphism

$$\rho_Y\colon\ \varinjlim \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathcal{M}) \to \underline{\Gamma}_Y\mathcal{M}$$

such that $\iota'\rho_Y = \Phi_Y$.

By [9, Lemma 1.47], the diagram

$$\begin{tikzcd}
(?)_i\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, ?) \arrow[r, "\Phi_n"] \arrow[d, "H"] & (?)_i \arrow[d, "\mathrm{id}"] \\
\underline{\mathrm{Hom}}_{\mathcal{O}_{X_i}}(\mathcal{O}_{X_i}/\mathcal{I}_i^n, ?)(?)_i \arrow[r, "\Phi_n"] & (?)_i
\end{tikzcd}$$

is commutative. Therefore,

$$\begin{tikzcd}
(?)_i \varinjlim \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, ?) \arrow[r, "\rho_Y"] \arrow[d, "\cong"] & (?)_i\underline{\Gamma}_Y \arrow[dd, "\hat{\gamma}_{Y;i}^{-1}"] \\
\varinjlim (?)_i\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, ?) \arrow[d, "H"] & \\
\varinjlim \underline{\mathrm{Hom}}_{\mathcal{O}_{X_i}}(\mathcal{O}_{X_i}/\mathcal{I}_i^n, ?)(?)_i \arrow[r, "\rho_{Y_i}"] & \underline{\Gamma}_{Y_i}(?)_i
\end{tikzcd} \tag{5}$$

is also commutative.

3.21. Lemma. *Let $X \in \mathcal{P}(I, \underline{\mathrm{Sch}})$ be locally Noetherian with flat arrows, and let Y be its Cartesian closed subdiagram. If $\mathcal{M} \in \mathrm{Lqc}(X)$, then*

$$\rho_Y\colon\ \varinjlim \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathcal{M}) \to \underline{\Gamma}_Y\mathcal{M}$$

is an isomorphism.

Proof. Since $\mathcal{O}_X/\mathcal{I}^n$ is coherent, H in (5) is an isomorphism by [9, Lemma 6.33]. Thus we may assume that X is a single scheme, and this is just a special case of [6, Thm. 2.8]. □

4. Local Cohomology for Diagrams

4.1. Let the notation be as in Section 3.1. For a complex $\mathbb{M}$ of $\mathrm{Mod}(X)$, the right derived functor $R^i\underline{\Gamma}_{U,V}\mathbb{M}$ is denoted by $\underline{H}^i_{U,V}(\mathbb{M})$, and we call it the ith *local cohomology sheaf* of $\mathbb{M}$.

For a Cartesian closed subdiagram Y of X and a Cartesian closed subdiagram Z of Y, $R^i\underline{\Gamma}_{Y;Z}\mathbb{M}$ is denoted by $\underline{H}^i_{Y;Z}(\mathbb{M})$ and $\underline{H}^i_{Y;\emptyset}(\mathbb{M})$ is denoted by $\underline{H}^i_Y(\mathbb{M})$.

4.2. Let $\mathbb{F} \in K(\mathrm{Mod}(X))$. We say that $\mathbb{F}$ is *K-flabby* if $\mathbb{F}_i$ $(i \in I)$ is K-flabby in the sense of [21]. By [9, Lemma 8.17], a weakly K-injective complex is K-flabby. By [9, Prop. 8.2], a K-flabby complex is K-limp. A single sheaf $\mathcal{M} \in \mathrm{Mod}(X)$ is said to be flabby if it is K-flabby as a complex. By [21, Prop. 5.13], $\mathcal{M}$ is flabby if and only if $\mathcal{M}_i$ is a flabby sheaf on the topological space X_i in the usual sense.

4.3. PROPOSITION. *With notation as before, suppose* $\mathbb{I}$ *is a K-flabby complex in* $\mathrm{Mod}(X)$. *Then* $\mathbb{I}$ *is* $\underline{\Gamma}_{U,V}$*-acyclic.*

Proof. Let $\varphi\colon \mathbb{I} \to \mathbb{J}$ be a K-injective resolution, which exists because $\mathrm{Mod}(X)$ is Grothendieck (see [3]). Note that $\mathbb{J}$ is K-flabby and so, replacing $\mathbb{I}$ by the mapping cone of φ, we may assume that $\mathbb{I}$ is exact; we need to prove that $\underline{\Gamma}_{U,V}(\mathbb{I})$ is exact. For this it suffices to prove, for any $i \in I$, that $(?)_i\underline{\Gamma}_{U,V}(\mathbb{I}) \cong \underline{\Gamma}_{U_i,V_i}(\mathbb{I}_i)$ is exact. So we may assume that X is a single scheme. To verify that $\underline{\Gamma}_{U,V}(\mathbb{I})$ is exact, it suffices to show that $\Gamma(W, \underline{\Gamma}_{U,V}(\mathbb{I}))$ is exact for any open subset W of X. Applying the functor $\Gamma(W, ?)$ to the exact sequence

$$0 \to \underline{\Gamma}_{U,V} \xrightarrow{\iota} f_*f^* \xrightarrow{u} f_*g_*g^*f^*,$$

we obtain the exact sequence

$$0 \to \Gamma(W, ?) \circ \underline{\Gamma}_{U,V} \to \Gamma(U \cap W, ?) \xrightarrow{\mathrm{res}} \Gamma(V \cap W, ?).$$

For $Z := (U \setminus V) \cap W$, we have that $\Gamma(W, ?)\underline{\Gamma}_{U,V}$ is isomorphic to $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_{Z\subset X}, ?)$. By [21, Prop. 5.21], $\Gamma(W, \underline{\Gamma}_{U,V}(\mathbb{I})) \cong \mathrm{Hom}^\bullet_{\mathcal{O}_X}(\mathcal{O}_{Z\subset X}, \mathbb{I})$ is exact. This is what we wanted to prove. $\square$

4.4. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathbb{F} \in K(\mathrm{Mod}(X))$. Then $\mathbb{F}$ is K-flabby if and only if $\mathbb{F}$ is K-flabby as a complex of sheaves of abelian groups. To verify this, it suffices to show that if $\mathbb{F}$ is a K-injective complex in $\mathrm{Mod}(X)$ then $\mathbb{F}$ is K-flabby as a complex of sheaves of abelian groups. Let $\mathbb{G}$ be an exact complex of sheaves of abelian groups that is bounded above, and assume that each term of $\mathbb{G}$ is a direct sum of sheaves of the form $\mathbb{Z}_{Z\subset X}$ for some locally closed subset Z of X. Since $\mathbb{G}$ is $\mathbb{Z}$-flat, $\mathbb{G}' = \mathcal{O}_X \otimes_{\mathbb{Z}} \mathbb{G}$ is again exact. Thus

$$\mathrm{Hom}^\bullet_{\mathbb{Z}}(\mathbb{G}, \mathbb{F}) \cong \mathrm{Hom}^\bullet_{\mathcal{O}_X}(\mathbb{G}', \mathbb{F})$$

is exact by the K-injectivity of $\mathbb{F}$. Hence $\mathbb{F}$ is K-flabby as a complex of sheaves of abelian groups.

Similarly, a complex of $\mathcal{O}_{\mathbb{X}}$-modules on a ringed site $(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ is K-limp if and only if it is K-limp as a complex of sheaves of abelian groups.

4.5. LEMMA [21, Prop. 5.15]. *Let* $f\colon X \to Y$ *be a continuous map between topological spaces. If* $\mathbb{F}$ *is a K-flabby complex of sheaves of abelian groups, then so is* $f_*\mathbb{F}$.

Similarly, if $f\colon \mathbb{Y} \to \mathbb{X}$ is an admissible continuous functor (see [9. (2.8)]) between sites and if $\mathbb{F}$ is a K-limp complex of sheaves of abelian groups on $\mathbb{X}$, then $f^\#\mathbb{F}$ is also K-limp; see [9, Lemma 3.31].

4.6. LEMMA. *Let X be a topological space, U an open subset of X, and $\mathbb{F}$ a K-flabby (resp. K-limp) complex of abelian groups. Then $\mathbb{F}|_U$ is again K-flabby (resp. K-limp).*

Proof. Let $\varphi\colon \mathbb{F} \to \mathbb{I}$ be a K-injective resolution, and let $i\colon U \hookrightarrow X$ be the inclusion. Since i^* has an exact left adjoint $i^!$, it follows that $i^*\mathbb{I}$ is K-injective. Because i^* is exact, $i^*\varphi\colon i^*\mathbb{F} \to i^*\mathbb{I}$ is a K-injective resolution. Let $\mathbb{J}$ be the mapping cone of φ. It suffices to show that, for any locally closed subset Z (resp. open subset V) of U, $\Gamma_Z(U, i^*\mathbb{J})$ (resp. $\Gamma(V, i^*\mathbb{J})$) is exact. But this is trivial, since $\Gamma_Z(U, i^*\mathbb{J}) \cong \Gamma_Z(X, \mathbb{J})$ (resp. $\Gamma(V, i^*\mathbb{J}) \cong \Gamma(V, \mathbb{J})$). $\square$

4.7. LEMMA. *Let X be a topological space, and let U, V, W, W' be open subsets of X such that $V \subset U$ and $W' \subset W$. Set $Z := W \setminus W'$. Let F be a flabby sheaf of abelian groups on X. Then the canonical map*

$$\Gamma_{Z\cap U}(X, F) \to \Gamma_{Z\cap V}(X, F)$$

is surjective.

Proof. Let $\alpha \in \Gamma_{Z\cap V}(X, F) = \operatorname{Ker}(\Gamma(W \cap V, F) \to \Gamma(W' \cap V, F))$. Then there is a unique section $\tilde{\alpha} \in \Gamma((W' \cap U) \cup (W \cap V), F)$ such that the restriction of $\tilde{\alpha}$ to $W \cap V$ is α and the restriction of $\tilde{\alpha}$ to $W' \cap U$ is zero. Because F is flabby, $\tilde{\alpha}$ is extended to an element β of $\Gamma(W \cap U, F)$. Then $\beta \in \operatorname{Ker}(\Gamma(W \cap U, F) \to \Gamma(W' \cap U, F)) = \Gamma_{Z\cap U}(X, F)$, and the restriction of β to $W \cap V$ is α. This shows that the canonical map

$$\Gamma_{Z\cap U}(X, F) \to \Gamma_{Z\cap V}(X, F)$$

is surjective.

4.8. LEMMA (cf. [6, Lemma 1.6]). *Let the notation be as in Section 3.1. Let $\mathbb{I}$ be a K-flabby complex in* $\operatorname{Mod}(X)$. *Then $\underline{\Gamma}_{U,V}\mathbb{I}$ is again K-flabby.*

Proof. We may assume that X is a single scheme.

Let $W' \subset W \subset X$ be open subsets and let $Z := W \setminus W'$. As in the proof of Proposition 4.3, it is easy to check that $\Gamma_Z(X, ?) \circ \underline{\Gamma}_{U,V}$ is isomorphic to the kernel of the map

$$\Gamma(U \cap W, ?) \to \Gamma(V \cap W, ?) \oplus \Gamma(U \cap W', ?).$$

Since this map factors through the injective map

$$\Gamma((V \cap W) \cup (U \cap W'), ?) \to \Gamma(V \cap W, ?) \oplus \Gamma(U \cap W', ?),$$

$\Gamma_Z(X, ?) \circ \underline{\Gamma}_{U,V}$ is isomorphic to $\Gamma_E(X, ?)$, where E is the locally closed subset $(U \cap W) \setminus ((V \cap W) \cup (U \cap W')) = (U \setminus V) \cap (W \setminus W')$.

First we consider the case where $\mathbb{I}$ is strictly injective (i.e., K-injective with each term injective). Then

$$0 \to \underline{\Gamma}_{U,V}\mathbb{I} \xrightarrow{\iota} f_*f^*\mathbb{I} \xrightarrow{u} f_*g_*g^*f^*\mathbb{I} \to 0 \tag{6}$$

is exact, since each term of $\mathbb{I}$ is flabby. By Lemma 4.5 and Lemma 4.6, $f_*f^*\mathbb{I}$ and $f_*g_*g^*f^*\mathbb{I}$ are K-flabby. Therefore, the (-1)-shift of the mapping cone of

$u\colon f_* f^* \mathbb{I} \to f_* g_* g^* f^* \mathbb{I}$ is a K-flabby resolution of $\underline{\Gamma}_{U,V}\mathbb{I}$. So to verify that $\underline{\Gamma}_{U,V}\mathbb{I}$ is K-flabby, it suffices to show that (6) remains exact after applying $\Gamma_Z(X, ?)$ for any locally closed subset Z of X. Applying $\Gamma_Z(X, ?)$ to (6) and letting $E = (U \setminus V) \cap Z$, we derive the sequence

$$0 \to \Gamma_E(X, \mathbb{I}) \to \Gamma_{U\cap Z}(X, \mathbb{I}) \to \Gamma_{V\cap Z}(X, \mathbb{I}) \to 0, \tag{7}$$

which is exact by Lemma 4.7, as can be seen easily. This finishes the case where $\mathbb{I}$ is strictly injective.

Next consider the general case. Let $\varphi\colon \mathbb{I} \to \mathbb{J}$ be a strictly injective resolution, which exists because $\mathrm{Mod}(X)$ is Grothendieck (see [3]). Since $\underline{\Gamma}_{U,V}\mathbb{J}$ is K-flabby, it suffices to show that, for any locally closed subset Z of X, $\Gamma_Z(X, \underline{\Gamma}_{U,V}\mathbb{I}) \to \Gamma_Z(X, \underline{\Gamma}_{U,V}\mathbb{J})$ is a quasi-isomorphism. Letting $\mathbb{K}$ be the mapping cone of φ, it thus suffices to show that $\Gamma_Z(X, \underline{\Gamma}_{U,V}\mathbb{K})$ is exact. But this is trivial, since $\Gamma_Z(X, \underline{\Gamma}_{U,V}\mathbb{K}) \cong \Gamma_E(X, \mathbb{K})$ and $\mathbb{K}$ is K-flabby exact (again, $E = (U \setminus V) \cap Z$). □

4.9. LEMMA. *Let X be a topological space, and let $\mathbb{F}$ be a complex of sheaves of abelian groups. If $\mathbb{F}$ is K-limp and if each term of $\mathbb{F}$ is flabby, then $\mathbb{F}$ is K-flabby.*

Proof. Let $\varphi\colon \mathbb{F} \to \mathbb{I}$ be a strictly injective resolution. Observe that $\mathbb{I}$ is K-limp and that each term of $\mathbb{I}$ is flabby. So, replacing $\mathbb{F}$ by the mapping cone of φ, we may assume that $\mathbb{F}$ is exact; we must prove that $\Gamma_Z(X, \mathbb{F})$ is exact for any locally closed subset Z of X. Let $V \subset U \subset X$ be open subsets of X such that $U \setminus V = Z$. Since each term of $\mathbb{F}$ is flabby,

$$0 \to \Gamma_Z(X, \mathbb{F}) \to \Gamma(U, \mathbb{F}) \to \Gamma(V, \mathbb{F}) \to 0$$

is a short exact sequence of complexes. Since $\mathbb{F}$ is K-limp exact, $\Gamma(U, \mathbb{F})$ and $\Gamma(V, \mathbb{F})$ are exact. Hence $\Gamma_Z(X, \mathbb{F})$ is also exact. □

4.10. LEMMA. *With notation as in Section 3.1, there exists a triangle of the form*

$$R\underline{\Gamma}_{U,V} \xrightarrow{\iota} Rf_* f^* \xrightarrow{u} Rf_* Rg_* g^* f^* \to R\underline{\Gamma}_{U,V}[1].$$

Proof. Let $\mathbb{I}$ be a K-limp complex with each term of $\mathbb{I}$ flabby. Then there is a short exact sequence of complexes

$$0 \to \underline{\Gamma}_{U,V}\mathbb{I} \xrightarrow{\iota} f_* f^* \mathbb{I} \xrightarrow{u} f_* g_* g^* f^* \mathbb{I} \to 0.$$

The lemma follows immediately. □

4.11. COROLLARY. *Let notation be as before. If f and g are quasi-compact, then $R\underline{\Gamma}_{U,V}(D_{\mathrm{Lqc}}(X)) \subset D_{\mathrm{Lqc}}(X)$. If f and g are quasi-compact Cartesian and if X has flat arrows, then $R\underline{\Gamma}_{U,V}(D_{\mathrm{Qch}}(X)) \subset D_{\mathrm{Qch}}(X)$.*

Proof. This follows from Lemma 4.10, [9, Lemma 8.5], [9, Lemma 8.7], and [9, Lemma 8.20]. □

4.12. LEMMA. *Given the notation of Section 3.1, assume that $f\colon U \hookrightarrow X$ and $g\colon V \hookrightarrow U$ are quasi-compact. If X is quasi-compact and if I is finite, then $R\underline{\Gamma}_{U,V}\colon D_{\mathrm{Lqc}}(X) \to D_{\mathrm{Lqc}}(X)$ is way-out in both directions* (*see* [7, (I.7)]).

Proof. The statement is obvious by [9, Lemma 8.5] and Lemma 4.10. □

4.13. Lemma. *Let J be a subcategory of I. Then the canonical functor*

$$\zeta\colon R(\underline{\Gamma}_{U_J,V_J}(?)_J) \to R\underline{\Gamma}_{U_J,V_J}(?)_J$$

is an isomorphism.

Proof. This follows because, if $\mathbb{I}$ is a strictly injective complex of $\mathrm{Mod}(X)$, then $\mathbb{I}_J$ is K-flabby. □

Lemma 4.13 yields the isomorphism

$$R\underline{\Gamma}_{U_J,V_J}(?)_J \xrightarrow{\zeta^{-1}} R(\underline{\Gamma}_{U_J,V_J}(?)_J) \xrightarrow{R\hat{\gamma}} R((?)_J\underline{\Gamma}_{U,V}) \xrightarrow{\zeta} (?)_J R\underline{\Gamma}_{U,V},$$

which we denote simply by $\hat{\gamma}$.

4.14. Lemma. *Let the notation be as in Section 3.5. Then the canonical map $\zeta\colon R(\underline{\Gamma}_{U,V}h_*) \to R\underline{\Gamma}_{U,V}Rh_*$ is an isomorphism.*

Proof. Let $\mathbb{I}$ be a K-injective complex of $\mathcal{O}_{X'}$-modules. Then $h_*\mathbb{I}$ is K-flabby by Lemma 4.5. Hence $h_*\mathbb{I}$ is $\underline{\Gamma}_{U,V}$-acyclic by Proposition 4.3, and the assertion follows. □

4.15. By Lemma 4.14, the canonical map

$$R\underline{\Gamma}_{U,V}Rh_* \xrightarrow{\zeta^{-1}} R(\underline{\Gamma}_{U,V}h_*) \xrightarrow{\bar{\gamma}} R(h_*\underline{\Gamma}_{U',V'}) \xrightarrow{\zeta} Rh_*R\underline{\Gamma}_{U,V'},$$

which we denote by $\bar{\gamma}$, is defined.

4.16. Lemma. *With notation as in Section 3.5, the canonical map*

$$\zeta\colon R(h_*\underline{\Gamma}_{U',V'}) \to Rh_*R\underline{\Gamma}_{U',V'}$$

is an isomorphism.

Proof. If $\mathbb{I}$ is a strictly injective complex of $\mathcal{O}_{X'}$-modules then, by Lemma 4.8, $\underline{\Gamma}_{U',V'}\mathbb{I}$ is K-flabby. The lemma follows immediately. □

4.17. Corollary (independence theorem; cf. [2, (4.2.1)]). *Let notation be as in Section 3.5, and assume that* (a) *and* (b) *in the diagram* (2) *are Cartesian. Then $\bar{\gamma}$: $R\underline{\Gamma}_{U,V}Rh_* \to Rh_*R\underline{\Gamma}_{U',V'}$ is an isomorphism.*

Proof. This follows immediately from Lemma 4.16 and Lemma 3.6. □

4.18. With notation as in Section 3.1, let $W \subset V$ be an open subdiagram of schemes and let $h\colon W \hookrightarrow V$ be the inclusion. Let $\mathbb{I}$ be a complex in $\mathrm{Mod}(X)$. Assume that each term of $\mathbb{I}_i$ is flabby for any $i \in I$. Then the diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[d] & \underline{\Gamma}_{V,W}\mathbb{I} \arrow[d, "d^{-1}c\iota"] & \\
0 \arrow[r] & \underline{\Gamma}_{U,V}\mathbb{I} \arrow[r, "\iota"] \arrow[d, "\bar{\gamma}"] & f_*f^*\mathbb{I} \arrow[r, "u"] \arrow[d, "\mathrm{id}"] & f_*g_*g^*f^*\mathbb{I} \arrow[r] \arrow[d, "u"] & 0 \\
0 \arrow[r] & \underline{\Gamma}_{U,W}\mathbb{I} \arrow[r, "\iota"] & f_*f^*\mathbb{I} \arrow[r, "uu"] \arrow[d] & f_*g_*h_*h^*g^*f^*\mathbb{I} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

is commutative with exact rows and columns, where $d^{-1}c\iota$ is the composite

$$\underline{\Gamma}_{V,W} \xrightarrow{\iota} (fg)_*(fg)^* \xrightarrow{c} f_*g_*(fg)^* \xrightarrow{d^{-1}} f_*g_*g^*f^*.$$

Utilizing the snake lemma, it is easy to see that the sequence

$$0 \to \underline{\Gamma}_{U,V}\mathbb{I} \xrightarrow{\bar{\gamma}} \underline{\Gamma}_{U,W}\mathbb{I} \xrightarrow{\bar{\gamma}} \underline{\Gamma}_{V,W}\mathbb{I} \to 0$$

is exact. Thus we have a triangle

$$R\underline{\Gamma}_{U,V} \xrightarrow{\bar{\gamma}} R\underline{\Gamma}_{U,W} \xrightarrow{\bar{\gamma}} R\underline{\Gamma}_{V,W} \xrightarrow{\hat{\delta}} R\underline{\Gamma}_{U,V}[1],$$

where $\hat{\delta}$ is induced by

$$\underline{\Gamma}_{V,W} \hookrightarrow \mathrm{Cone}(\underline{\Gamma}_{U,W} \xrightarrow{\bar{\gamma}} \underline{\Gamma}_{V,W}) \underset{\simeq}{\xleftarrow{\bar{\gamma}}} \underline{\Gamma}_{U,V}[1]$$

and $\simeq$ denotes a quasi-isomorphism.

5. Quasi-flabby Sheaves

5.1. The following definition is due to Kempf [16], although we make a slight modification here.

5.2. DEFINITION. Let X be a topological space. A presheaf $\mathcal{M}$ of abelian groups on X is said to be *quasi-flabby* if the restriction map $\Gamma(U, \mathcal{M}) \to \Gamma(V, \mathcal{M})$ is surjective for any quasi-compact open subsets U and V such that $U \supset V$.

Note that a flabby sheaf is quasi-flabby. For the sake of completeness, we list Kempf's results for this modified definition.

5.3. LEMMA [16]. *Let X be a topological space such that the intersection of two quasi-compact open subsets is again quasi-compact. Let*

$$0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0$$

be a short exact sequence of sheaves of abelian groups. If $\mathcal{L}$ is quasi-flabby and U is a quasi-compact open subset of X, then the sequence

$$0 \to \Gamma(U, \mathcal{L}) \to \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N}) \to 0$$

is exact.

5.4. COROLLARY. *Let X be as in the lemma. Let*

$$0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0$$

be a short exact sequence of sheaves of abelian groups. If $\mathcal{L}$ and $\mathcal{M}$ are quasi-flabby, then so is $\mathcal{N}$.

5.5. COROLLARY. *Let X be as in Lemma 5.3. If $\mathcal{L}$ is quasi-flabby and U is a quasi-compact open subset, then $H^i(U, \mathcal{L}) = 0$ for $i > 0$.*

5.6. LEMMA. *Let $f: X \to Y$ be a continuous map of topological spaces. Assume that Y has an open basis consisting of quasi-compact open subsets and that $f^{-1}(U)$ is quasi-compact if U is a quasi-compact open subset of Y. Assume, moreover, that Y has an open covering (U_λ) such that, for any λ and quasi-compact open subsets V, V' of $f^{-1}(U_\lambda)$, the intersection $V \cap V'$ is again quasi-compact. Then, for a short exact sequence*

$$0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0$$

of sheaves of abelian groups on X with $\mathcal{L}$ quasi-flabby, the sequence

$$0 \to f_*\mathcal{L} \to f_*\mathcal{M} \to f_*\mathcal{N} \to 0$$

is exact.

Proof. It suffices to show that $(f|_{f^{-1}(U_\lambda)})_*\mathcal{M}|_{f^{-1}(U_\lambda)} \to (f|_{f^{-1}(U_\lambda)})_*\mathcal{N}|_{f^{-1}(U_\lambda)}$ is surjective for each λ. Since $\mathcal{L}|_{f^{-1}(U_\lambda)}$ is quasi-flabby for each λ, we may assume that, for any two quasi-compact open subsets V, V' of X, the intersection $V \cap V'$ is quasi-compact, replacing $f: X \to Y$ by $f|_{f^{-1}(U_\lambda)}: f^{-1}(U_\lambda) \to U_\lambda$.

Because there is an open basis of Y consisting of quasi-compact open subsets, it suffices to show that $\Gamma(U, f_*\mathcal{M}) \to \Gamma(U, f_*\mathcal{N})$ is surjective for any quasi-compact open subset U of Y. Since $f^{-1}(U)$ is quasi-compact, this is Lemma 5.3. $\square$

5.7. COROLLARY. *Let $f: X \to Y$ be as in the lemma. If $\mathcal{L}$ is a quasi-flabby sheaf of abelian groups on X, then $R^i f_*\mathcal{L} = 0$ for $i > 0$.*

Proof. The question is local on Y, and we may assume that, for any two quasi-compact open subsets V, V' of X, $V \cap V'$ is again quasi-compact. Take a short exact sequence of the form

$$0 \to \mathcal{L} \to \mathcal{I} \xrightarrow{p} \mathcal{L}' \to 0$$

with $\mathcal{I}$ injective. Because an injective sheaf is quasi-flabby, $\mathcal{L}'$ is quasi-flabby by Corollary 5.4.

We now use the induction on i. Note that

$$f_*\mathcal{I} \xrightarrow{f_*p} f_*\mathcal{L}' \to R^1 f_*\mathcal{L} \to R^1 f_*\mathcal{I}$$

is exact. Since $\mathcal{I}$ is injective, $R^1 f_*\mathcal{I} = 0$. On the other hand, f_*p is surjective by the lemma, so $R^1 f_*\mathcal{L} = 0$.

Consider the case $i \geq 2$. Then $R^i\mathcal{L} \cong R^{i-1}\mathcal{L}' = 0$ by induction. □

5.8. LEMMA. *Let X be a topological space. Assume that X has an open basis consisting of quasi-compact open subsets. Let U be a quasi-compact open subset of X, and let $(\mathcal{M}_\lambda)$ be a pseudo-filtered inductive system of sheaves of abelian groups on X. Then the canonical map*

$$\varinjlim \Gamma(U, \mathcal{M}_\lambda) \to \Gamma(U, \varinjlim \mathcal{M}_\lambda)$$

is an isomorphism.

5.9. COROLLARY. *Let X be as in the lemma. Then a filtered inductive limit of quasi-flabby sheaves is quasi-flabby.*

5.10. COROLLARY. *Let $f: X \to Y$ be a quasi-compact morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}})$. If $(\mathcal{M}_\lambda)$ is a pseudo-filtered inductive system of sheaves of abelian groups on X, then the canonical map*

$$\varinjlim f_*\mathcal{M}_\lambda \to f_* \varinjlim \mathcal{M}_\lambda$$

is an isomorphism.

Proof. By restriction, we may assume that the problem is on single schemes. Since Y has an open basis consisting of quasi-compact open subsets, it suffices to show that, for a quasi-compact open subset U of Y,

$$\Gamma(U, \varinjlim f_*\mathcal{M}_\lambda) \to \Gamma(U, f_* \varinjlim \mathcal{M}_\lambda) \tag{8}$$

is an isomorphism. Because U and $f^{-1}(U)$ are quasi-compact, the canonical maps

$$\varinjlim \Gamma(U, f_*\mathcal{M}_\lambda) \to \Gamma(U, \varinjlim f_*\mathcal{M}_\lambda)$$

and

$$\varinjlim \Gamma(f^{-1}(U), \mathcal{M}_\lambda) \to \Gamma(f^{-1}(U), \varinjlim \mathcal{M}_\lambda)$$

are isomorphisms by Lemma 5.8. Hence the map (8) is also an isomorphism, as required. □

5.11. LEMMA. *Let $X \in \mathcal{P}(I, \underline{\mathrm{Sch}})$. Let U be an open subdiagram of X, and let V be an open subdiagram of U. Assume that the inclusions $f: U \hookrightarrow X$ and $g: V \to U$ are quasi-compact. Then, for a pseudo-filtered inductive system $(\mathcal{M}_\lambda)$ of $\mathcal{O}_X$-modules, the canonical map*

$$\varinjlim \underline{\Gamma}_{U,V}\mathcal{M}_\lambda \to \underline{\Gamma}_{U,V} \varinjlim \mathcal{M}_\lambda$$

is an isomorphism.

Proof. Consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim \underline{\Gamma}_{U,V}\mathcal{M}_\lambda \arrow[r, "\iota"] \arrow[d] & \varinjlim f_* f^*\mathcal{M}_\lambda \arrow[r, "u"] \arrow[d] & \varinjlim f_* g_* g^* f^*\mathcal{M}_\lambda \arrow[d] \\
0 \arrow[r] & \underline{\Gamma}_{U,V} \varinjlim \mathcal{M}_\lambda \arrow[r, "\iota"] & f_* f^* \varinjlim \mathcal{M}_\lambda \arrow[r, "u"] & f_* g_* g^* f^* \varinjlim \mathcal{M}_\lambda .
\end{tikzcd}$$

The middle and the right vertical arrows are isomorphisms by Corollary 5.10. By the five lemma, we are done. □

5.12. Lemma. *Let $f: X \to Y$ be a concentrated morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}})$. Let $\mathcal{I}$ be an $\mathcal{O}_X$-module such that $\mathcal{I}_i$ is quasi-flabby for each $i \in I$. Then $\mathcal{I}$ is f_*-acyclic.*

Proof. We may assume that the problem is on a single scheme. This case is Corollary 5.7. □

5.13. Lemma. *Let $X \in \mathcal{P}(I, \underline{\mathrm{Sch}})$. Let U be an open subdiagram of schemes of X, and let V be an open subdiagram of schemes of U. Let $f: U \hookrightarrow X$ and $g: V \hookrightarrow U$ be inclusions. Assume that f and g are concentrated. If $\mathcal{M}$ is a quasi-flabby sheaf of abelian groups on X, then*

$$0 \to \underline{\Gamma}_{U,V}\mathcal{M} \xrightarrow{\iota} f_* f^*\mathcal{M} \xrightarrow{u} f_* g_* g^* f^*\mathcal{M} \to 0$$

is exact.

Proof. It suffices to show that, for a quasi-compact open subset W of X, the restriction $\Gamma(U \cap W, \mathcal{M}) \to \Gamma(V \cap W, \mathcal{M})$ is surjective. This is trivial. □

5.14. Corollary. *Let notation be as in the lemma. Then $\mathcal{M}$ is $\underline{\Gamma}_{U,V}$-acyclic.*

Proof. Note that $f^*\mathcal{M}$ is quasi-flabby and hence is f_*-acyclic. Similarly, $g^* f^*\mathcal{M}$ is $(fg)_*$-acyclic. The lemma follows from the long exact sequence

$$\begin{aligned} 0 \to \underline{\Gamma}_{U,V}\mathcal{M} &\xrightarrow{\iota} f_* f^*\mathcal{M} \xrightarrow{u} f_* g_* g^* f^*\mathcal{M} \\ &\to \underline{H}^1_{U,V}\mathcal{M} \to R^1 f_* f^*\mathcal{M} \to R^1(fg)_*(fg)^*\mathcal{M} \to \cdots, \end{aligned}$$

in which $u: f_* f^*\mathcal{M} \to f_* g_* g^* f^*\mathcal{M}$ is surjective. □

5.15. Let $\mathcal{A}$ be an abelian category and C a complex in $\mathcal{A}$. For $n \in \mathbb{Z}$, we define $\tau^{\leq n} C$ to be the truncated complex

$$\cdots \to C^{n-2} \to C^{n-1} \to \operatorname{Ker} d^n \to 0.$$

Similarly, $\tau^{\geq n} C$ is defined to be the complex

$$0 \to \operatorname{Coker} d^{n-1} \to C^{n+1} \to C^{n+2} \to \cdots,$$

which is quasi-isomorphic to $C/\tau^{\leq n-1} C$.

5.16. LEMMA (cf. [17, (3.9.3.1)]). *Let X, $f: U \to X$, and $g: V \to U$ be as in Lemma 5.13. Let (C_α) be a pseudo-filtered inductive system of complexes of $\mathcal{O}_X$-modules such that, for each $j \in I$, there exists some $n_j \in \mathbb{Z}$ such that $\tau^{\leq n_j - 1}(C_\alpha)_j$ is exact for any α. Set $C = \varinjlim C_\alpha$. Then the canonical map*

$$\varinjlim \underline{H}^i_{U,V} C_\alpha \to \underline{H}^i_{U,V} C \tag{9}$$

is an isomorphism for $i \in \mathbb{Z}$.

Proof. We may assume that the problem is on single schemes. Let n be an integer such that $\tau^{\leq n-1} C_\alpha$ is exact for any α.

As in the proof of [17, (3.9.3.1)], let $\tau^{\geq n} C_\alpha \to F_\alpha$ be the Godement resolution so that we have a composite quasi-isomorphism $C_\alpha \to \tau^{\geq n} C_\alpha \to F_\alpha$. Observe that each term of F_α is flabby. In particular, this is a $\underline{\Gamma}_{U,V}$-acyclic resolution. Then, taking the inductive limit, we have a quasi-isomorphism $C \to F := \varinjlim F_\alpha$. Note that each term of F is quasi-flabby by Corollary 5.9. Hence this is also a $\underline{\Gamma}_{U,V}$-acyclic resolution.

As a result, the map (9) is nothing but the composite

$$\varinjlim H^i(\underline{\Gamma}_{U,V} F_\alpha) \xrightarrow{\cong} H^i(\varinjlim \underline{\Gamma}_{U,V} F_\alpha) \xrightarrow{\cong} H^i(\underline{\Gamma}_{U,V} \varinjlim F_\alpha) = H^i(\underline{\Gamma}_{U,V} F),$$

where the second $\cong$ is an isomorphism by Lemma 5.11. This is what we wanted to prove. □

6. Flat Base Change of Local Cohomology of Diagrams

6.1. Let the commutative diagram (2) be as in Section 3.5. Assume that h is flat. Then there is a canonical composite map

$$h^* R\underline{\Gamma}_{U,V} \xrightarrow{\zeta^{-1}} R(h^* \underline{\Gamma}_{U,V}) \xrightarrow{R\bar{\delta}} R(\underline{\Gamma}_{U',V'} h^*) \xrightarrow{\zeta} R\underline{\Gamma}_{U',V'} h^*,$$

which we denote by $\bar{\delta}$.

6.2. LEMMA. *Let notation be as before. If h is an open immersion, then*

$$\zeta : R(\underline{\Gamma}_{U',V'} h^*) \to R\underline{\Gamma}_{U',V'} h^*$$

is an isomorphism.

Proof. Let $\mathbb{I}$ be a K-injective complex of $\mathcal{O}_X$-modules. Then, by Lemma 4.6, $h^*\mathbb{I}$ is K-flabby and hence is $\underline{\Gamma}_{U',V'}$-acyclic. The assertion follows. □

6.3. COROLLARY. *Let the commutative diagram* (2) *be as in Section 3.5. If h is locally an open immersion and if* (a) *and* (b) *are Cartesian in* (2)*, then $\bar{\delta}: h^* R\underline{\Gamma}_{U,V} \to R\underline{\Gamma}_{U',V'} h^*$ is an isomorphism.*

Proof. For $i \in I$, the diagram

$$\begin{array}{ccccc} h_i^* R\underline{\Gamma}_{U_i,V_i}(?)_i & \xrightarrow{\hat{\gamma}} & h_i^*(?)_i R\underline{\Gamma}_{U,V} & \xrightarrow{\theta} & (?)_i h^* R\underline{\Gamma}_{U,V} \\ \downarrow{\scriptstyle \bar{\delta}} & & & & \downarrow{\scriptstyle (?)_i\bar{\delta}} \\ R\underline{\Gamma}_{U_i',V_i'} h_i^*(?)_i & \xrightarrow{\theta} & R\underline{\Gamma}_{U_i',V_i'}(?)_i h^* & \xrightarrow{\hat{\gamma}} & (?)_i R\underline{\Gamma}_{U',V'} h^* \end{array}$$

is commutative by Lemma 3.8. It suffices to show that the right vertical arrow $(?)_i\bar{\delta}$ is an isomorphism. For this we need only show that the left vertical arrow $\bar{\delta}\colon h_i^* R\underline{\Gamma}_{U_i,V_i}(?)_i \to R\underline{\Gamma}_{U_i',V_i'} h_i^*(?)_i$ is an isomorphism. Hence we may assume that the problem is on single schemes.

First assume that h is an open immersion. Then the assertion follows immediately from Lemma 6.2 and Lemma 3.15.

Now consider the general case. Take an open covering $\bigcup_\lambda W_\lambda$ of X' such that $h|_{W_\lambda}$ is an open immersion for each λ. It suffices to show that $j^*\bar{\delta}\colon j^*h^*R\underline{\Gamma}_{U,V} \to j^*R\underline{\Gamma}_{U',V'}h^*$ is an isomorphism for each λ, where $j\colon W = W_\lambda \to X'$ is the inclusion. However, the diagram

$$\begin{array}{ccccc} j^*h^*R\underline{\Gamma}_{U,V} & & \xrightarrow{\quad d^{-1} \quad} & & (hj)^*R\underline{\Gamma}_{U,V} \\ \downarrow{\scriptstyle j^*\bar{\delta}} & & & & \downarrow{\scriptstyle \bar{\delta}} \\ j^*R\underline{\Gamma}_{U',V'}h^* & \xrightarrow{\bar{\delta}} & R\underline{\Gamma}_{W\cap U',W\cap V'}j^*h^* & \xrightarrow{d^{-1}} & R\underline{\Gamma}_{W\cap U',W\cap V'}(hj)^* \end{array}$$

is commutative by Lemma 3.17, and the all arrows except for $j^*\bar{\delta}$ are isomorphisms by what we have already proved. Hence $j^*\bar{\delta}$ is also an isomorphism, as desired. □

6.4. Lemma (cf. [17, (3.9.3.2)]). *Let X, $f\colon U \to X$, and $g\colon V \to U$ be as in Lemma 5.13. Let (C_α) be a pseudo-filtered inductive system of complexes in* $\mathrm{Mod}(X)$. *Assume one of the following*:

(a) *U is locally Noetherian and, for each $i \in I$, U_i admits an open covering (U_α) such that each U_α is of finite Krull dimension*;
(b) *C_α has locally quasi-coherent cohomology groups for each α*;
(c) *for each $j \in I$, there exists some $n_j \in \mathbb{Z}$ such that $\tau^{\leq n_j - 1}(C_\alpha)_j$ is exact for any α.*

Then the canonical map

$$\varinjlim \underline{H}^i_{U,V} C_\alpha \to \underline{H}^i_{U,V} C$$

is an isomorphism for $i \in \mathbb{Z}$, where $C = \varinjlim C_\alpha$.

Proof. The case when (c) is satisfied is Lemma 5.16. We consider the case where (a) or (b) is satisfied. By restriction, we may assume that the problem is on a single scheme. Also, we may assume by Corollary 6.3 that X is an affine scheme.

If we assume (a) (resp. (b)) then there exists some $d_0 \in \mathbb{Z}$ such that—for any $i \in \mathbb{Z}$, any $d \geq d_0$, and any complex D in $\mathrm{Mod}(X)$ (resp. any complex D in $\mathrm{Mod}(X)$ with quasi-coherent cohomology groups)—$R^i f_* f^*(\tau^{\leq i-d} D) = 0$ and $R^i(gf)_*(gf)^*(\tau^{\leq i-d} D) = 0$; see [17, Remarks in (3.9.3.2)]. This implies that $\underline{H}^i_{U,V}(\tau^{\leq i-d} D) = 0$ for any $i \in \mathbb{Z}$, any $d \geq d_0 + 1$, and any complex D in $\mathrm{Mod}(X)$ (resp. any complex D in $\mathrm{Mod}(X)$ with quasi-coherent cohomology). Hence $\underline{H}^i_{U,V}(D) \to \underline{H}^i_{U,V}(\tau^{\geq i-d} D)$ is an isomorphism for $d \geq d_0$. The square

$$\begin{array}{ccc} \varinjlim \underline{H}^i_{U,V}(C_\alpha) & \xrightarrow{\cong} & \varinjlim \underline{H}^i_{U,V}(\tau^{\geq i-d_0} C_\alpha) \\ \downarrow & & \downarrow \\ \underline{H}^i_{U,V}(C) & \xrightarrow{\cong} & \underline{H}^i_{U,V}(\tau^{\geq i-d_0} C) \end{array}$$

is commutative and so, replacing C_α by $\tau^{\geq i-d_0} C_\alpha$, we may assume that there exists some $n \in \mathbb{Z}$ such that $\tau^{\leq n-1} C_\alpha$ is exact for each α. This is the case where (c) is assumed, and we are done. □

6.5. COROLLARY (cf. [17, (3.9.3.3)]). *Let X, $f : U \to X$, and $g : V \to U$ be as in Lemma 6.4. Let (C_α) be a small family of complexes in* $\mathrm{Mod}(X)$. *If one of* (a), (b), *or* (c) *in the lemma is satisfied, then the canonical map*

$$\bigoplus_\alpha R\underline{\Gamma}_{U,V} C_\alpha \to R\underline{\Gamma}_{U,V}\left(\bigoplus_\alpha C_\alpha\right)$$

is an isomorphism.

6.6. COROLLARY. *Let X, $f : U \to X$, and $g : V \to U$ be as in Lemma 6.4. If X is concentrated, then $R\underline{\Gamma}_{U,V} : D_{\mathrm{Lqc}}(X) \to D_{\mathrm{Lqc}}(X)$ has a right adjoint.*

Proof. Note that $D_{\mathrm{Lqc}}(X)$ is compactly generated by [9, Lemma 17.1]. The corollary follows from Corollary 6.5 and Neeman's theorem [19, Theorem 4.1]. □

6.7. COROLLARY (cf. [17, 3.9.3.4)]). *Under the assumptions of Lemma 6.4, if each C_α is $\underline{\Gamma}_{U,V}$-acyclic then C is $\underline{\Gamma}_{U,V}$-acyclic.*

Proof. By assumption, $H^i(\underline{\Gamma}_{U,V} C_\alpha) \to \underline{H}^i_{U,V} C_\alpha$ is an isomorphism for each $i \in \mathbb{Z}$ and α. Taking the inductive limit, the composite

$$H^i(\underline{\Gamma}_{U,V} C) \cong \varinjlim H^i(\underline{\Gamma}_{U,V} C_\alpha) \cong \varinjlim \underline{H}^i_{U,V} C_\alpha \cong \underline{H}^i_{U,V} C$$

is an isomorphism, where the first $\cong$ is an isomorphism by Lemma 5.11 and the last $\cong$ is an isomorphism by Lemma 6.4. Therefore, C is $\underline{\Gamma}_{U,V}$-acyclic. □

6.8. COROLLARY (cf. [17, (3.9.3.5)]). *Let X, $f : U \to X$, and $g : V \to U$ be as in Lemma 5.13. Let C be a complex in* $\mathrm{Mod}(X)$, *and assume one of the following.*

(a) *U is locally Noetherian, U_i $(i \in I)$ admits an open covering (U_α) such that each U_α is of finite Krull dimension, and each term of C_i $(i \in I)$ is quasi-flabby*;
(b) *X is locally Noetherian and, for each $i \in I$, each term of C_i is an injective object of* $\mathrm{Qch}(X_i)$.

Then C is $\underline{\Gamma}_{U,V}$-acyclic.

Proof. Let $C \to \mathbb{I}$ be a K-injective resolution. Then $\mathbb{I}_i$ is $\underline{\Gamma}_{U_i,V_i}$-acyclic for each $i \in I$, since $\mathbb{I}_i$ is K-flabby. So it suffices to show that each C_i is $\underline{\Gamma}_{U_i,V_i}$-acyclic, and we may assume that the problem is on single schemes.

In every case, each term C^n of C is $\underline{\Gamma}_{U,V}$-acyclic. Indeed, in case (a), this is Corollary 5.14. In case (b) this is obvious, since an injective object of $\mathrm{Qch}(X)$ is an injective object of $\mathrm{Mod}(X)$ [7, (II.7)]. Thus the truncated subcomplex

$$\sigma^{\geq n}C\colon\ \cdots \to 0 \to 0 \to C^n \to C^{n+1} \to \cdots$$

of C is $\underline{\Gamma}_{U,V}$-acyclic for any $n \in \mathbb{Z}$. Since $C \cong \varinjlim \sigma^{\geq n}C$, the assertion follows from Corollary 6.7. □

6.9. Lemma. *Let $h\colon X' \to X$ be a flat morphism between locally Noetherian schemes. Let Y be a closed subscheme of X, and let $\mathbb{I}$ be an injective object of* $\mathrm{Qch}(X)$. *Then $h^*\mathbb{I}$ is $\underline{\Gamma}_{Y'}$-acyclic, where $Y' := h^{-1}(Y)$.*

Proof. By [7, Thm. II.7.18], $h^*\mathbb{I}$ has an injective resolution $\mathbb{J}$ in $\mathrm{Qch}(X')$; it is an injective resolution in $\mathrm{Mod}(X')$ as well (see [7, (II.7)]). Let $\mathcal{I}$ be the defining ideal sheaf of Y. Then Y' is defined by $\mathcal{I}\mathcal{O}_{Y'}$. So, by Lemma 3.21 and [7, Prop. II.5.8],

$$\begin{aligned} R^i\underline{\Gamma}_{Y'}(h^*\mathbb{I}) = H^i(\underline{\Gamma}_{Y'}(\mathbb{J})) &\cong H^i(\varinjlim \underline{\mathrm{Hom}}_{\mathcal{O}_{X'}}(h^*(\mathcal{O}_X/\mathcal{I}^n), \mathbb{J})) \\ &\cong \varinjlim \underline{\mathrm{Ext}}^i_{\mathcal{O}_{X'}}(h^*(\mathcal{O}_X/\mathcal{I}^n), h^*\mathbb{I}) \\ &\cong \varinjlim h^*(\underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathbb{I})) = 0 \end{aligned}$$

for $i > 0$. □

6.10. Theorem (flat base change; cf. [2, Thm. 4.3.2]). *Let $h\colon X' \to X$ be a flat morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}})$. Assume that X and X' are locally Noetherian. Let Y be a Cartesian closed subdiagram of schemes of X, and let Z be a Cartesian closed subdiagram of schemes of Y. Then the canonical map $\bar{\delta}\colon h^*R\underline{\Gamma}_{Y;Z} \to R\underline{\Gamma}_{Y';Z'}h^*$ is an isomorphism of functors from $D_{\mathrm{Lqc}}(X)$ to $D_{\mathrm{Lqc}}(X')$, where $Y' = h^{-1}(Y)$ and $Z' = h^{-1}(Z)$.*

Proof. By an argument similar to the proof of Corollary 6.3, we may assume that the problem is on single schemes. Moreover, the question is local both on X and X' by Corollary 6.3, so we may assume that both $X = \operatorname{Spec} A$ and $X' = \operatorname{Spec} B$ are affine.

Now, by Lemma 4.12, $\underline{\Gamma}_{Y;Z}\colon D_{\mathrm{Lqc}}(X) \to D_{\mathrm{Lqc}}(X)$ and $\underline{\Gamma}_{Y';Z'}\colon D_{\mathrm{Lqc}}(X') \to D_{\mathrm{Lqc}}(X')$ are way-out in both directions. By the way-out lemma [7, Prop. I.7.1], it suffices to show that $\bar{\delta}\colon h^*R\underline{\Gamma}_{Y;Z}\mathcal{I} \to R\underline{\Gamma}_{Y';Z'}h^*\mathcal{I}$ is an isomorphism for an injective object $\mathcal{I}$ of $\mathrm{Qch}(X)$.

Observe that $\bar{\delta}$ is the composite

$$h^*R\underline{\Gamma}_{Y;Z}\mathcal{I} \xrightarrow{\zeta^{-1}} R(h^*\underline{\Gamma}_{Y;Z})\mathcal{I} \xrightarrow{R\bar{\delta}} R(\underline{\Gamma}_{Y';Z'}h^*)\mathcal{I} \xrightarrow{\zeta} R\underline{\Gamma}_{Y';Z'}h^*\mathcal{I}.$$

Hence it suffices to show that $R\bar{\delta}$ and ζ are isomorphisms.

By Lemma 3.15, $\bar{\delta}\colon h^*\underline{\Gamma}_{Y;Z}\mathcal{I} \to \underline{\Gamma}_{Y';Z'}h^*\mathcal{I}$ is an isomorphism. Since $\mathcal{I}$ is injective in $\mathrm{Mod}(X)$ [7, (II.7)], it follows that $R\bar{\delta}$ is an isomorphism.

To prove that ζ is an isomorphism, we need only prove that $h^*\mathcal{I}$ is $\underline{\Gamma}_{Y';Z'}$-acyclic. By Section 4.18, there is an exact sequence

$$\cdots \to \underline{H}^i_Z(h^*\mathcal{I}) \to \underline{H}^i_Y(h^*\mathcal{I}) \to \underline{H}^i_{Y;Z}(h^*\mathcal{I}) \to \underline{H}^{i+1}_Z(h^*\mathcal{I}) \to \cdots.$$

By Lemma 6.9, $\underline{H}^i_Y(h^*\mathcal{I}) = 0$ $(i > 0)$ and $\underline{H}^i_Z(h^*\mathcal{I}) = 0$ $(i > 0)$. So

$$\underline{H}^i_{Y;Z}(h^*\mathcal{I}) = 0 \quad \text{for } i > 0,$$

as desired. □

7. Compatibility with G-invariance

7.1. Let S be a scheme, G a flat S-group scheme, and X an S-scheme with a trivial G-action. As in [9, (30.1)], we denote the G-invariance functor $\mathrm{Mod}(G,X) \to \mathrm{Mod}(X)$ by $(?)^G$. By [9, Lemma 30.3], $(?)^G$ agrees with $(?)_{-1}R_{\Delta_M}$, where $R_{\Delta_M}\colon \mathrm{Mod}(G,X) = \mathrm{Mod}(B^M_G(X)) \to \mathrm{Mod}(\tilde{B}^M_G(X))$ is the right induction for $\tilde{B}^M_G(X)$ the augmented diagram described in [9, (30.2)]. If G is concentrated over S, then $(?)^G(\mathrm{Lqc}(G,X)) \subset \mathrm{Qch}(X)$. Note that $\tilde{B}^M_G(X)_{\Delta_M} = B^M_G(X)$. As in [9, Sec. 29], for a G-morphism f we let $B^M_G(f)_*$ be simply denoted by f_* and $B^M_G(f)^*$ by f^*, et cetera.

We know that $(?)^G = (?)_{-1}R_{\Delta_M}$ has an exact left adjoint $(?)_{\Delta_M}L_{-1}$. Hence

$$(?)^G\colon \mathrm{Mod}(G,X) \to \mathrm{Mod}(X)$$

preserves injectives and $R(?)^G\colon D(G,X) \to D(X)$ preserves K-injectives.

It seems that the following question is fundamental.

7.2. QUESTION. Let $\mathcal{I}$ be an injective object of $\mathrm{Qch}(G,X)$. Then, for $i > 0$, does $R^i(?)^G\mathcal{I} = 0$?

This is not obvious a priori, since the derived functor is computed in $D(G,X)$.

7.3. Let $f\colon X \to Y$ be a morphism of S-schemes with trivial G-actions. Then $\tilde{B}^M_G(f)\colon \tilde{B}^M_G(X) \to \tilde{B}^M_G(Y)$ is induced. Note that $\tilde{B}^M_G(f)$ is Cartesian. The composite isomorphism

$$\begin{aligned} e = e_f : f_*(?)^G = f_*(?)_{-1}R_{\Delta_M} &\xrightarrow{c^{-1}} (?)_{-1}\tilde{B}^M_G(f)_*R_{\Delta_M} \\ &\xrightarrow{\xi} (?)_{-1}R_{\Delta_M}B^M_G(f)_* = (?)^G f_* \end{aligned}$$

is induced (see [9, Cor. 6.26]).

7.4. Moreover, the natural map

$$\begin{aligned}\epsilon = \epsilon^f : f^*(?)^G = f^*(?)_{-1}R_{\Delta_M} &\xrightarrow{\theta} (?)_{-1}\tilde{B}_G^M(f)^*R_{\Delta_M} \\ &\xrightarrow{\mu} (?)_{-1}R_{\Delta_M}B_G^M(f)^* = (?)^G f^*\end{aligned}$$

is induced; see [9, (6.27)]. We distinguish ϵ from ε. Note that θ is an isomorphism by [9, (6.25)]. Exactly the same proof as in [9, (10.7)] shows that μ is an isomorphism of functors from $\mathrm{Lqc}(G, Y)$ to $\mathrm{Qch}(X)$, provided f is flat and G is concentrated over S. Similarly, μ is an isomorphism of functors from $\mathrm{Mod}(G, Y)$ to $\mathrm{Mod}(X)$ if f is locally an open immersion. Thus we have the following result.

7.5. Lemma. *Let $f : X \to Y$ be an S-morphism between S-schemes with trivial G-actions. If f is flat and G is concentrated over S, then*

$$\epsilon : f^*(?)^G \to (?)^G f^*$$

is an isomorphism between functors from $\mathrm{Lqc}(G, Y)$ *to* $\mathrm{Qch}(X)$. *If f is locally an open immersion, then ϵ is an isomorphism between functors from* $\mathrm{Mod}(G, Y)$ *to* $\mathrm{Mod}(X)$.

7.6. Lemma. *Let $f : X \to Y$ be as in Section 7.3. Then the diagram*

$$\begin{array}{ccc} (?)^G & \xrightarrow{u} & f_*f^*(?)^G \\ \downarrow \mathrm{id} & & \downarrow e\epsilon \\ (?)^G & \xrightarrow{u} & (?)^G f_*f^* \end{array}$$

is commutative.

Proof. We need to prove that the composite

$$\begin{aligned}(?)_{-1}R_{\Delta_M} \xrightarrow{u} f_*f^*(?)_{-1}R_{\Delta_M} \xrightarrow{\theta} f_*(?)_{-1}\tilde{B}_G^M(f)^*R_{\Delta_M} &\xrightarrow{\mu} f_*(?)_{-1}R_{\Delta_M}f^* \\ \xrightarrow{c^{-1}} (?)_{-1}\tilde{B}_G^M(f)_*R_{\Delta_M}f^* &\xrightarrow{\xi} (?)_{-1}R_{\Delta_M}f_*f^*\end{aligned}$$

agrees with u. Since $c^{-1}\mu$ in the displayed composition agrees with μc^{-1} by the naturality of c^{-1}, it suffices to show that the composite

$$(?)_{-1} \xrightarrow{u} f_*f^*(?)_{-1} \xrightarrow{\theta} f_*(?)_{-1}\tilde{B}_G^M(f)^* \xrightarrow{c^{-1}} (?)_{-1}\tilde{B}_G^M(f)_*\tilde{B}_G^M(f)^* \tag{10}$$

agrees with u and that the composite

$$R_{\Delta_M} \xrightarrow{u} \tilde{B}_G^M(f)_*\tilde{B}_G^M(f)^*R_{\Delta_M} \xrightarrow{\mu} \tilde{B}_G^M(f)_*R_{\Delta_M}f^* \xrightarrow{\xi} R_{\Delta_M}f_*f^* \tag{11}$$

agrees with u.

By [9, Lemma 1.24], (10) agrees with u; (11) agrees with u by the commutativity of the diagram

$$
\begin{tikzcd}
 & R_{\Delta_M} \arrow[r, "u"] \arrow[dl, "u"'] & R_{\Delta_M} f_* f^* \arrow[dl, "u"'] \arrow[dd, "\xi^{-1}"] \\
g_* g^* R_{\Delta_M} \arrow[r, "u"] \arrow[ddr, bend right, "\mu"] & g_* g^* R_{\Delta_M} f_* f^* \arrow[d, "\xi^{-1}"] & \\
 & g_* g^* g_* R_{\Delta_M} f^* \arrow[d, "\varepsilon"] & g_* R_{\Delta_M} f^* \arrow[l, "u"'] \arrow[d, "\xi"] \arrow[dl, "\mathrm{id}"'] \\
 & g_* R_{\Delta_M} f^* \arrow[r, "\xi"] & R_{\Delta_M} f_* f^*,
\end{tikzcd}
$$

where $g = \tilde{B}_G^M(f)$. □

7.7. Let X be a G-scheme, U a G-stable open subscheme of X, and V a G-stable open subscheme of U. The local section functor $\underline{\Gamma}_{B_G^M(U), B_G^M(V)} \colon \operatorname{Mod}(G, X) \to \operatorname{Mod}(G, X)$ is simply denoted by $\underline{\Gamma}_{U,V}$ and is called the *equivariant local section functor*. The right derived functor $R^i \underline{\Gamma}_{U,V}$ is denoted by $\underline{H}^i_{U,V}$ and is called the *equivariant local cohomology*. For a G-stable closed subscheme Y of X and a G-stable closed subscheme Z of Y, the local section functor $\underline{\Gamma}_{B_G^M(Y); B_G^M(Z)}$ is simply denoted by $\underline{\Gamma}_{Y;Z}$. As usual, $\underline{\Gamma}_{Y;\emptyset}$ is denoted by $\underline{\Gamma}_Y$. The derived functor $R^i \underline{\Gamma}_{Y;Z}$ is denoted by $\underline{H}^i_{Y;Z}$, and $R^i \underline{\Gamma}_Y$ is denoted by $\underline{H}^i_Y$.

7.8. Let X be an S-scheme with a trivial G-action. Let U be an open subscheme of X, and let V be an open subscheme of U. Let $f \colon U \hookrightarrow X$ be the inclusion, and let $g \colon V \hookrightarrow U$ be the inclusion.

By Lemma 7.6, we have a commutative diagram with exact rows:

$$
\begin{tikzcd}
0 \arrow[r] & \underline{\Gamma}_{U,V}(?)^G \arrow[r, "\iota"] & f_* f^*(?)^G \arrow[r, "u"] \arrow[d, "e\epsilon"] & f_* g_* g^* f^*(?)^G \arrow[d, "ee\epsilon\epsilon"] \\
0 \arrow[r] & (?)^G \underline{\Gamma}_{U,V} \arrow[r, "\iota"] & (?)^G f_* f^* \arrow[r, "u"] & (?)^G f_* g_* g^* f^*.
\end{tikzcd}
\tag{12}
$$

Hence there is a unique natural map

$$E \colon \underline{\Gamma}_{U,V}(?)^G \to (?)^G \underline{\Gamma}_{U,V}$$

such that $\iota E = e\epsilon\iota$.

By Lemma 7.5, the vertical maps in (12) are isomorphisms. Consequently, E is an isomorphism.

8. *G*-local *G*-schemes

Let S be a scheme, G a flat S-group scheme concentrated over S, and X a G-scheme (i.e., an S-scheme with a left G-action).

8.1. Let $\iota\colon Y \hookrightarrow X$ be a subscheme. We denote the composite

$$G \times Y \xrightarrow{1_G \times \iota} G \times X \xrightarrow{a} X$$

by a_Y, where a is the action. If a_Y factors through Y, then we say that Y is *G-stable*. In this case, Y has a unique G-scheme structure such that ι is a G-morphism.

The scheme-theoretic image of a_Y is denoted by Y^*. If ι is quasi-compact, then Y^* is the smallest closed G-stable subscheme of X containing Y (see [8, Lemma 2.1.5]).

8.2. A closed subscheme Y of X is G-stable if and only if $Y = Y^*$. Let $(Y_\lambda)_{\lambda\in\Lambda}$ be a family of closed subschemes of X. If Y_λ is defined by a quasi-coherent ideal sheaf $\mathcal{I}_\lambda$, then the sum $\sum_\lambda \mathcal{I}_\lambda$ is also a quasi-coherent ideal sheaf and it defines the intersection $\bigcap_\lambda Y_\lambda$ (i.e., the direct product of Y_λ in the category of X-schemes; it is also the usual intersection, set theoretically). If each Y_λ is G-stable, then $\bigcap Y_\lambda$ is also G-stable. The complement of a G-stable closed subscheme is a G-stable open subset.

8.3. The intersection of finitely many G-stable open subsets is G-stable. Moreover, the union of G-stable open subsets is G-stable. Letting a G-stable open subset open, we can define a topology on X. We call this topology the *G-Zariski topology.*

If X is quasi-compact with respect to the G-Zariski topology, we say that X is *G-quasi-compact*. Since the G-Zariski topology is coarser than the Zariski topology, a quasi-compact G-scheme is G-quasi-compact.

Let U be a G-stable open subset of X, and let Y be $X \setminus U$ with the reduced structure. It is easy to verify that Y^* does not intersect U (so $Y^* = Y$, set theoretically). Note that Y^* is G-stable, and hence U has a G-stable complement Y^*. Thus a closed subset in the G-Zariski topology is nothing but an underlying subset of some G-stable closed subscheme. If Y is an open or closed G-stable subscheme of X, then the G-Zariski topology of Y agrees with the induced topology of Y induced by the G-Zariski topology of X. If $f\colon X \to X'$ is a G-morphism of G-schemes, then f is continuous with respect to the G-Zariski topologies.

8.4. Lemma. *If X is G-quasi-compact and if Y is a G-stable closed subscheme of X, then there exists a minimal nonempty closed G-subscheme of Y.*

Proof. Observe that Y is G-quasi-compact, since it is a closed subset of quasi-compact X, with respect to the G-Zariski topology. Let Ω be the set of nonempty G-stable closed subschemes of Y. For $Z, Z' \in \Omega$, we say that $Z \leq Z'$ if $Z \supset Z'$. Then, by Zorn's lemma, Ω has a maximal element and the proof is complete. □

8.5. Lemma. *Assume that $G \to S$ is universally open. Then any $x \in X$ has a quasi-compact G-stable open neighborhood.*

Proof. Let U be an affine open neighborhood of x. Because the action $a\colon G \times X \to X$ is an open map, $U^\star := a(G \times U)$ is open; it is also G-stable, as can be seen easily. Since U is quasi-compact and since G is quasi-compact over S, it follows that $G \times U$ is quasi-compact. Hence $U^\star$ is quasi-compact. Now $U^\star \supset U$ is obvious, so $U^\star$ is the desired open neighborhood of x. □

Since we assume that G is flat, if G is locally of finite presentation over S then $G \to S$ is universally open (see [5, (I.10.4)]).

8.6. COROLLARY. *Let $G \to S$ be universally open. If X is G-quasi-compact, then X is quasi-compact.*

Proof. By Lemma 8.5, X has an open covering U_λ consisting of quasi-compact G-stable open subschemes. By assumption, there exist $\lambda_1, \dots, \lambda_n$ such that $X = \bigcup_{i=1}^n U_{\lambda_i}$. Because each U_{λ_i} is quasi-compact, X is quasi-compact. □

8.7. A topological space Γ is said to be *local* if it has a unique minimal nonempty closed subset—say, Θ. In this case, we say that (Γ, Θ) is local.

8.8. LEMMA. *Let Γ be a topological space. Then the following statements are equivalent.*

(i) Γ *is local.*
(ii) Γ *is nonempty and, if (F_λ) is a nonempty family of nonempty closed subsets of* Γ, *then $\bigcap F_\lambda$ is nonempty.*
(iii) Γ *is nonempty and, for any open covering (U_λ) of* Γ, *there exists some λ such that $X = U_\lambda$.*

In particular, a local topological space is nonempty and quasi-compact.

Proof. (i) $\Rightarrow$ (ii): Let (Γ, Θ) be local. Then $\Gamma \supset \Theta \neq \emptyset$. Moreover, $\bigcap_\lambda F_\lambda \supset \Theta \neq \emptyset$.

(ii) $\Rightarrow$ (i): Let Ω be the set of nonempty closed subsets of Γ. Then $\bigcap_{F \in \Omega} F$ is the desired unique minimal nonempty closed subset of Γ.

(ii) $\Leftrightarrow$ (iii): This is trivial. □

8.9. COROLLARY. *If $f : \Gamma \to \Gamma'$ is a surjective continuous map of topological spaces and if Γ is local, then Γ' is local. If (Γ, Θ) is local then (Γ', Θ') is local, where Θ' is the closure of $f(\Theta)$.*

Proof. Since f is a map and Γ is nonempty, it follows that Γ' is nonempty. Let Ω' be a nonempty set of nonempty closed subsets of Γ'. Then $f^{-1}(F') \neq \emptyset$ for $F' \in \Omega'$ by the surjectivity of f. Hence $f^{-1}\bigl(\bigcap_{F' \in \Omega'} F'\bigr) = \bigcap f^{-1}F' \neq \emptyset$ by the localness of Γ, so Γ' is local.

We prove the last assertion. Let (Γ', Θ') be local. Because f is surjective, $f^{-1}(\Theta')$ is a nonempty closed subset of Γ and hence $f^{-1}(\Theta') \supset \Theta$. Therefore, the closure of $f(\Theta)$ is a nonempty closed subset of Θ'. By minimality, they agree. □

8.10. LEMMA. *A T_0-space Γ is local if and only if Γ is quasi-compact and has exactly one closed point γ. In this case, (Γ, γ) is local.*

Proof. We first prove the "only if" part. By Lemma 8.8, Γ is nonempty and quasi-compact. A nonempty quasi-compact T_0-space has a closed point, so Γ has at least one closed point γ. However, a closed point is minimal nonempty closed. Such a point must be unique, and the last assertion is also obvious.

For the "if" part, let F be a nonempty closed subset of Γ. Then F is a nonempty quasi-compact T_0-space and has a closed point. This closed point must be γ, so γ is the unique minimal nonempty closed subset of Γ. □

For $x, y \in \Gamma$, we define $x \equiv y$ if $\bar{x} = \bar{y}$, where the bar denotes closure. The quotient space $\Gamma/\equiv$ is called the T_0-*ification* of Γ.

8.11. LEMMA. *Let $\pi : \Gamma \to \Gamma_0$ be the T_0-ification. Then Γ is local if and only if Γ_0 is local. If (Γ, Θ) and (Γ_0, Θ_0) are local, then $\pi(\Theta) = \Theta_0$ and $\Theta = \pi^{-1}(\Theta_0)$.*

Proof. Because π is surjective and continuous, if Γ is local then Γ_0 is local by Corollary 8.9.

We prove the converse. Since Γ_0 is nonempty and π is surjective, it follows that Γ is nonempty. Let Ω be a nonempty set of nonempty closed subsets of Γ. Then, for each $F \in \Omega$, we have $F = \pi^{-1}(\pi(F))$. Because π is submersive (i.e., for any subset F' of Γ_0, F' is closed if and only if $\pi^{-1}(F')$ is closed), $\pi(F)$ is both closed and nonempty. Therefore,

$$\pi\Big(\bigcap_{F\in\Omega} F\Big) = \pi\Big(\pi^{-1}\Big(\bigcap \pi(F)\Big)\Big) = \bigcap \pi(F) \neq \emptyset.$$

Hence $\bigcap F$ is nonempty and Γ is local.

Now $\pi(\Theta) = \Theta_0$ follows from Corollary 8.9, since Θ_0 is a point by Lemma 8.10. Since Θ is closed, $\Theta = \pi^{-1}(\pi(\Theta)) = \pi^{-1}(\Theta_0)$. □

8.12. LEMMA. *For a scheme Z, the following statements are equivalent.*

(i) *The underlying topological space of Z is local.*
(ii) *Z is local; that is, $Z \cong \operatorname{Spec} A$ for some local ring $(A, \mathfrak{m})$.*
(iii) *Z is quasi-compact and has a unique closed point z.*

In this case, $(Z, z) \cong (\operatorname{Spec} A, \mathfrak{m})$ are local topological spaces.

Proof. (i) ⇒ (ii): Let (U_λ) be an affine open covering of Z. Then $Z = U_\lambda$ for some λ by Lemma 8.8, so $Z \cong \operatorname{Spec} A$ is affine. Since Z is nonempty, A is nonzero and has a maximal ideal. If A has two or more maximal ideals, then Z has two or more closed points and then Z cannot be local. Hence A is a local ring.

(ii) ⇒ (iii): This is obvious.

(iii) ⇒ (i): This follows from Lemma 8.10, since a scheme is T_0.

The last assertion is obvious. □

8.13. DEFINITION. We say that a G-scheme X is G-*local* if there is a unique minimal nonempty G-stable closed subscheme of X. If X is G-local and if Y is the unique minimal nonempty G-stable closed subscheme, then we say that (X, Y) is G-local.

8.14. LEMMA. *Let X be a G-scheme. Then the following are equivalent*:

(i) *X is G-local*;
(ii) *X is local in the G-Zariski topology.*

In particular, a G-local G-scheme is G-quasi-compact. Moreover, if (X, Y) is G-local, then (X, Y) is local in the G-Zariski topology.

Proof. (i) $\Rightarrow$ (ii): Let (X, Y) be G-local. If F is a nonempty closed subset of X in the G-Zariski topology, then F is the underlying set of some nonempty G-stable closed subscheme of X. So $F \supset Y$, and (X, Y) is local in the G-Zariski topology.

(ii) $\Rightarrow$ (i): Let $Y = \bigcap_{F \in \Omega} F$, where Ω is the set of all nonempty G-stable closed subschemes of X. Then Y is nonempty by assumption, and (X, Y) is G-local. $\square$

8.15. COROLLARY. *If $G \to S$ is universally open, then a G-local G-scheme is quasi-compact.*

Proof. This follows immediately from the lemma and Corollary 8.6. $\square$

8.16. COROLLARY. *Let $f : X \to X'$ be a surjective G-morphism of G-schemes. If X is G-local, then X' is G-local. Moreover, if f is concentrated, (X, Y) is G-local, and (X', Y') is G-local, then the scheme-theoretic image of $f|_Y$ is Y'.*

Proof. The first assertion is an immediate consequence of the theorem and Corollary 8.9. We prove the last assertion. Since f is surjective, $Y \subset f^{-1}(Y')$. Thus the scheme-theoretic image of $f|_Y$ is contained in Y'. Because f is concentrated, $f|_Y$ is also concentrated and hence the scheme-theoretic image of $f|_Y$ is G-stable closed, since $(f|_Y)_*\mathcal{O}_Y \in \mathrm{Qch}(G, X')$. By the minimality of Y', the scheme-theoretic image of $f|_Y$ agrees with Y'. $\square$

Here are some examples of G-local G-schemes.

8.17. EXAMPLE. Assume that G is trivial. Then the G-Zariski topology agrees with the usual Zariski topology and, by Lemma 8.12, X is G-local if and only if X is a local scheme.

8.18. EXAMPLE. If $S = \operatorname{Spec} k$ for k a field, then (G, G) is G-local, where G acts on G left regularly.

Proof. It suffices to show that, if Y is a nonempty G-stable closed subscheme of G, then $Y = G$. Since Y is nonempty, Y has a geometric point $\eta : \operatorname{Spec} K \to Y$. Taking the base change and replacing k by K, we may assume that Y has a k-rational point y. Then $G \to G$ $(g \mapsto gy)$ is an isomorphism and hence $Y = Y^* \supset \{y\}^* = G$. $\square$

8.19. EXAMPLE. Let k be a field, let G be affine and of finite type, and let X be a homogeneous space G/H for some closed subgroup scheme H of G. If $S = \operatorname{Spec} k$, then (X, X) is G-local. This example shows that, even if S and G are affine, a G-local G-scheme X need not be affine in general.

Proof. Let $p\colon G \to X = G/H$ be the canonical projection. Then p is faithfully flat and is surjective. Since (G, G) is G-local by Example 8.18, (X, X) is G-local by Corollary 8.16. $\square$

8.20. EXAMPLE. Let $S = \operatorname{Spec} \mathbb{Z}$ and let $G = \mathbb{G}_m^n$, the split torus over S. Let $X = \operatorname{Spec} A$ be affine. Then A is a $\mathbb{Z}^n$-graded ring in a natural way [8, (II.1.2)]. By definition, X is G-local if and only if A is H-local in the sense of Goto and Watanabe [4].

8.21. EXAMPLE. Let $S = \operatorname{Spec} k$ with k an algebraically closed field, and let G be a reductive group, B a Borel subgroup of G, and P a parabolic subgroup of G containing B. A Schubert subvariety of G/P is a B-stable closed subvariety by definition. The point P/P is the unique minimal Schubert subvariety (see [15, Chap. 13]), and we have that $(G/P, P/P)$ is B-local.

8.22. Let $S = \operatorname{Spec} k$ with k a field. We say that G is *geometrically reductive* if G is affine of finite type and if, for any finite-dimensional G-module V and any $v \in V^G \setminus 0$, there exist an $r > 0$ and an $f \in (\operatorname{Sym}_r V^*)^G$ such that $f(v) \neq 0$. Moreover, if we can take r to be 1 (for any V and v), then we say that G is *linearly reductive*. If r can be taken to be 1 if the characteristic of k is zero, and a power of p if the characteristic p of k is positive, then we say that G is *strongly geometrically reductive* (SGR for short). By definition, a linearly reductive group scheme is SGR. We can prove that G is geometrically reductive if and only if G is SGR if and only if the radical of the linear algebraic group $(\bar{k} \otimes_k G)_{\mathrm{red}}$ is a torus if and only if, for any finitely generated k-algebra A with a G-action, A^G is finitely generated [10]. This last fact is probably well known for linear algebraic groups, but we will not use it here and mainly consider the SGR property.

8.23. Assume that G and $S = \operatorname{Spec} R$ are affine. We say that A is a *G-algebra* if A is an R-algebra and if a G-scheme structure of $\operatorname{Spec} A$ is given. This is equivalent to saying that A is both an R-algebra and a G-module (whose underlying R-module structures agree) and that the product $A \otimes_R A \to A$ is G-linear. An ideal I of A is called a *G-ideal* if $\operatorname{Spec} A/I$ is a G-stable closed subscheme of $\operatorname{Spec} A$ or, equivalently, if I is a (G, A)-submodule of A.

8.24. LEMMA. *Let $S = \operatorname{Spec} k$ with k a field, and let G be an SGR k-group scheme. Let A be a G-algebra, and let $f \in \left(\sum_\lambda I_\lambda\right)^G$. If I_λ is a family of G-ideals, then there exists some q such that $f^q \in \sum_\lambda I_\lambda^G$, where q is a power of p if the characteristic of k is $p > 0$ and $q = 1$ if k is of characteristic* 0.

Proof. See [18, Apx. to Chap. 1, C]. $\square$

8.25. Let S and G be affine, and let A be a G-algebra. A maximal element of

$$\{I \mid I \text{ is a } G\text{-ideal and } I \neq A\}$$

is said to be *G-maximal*. We say that A is *G-local* if A has a unique G-maximal G-ideal. Note that A is G-local if and only if $\operatorname{Spec} A$ is G-local.

8.26. LEMMA. *Let S and G be affine. If A is a G-algebra and if $I \neq A$ is a G-ideal, then there is a G-maximal ideal of A containing I.*

Proof. Since $X = \operatorname{Spec} A$ is quasi-compact, it is also G-quasi-compact. Now apply Lemma 8.4. $\square$

8.27. PROPOSITION. *Let $S = \operatorname{Spec} k$ with k a field, and let G be SGR. Let A be a G-algebra. If $\mathfrak{p} \in \operatorname{Spec} A^G$, then $A_\mathfrak{p} := A \otimes_{A^G} A^G_\mathfrak{p}$ is G-local.*

Proof. Observe that $(A_\mathfrak{p})^G = A^G_\mathfrak{p}$. Replacing A by $A_\mathfrak{p}$, we may assume that $(A^G, \mathfrak{m})$ is a local ring; we are to prove that A is G-local.

Because A^G is nonzero, A is nonzero. By Lemma 8.26, A has a G-maximal ideal. Assume that A has two different G-maximal ideals I and J. Since $1 \notin I$ and $1 \notin J$, it follows that $I^G \subset \mathfrak{m}$ and $J^G \subset \mathfrak{m}$. On the other hand, $I + J = A$ by maximality. By Lemma 8.24, $1 \in I^G + J^G \subset \mathfrak{m}$. This is a contradiction, so A is G-local. $\square$

Note that, in the proposition, $\mathfrak{p}A_\mathfrak{p}$ may not be the G-maximal ideal. Indeed, if $G = \mathbb{G}_m$ and $A = k[x]$ with $\deg x = 1$ and $\mathfrak{p} = 0 \subset A^G = k$, then $0 = \mathfrak{p}A_\mathfrak{p}$ is not G-maximal, since $(x) \subset A$ is a G-ideal.

9. A Generalization of a Special Case of a Theorem of Hochster and Eagon

Let S, G, and X be as in Section 8. In this section, we give an application of equivariant local cohomology on a G-local G-scheme to invariant theory.

9.1. LEMMA. *Let $S = \operatorname{Spec} k$ with k a field, and let G be SGR. Let A be a G-algebra. Assume that the canonical map $\pi\colon \operatorname{Spec} A \to \operatorname{Spec} A^G$ is a geometric quotient in the sense of* [18]. *Then, for any prime ideal $\mathfrak{p}$ of A^G, $\mathfrak{p}A_\mathfrak{p}$ and the G-maximal ideal P of the G-local ring $A_\mathfrak{p}$ have the same radical.*

Proof. Because $\mathfrak{p}A_\mathfrak{p}$ is a G-ideal of $A_\mathfrak{p}$, we have $P \supset \mathfrak{p}A_\mathfrak{p}$. Assume that $\sqrt{P} \neq \sqrt{\mathfrak{p}A_\mathfrak{p}}$. Then there is an algebraically closed extension field K of $\kappa(\mathfrak{p})$ such that (a) there are K-valued points ξ of $V(P)$ and η of $V(\mathfrak{p}A_\mathfrak{p}) \setminus V(P)$ and (b) the set of K-valued points of $V(\mathfrak{p}A_\mathfrak{p})$ constitutes one orbit with respect to the action of $G(K)$. But since $V(P)$ is G-stable and since $\xi \in V(P)(K)$ and $\eta \notin V(P)(K)$, it follows that ξ and η cannot be on the same orbit. This is a contradiction; hence $\sqrt{P} = \sqrt{\mathfrak{p}A_\mathfrak{p}}$. $\square$

9.2. Let X be a locally Noetherian G-scheme and $\mathcal{M}$ a coherent $(G, \mathcal{O}_X)$-module. Then $\underline{\operatorname{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{M})$ is also a coherent $(G, \mathcal{O}_X)$-module, as can be seen easily from [9, Lemma 6.33] and [9, Lemma 7.11]. The canonical map

$$\mathcal{O}_X \to \underline{\operatorname{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{M})$$

is $(G, \mathcal{O}_X)$-linear. Hence the kernel $\underline{\operatorname{ann}}\, \mathcal{M}$ is a coherent G-ideal. Therefore, $\operatorname{Supp} \mathcal{M} = V(\underline{\operatorname{ann}}\, \mathcal{M})$ is a G-stable closed subscheme of X.

9.3. Let (X, Y) be a G-local G-scheme and assume that X is Noetherian. Let Z be any irreducible component of Y, and let ζ be the generic point of Z.

9.4. LEMMA. *The functor* $(?)_\zeta : \mathrm{Qch}(G, X) \to \mathrm{Mod}(\mathcal{O}_{X,\zeta})$ *is faithfully exact.*

Proof. Clearly, the restriction $\mathrm{Qch}(G, X) \to \mathrm{Qch}(X)$ is exact. Moreover, the stalk functor $(?)_\zeta : \mathrm{Qch}(X) \to \mathrm{Mod}(\mathcal{O}_{X,\zeta})$ is exact. Hence the composite is exact.

We prove that the functor in question is faithful. Assume the contrary, and let $\mathcal{M} \in \mathrm{Qch}(G, X)$, $\mathcal{M} \neq 0$, and $\mathcal{M}_\zeta = 0$. Then, since $\mathrm{Qch}(G, X)$ is locally Noetherian and since a Noetherian object of $\mathrm{Qch}(G, X)$ is nothing more than a coherent $(G, \mathcal{O}_X)$-module by [9, Cor. 11.8], there exists some nonzero coherent $(G, \mathcal{O}_X)$-submodule $\mathcal{N}$ of $\mathcal{M}$. Let $V := \mathrm{Supp}\,\mathcal{N}$. Then V is nonempty, closed, and G-stable, so $V \supset Y \supset Z \ni \zeta$. Hence $0 = \mathcal{M}_\zeta \supset \mathcal{N}_\zeta \neq 0$, and this is a contradiction. □

9.5. THEOREM. *Let k be a field, G a linearly reductive k-group scheme, and X a Cohen–Macaulay Noetherian G-scheme. Let $\pi : X \to Y$ be a geometric quotient under the action of G in the sense of* [18]. *Assume that π is an affine morphism. Then Y is Noetherian and Cohen–Macaulay.*

Proof. Since π is surjective, Y is quasi-compact. It therefore suffices to show that Y is locally Noetherian and Cohen–Macaulay. The question is local on Y, and we may assume that $Y = \mathrm{Spec}\, A$ is affine.

Since π is affine, it follows that $X = \mathrm{Spec}\, B$ is also affine, and $A = B^G$ by assumption. We remark that A is a direct summand subring of B because G is linearly reductive. In particular, A is Noetherian since B is (see [14, Prop. 6.15]).

It remains to show that A is Cohen–Macaulay. Toward this end, we localize A at one of its maximal ideals and so may further assume that $(A, \mathfrak{m})$ is local. Note that π is still submersive after localization, since G is linearly reductive and $A = B^G$ (see the proof of [18, Thm. 1.1]). By Proposition 8.27, X is G-local. Let Z be the unique minimal nonempty closed G-subscheme of X.

Let y be the closed point of Y. Then

$$\underline{H}^i_y(\mathcal{O}_Y) \cong H^i(R\underline{\Gamma}_y((\pi_*\mathcal{O}_X)^G)).$$

Let $\mathbb{J}$ be the injective resolution of $\pi_*\mathcal{O}_X$ in $\mathrm{Qch}(G, Y)$. Then $\mathbb{J}^G$ is an injective resolution of $(\pi_*\mathcal{O}_X)^G$ in $\mathrm{Qch}(Y)$, because $(?)^G : \mathrm{Qch}(G, Y) \to \mathrm{Qch}(Y)$ is exact and preserves injectives (since it has an exact left adjoint $(?)_{\Delta_M} L_{-1}$). Any injective object of $\mathrm{Qch}(Y)$ is injective in $\mathrm{Mod}(Y)$ by [7, (II.7)]. Hence we have isomorphisms

$$\underline{H}^i_y(\mathcal{O}_Y) \cong H^i(\underline{\Gamma}_y \mathbb{J}^G) \cong H^i((\underline{\Gamma}_y \mathbb{J})^G) \cong (H^i(\underline{\Gamma}_y \mathbb{J}))^G,$$

where the second isomorphism is by Section 7.8 and the third isomorphism is by the exactness of $(?)^G$ on $\mathrm{Qch}(G, Y)$ (note that $\underline{\Gamma}_y \mathbb{J}$ is a complex in $\mathrm{Qch}(G, Y)$ by Corollary 4.11).

Thus, to show that Y is Cohen–Macaulay it suffices to show that the cohomology of the complex $\underline{\Gamma}_y \mathbb{J}$ is concentrated in one place. By Lemma 9.1, $\pi^{-1}(y)$ and Z agree set theoretically. So, by Corollary 4.17,

$$H^i(\underline{\Gamma}_y \mathbb{J}) \cong H^i(R\underline{\Gamma}_y(\pi_* \mathcal{O}_X)) \cong H^i(R\underline{\Gamma}_y R\pi_* \mathcal{O}_X)$$
$$\cong H^i(R\pi_* R\underline{\Gamma}_{\pi^{-1}(y)} \mathcal{O}_X) = H^i(R\pi_* R\underline{\Gamma}_Z \mathcal{O}_X).$$

Observe that

$$(?)_\zeta \underline{H}^i_Z(\mathcal{O}_X) \cong H^i((?)_\zeta R\underline{\Gamma}_Z \mathcal{O}_X) \cong H^i_\zeta(\mathcal{O}_{X,\zeta})$$

by Theorem 6.10. We have $H^i_\zeta(\mathcal{O}_{X,\zeta}) = 0$ for $i \neq d$ ($d := \dim \mathcal{O}_{X,\zeta}$), since $\mathcal{O}_{X,\zeta}$ is a Cohen–Macaulay local ring. Since $(?)_\zeta$ is faithfully exact by Lemma 9.4, $\underline{H}^i_Z(\mathcal{O}_X) = 0$ for $i \neq d$. Let $\mathcal{M} := \underline{H}^d_Z(\mathcal{O}_X)$, and note that $\mathcal{M}$ is quasi-coherent. Then

$$R\pi_* R\underline{\Gamma}_Z \mathcal{O}_X \cong R\pi_* \mathcal{M}[-d] \cong \pi_* \mathcal{M}[-d].$$

As a result,

$$H^i(\underline{\Gamma}_y \mathbb{J}) \cong H^i(R\pi_* R\underline{\Gamma}_Z \mathcal{O}_X) = H^{d-i}(\pi_* \mathcal{M}) = 0$$

for $i \neq d$. This is what we wanted to prove. □

9.6. COROLLARY. *Let k be an algebraically closed field, G a linearly reductive k-group scheme, and $X = \operatorname{Spec} B$ a Cohen–Macaulay affine G-scheme of finite type. Let $\pi : X \to Y = \operatorname{Spec} B^G$ be the canonical morphism, and set*

$$U := \{x \in X \mid \dim O(x) \text{ is maximal and } O(x) \text{ is closed}\},$$

where $O(x)$ is the G-orbit of x. Then U is a G-stable open subset of X, and $\pi(U)$ is Cohen–Macaulay.

Proof. This follows easily from the theorem and [20, Prop. 3.8]. □

9.7. COROLLARY. *Let k be a field, G a linearly reductive finite k-group scheme, and B a Noetherian and Cohen–Macaulay G-algebra. Then B^G is Noetherian and Cohen–Macaulay.*

The corollary is an immediate consequence of a theorem of Hochster and Eagon [11, Prop. 12] (note that B is integral over B^G; see the proof of Lemma 9.8 to follow). Indeed, the case of G a finite group is stated in [11, Prop. 13] (however, they do not assume that B contains a field, and our corollary is not a complete generalization of [11, Prop. 13]). Corollary 9.7 is also obvious by Theorem 9.5 and the following lemma.

9.8. LEMMA. *Let k be a field and let G be a finite k-group scheme. Let B be a G-algebra. Then the canonical map $\pi : \operatorname{Spec} B \to \operatorname{Spec} B^G$ is a geometric quotient.*

Proof. Since G° (the identity component of G) is normal in G, it suffices to prove that $\operatorname{Spec} B \to \operatorname{Spec} B^{G^\circ}$ and $\operatorname{Spec} B^{G^\circ} \to \operatorname{Spec} (B^{G^\circ})^{G/G^\circ}$ are geometric quotients. Thus we may assume that G is either infinitesimal or étale.

Consider the case where G is infinitesimal. We may assume that the characteristic p of k is positive, since any group scheme over a field of characteristic 0 is reduced [22, Thm. 11.4]. Let H be the coordinate ring of G°. Since G° is a point (set theoretically), H is an Artinian local ring. Let $\mathfrak{m}$ be the maximal ideal of H, and take $e \geq 1$ sufficiently large so that $\mathfrak{m}^{p^e} = 0$. Then it is easy to see that $b^{p^e} \in B^G$ for any $b \in B$. This shows that any base change of π is a homeomorphism (note also that B is integral over B^G). Hence π is a geometric quotient, as can be checked easily.

Next consider the case where G is étale. We show that B is integral over B^G. To verify this, we may assume (by virtue of the base change) that k is algebraically closed. In this case, G is a finite group. Then $b \in B$ is integral over B^G, since b is a root of the monic polynomial $\prod_{g \in G}(t - gb) \in B^G[t]$.

It remains to show that π is an orbit space. To verify this, we may assume that k is algebraically closed again. Thus G is a finite group, and it must be SGR. Indeed, let V be a finite-dimensional G-module and let $v \in V^G \setminus 0$; then there is a linear form $\varphi \in V^*$ such that $\varphi(v) \neq 0$. Let H be the trivial subgroup of G if the characteristic of k is 0, and let H be a p-Sylow subgroup of G if the characteristic p of k is positive. Let r be the order of H, and let $\{g_1, \dots, g_l\}$ be a complete set of representatives of G/H. Note that l is nonzero in k. Then $f := \sum_{i=1}^{l} g_i \bigl(\prod_{h \in H} h\varphi\bigr)$ is in $(\mathrm{Sym}_r V^*)^G$, and $f(v) = l\varphi(v)^r \neq 0$.

Now assume that π is not a geometric quotient. Then there is an algebraically closed field K with $\operatorname{Spec} K \to \operatorname{Spec} B^G$ such that the geometric fiber $\operatorname{Spec} C$ has two K-rational points x and y on two different $G(K)$-orbits, where $C := K \otimes_{B^G} B$.

For any $c \in C^G$, there exists some q such that $c^q \in K$, where $q = 1$ when the characteristic of k is 0 and where q is a power of p when the characteristic p of k is positive. This can be seen easily from Lemma A.1.2 of [18, Apx. to Chap. 1, C]. Therefore, C^G is a ring with only one prime ideal.

On the other hand, Gx and Gy are closed orbits in $\operatorname{Spec} C$, since x and y are closed points and G is finite. By the choice of x and y, we have $Gx \cap Gy = \emptyset$. By Lemma 8.24 and the proof of [18, Thm. 1.1], x and y are mapped to different points in $\operatorname{Spec} C^G$. This contradicts the fact that C^G has only one prime ideal. □

9.9. Assume that the characteristic of k is 0. In addition to the assumption of Theorem 9.5, assume that X is of finite type over k and has rational singularities. Then Y is of finite type and has rational singularities by Boutot's theorem [1], which makes Theorem 9.5 unnecessary. Similarly, if the characteristic is positive and X is F-regular, then Y is F-regular by Corollary 9.11.

However, if D is a nonreduced Artinian local G-algebra with residue field k that is finite over k, then $\operatorname{Spec} D \times X$ is still of finite type and Cohen–Macaulay but does not have rational singularities, since it is not even reduced. By [18, Prop. 1.9], $\operatorname{Spec} D \times X$ admits an affine geometric quotient, which is Cohen–Macaulay by Theorem 9.5.

The following theorem and its corollary are due to Hochster. We include proofs because there is no appropriate reference.

9.10. THEOREM. *Let B be a ring and A its pure subring. If A is Noetherian then, for any maximal ideal $\mathfrak{m}$ of A, there exists some maximal ideal $\mathfrak{M}$ of B such that $A_{\mathfrak{m}} \to B_{\mathfrak{M}}$ is pure.*

Proof. We remark that $A_{\mathfrak{m}} \to B_{\mathfrak{m}}$ is pure. By [13, (2.2)], there exists some maximal ideal $\mathfrak{M}' = \mathfrak{M}B_{\mathfrak{m}}$ of $B_{\mathfrak{m}}$ ($\mathfrak{M} = \mathfrak{M}' \cap B$) such that $A_{\mathfrak{m}} \to (B_{\mathfrak{m}})_{\mathfrak{M}'} = B_{\mathfrak{M}}$ is pure. Let M be a maximal ideal of B containing $\mathfrak{M}$. Since $\mathfrak{M}$ lies on $\mathfrak{m}$ (by the purity) and since $\mathfrak{m}$ is maximal, it follows that M also lies on $\mathfrak{m}$. So $\mathfrak{M}' = \mathfrak{M}B_{\mathfrak{m}} \subset MB_{\mathfrak{m}} \neq B_{\mathfrak{m}}$. Since $\mathfrak{M}'$ is maximal, $\mathfrak{M}B_{\mathfrak{m}} = MB_{\mathfrak{m}}$ and hence $\mathfrak{M} = M$ is maximal. □

9.11. COROLLARY. *Let B be a Noetherian ring and A its pure subring. If B is normal* (*resp. of prime characteristic and weakly F-regular or of prime characteristic and F-regular*), *then so is A.*

Proof. Recall that A is Noetherian [14, Prop. 6.15]. The assertion for F-regularity follows from that for weak F-regularity by localization, so we consider normality and weak F-regularity. Note that each property in the problem is local on maximal ideals (see [12, (4.15)]). Hence by Theorem 9.10 we may assume that both A and B are local. Because weakly F-regular implies normal by [12, (5.11)], B is a normal domain. Now the assertion for normality follows from [14, Prop. 6.15], and the assertion for weak F-regularity follows from [12, (4.12)]. □

Added in proof. Related to Section 8.22, we refer the reader to Section 2 of W. van der Kallen, *A reductive group with finitely generated cohomology algebras,* Algebraic groups and homogeneous spaces, pp. 301–314, Tata Inst. Fund. Res. Stud. Math., Tata Inst. Fund. Res., Mumbai, 2007.

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