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# On Elliptic Dunkl Operators

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## 1. Introduction

Elliptic Dunkl operators for Weyl groups were introduced in [BuFV]. Another version of such operators was considered by Cherednik [C1], who used them to prove the quantum integrability of the elliptic Calogero–Moser systems. The goal of the present paper is to define elliptic Dunkl operators for any finite group $W$ acting on a (compact) complex torus $X$. (Although we work with a general finite group $W$, the theory essentially reduces to the case when $W$ is a crystallographic reflection group [GeM, 5.1] because $W$ can be replaced by its subgroup generated by reflections.) We attach such a set of operators to any topologically trivial holomorphic line bundle $\mathcal{L}$ on $X$ with trivial stabilizer in $W$ and to any flat holomorphic connection $\nabla$ on this bundle. When $W$ is the Weyl group of a root system and $X$ is the space of homomorphisms from the root lattice to the elliptic curve, our operators coincide with those of [BuFV]. We prove that the elliptic Dunkl operators commute, and we show that the monodromy of the holonomic system of differential equations defined by them gives rise to a family of $|W|$-dimensional representations of the Hecke algebra $\mathcal{H}_\tau(X,W)$ of the orbifold $X/W$ defined in [E]; conjecturally, this gives generic irreducible representations of this algebra. In the case of Weyl groups, the algebra $\mathcal{H}_\tau(X,W)$ is the double affine Hecke algebra (DAHA) of Cherednik [C2], while in the case $W = S_n \ltimes (\mathbb{Z}/\ell Z)^n$ ($\ell = 2,3,4,6$) it is the generalized DAHA introduced in [EGO]. We reproduce known families of representations of these algebras and also explain how to use the elliptic Dunkl operators to construct representations from category $\mathcal{O}$ over the elliptic Cherednik algebra—that is, the Cherednik algebra of the orbifold $X/G$ defined in [E].

In future work we plan to use the elliptic Dunkl operators to construct new quantum integrable systems. Namely, we expect that for any $X$ and a complex reflection group $W$ acting on $X$ there exists a commuting system of differential operators $L_1,\dots,L_d$, where $d = \dim X$, whose symbols are generators of the ring of $W$-invariant polynomials on the tangent space to $X$ at the origin. This system is supposed to depend on the same collection of parameters ("coupling constants") as the elliptic Dunkl operators (except for the bundle $\mathcal{L}$, on which it should be independent), and it should be obtained by appropriately symmetrizing the elliptic

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Dunkl operators and then degenerating the bundle $\mathcal{L}$ into the trivial bundle (following the idea outlined in [BuFV]). In the case when $W$ is a Weyl group, this system is the elliptic quantum Calogero–Moser system. We note that if $W$ is not a Weyl group then the integrable system $L_1, \dots, L_d$ will be somewhat "unphysical", since it won't have a second-order Hamiltonian.

## 2. Preliminaries

### *2.1. Finite Group Actions on Complex Tori*

Let $V$ be a finite-dimensional complex vector space. A nontrivial element $g \in \mathrm{GL}(V)$ is called a *reflection* if it is semisimple and fixes a hyperplane in $V$ pointwise.

Let $W$ be a finite subgroup of $\mathrm{GL}(V)$, and let $\Gamma \subset V$ be a lattice that is preserved by $W$. Then we have a $W$-action on the complex torus $X = V/\Gamma$. (Note that we don't assume that $X$ is algebraic, i.e., an abelian variety. For example, if $W$ is a trivial group then $\Gamma$ can be any lattice. However, in interesting examples $X$ is an abelian variety and, moreover, a power of an elliptic curve.) For any reflection $g \in W$, let $X^g$ be the set of $x \in X$ s.t. $gx = x$. A *reflection hypertorus* is any connected component of $X^g$ that has codimension 1. Let $X_{\mathrm{reg}}$ be the complement of reflection hypertori in $X$.

Let $H$ be a reflection hypertorus, and let $W_H \subset W$ be the stabilizer of a generic point in $H$. Then $W_H$ is a cyclic group with order $n_H$. The generator $g_H$ is the element in $W_H$ with determinant $\exp(2\pi \mathrm{i}/n_H)$. Let $\mathcal{S}$ denote the set of pairs $(H, j)$, where $H$ is a reflection hypertorus and $j = 1, \dots, n_H - 1$.

Under the $g_H$-action, we have the decomposition

$$V = V^{g_H} \oplus V_H,$$

where $V^{g_H}$ is the codimension-1 subspace of $V$ with a trivial action of $g_H$ and where $V_H = ((V^*)^{g_H})^{\perp}$, which is a $g_H$-invariant 1-dimensional space. We also have a similar decomposition on the dual space: $V^* = (V^*)^{g_H} \oplus V_H^*$.

### *2.2. Holomorphic Line Bundles on Complex Tori*

Let us recall the theory of holomorphic line bundles on complex tori (see [Mu] or [La] for more details).

Let $X = V/\Gamma$ be a complex torus. Any holomorphic line bundle $\mathcal{L}$ on $X$ is a quotient of $V \times \mathbb{C}$ by the $\Gamma$ action:

$$\gamma \colon (z, \xi) \mapsto (z + \gamma, \chi(\gamma, z)\xi),$$

where $\chi(\gamma, \cdot) \colon V \to \mathbb{C}^*$ is a holomorphic function s.t.

$$\chi(\gamma_1 + \gamma_2, z) = \chi(\gamma_1, z + \gamma_2)\chi(\gamma_2, z).$$

Denote by $\mathcal{L}(\chi)$ the line bundle corresponding to $\chi$.

Let $V^\vee = \mathrm{Hom}_{\bar{\mathbb{C}}}(V, \mathbb{C})$ be the vector space of $\mathbb{C}$-antilinear forms on $V$. We have a nondegenerate $\mathbb{R}$-bilinear form

$$\omega\colon V^\vee \times V \to \mathbb{R}, \quad \omega(\alpha, v) = \operatorname{Im}\alpha(v).$$

Then we define $\Gamma^\vee = \{\alpha \in V^\vee \mid \omega(\alpha, \Gamma) \subset \mathbb{Z}\}$. It is easy to see that $\Gamma^\vee$ is a lattice in $V^\vee$ and that we have the dual torus $X^\vee = V^\vee/\Gamma^\vee$.

To any element $\alpha \in X^\vee$ we can associate a line bundle $\mathcal{L}_\alpha = \mathcal{L}(\chi_\alpha)$, where $\chi_\alpha(\gamma, z) = \exp(2\pi i \omega(\alpha, \gamma))$. This is a topologically trivial line bundle on $X$.

PROPOSITION 2.1. *The map $\alpha \mapsto \mathcal{L}_\alpha$ is an isomorphism of groups*

$$X^\vee \to \operatorname{Pic}^0(X).$$

Now suppose that a finite group $W$ acts faithfully on $V$ and preserves a lattice $\Gamma$. By using the bilinear form $\omega$, we can define the dual $W$-action on $V^\vee$ that preserves the dual lattice $\Gamma^\vee$. Hence we have an action of $W$ on the complex torus $X$ and its dual $X^\vee$.

We define a $W$-action on $\operatorname{Pic}^0(X)$ by

$$w\colon \mathcal{L}_\alpha \mapsto \mathcal{L}_\alpha^w = \mathcal{L}_{w\alpha}.$$

We have $(\mathcal{L}^g)^h = \mathcal{L}^{hg}$.

### 2.3. The Poincaré Residue

Suppose $\alpha$ is a meromorphic 1-form on an $n$-dimensional complex manifold $X$ with a simple pole on a smooth hypersurface $Z \subset X$ and no other singularities. Near any point of $Z$, we can choose local coordinates $(z_1, \dots, z_n)$ on $X$ s.t. $Z$ is locally defined by the equation $z_1 = 0$. Then $\alpha$ can be locally expressed as

$$\alpha = \frac{1}{z_1} \sum_{i=1}^{n} \beta_i(z_1, \dots, z_n)\, \mathrm{d}z_i,$$

where the $\beta_i$ are holomorphic. Then $\beta_1|_Z$ is a holomorphic function on $Z$ and does not depend on the choice of coordinates. We define the Poincaré residue of $\alpha$ at $Z$ to be $\operatorname{Res}_Z(\alpha) = \beta_1|_Z$.

More generally, let $\mathcal{E}$ be a holomorphic vector bundle on $X$, and let $s$ be a meromorphic section of $\mathcal{E} \otimes T^*X$ that has a simple pole on a smooth hypersurface $Z \subset X$ and no other singularities. Similarly as before, we define the Poincaré residue of $s$ to be an element in $\Gamma(Z, \mathcal{E}|_Z)$ denoted by $\operatorname{Res}_Z(s)$.

## 3. Construction of Elliptic Dunkl Operators

### 3.1. The Sections $f^{\mathcal{L}}_{H,j}$

The goal of this subsection is to define certain meromorphic sections $f^{\mathcal{L}}_{H,j}$ of the bundle $(\mathcal{L}^{g_H^j})^* \otimes \mathcal{L} \otimes V_H^*$ that are used in the definition of elliptic Dunkl operators.

The line bundle $(\mathcal{L}_\alpha^w)^* \otimes \mathcal{L}_\alpha$ is topologically trivial, and it is holomorphically trivial if and only if $\alpha$ is a fixed point of $w$. Because $W$ acts faithfully on $V$, we can always find a point $\alpha \in X^\vee$ that is not fixed by any $w \in W$; in other words, there exists a topologically trivial line bundle $\mathcal{L} := \mathcal{L}_\alpha$ such that $(\mathcal{L}^w)^* \otimes \mathcal{L}$ is nontrivial for any $w \in W$. From now on, we fix such a line bundle.

Let $H \subset X$ be a reflection hypertorus.

LEMMA 3.1. *For $j = 1, \ldots, n_H - 1$, the holomorphic line bundle $(\mathcal{L}^{g_H^j})^* \otimes \mathcal{L}$ has a global meromorphic section $s$ with a simple pole on $H$ and no other singularities. Such $s$ is unique up to a scalar.*

*Proof.* Let $\bar{H} = \{x \in X \mid x + H = H\}$. Then $\bar{H}$ is a complex torus. It is sufficient to assume in the proof that $H = \bar{H}$.

We have a short exact sequence of complex tori:

$$0 \longrightarrow H \xrightarrow{\mu} X \xrightarrow{\nu} E \longrightarrow 0,$$

where $E = X/H$ is an elliptic curve. This sequence induces in turn a short exact sequence for the dual tori:

$$0 \longrightarrow E^\vee \longrightarrow X^\vee \longrightarrow H^\vee \longrightarrow 0,$$

which can be written, using the isomorphism of Proposition 2.1, as

$$1 \longrightarrow \mathrm{Pic}^0(E) \xrightarrow{\nu^*} \mathrm{Pic}^0(X) \xrightarrow{\mu^*} \mathrm{Pic}^0(H) \longrightarrow 1.$$

Since $\mu^*((\mathcal{L}^{g_H^j})^* \otimes \mathcal{L})$ is trivial, there exists a topologically trivial line bundle $\mathcal{L}'$ on $E$ such that $\nu^*\mathcal{L}' = (\mathcal{L}^{g_H^j})^* \otimes \mathcal{L}$. It is well known that $\mathcal{L}'$ has a unique meromorphic section, up to a scalar, which has simple pole at zero. Then $s = \nu^* s'$ is the required section of the bundle $(\mathcal{L}^{g_H^j})^* \otimes \mathcal{L}$ on $X$.

Now we prove the uniqueness of $s$ up to a scalar. The section $s$ can be viewed as a global holomorphic section of the line bundle $\mathcal{F} = (\mathcal{L}^{g_H^j})^* \otimes \mathcal{L} \otimes \mathcal{O}(H)$. Since $\mathcal{O}(H)$ is the pullback of $\mathcal{O}(0)$ on $E$, it follows that $\mathcal{F}$ is the pullback of the bundle $\mathcal{L}' \otimes \mathcal{O}(0)$ on $E$. Therefore, $H^0(X, \mathcal{F}) \simeq H^0(E, \mathcal{L}' \otimes \mathcal{O}(0)) = \mathbb{C}$ and $s$ is unique up to a scalar. □

Now choose a nonzero element $\alpha \in V_H^*$ and consider $s \otimes \alpha$, where $s$ is the global meromorphic section described in Lemma 3.1. Then $s \otimes \alpha$ is a global section of the bundle $(\mathcal{L}^{g_H^j})^* \otimes \mathcal{L} \otimes V_H^*$. Its only singularity is a simple pole at $H$, and it is defined by this condition uniquely up to scaling.

Next, observe that since $X$ is a torus, the bundle $T^*X$ is canonically trivial, and we can canonically identify the fibers of $T^*X$ with $V^*$. Thus we may consider the Poincaré residue $\mathrm{Res}_H(s \otimes \alpha)$, which is an element in $\Gamma(H, ((\mathcal{L}^{g_H^j})^* \otimes \mathcal{L})|_H)$. Since $((\mathcal{L}^{g_H^j})^* \otimes \mathcal{L})|_H$ is trivial, $\mathrm{Res}_H(s \otimes \alpha)$ is a holomorphic function on $H$; since $H$ is compact, $\mathrm{Res}_H(s \otimes \alpha)$ is a constant. By fixing this constant, we can fix $s \otimes \alpha$ uniquely—that is, we have the following lemma.

LEMMA 3.2. *For any $(H, j) \in \mathcal{S}$, there exists a unique global meromorphic section $f_{H,j}^{\mathcal{L}}$ of the bundle $(\mathcal{L}^{g_H^j})^* \otimes \mathcal{L} \otimes V_H^*$ with a simple pole on $H$, no other singularities, and residue $1$ on $H$.*

### 3.2. Elliptic Dunkl Operators

For any $g \in W$ there is a $W$-action on $\mathcal{S}$: $g(H, j) = (gH, j)$. Let $C$ be a $W$-invariant function on $\mathcal{S}$. Choose a holomorphic flat connection $\nabla$ on $\mathcal{L}$.

DEFINITION 3.1 (Elliptic Dunkl operators). For any $v \in V$, the *elliptic Dunkl operator* corresponding to $v$ is defined as the following operator acting on the local meromorphic sections of $\mathcal{L}$:

$$\mathcal{D}_{v,C}^{\mathcal{L},\nabla} = \nabla_v - \sum_{(H,j)\in\mathcal{S}} C(H,j)\langle f_{H,j}^{\mathcal{L}}, v\rangle g_H^j,$$

where $\nabla_v$ is the covariant derivative along $v$ corresponding to the connection $\nabla$ and where $\langle\cdot,\cdot\rangle$ is the natural pairing between $V$ and $V^*$.

REMARK 3.1. Let $\nabla, \nabla'$ be two flat holomorphic connections on $\mathcal{L}$. Then $\nabla - \nabla' = \xi$, where $\xi \in V^*$ is a holomorphic 1-form on $X$. Therefore,

$$\mathcal{D}_{v,C}^{\mathcal{L},\nabla} - \mathcal{D}_{v,C}^{\mathcal{L},\nabla'} = \xi(v).$$

Hence elliptic Dunkl operators attached to different flat connections on the same line bundle $\mathcal{L}$ differ by additive constants.

For simplicity, we will use the same notation $\nabla$ for the connection on each bundle $\mathcal{L}^w$ obtained from the connection $\nabla$ on $\mathcal{L}$ by the action of $w \in W$. Then we have the following result on the equivariance of the elliptic Dunkl operators under the action of $W$.

PROPOSITION 3.1. *One has*

$$w \circ \mathcal{D}_{v,C}^{\mathcal{L},\nabla} \circ w^{-1} = \mathcal{D}_{wv,C}^{\mathcal{L}^w,\nabla}.$$

## 4. The Commutativity Theorem

### 4.1. The Elliptic Cherednik Algebra

Let $c(H,j) = \frac{1}{2}(e^{-2\pi i j/n_H} - 1)C(H,j)$, and set $c(H,0) = 0$. Recall from [E] that the sheaf of algebras $H_{1,c,0,X,W}$ on $X/W$ is defined as follows. Let $\bar{U}$ be a small open set in $X/W$, and let $U$ be its preimage in $X$. Then the algebra $H_{1,c,0,X,W}(U) = H_{1,c,0}(U,W)$ is generated by the algebra of holomorphic functions $\mathcal{O}(U)$, the group $W$, and the Dunkl–Opdam operators

$$D_{v,\phi} = \partial_v - \sum_{(H,j)\in\mathcal{S}} C(H,j)\langle\phi_H, v\rangle g_H^j.$$

Here $\phi = (\phi_H)$ is a collection of 1-forms on $U$ that locally near $H$ have the form $\phi_H = \mathrm{d}\log \ell_H + \phi'_H$; $\ell_H$ is a nonzero holomorphic function with a simple zero along $H$, and $\phi'_H$ is holomorphic. For brevity we will denote this sheaf by $H_{c,X,W}$. It is called the Cherednik algebra of the orbifold $X/W$ attached to the parameter $c$, or the *elliptic Cherednik algebra*.

The sheaf $H_{c,X,W}$ sits inside $W \ltimes D_{X_{\mathrm{reg}}}$, where $D_{X_{\mathrm{reg}}}$ is the sheaf of differential operators on $X$ with poles on the reflection hypertori. Thus the sheaf $H_{c,X,W}$ has a filtration by order of differential operators. It is known [E] that $F^0 H_{c,X,W} = W \ltimes \mathcal{O}_X$.

### *4.2. The Commutativity Theorem*

One of the main theorems of this paper is the following result.

THEOREM 4.1. *The elliptic Dunkl operators commute*; *that is,* $[\mathcal{D}^{\mathcal{L},\nabla}_{v,C}, \mathcal{D}^{\mathcal{L},\nabla}_{u,C}] = 0$.

*Proof.* Since $\langle f^{\mathcal{L}}_{H,j}, v\rangle$ depends only on the projection of $v$ to $V_H$, which is a 1-dimensional space, it is easy to check that the commutator $[\mathcal{D}^{\mathcal{L},\nabla}_{v,C}, \mathcal{D}^{\mathcal{L},\nabla}_{u,C}]$ does not have differential terms. In other words, we have

$$[\mathcal{D}^{\mathcal{L},\nabla}_{v,C}, \mathcal{D}^{\mathcal{L},\nabla}_{u,C}] = \sum_{g\in W} \varphi_g g,$$

where $\varphi_g$ is a meromorphic section of the line bundle $(\mathcal{L}^g)^* \otimes \mathcal{L}$.

We claim that $\varphi_1 = 0$. Indeed, write $\mathcal{D}^{\mathcal{L},\nabla}_{v,C}$ in the form

$$\mathcal{D}^{\mathcal{L},\nabla}_{v,C} = \nabla_v - \sum_H \langle F_H, v\rangle,$$

where $F_H = \sum_{j=1}^{n_H-1} C(H,j) f^{\mathcal{L}}_{H,j} g^j_H$. To show that $\varphi_1 = 0$, it suffices to show that

$$[\langle F_H, v\rangle, \langle F_K, u\rangle] + [\langle F_K, v\rangle, \langle F_H, u\rangle] = 0$$

if $W_H \cap W_K \neq 1$. But this is obvious, given that $\langle F_H, v\rangle$ depends only on the projection of $v$ to $V_H$, which is 1-dimensional, and $V_H = V_K$ once $W_H \cap W_K \neq 1$.

The rest of the proof is based on the following key lemma.

LEMMA 4.1. *The sections $\varphi_g$ are holomorphic.*

The lemma clearly implies the theorem, since the bundle $(\mathcal{L}^g)^* \otimes \mathcal{L}$ is a topologically, but not holomorphically, trivial bundle and hence every holomorphic section of this bundle is zero. □

*Proof of Lemma 4.1.* The lemma is proved by local analysis—that is, essentially, by reduction to the case of usual (rational) Dunkl–Opdam operators [DOp]. It is sufficient to show that the $\varphi_g$ are regular when restricted to a small $W$-invariant neighborhood $X_b$ of $Wb$, where $b \in X$ is an arbitrary point. Let $W_b$ be the stabilizer of $b$ in $W$. Then $X_b$ is a union of $|W/W_b|$ small balls around the points of the orbit $Wb$. Let us pick a trivialization of $\mathcal{L}$ on $X_b$. This trivialization defines a trivialization of the line bundle $\mathcal{L}^w$ for every $w \in W$. With these trivializations, the elliptic Dunkl operators $\mathcal{D}^{\mathcal{L},\nabla}_{v,C}$ become operators acting on meromorphic functions on $X_b$.

The remainder of the proof is based on the theory of Cherednik algebras for orbifolds. Namely, it is clear from the definition of the elliptic Dunkl operators that they belong to the algebra $H_{c,X,W}(X_b)$. Since $F^0 H_{c,X,W} = W \ltimes \mathcal{O}_X$, this implies that the sections $\varphi_g$, upon trivialization, become holomorphic functions on $X_b$. This proves the lemma. □

EXAMPLE 4.1 [BuFV, Sec. 3]. Let $W$ be the Weyl group of a root system $R$ with root lattice $Q$, let $V$ be the complexified reflection representation of $W$, and let $\Gamma = Q \oplus \eta Q$, where $\eta$ is a complex number with positive imaginary part. In this case we have $X = Q \otimes_{\mathbb{Z}} E$, where $E = \mathbb{C}/\langle 1, \eta\rangle$ is the elliptic curve defined by $\eta$.

Let $\theta_1$ be the standard Jacobi $\theta$-function

$$\theta_1(z) = -\sum_{n=-\infty}^{\infty} e^{2\pi i(z+1/2)(n+1/2)+\pi i\eta(n+1/2)^2};$$

it represents a section of the bundle $\mathcal{O}(1)$ over $E$. Consider the function of two variables,

$$\sigma_w(z) = \frac{\theta_1(z-w)\theta_1'(0)}{\theta_1(z)\theta_1(-w)},$$

which has the following defining properties:

(i) $\sigma_w(z+1) = \sigma_w(z)$;
(ii) $\sigma_w(z+\eta) = e^{2\pi i w}\sigma_w(z)$;
(iii) $\sigma_w$ is meromorphic with poles on the lattice generated by $1, \eta$ and residue 1 at zero.

Now let $H_\alpha$ be the reflection hypertorus in $X$ through the origin defined by a root $\alpha$. Also, let $\mathcal{L}$ be a line bundle on $X$ defined by the weight $\lambda \in V^*$. Then it follows from the above that

$$f^{\mathcal{L}}_{H_\alpha,1}(x) = \sigma_{(\lambda,\alpha^\vee)}((x,\alpha))\alpha.$$

Thus, the elliptic Dunkl operators have the form

$$\mathcal{D}^{\mathcal{L},\nabla}_{v,C} = \nabla_v - \sum_{\alpha>0} C_\alpha \sigma_{(\lambda,\alpha^\vee)}((x,\alpha))\alpha(v)s_\alpha,$$

where $C_\alpha$ is a $W$-invariant function on roots and $s_\alpha$ is the reflection corresponding to $\alpha$. These are exactly the elliptic Dunkl operators from [BuFV].

## 5. Representations of Elliptic Cherednik Algebras Arising from Elliptic Dunkl Operators

In this section we will use elliptic Dunkl operators to construct representations of the sheaf of elliptic Cherednik algebras $H_{c,X,W}$ on the sheaf $\mathcal{F} := \bigoplus_{w\in W} \mathcal{L}^{*w}$.

Let us write the elliptic Dunkl operator in the form

$$\mathcal{D}^{\mathcal{L},\nabla}_{v,C} = \nabla_v - \sum_{g\in W}\langle F^{\mathcal{L}}_{C,g}, v\rangle g,$$

where

$$F^{\mathcal{L}}_{C,g} = \sum_{(H,j):g_H^j=g} C(H,j) f^{\mathcal{L}}_{H,j}$$

is a section of $(\mathcal{L}^g)^* \otimes \mathcal{L} \otimes V^*$. Note that $F^{\mathcal{L}}_{C,g} = 0$ unless $g$ is a reflection.

LEMMA 5.1.

(i) $\mathrm{Ad}\, w(F^{\mathcal{L}}_{C,g}) = F^{\mathcal{L}^w}_{C,wgw^{-1}}$, *where* Ad $w$ *denotes the adjoint action of* $w$;

(ii) $\nabla_u \langle F^{\mathcal{L}}_{C,g}, v\rangle = \nabla_v \langle F^{\mathcal{L}}_{C,g}, u\rangle$;

(iii) $\sum_{h,g:hg=k} \langle F^{\mathcal{L}}_{C,g}, v\rangle\langle F^{\mathcal{L}^g}_{C,h}, gu\rangle = \sum_{h,g:hg=k} \langle F^{\mathcal{L}}_{C,g}, u\rangle\langle F^{\mathcal{L}^g}_{C,h}, gv\rangle$;

(iv) $\sum_{h,g:hg=k} \langle F^{\mathcal{L}}_{C,g}, v\rangle\langle F^{\mathcal{L}^g}_{C,h}, u\rangle = \sum_{h,g:hg=k} \langle F^{\mathcal{L}}_{C,g}, u\rangle\langle F^{\mathcal{L}^g}_{C,h}, v\rangle$.

*Proof.* Statement (i) follows from Proposition 3.1. Statements (ii) and (iii) follow from the commutativity of the elliptic Dunkl operators, using (i). Statement (iv), using (iii), reduces to the identity

$$\sum_{h,g:hg=k} \big(\langle F^{\mathcal{L}}_{C,g}, v\rangle\langle F^{\mathcal{L}^g}_{C,h}, u-gu\rangle - \langle F^{\mathcal{L}}_{C,g}, u\rangle\langle F^{\mathcal{L}^g}_{C,h}, v-gv\rangle\big) = 0.$$

Every summand in this sum is a skew symmetric bilinear form in $u, v$ that factors through $\mathrm{Im}(1-g)$. But if $F^{\mathcal{L}}_{C,g}$ is nonzero then $g$ is a reflection, so $\mathrm{Im}(1-g)$ is a 1-dimensional space. This means that every summand in this sum is zero, and the identity follows. □

Now we will define the representation of the elliptic Cherednik algebra. We start by defining an action $\rho = \rho_{\mathcal{L},\nabla}$ of the sheaf $W \ltimes D_{X_{\mathrm{reg}}}$ on (local) sections of $\mathcal{F}$ (with poles on reflection hypertori).

For a section $\beta$ of $(\mathcal{L}^*)^w$ we define:

$$\forall g \in W, \quad (\rho(g)\beta)(x) = \beta(gx)$$

(a section of $(\mathcal{L}^*)^{gw}$); if $f$ is a section of $\mathcal{O}_X$, then

$$\rho(f)\beta = f\beta;$$

and, for $v \in V$,

$$\rho(\partial_v)\beta = \Big(\nabla_v + \sum_{g\in W} \langle F^{\mathcal{L}^w}_{C,g}, v\rangle\Big)\beta.$$

PROPOSITION 5.1. *These formulas define a representation of* $W \ltimes D_{X_{\mathrm{reg}}}$ *on* $\mathcal{F}|_{X_{\mathrm{reg}}}$.

*Proof.* The only relations whose compatibility with $\rho$ needs to be checked are $[\partial_v, \partial_u] = 0$. This compatibility follows from statements (ii) and (iv) of Lemma 5.1. □

COROLLARY 5.1. *The restriction of* $\rho$ *to* $H_{c,X,W} \subset W \ltimes D_{X_{\mathrm{reg}}}$ *is a representation of* $H_{c,X,W}$ *on* $\mathcal{F}$.

*Proof.* We need to show that, for any section $D$ of $H_{c,X,W}$, $\rho(D)$ preserves holomorphic sections of $\mathcal{F}$. It is sufficient to check this for $D = D_{v,\phi}$, a Dunkl–Opdam operator. Clearly, we have

$$\rho(D_{v,\phi})|_{\mathcal{L}^w} = \nabla_v + \sum_{(H,j)\in\mathcal{S}} C(H,j)(\langle f^{\mathcal{L}^w}_{H,j}, v\rangle - \langle \phi_H, v\rangle g^j_H).$$

It is easy to see that each operator in parentheses preserves holomorphic sections, so the result follows. □

Note that the representation $\rho$ of $H_{c,X,W}$ belongs to category $\mathcal{O}$, which is the category of representations of $H_{c,X,W}$ on coherent sheaves on $X$.

## 6. Monodromy Representation of Orbifold Hecke Algebras

### *6.1. Orbifold Fundamental Group and Hecke Algebra*

The quotient $X/W$ is a complex orbifold. Thus, for any $x \in X$ with trivial stabilizer, we can define the orbifold fundamental group $\pi_1^{\mathrm{orb}}(X/W, x)$. It is the group consisting of the homotopy classes of paths on $X$ connecting $x$ and $gx$ for $g \in W$, with multiplication defined by the following rule: $\gamma_1 \circ \gamma_2$ is $\gamma_2$ followed by $g\gamma_1$, where $g$ is such that $gx$ is the endpoint of $\gamma_2$. It is clear that the orbifold fundamental group of $X/W$ is naturally isomorphic to the semidirect product $W \ltimes \Gamma$.

The braid group of $X/W$ is the orbifold fundamental group $\pi_1^{\mathrm{orb}}(X_{\mathrm{reg}}/W, x)$. It can also be defined as $\pi_1(X'/W, x)$, where $X'$ is the set of all points of $X$ with trivial stabilizer.

Now let $H$ be a reflection hypertorus. Let $C_H$ be the conjugacy class in the braid group $\pi_1^{\mathrm{orb}}(X_{\mathrm{reg}}/W, x)$ corresponding to a small circle going counterclockwise around the image of $H$ in $X/W$. Then we have the following result (see e.g. [BMRo]).

PROPOSITION 6.1. *The group $\pi_1^{\mathrm{orb}}(X/W, x) = W \ltimes \Gamma$ is a quotient of the braid group $\pi_1^{\mathrm{orb}}(X_{\mathrm{reg}}/W, x)$ by the relations $T^{n_H} = 1$ for all $T \in C_H$.*

Now, for any conjugacy class of $H$ we introduce complex parameters $\tau_{H,1}, \ldots, \tau_{H,n_H}$. The entire collection of these parameters will be denoted by $\tau$.

The Hecke algebra of $(X, W)$, denoted $\mathcal{H}_\tau(X, W, x)$, is the quotient of the group algebra of the braid group, $\mathbb{C}[\pi_1^{\mathrm{orb}}(X_{\mathrm{reg}}/W, x)]$, by the relations

$$\prod_{m=1}^{n_H}(T - e^{2\pi \mathrm{i} m/n_H} e^{\tau_{H,m}}) = 0, \quad T \in C_H.$$

(This relation is a deformation of the relation $T^{n_H} = 1$, which can be written as $\prod_{m=1}^{n_H}(T - e^{2\pi \mathrm{i} m/n_H}) = 0$.) The Hecke algebra is independent on the choice of $x$, so we will drop $x$ from the notation.

REMARK 6.1. It is known from [E] that, if $\tau$ is a formal (rather than complex) parameter, then the algebra $\mathcal{H}_\tau(X, W)$ is a flat deformation of the group algebra $\mathbb{C}[W \ltimes \Gamma]$.

EXAMPLE 6.1. Let $W$ be a Weyl group, let $V$ be its reflection representation, and let $\Gamma = Q^\vee \oplus \eta Q^\vee$, where $Q^\vee$ is the dual root lattice of $W$ and $\eta \in \mathbb{C}^+$. Then

$\mathcal{H}_\tau(X, W)$ is the double affine Hecke algebra (DAHA) of Cherednik [C2]. (In the type-$BC$ case, the result is Sahi's [S] 6-parameter version of the double affine Hecke algebra.)

EXAMPLE 6.2. Let $W = S_n \ltimes (\mathbb{Z}/\ell Z)^n$ for $\ell = 2, 3, 4, 6$, let $V = \mathbb{C}^n$, and let $\Gamma = \Lambda^n$, where $\Lambda \subset \mathbb{C}$ is a lattice invariant under $\mathbb{Z}/\ell Z$ (any lattice for $\ell = 2$, triangular for $\ell = 3, 6$, square for $\ell = 4$). Then $\mathcal{H}_\tau(X, W)$ is the generalized double affine Hecke algebra of higher rank of type $D4$, $E6$, $E7$, $E8$, respectively, defined in [EGO]. We remark that (a) if $\ell = 2$ then this reproduces the BC case from Example 6.1 (Sahi's algebra) and for (b) $n = 1$ these Hecke algebras were studied earlier in [EOR] in connection with quantization of del Pezzo surfaces.

### *6.2. The Monodromy Representation*

The representation $\rho$ defines a structure of a $W$-equivariant holonomic $\mathcal{O}$-coherent $D$-module (i.e., a $W$-equivariant local system) on the restriction of the vector bundle $\bigoplus_{w\in W}(\mathcal{L}^*)^w$ to $X_{\text{reg}}$. This local system yields a monodromy representation $\pi_{\mathcal{L},\nabla}$ of the braid group $\pi_1^{\text{orb}}(X_{\text{reg}}/W, x)$ (of dimension $|W|$). By Corollary 5.1, this local system is obtained by localizing to $X_{\text{reg}}$ an $\mathcal{O}_X$-coherent $H_{c,X,W}$-module. Hence, by [E, Prop. 3.4], the representation $\pi_\rho$ factors through the Hecke algebra $\mathcal{H}_\tau(X, W)$, where $\tau$ is given by the formula

$$\tau_{H,m} = -\frac{2\pi\mathrm{i}}{n_H}\sum_{j=1}^{n_H-1} C(H, j)e^{-2\pi\mathrm{i}jm/n_H}.$$

Thus, for any collection of parameters $\tau_{H,j}$ with $\sum_j \tau_{H,j} = 0$ for all $H$, we have constructed a family of $|W|$-dimensional representations $\pi_{\mathcal{L},\nabla}$ of the Hecke algebra $\mathcal{H}_\tau(X, W)$ that is parameterized by pairs $(\mathcal{L}, \nabla)$; this family has $2\dim V$ parameters.

Here is another version of the definition of the representation $\pi_{\mathcal{L},\nabla}$ of $\mathcal{H}_\tau(X, W)$, one that refers directly to elliptic Dunkl operators and does not mention elliptic Cherednik algebras. Consider the system of differential reflection equations

$$\mathcal{D}_{v,C}^{\mathcal{L},\nabla}\psi = 0, \quad v \in V. \tag{1}$$

Let $\mathcal{E}$ be the sheaf of solutions of this equation on $X'/W$ (sections of this sheaf over $\bar{U} = U/W$ are, by definition, solutions of this system on $U$). Then $\mathcal{E}$ is a local system of rank $|W|$ and so has a monodromy representation $\xi_{\mathcal{L},\nabla}$. It is easy to see that $\pi_{\mathcal{L},\nabla} = \xi^*_{\mathcal{L},\nabla}$.

REMARK 6.2. We could generalize the preceding construction by replacing equations (1) by the eigenvalue equations

$$\mathcal{D}_{v,C}^{\mathcal{L},\nabla}\psi = \lambda(v)\psi, \quad v \in V,$$

where $\lambda \in V^*$, but this does not really give anything new because it is equivalent to changing the connection $\nabla$.

EXAMPLE 6.3. If $W$ is a Weyl group (Example 6.1), then the relation $\sum_j \tau_{H,j} = 0$ corresponds to the "classical" case of double affine Hecke algebras ($q = 1$). In this case the DAHA is finitely generated over its center, and generically over the spectrum of the center is an Azumaya algebra of rank $|W|$. Then our construction yields generic irreducible representations of this algebra. In this case, such representations can also be constructed by using classical analogues of difference Dunkl–Cherednik operators [C2].

If $W = S_n \ltimes (\mathbb{Z}/\ell\mathbb{Z})^n$ (Example 6.2), then we obtain (an open part of) the $2n$-parameter family of $|W|$-dimensional representations of generalized DAHA that was constructed in [EGO] by another method.

In other cases of crystallographic reflection groups, however, the family of representations constructed here appears to be new.

CONJECTURE 6.1. *For any $W, V, \Gamma$, if $\sum_m \tau_{H,m} = 0$ for all $H$ then the Hecke algebra $\mathcal{H}_\tau(X, W)$ is finitely generated as a module over its center $Z_\tau(X, W)$, which is the algebra of functions on an irreducible affine algebraic variety $M_\tau(X, W)$ of dimension $2 \dim V$. Moreover, this algebra is an Azumaya algebra of rank $|W|$ at the generic point of $M_\tau(X, W)$, and the family of representations $\pi_{\mathcal{L},\nabla}$ provides generic irreducible representations of $\mathcal{H}_\tau(X, W)$.*

This conjecture is known only in the case of Weyl groups (Example 6.1); see [C2] and, for the case $\dim V = 1$, [EOR]. In particular, the conjecture is open in the case of Example 6.2 for $\ell = 3, 4, 6$.

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