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Hyperplane Arrangements and Box Splines

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(with an Appendix by A. Björner)

Dedicated to Mel Hochster on the occasion of his 65th birthday

1. Introduction

The purpose of this paper is to explain some explicit formulas that one can develop in the theory of the box spline and the corresponding algorithms of approximation by functions—in particular, (2.11) and (3.2). The theory of splines is a large subject, and even the part on the box spline is rather well developed. The reader should consult the fundamental book by de Boor, Höllig, and Riemenschneider [7] or the recent notes by Ron [19]. Of this large theory we concentrate on some remarkable theorems of Dahmen and Micchelli (see [3; 4; 5]) and on the theory of quasi-interpolants and the Strang–Fix conditions, for which we refer to de Boor [6].

In essence, here we make explicit certain constructs that are already present in [6]. Thus, from a purely computational point of view, there is probably no real difference with that paper, yet we believe that the explicit formulas (2.11) and (3.2) shed a light on the whole procedure. In fact, the main new formula is (3.2) since (2.11) is essentially in Dahmen–Micchelli (although not so explicit).

We also show how some facts about matroids, which are recalled in an appendix written by A. Björner, give a proof of one of the basic theorems of the theory on the dimension of a certain space of polynomials describing box splines locally.

2. Preliminaries

2.1. Box Splines

The theory has been developed in the general framework of approximation theory by splines—in particular, two special classes of functions: the *multivariate spline* $T_X(x)$ and the *box spline* $B_X(x)$.

Take a finite list $X := \{a_1, \dots, a_m\}$ of nonzero vectors $a_i \in V = \mathbb{R}^s$, thought of as the columns of a matrix A. If X spans $\mathbb{R}^s$, one builds an important function for numerical analysis, the *box spline* $B_X(x)$, which is implicitly defined by the formula

$$\int_{\mathbb{R}^s} f(x) B_X(x)\,dx := \int_0^1 \cdots \int_0^1 f\Big(\sum_{i=1}^m t_i a_i\Big)\,dt_1 \dots dt_m, \tag{2.1}$$

where $f(x)$ varies in a suitable set of test functions.

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If 0 is not in the convex hull of the vectors a_i then one has a simpler function $T_X(x)$, the *multivariate spline* (cf. [7]), which is characterized by the formula

$$\int_{\mathbb{R}^s} f(x)T_X(x)\,dx = \int_0^\infty \cdots \int_0^\infty f\Big(\sum_{i=1}^m t_i a_i\Big)\,dt_1 \dots dt_m, \tag{2.2}$$

where $f(x)$ varies in a suitable set of test functions (usually continuous functions with compact support or, more generally, exponentially decreasing on the cone $C(X)$). From now on, in any statement regarding the multivariate spline $T_X(x)$ we shall tacitly assume that the convex hull of the vectors in X does not contain 0.

An efficient way of studying these functions is through the *Laplace transform*:

$$Lf(u) := \int_V e^{-\langle u|v\rangle} f(v)\,dv. \tag{2.3}$$

We think of each $a \in X$ as a linear function of $U := V^*$, and we have

$$LB_X = \prod_{a\in X} \frac{1-e^{-a}}{a}, \qquad LT_X = \prod_{a\in X} \frac{1}{a}. \tag{2.4}$$

These functions are splines in the sense that there is a polyhedral decomposition of the ambient space such that, on each polyhedron, the functions coincide with a polynomial lying in a finite-dimensional space $D(X)$ of polynomials. One of the important theorems characterizes $D(X)$ by differential equations.

In order to describe these equations, we need a definition from the theory of matroids (see the Appendix).

Definition 1. We say that a sublist $Y \subset X$ is a *cocircuit* if the elements in $X \setminus Y$ do not span V.

If a is a vector, we denote by D_a the directional derivative relative to a. If $Y \subset X$, let

$$D_Y := \prod_{a\in Y} D_a.$$

We denote the set of cocircuits by $\mathcal{E}(X)$.

Theorem 2.1. (1) *The space $D(X)$ is given by*

$$D(X) := \{f \in S[U] \mid D_Y f = 0\ \forall Y \in \mathcal{E}(X)\}. \tag{2.5}$$

(2) *The dimension of $D(X)$ is the number $d(X)$ of bases that can be extracted from X.*

Although this theorem originates from the theory of the box spline, it is also of interest for commutative algebra and algebraic geometry—particularly in the theory of hyperplane arrangements and partition functions.

A basic invariant of X is the minimum length of a cocircuit, which we denote by $m(X)$. By Theorem 2.1, all the polynomials of degree $< m(X)$ lie in $D(X)$. Moreover, if $m(X) \geq 2$ then B_X must be a continuous function of class $C^{m(X)-2}$.

The main relationship between this theory and our hyperplane arrangements is explained in [10]. The hyperplane arrangement in V^* is that given by the hyperplanes whose linear equations are determined by the vectors in X. There we show how the basic function T_X is built out of special polynomials that appear as suitable residues for certain points at infinity of the wonderful model associated to the arrangement. A similar theory is developed later for the arithmetic function and the formulas for the partition functions; for a survey, see [11].

The connection with commutative algebra has been explored in several papers (see [8; 9; 15; 21]). One point—the connection with the theory of Reisner–Stanley algebras—is discussed in the Appendix by Anders Björner.

In particular, this discussion gives a matroid-theoretic proof of the following more precise statement (see [12]).

THEOREM 2.2. *The graded dimension of $D(X)$ is given by*

$$H_X(q) := \sum_{\underline{b} \in \mathcal{B}(X)} q^{m-s-e(\underline{b})} = q^{m-s} T(X, 1, q^{-1}), \tag{2.6}$$

where $e(\underline{b})$ is the external activity of a basis $\underline{b} \in B(X)$ (see the Appendix for a definition) and $T(X,x,y)$ is the Tutte polynomial [25].

Proof. Consider the algebra $S[V]/I$ for I the ideal generated by the products $\prod_{i \in I} a_i$, where the a_i $(i \in I)$ form a cocircuit in X.

We claim that the variety defined by the ideal I reduces to the unique point 0. For this, note that the variety of zeros of a set of equations—each one of which is itself a product of equations—is the union of the subvarieties defined by selecting an equation out of each product.

If we now take one element b_Y from each cocircuit Y, then the resulting set of vectors $\{b_Y\}$ spans V. Indeed, if (by contradiction) these elements do not span V then their complement is a cocircuit. Because we selected an element from each cocircuit, this is not possible.

It follows that the equations $b_Y = 0$ for $Y \in \mathcal{E}(X)$ define the subvariety consisting of the point 0, and hence our claim follows. As a consequence, we deduce using standard facts of commutative algebra that $S[V]/I$ is finite-dimensional and is dual to the space $D(X)$. Thus $S[V]/I$ has the same dimension and Hilbert series as $D(X)$.

The algebra $S[V]/I$ can be also constructed as follows. First take a variable x_i corresponding to each vector a_i in X and define (as explained in the Appendix) the face ring (denoted by $\mathbf{k}[M]$), which is the polynomial algebra in the variables x_i modulo the ideal generated by the monomials $\prod_{i \in I} x_i$; here the a_i $(i \in I)$ form a cocircuit in X.

We clearly have a homomorphism $\rho \colon k[M] \to S[V]/I$, from the face ring to $S[V]/I$, mapping $x_i \mapsto a_i$. In fact, the linear map $j \colon \bigoplus_{i=1}^m F x_i \to V$, $j : x_i \mapsto a_i$ is surjective with kernel an $(m-s)$-dimensional subspace K. It is clear that the ideal generated by K (or, equivalently, by any basis of K) in $\mathbf{k}[M]$ is the kernel of ρ.

We now apply Theorem A1, which shows that $\mathbf{k}[M]$ is Cohen–Macaulay of dimension $m-s$. Since $S[V]/I$ is finite-dimensional, it follows from the theory of Cohen–Macauly rings that any basis of K is a regular sequence in $\mathbf{k}[M]$. At this point, formula (2.6) follows from Theorem A3. □

2.2. *Discrete Convolution*

Box splines are functions of compact support; in fact, the support of B_X is the *zonotope* $B(X) := \sum_{i=1}^{m} t_i a_i$ for $0 \le t_i \le 1$. Particularly interesting is the case in which X is a list of vectors in a lattice Λ (as $\mathbb{Z}^s$). In this case, for $a \in \Lambda$ the translates $B_X(x-a)$ of B_X form a partition on unity, and they are used in approximation theory and, in particular, in the finite element method. Here the important idea comes from the Strang–Fix conditions (see [23]). We present a new approach to the construction of quasi-interpolants by using in a systematic way the concept of a super function.

2.3. *Scaling*

We use the notation of Section 2.2. Theorem 2.10 tells us that the space $D(X)$ coincides with the space of polynomials in the cardinal spline space $\mathcal{S}_X$. This has a useful application for approximation theory. In order to state the results, we need to introduce some notation.

For every real number h, we define the *scale operator*

$$(\sigma_h f)(x) := f(x/h).$$

In particular, we shall apply this when $h = n^{-1}$ for $n \in \mathbb{N}$, so that $h\Lambda \supset \Lambda$ is a *refinement* of Λ.

We remark that, if U is a domain, then

$$\int_U f(x)\,dx = h^{-s} \int_{hU} \sigma_h f\,dx.$$

Moreover, if f has as support a set C, then $\sigma_h f$ has as support the set hC.

We define the scaling operator on distributions by duality. Thus, on a test function f,

$$\langle \sigma_h(T) \mid f \rangle := \langle T \mid \sigma_h^{-1}(f) \rangle = \langle T \mid \sigma_{1/h}(f) \rangle. \tag{2.7}$$

In particular, $\langle \sigma_h \delta_a \mid f(x) \rangle = \langle \delta_a \mid f(hx) \rangle = f(ha)$ and so

$$\sigma_h \delta_a = \delta_{ha}, \qquad \sigma_h \sum_a f(a)\delta_a = \sum_a f\left(\frac{a}{h}\right)\delta_a.$$

Observe that σ_h acts as an automorphism with respect to convolution.

If T is represented by a function g, so that $\langle T \mid f \rangle = \int_V g(x) f(x)\,dx$, then

$$\langle \sigma_h(T) \mid f \rangle = \int_V g(x) f(hx)\,dx = h^{-s} \int_V g(h^{-1}x) f(x)\,dx;$$

hence $\sigma_h(T)$ is represented by the function $h^{-s}\sigma_h(g)$.

The relation between scaling and the Laplace transform is

$$L(\sigma_h f) = h^s \sigma_{1/h} L(f). \tag{2.8}$$

In fact,

$$L(\sigma_h f)(y) = \int_V e^{-\langle x|y\rangle} f\left(\frac{x}{h}\right) dx = h^s \int_V e^{-\langle u|hy\rangle} f(u)\, du = h^s \sigma_{1/h} L(f)(y).$$

To simplify the notation, we shall denote the cardinal space $\mathcal{S}_{B_X}$ by $\mathcal{S}_X$.

LEMMA 2.3. *The space $\sigma_h(\mathcal{S}_X)$ equals the cardinal space $\mathcal{S}_{hX}$ with respect to the lattice $h\Lambda$.*

Proof. Let f be a function on Λ. Then

$$\sigma_h(B_X * f) = \sum_{\lambda\in\Lambda} B_X\left(\frac{x}{h} - \lambda\right) f(\lambda) = \sum_{\mu\in h\Lambda} B_X\left(\frac{x-\mu}{h}\right) f\left(\frac{\mu}{h}\right).$$

One easily verifies that $\sigma_h B_X = h^s B_{hX}$, so we deduce that

$$\sigma_h(B_X * f) = h^s \sum_{\mu\in h\Lambda} B_{hX}(x-\mu)(\sigma_h f)(\mu) \in \mathcal{S}_{hX}^{h\Lambda}.$$

The claim follows because the operator $f \mapsto h^s \sigma_h f$ induces a linear isomorphism between the space of functions on Λ and that of functions on $h\Lambda$. □

We have the following commutation relations between σ_h and a derivative D_v, a difference operator ∇_a, and a translation τ_a:

$$D_v\sigma_h = h^{-1}\sigma_h D_v, \quad \sigma_h\nabla_a = \nabla_{ha}\sigma_h, \quad \tau_a\sigma_h = \sigma_h\tau_{a/h}. \tag{2.9}$$

2.4. Approximation Power

Let us start with some definitions.

DEFINITION 2. (A) A function on Λ is called a *mesh function*.

(B) Let $a\colon \Lambda \to \mathbb{C}$ be a mesh function and let $M(x)$ be any function with compact support. We set $M * a$ to be the function

$$(M * a)(x) := \sum_{\lambda\in\Lambda} M(x-\lambda)a(\lambda).$$

This is well-defined because $M(x)$ has compact support, and $M * a$ is called the *discrete convolution* of M with a.

(C) The vector space $\mathcal{S}_M^\Lambda$ formed by all the convolutions $M * a$, where a is a mesh function, is called the *cardinal space* associated to M and Λ.

(D) A function $a(x)$ on $\mathbb{R}^s$ restricts to a mesh function $a|_\Lambda$ on Λ. We set

$$M *' a := M * a|_\Lambda$$

and call this a *semi-discrete convolution* of M with a.

The goal of this and the following sections is to approximate a function by a sequence of functions obtained by rescaling semi-discrete convolutions with B_X.

DEFINITION 3. We say that M has *approximation order* $\geq r$ in the L^p norm $(1 \leq p \leq \infty)$ if, for any bounded domain G and a smooth function f in a neighborhood of G, there exists a sequence $f_h \in \sigma_h(\mathcal{S}_M)$ with $h = 1/n$ and $n \in \mathbb{N}$ such that

$$\| f_h - f \|_{L^p(G)} = O(h^r).$$

When M has approximation order $\geq r$ but M does not have approximation order $\geq r + 1$, we say that M has approximation order r.

The Strang–Fix conditions consist of a general statement that we shall discuss only in the case of box splines.

STRANG–FIX CONDITIONS. *The approximation order of a function M is the maximum r such that the space of all polynomials of degree $< r$ is contained in the cardinal space $\mathcal{S}_M$.*

Our goal is to present a different approach to the following classical theorem.

THEOREM 2.4. *B_X has approximation order equal to $m(X)$.*

By [7] we know that B_X does not have approximation order larger than $m(X)$. However, we want rather to explain an explicit approximation algorithm producing the required approximation order $\geq m(X)$. Note that it will be sufficient for us to work with the L^∞ norm because $\|g\|_{L^p(G)} \leq (\mathrm{Vol}\, G)^{1/p} \|g\|_{L^\infty(G)}$ for any $1 \leq p < \infty$ and any L^∞ function g on a bounded domain G.

2.5. The Space $D(X)$

Consider all hyperplanes generated by subsets of X and all of their translates. One obtains a locally finite set of hyperplanes called the *cut locus,* and each connected component of its complement is called a *chamber.*

PROPOSITION 2.5. *Each translate of B_X, and hence each element of the cardinal spline space, is a polynomial in $D(X)$ on each chamber.*

One of the main results of the theory is as follows.

THEOREM 2.6. *If $p \in D(X)$ then also $B_X *' p \in D(X)$. This defines a linear isomorphism F of $D(X)$ to itself that is given explicitly by the invertible differential operator*

$$F_X := \prod_{a \in X} \frac{1 - e^{-D_a}}{D_a}.$$

Proof. It is not hard to see that we can reduce to the case in which $m(X) \geq 2$ (we call this case *nondegenerate*). If $X = \{a\}$ is a number and if $s = 1$, then clearly $D(X)$ reduces to the constants and so the statement reduces to the fact that the translates of $B_{\{a\}}$ sum to a constant function.

In the other cases, B_X is a continuous function on $\mathbb{R}^s$. One way of understanding $B_X *' p$ is by applying the Poisson summation formula to the function of y: $B_X(x+y)p(-y)$.

Given our definition of the Laplace transform, we have

$$Lf(\xi) = (2\pi)^{n/2}\hat{f}(i\xi),$$

where $\hat{f}$ denotes the usual Fourier transform and, in order to avoid confusion, we use ξ to denote the variables in the Laplace transform. The Laplace transform of $B_X(x+y)p(-y)$ is obtained from the Laplace transform $e^x \prod_{a\in X}(1-e^{-a})/a$ of $B_X(x+y)$ by applying the polynomial $\hat{p} = p\left(\frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial \xi_s}\right)$ as differential operator.

We now want to apply the classical Poisson summation formula (cf. [27]), which gives—for a function ϕ with suitable conditions and, in particular, if ϕ is continuous with compact support—

$$\sum_{\mu\in\Lambda^*} L\phi(\mu) = \sum_{\lambda\in\Lambda}\phi(\lambda),$$

where μ runs in the *dual lattice* Λ^* of elements for which $\langle \mu \mid \lambda\rangle \in 2\pi i\mathbb{Z}$ for all $\lambda \in \Lambda$.

As a result, when $L\phi(\mu) = 0$ for all $\mu \neq 0$ and $\mu \in \Lambda^*$, we have

$$L\phi(0) = \sum_{\lambda\in\Lambda}\phi(\lambda).$$

This is the key result of Dahmen and Micchelli that we shall prove in our setting and that will imply all the main results.

Before proving the main lemma (Lemma 2.8), we write in a suitable form the action of differential operators of some degree k given by a polynomial $p\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_s}\right)$ on a power series $F(x_1, \dots, x_s)$.

LEMMA 2.7. *Introducing the auxiliary variables $t_1, \dots, t_s$, we have*

$$p\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_s}\right)[F(x_1,\dots,x_s)] = F\left(x_1 + \frac{\partial}{\partial t_1}, \dots, x_s + \frac{\partial}{\partial t_s}\right)[p(t_1,\dots,t_s)]_{t_1=\dots=t_s=0}. \quad (2.10)$$

Proof. Start from the obvious identity:

$$p\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_s}\right)[F(x_1+t_1,\dots,x_s+t_s)] = p\left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_s}\right)[F(x_1+t_1,\dots,x_s+t_s)],$$

from which it follows that

$$p\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_s}\right)[F(x_1,\dots,x_s)] = p\left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_s}\right)[F(x_1+t_1,\dots,x_s+t_s)]_{t_1=\dots=t_s=0}.$$

Now use the fact that, if p, q are two polynomials in the variables t_i, then

$$p\left(\frac{\partial}{\partial t_1},\dots,\frac{\partial}{\partial t_s}\right)[q(t_1,\dots,t_s)]_{t_1=\cdots=t_s=0} = q\left(\frac{\partial}{\partial t_1},\dots,\frac{\partial}{\partial t_s}\right)[p(t_1,\dots,t_s)]_{t_1=\cdots=t_s=0}. \quad \square$$

The main observation of Dahmen and Micchelli is as follows.

LEMMA 2.8. *If $p(x) \in D(X)$, then the Laplace transform of $B_X(x+y)p(-y)$ (viewed as a function of y) vanishes at all points $\mu \neq 0$ with $\mu \in \Lambda^*$.*

Proof. If $a = (a_1,\dots,a_s)$ then we use the notation

$$a = \langle a \mid \xi\rangle = \sum_i a_i \xi_i, \qquad e^a = \exp\left\{\sum_i a_i\xi_i\right\}, \quad e^x = \exp\left\{\sum_i x_i \xi_i\right\}.$$

With this notation, the Laplace transform of $B_X(x+y)p(-y)$ (as a function of y) equals

$$p\left(\frac{\partial}{\partial \xi_1},\dots,\frac{\partial}{\partial \xi_s}\right)\left[e^x \prod_{a\in X}\frac{1-e^{-a}}{a}\right].$$

We may assume that $p(y)$ is homogeneous of some degree k.

Now we can apply formula (2.10). Replace ξ_i with $\xi_i + \frac{\partial}{\partial t_i}$ to obtain

$$\frac{1-e^{-\langle a \mid \xi+\partial_t\rangle}}{\langle a \mid \xi+\partial_t\rangle} = \frac{1-e^{-\langle a\mid\xi\rangle}}{\langle a\mid\xi\rangle} + H_a(\xi,\partial_t)D_a,$$

where we have set $\langle a \mid \partial_t\rangle = D_a$. Then

$$\begin{aligned} L_X(\xi+\partial_t) &= e^{\langle x \mid \xi+\partial_t\rangle}\prod_{a\in X}\left(\frac{1-e^{-\langle a\mid\xi\rangle}}{\langle a\mid\xi\rangle} + H_a(\xi,\partial_t)D_a\right) \\ &= \sum_{Y\subset X} e^{\langle x\mid\xi+\partial_t\rangle}\prod_{a\notin Y}\left(\frac{1-e^{-\langle a\mid\xi\rangle}}{\langle a\mid\xi\rangle}\right)\prod_{a\in Y} H_a(\xi,\partial_t)D_a. \end{aligned}$$

Take a summand relative to Y and the corresponding function:

$$e^{\langle x\mid\xi+\partial_t\rangle}\prod_{a\notin Y}\left(\frac{1-e^{-\langle a\mid\xi\rangle}}{\langle a\mid\xi\rangle}\right)\prod_{a\in Y} H_a(\xi,\partial_t)D_a[p(t_1,\dots,t_s)]_{t_1=t_2=\cdots=t_s=0}.$$

Then we have either that Y is a cocircuit or that $X \setminus Y$ contains a basis. In the former case, since $p(t) \in D(X)$ we have $\prod_{a\in Y} D_a p(t) = 0$. These terms are identically zero. In the latter case, the elements of $X\setminus Y$ span the vector space and so at least one of the $a \in X \setminus Y$ does not vanish at μ whenever $\mu \neq 0$. Hence, if μ is in the lattice Λ^*, then $(1-e^{-a})/a$ vanishes at μ and thus the entire product vanishes. $\square$

We return now to the proof of Theorem 2.6. By Lemma 2.8, Poisson summation degenerates to the computation at 0.

Recall that $e^x = e^{\langle x \mid \xi \rangle} = \exp\{\sum_{i=1}^s x_i \xi_i\}$. Consider the duality $\langle p \mid f \rangle$ defined as follows. Take a polynomial $p(\xi_1, \dots, \xi_s)$, compute it in the derivatives $\frac{\partial}{\partial \xi_i}$, apply it as differential operator to the function f, and then evaluate the resulting function at 0. We have for each i that

$$\langle p \mid \xi_i f \rangle = \left\langle \frac{\partial p}{\partial \xi_i} \mid f \right\rangle.$$

Thus, setting $F_X := \prod_{a \in X} \frac{1-e^{-a}}{a}$, we have

$$p\left(\frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial \xi_s}\right)\left(e^x \prod_{a \in X} \frac{1-e^{-a}}{a}\right)(0) = \left\langle p \mid e^x \prod_{a \in X} \frac{1-e^{-a}}{a} \right\rangle$$
$$= \langle F_X p \mid e^x \rangle = F_X p(x).$$

This follows because, for any polynomial q,

$$\left\langle q\left(\frac{\partial}{\partial \xi}\right) \mid e^{\langle x \mid \xi \rangle} \right\rangle = q(x).$$

Since $D(X)$ is stable under derivatives and since F_X is clearly invertible, both our claims follow. □

COROLLARY 2.9. *On $D(X)$, the inverse of F_X is given by the following differential operator of infinite order*:

$$Q := \prod_{x \in X} \frac{D_x}{1 - e^{-D_x}}. \tag{2.11}$$

Observe that Q is like a *Todd operator* in that its factors can be expanded using the Bernoulli numbers B_n by the defining formula

$$\frac{D_x}{1 - e^{-D_x}} = \sum_{k=0}^{\infty} \frac{B_n}{n!}(-D_x)^n;$$

thus Q acts on $D(X)$ as an "honest" differential operator.

Theorem 2.6 tells us that $D(X) \subset \mathcal{S}_X$. On the other hand, we have the following result.

THEOREM 2.10. *If p is a polynomial in $\mathcal{S}_X$, then $p \in D(X)$. Thus, $D(X)$ coincides with the space of all polynomial in $\mathcal{S}_X$.*

Proof. By Proposition 2.5 we know that each function f in the space $\mathcal{S}_X$, once restricted to a chamber $\mathfrak{c}$, coincides with a polynomial $f_{\mathfrak{c}}$ in $D(X)$. Hence, if f is itself a polynomial, then it must coincide with $f_{\mathfrak{c}}$ everywhere. □

Given an element $v \in V$ and a function F on V, we define the difference operator $\nabla_v F$ and the translation operator τ_v by

$$\nabla_v F(u) = F(u) - F(u - v), \qquad \tau_v F(u) = F(u - v).$$

PROPOSITION 2.11. *If* $X = (Y, z)$ *then we have* $D_z(B_X) = \nabla_z B_Y$ (*in the sense of distributions*).

Proof. Given a function f in Schwartz space, we have

$$\begin{aligned}\langle D_z(B_X) \mid f\rangle &= \langle B_X \mid -D_z(f)\rangle \\ &= -\int_0^1 \cdots \int_0^1 \left(\int_0^1 D_z(f)\left(\sum_{i=1}^{m-1} t_i a_i + tz\right) dt\right) dt_1 \ldots dt_{m-1} \\ &= \int_0^1 \cdots \int_0^1 \left[f\left(\sum_{i=1}^{m-1} t_i a_i\right) - f\left(\sum_{i=1}^{m-1} t_i a_i + z\right)\right] dt_1 \ldots dt_{m-1} \\ &= \int_0^1 \cdots \int_0^1 (\nabla_{-z} f)\left(\sum_{i=1}^{m-1} t_i a_i\right) dt_1 \ldots dt_{m-1} = \langle \nabla_z B_Y \mid f\rangle. \qquad \square\end{aligned}$$

Because it is also clear that $(\nabla_z B) * a = B * \nabla_z a$, for any subset $Y \subset X$ we obtain

$$D_Y(B_X * a) = B_{X \setminus Y} * \nabla_Y a. \tag{2.12}$$

This gives another insight into Theorem 2.10. Assume that a polynomial p is of the form $p = B_X * a$ for some function a on Λ, and let Y be a cocircuit. Then $D_Y p = D_Y(B_X * a) = B_{X\setminus Y} * \nabla_Y a$ is a distribution supported on the subspace $\langle X \setminus Y\rangle$. Being a polynomial, $D_Y p$ must equal 0; this shows again that $p \in D(X)$.

3. Approximation Theory

As usual, we take an s-dimensional real vector space V in which we fix a Euclidean structure, and we denote by dx the corresponding Lebesgue measure. We also fix a lattice $\Lambda \subset V$ and a list X of vectors in Λ spanning V as a vector space.

3.1. An Algorithm

We shall use the standard remainder estimate in the Taylor series as follows.

THEOREM 3.1. *Let* G *be a bounded domain in* $\mathbb{R}^s$, *and let* f *be a* C^r *function on* G. *Choose* $x_0 \in G$, *and let* q_r *be the polynomial expressing the Taylor expansion of* f *at* x_0 *up to order* r. *For every* $1 \le p \le \infty$ *there is a constant* C, *dependent on* p, r, s *but independent of* f, *such that*

$$\|f - q_{r-1}\|_{L^p(G)} \le C \sum_{|\alpha|=r} \|D^\alpha f\|_{L^p(G)} \operatorname{diam}(G)^r. \tag{3.1}$$

Recall (Corollary 2.9) that the operator

$$Q := \prod_{a\in X} \frac{D_a}{1 - e^{-D_a}}$$

has the property that

$$Q(B_X *' q) = q$$

for every polynomial $q \in D(X)$. We formally write Q as a difference operator as follows. Start with the identity $\nabla_x = 1 - e^{-D_x}$. The meaning of this identity in general resides in the fact that the vector field D_x is the infinitesimal generator of the 1-parameter group of translations $v \mapsto v + tx$. On the space of polynomials, this is an identity as operators because both D_x and ∇_x are locally nilpotent.

This identity gives

$$D_x = -\log(1 - \nabla_x) = \sum_{k=1}^{\infty} \frac{\nabla_x^i}{i},$$

from which we deduce that

$$\frac{D_x}{1 - e^{-D_x}} = \sum_{k=0}^{\infty} \frac{\nabla_x^i}{i+1}.$$

Therefore,

$$Q := \prod_{a \in X} \Big(\sum_{i=0}^{\infty} \frac{\nabla_a^i}{i+1} \Big). \tag{3.2}$$

REMARK 3.2. Recall that

$$\frac{D_x}{1 - e^{-D_x}} = \sum_{i \geq 0} \frac{B_i}{i!} (-D_x)^i,$$

where B_i is the ith Bernoulli number. Thus, for each $n \geq 0$,

$$B_n = (-1)^n \Big[\sum_{i \geq 0} \frac{B_i}{i!} (-D_x)^i \Big] (x^n) \,|_{x=0} = \sum_{i=0}^{n} \frac{\nabla_1^i}{i+1} x^n \,|_{x=0} .$$

Indeed, $\sum_{i=0}^{n} (\nabla_1^i/(i+1)) x^n$ is a variant of the nth Bernoulli polynomial. As a matter of fact, the difference between the nth Bernoulli polynomial and the nth previously defined polynomial equals $nx^{n-1} = D_x(x^n)$.

Define Q_X to be the difference operator expressed by Q truncated at order $m(X)-1$ (or perhaps higher). We have that Q_X also acts as the inverse of $q \mapsto B_X *' q$ on the polynomials of degree $\leq m(X) - 1$.

The theory is already interesting in the 1-dimensional case. When X consists of 1 repeated $m + 1$ times, we denote by $b_m(x)$ the corresponding box spline.

EXAMPLE 3.3. The hat function $b_1(x)$:

$$b_1(x) = \begin{cases} x & \text{for all } 0 \leq x \leq 1, \\ 2 - x & \text{for all } 1 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

In this case, $X = \{1, 1\}$ and $m(X) = 2$. One easily verifies that the cardinal spline space coincides with the space of all continuous functions on $\mathbb{R}$ that are linear on the intervals $[i, i + 1]$ for $i \in \mathbb{Z}$. Such a function is completely determined by its values $f(i)$ on the integers. Moreover, given any function f on $\mathbb{Z}$, we see that

$b_1 * f$ at i equals $f(i-1) =: \tau_1 f(i)$. As for the operator Q_X, it equals $1 + \nabla_1 = 2 - \tau_1$. However, on the space of linear polynomials we see that $2 - \tau_1 = \tau_{-1}$, so in this particular case the most convenient choice of Q_X is τ_{-1}. In general it is difficult to describe, in the simplest possible way, the operator Q_X as a linear combination of translations.

Let $Q_X = \sum_i^{\ell} c_i \tau_{-b_i}$ and set, for $h \leq 1$ and g a function on V:

$$g_h := \sigma_h[Q_X B_X *' (\sigma_{1/h} g)]; \tag{3.3}$$

$$\begin{aligned} g_h(x) &= \sum_{\underline{i} \in \Lambda} (Q_X B_X)\left(\frac{x}{h} - \underline{i}\right) g(h\underline{i}) \\ &= \sum_{\underline{i} \in \Lambda} B_X\left(\frac{x}{h} - \underline{i}\right)\left[\sum_i c_i g(h\underline{i} + hb_i)\right]. \end{aligned} \tag{3.4}$$

We apply this definition to $h := n^{-1}$ for n a positive integer and so obtain an *interpolation algorithm* A_X^n given by $A_X^n(g) = g_{1/n}$.

We easily see that the algorithm is *local* in the sense that the value of g_h at a point x depends only on the values of g in a neighborhood that tends to x as $h \to 0$. Thus there is no harm in assuming that the functions we analyze are defined everywhere. We shall also assume that their derivatives (up to some required order) are defined and bounded everywhere.

Let Δ_R denote the closed disk centered at the origin of radius R. Define the linear functional $\gamma(f) := Q_X f(0)$. Then

$$Q_X(f) = \sum_{i=1}^{\ell} c_i f(x + b_i), \qquad \gamma(f) = \sum_{i=1}^{\ell} c_i f(b_i).$$

If we choose R so that $b_i \in \Delta_R$ for each $i = 1, \ldots, \ell$, then γ is clearly a continuous functional on the space $C^0(\Delta_R)$ with L^∞ norm. Its norm $\|\gamma\|$ is at most equal to $\sum_{i=1}^{\ell} |c_i|$; in fact, $|\gamma(f)| \leq \sum_{i=1}^{\ell} |c_i| \|f\|_{L^\infty(\Delta_R)}$. (It is easy to see that $\|\gamma\| = \sum_{i=1}^{\ell} |c_i|$.)

Take a point x and consider the finite set $b(x \mid X) = \Lambda \cap (x - B(X))$. Using that B_X is supported in $B(X)$, we derive that the condition that $B_X(x/h - j) \neq 0$ implies $j \in b(x/h \mid X)$. It follows from formula (3.3) that, for any function g,

$$g_h(x) = \sum_{j \in b(x/h \mid X)} B_X\left(\frac{x}{h} - j\right) \gamma(\tau_{-j} \sigma_{1/h} g). \tag{3.5}$$

Moreover,

$$|\gamma(\tau_{-j}\sigma_{1/h} g)| \leq \|\gamma\| \|\tau_{-j}\sigma_{1/h} g\|_{L^\infty(\Delta_R)} = \|\gamma\| \|g\|_{L^\infty(h(\Delta_R + j))}. \tag{3.6}$$

It is well known (see [7]) that B_X is nonnegative and that, for any x, we have

$$\sum_{j \in b(x/h \mid X)} B_X\left(\frac{x}{h} - j\right) = 1.$$

Hence we may use (3.5) and (3.6) to obtain

$$|g_h(x)| \leq \|\gamma\| \|g\|_{L^\infty(h(\Delta_R + x/h - B(X)))}. \tag{3.7}$$

Set $\Delta := \Delta_R - B(X)$. Since $Q_X B_X *' -$ is the identity on polynomials of degree at most $m(X) - 1$, we immediately deduce that $q_h = q$ for every polynomial q of degree $< m(X)$.

Because $0 \in \Delta$, using (3.7) for $g = f - q$ yields

$$|f_h(x) - f(x)| = |g_h(x) - g(x)| \leq (\|\gamma\| + 1)\|f - q\|_{L^\infty(x+h\Delta)}.$$

Theorem 3.1 then implies that we can choose q in such a way that

$$\|f - q\|_{L^\infty(x+h\Delta))} \leq K h^{m(X)} \sum_{|\underline{j}|=m(X)} \|D^{\underline{j}} f\|_{L^\infty(x+h\Delta))},$$

with K an absolute constant depending only on X, which can be bounded by $(st)^{m(X)}/m(X)!$ for t any number such that Δ is contained in a disk of radius t. Therefore,

$$|f_h(x) - f(x)| \leq (\|\gamma\| + 1) K h^{m(X)} \sum_{|\underline{j}|=m(X)} \|D^{\underline{j}} f\|_{L^\infty(x+h\Delta))}.$$

Finally, over any domain G we have

$$\|f_h - f\|_{L^\infty(G)} \leq (\|\gamma\| + 1) K h^{m(X)} \sum_{|\underline{j}|=m(X)} \|D^{\underline{j}} f\|_{L^\infty(G+h\Delta)}.$$

Thus we have proved the following theorem.

THEOREM 3.4. *Under the explicit algorithm previously constructed, for any domain G we have*

$$\|f_h - f\|_{L^\infty(G)} = O(h^{m(X)}).$$

In particular, this yields our new proof of Theorem 2.4.

REMARK 3.5. The algorithm is not too sensitive to the way we truncate Q. Suppose we take two truncations that differ by a difference operator T with terms only of order $N \geq m(X)$. The difference between the resulting approximating functions is then

$$\sigma_h[B_X *' T\sigma_{1/h}(g(x))].$$

The following lemma tells us that these terms contribute to order $O(h^N)$ and hence do not change the statement of the theorem.

Let $A = \{a_1, \ldots, a_m\}$ be a list of m nonzero vectors in $\mathbb{R}^s$. We use the notation $|f|_\infty$ for the L^∞ norm on the whole space.

LEMMA 3.6. *Let f be a function of class C^m with bounded derivatives on the space V. Then, for any positive h,*

$$|\nabla_A \sigma_{1/h} f|_\infty \leq h^m |D_A f|_\infty. \tag{3.8}$$

Proof. By elementary calculus, for any vector a we have

$$|\nabla_a \sigma_{1/h} f|_\infty \leq h \left|\frac{\partial f}{\partial a}\right|_\infty.$$

By induction, set $A := \{B, a\}$. We have:

$$\begin{aligned}|\nabla_B \nabla_a \sigma_{1/h} f| &= |\nabla_B \sigma_{1/h} \nabla_{ha} f| \leq h^{m-1} |D_B \nabla_{ha} f|_\infty \\ &= h^{m-1} |\sigma_{1/h} D_B \nabla_{ha} f|_\infty \\ &= h^{m-1} |\nabla_a \sigma_{1/h} D_B f|_\infty \leq h^m |D_A f|_\infty.\end{aligned}$$ □

EXAMPLE 3.7. Let $s = 1$, $X = 1^m$ (so $B_X = b_{m-1}$), $m(X) = m$, and $B(X) = [0, m]$. If $\nabla = \nabla_1 = 1 - \tau_1$, then Q_X is the truncation of

$$\left(\sum_{i=0}^{m-1} \frac{\nabla^i}{i+1}\right)^m = \left(\sum_{i=0}^{m-1} \frac{(1-\tau_1)^i}{i+1}\right)^m.$$

An estimate of $\|\gamma\|$ is $\left(\sum_{i=0}^{m-1} \frac{2^i}{i+1}\right)^m$, as for $R = m(m-1)$, and then we have $\Delta = [-m^2, m(m-1)]$. The chambers are the open intervals $(n, n+1)$ with $n \in \mathbb{Z}$.

3.2. Derivatives in the Algorithm

We want to see next how this algorithm behaves with respect to derivatives. In fact, the theory of Strang–Fix ensures that, when we have approximation power m, we can approximate a function f with a spline to order $O(h^m)$ and at the same time its derivatives of order $k < m$ to order $O(h^{m-k})$. We want to show that our proposed algorithm indeed satisfies this property. Since we shall work also with subsets Y of X, we put $A_X^h(f) := f_h$, the approximant to f in the algorithm associated to X at the step $n = 1/h$.

Again the algorithm is local, so we may assume that f and all of its derivatives up to order $m(X)$ are continuous and bounded everywhere by some constant C. (If we assume that f has only bounded derivatives of order $\leq t$, then we get the same results but only for these derivatives.)

By Remark 3.5, the choice of truncation is not essential for the final Theorem 3.10. To make our induction simpler, we establish some notations and normalize the truncation as follows:

$$T_a^{(m)} := \sum_{i=0}^{m-1} \frac{\nabla_a^i}{i+1}, \qquad \nabla_a^{[m]} := \sum_{i=1}^{m} \frac{\nabla_a^i}{i} = \nabla_a T_a^{(m)};$$

$$Q_X := \prod_{a \in X} \left(\sum_{i=0}^{m(X)-1} \frac{\nabla_a^i}{i+1} \right) = \prod_{a \in X} T_a^{(m(X))}. \tag{3.9}$$

Note that this truncation is not the optimal one but rather the most convenient one for carrying out the proof.

EXAMPLE 3.8. For the 1-dimensional case $X = 1^m$ we have $m(X) = m$. In this case a natural truncation is $Q_X = \left(\sum_{i=0}^{m-1} \frac{\nabla^i}{i+1}\right)^m$ with $\nabla_1 = \nabla$. For instance, for

$m = 2$ we have $(1 + \nabla/2)^2 = 1 + \nabla + \nabla^2/4$ though the natural truncation is $1 + \nabla$, which in any case is also not the optimal choice. The optimal choice is given by τ_{-1}.

LEMMA 3.9. *Given a list $A = \{a_1, \ldots, a_k\}$ of nonzero vectors in V and $m \geq k$, consider the two operators*

$$D_A := D_{a_1} \cdots D_{a_k}, \qquad \nabla_A^{[m]} := \nabla_{a_1}^{[m]} \cdots \nabla_{a_k}^{[m]}.$$

Then, for $x_0 \in V$ and any function g of class C^{m+1}, we have

$$|([\nabla_A^{[m]} - D_A]\sigma_{1/h} g)(x_0)| \leq h^{m+1} K \sum_{|\alpha|=m+1} \|D^\alpha g\|_{L^\infty(x_0+\Delta_{hr})},$$

where K is a constant that is independent of x_0 and g.

Proof. On polynomials of degree $\leq m$, every difference operator consisting of terms of order $> m$ is zero. Hence, on such polynomials

$$\nabla_A^{[m]} = \prod_{i=1}^{k} \sum_{i=1}^{\infty} \frac{\nabla_{a_i}^i}{i} = D_A.$$

Since $\nabla_A^{[m]}$ is a finite difference operator, it follows that

$$(\nabla_A^{[m]}) f = \sum_{i=1}^{k} c_i f(x_0 - u_i), \quad u_i \in \Lambda,$$

for any function f. Hence one has the uniform estimate

$$|(\nabla_A^{[m]} f)(x_0)| \leq c \|f\|_{L^\infty(x_0+\Delta_r)}, \tag{3.10}$$

where $c = \max(|c_i|)$, $r = \max(|u_i|)$, and Δ_r denotes the disk of radius r centered at 0.

Given f, let q be the Taylor series of f at x_0 that is truncated at degree $\leq m$; then $[\nabla_A^{[m]} - D_A](f) = [\nabla_A^{[m]} - D_A](f - q)$. Since $m \geq k$, we also have that $D_A(f - q)(x_0) = 0$ and thus

$$[\nabla_A^{[m]} - D_A](f)(x_0) = (\nabla_A^{[m]}(f - q))(x_0). \tag{3.11}$$

We now apply (3.11) to the function $f = \sigma_{1/h} g$ at the point x_0/h. Denote by q the Taylor series of g at x_0 truncated at degree $\leq m$. Using (3.10) and (3.11) together with (3.1), we obtain the estimate

$$\begin{aligned} &|[\nabla_A^{[m]} - D_A](\sigma_{1/h} g)(x_0)| \\ &\quad = (\nabla_A^{[m]} \sigma_{1/h}(g - q))(x_0) \leq c \|\sigma_{1/h}(g - q)\|_{L^\infty(x_0/h+D(r))} \\ &\quad = c\|g - q\|_{L^\infty(x_0+D(hr))} \leq h^{m+1} K \sum_{|\alpha|=m+1} \|D^\alpha g\|_{L^\infty(x_0+D(hr))}. \end{aligned}$$

The constant $K = c\,Cr^{m+1}$ is an absolute constant that is independent of g and x_0 (but depends on m and A). □

Theorem 3.10. *Under the algorithm $f_h = A^h_X(f)$, for any domain G and for every multi-index $\alpha \in \mathbb{N}^s$ with $|\alpha| \le m(X) - 1$ we have*

$$\|\partial^\alpha f_h - \partial^\alpha f\|_{L^\infty(G)} = \|\partial^\alpha(f_h - f)\|_{L^\infty(G)} = O(h^{m(X)-|\alpha|}).$$

Proof. Given any domain H whose closure lies in G, we may extend the restriction of f to H to a function defined on V having uniformly bounded derivatives up to order $m(X)-1$. By proving the estimate for such functions we are easily done.

We prove first the estimate

$$\|D_Y f_h - D_Y f\|_\infty = O(h^{m(X)-|Y|}) \tag{3.12}$$

for the differential operators D_Y, where Y is a sublist in X that is not a cocircuit, and then show how the required estimate follows easily from this. Recall (cf. (2.11)) that $D_Y(B_X) = \nabla_Y B_{X\setminus Y}$ as distributions. Under our assumptions, $D_Y(B_X)$ as well as $D_Y g$ for any $g \in \mathcal{S}_X$ is in fact a function, although when $|Y| = m(X)$ it need not be continuous.

Let $(D_Y f)_h = A^h_{X\setminus Y}(D_Y f)$. Because $m(X\setminus Y) \ge m(X) - |Y|$, we may deduce from Theorem 3.4 that $\|(D_Y f)_h - D_Y f\|_\infty = O(h^{m(X)-|Y|})$. Hence Theorem 3.10 is proved once we show the estimate $\|D_Y f_h - (D_Y f)_h\|_\infty = O(h^{m(X)-|Y|})$.

Using the commutation rules (2.9) and Q_X as in (3.9), we obtain

$$D_Y f_h = h^{-|Y|}\sigma_h(D_Y B_X *' Q_X \sigma_{1/h} f) = h^{-|Y|}\sigma_h(\nabla_Y B_{X\setminus Y} *' Q_X \sigma_{1/h} f).$$

Set $\tilde{Q}_{X\setminus Y} = \prod_{z\in X\setminus Y} T_z^{(m(X))}$ and observe that $\nabla_Y \prod_{z\in Y} T_z^{(m(X))} = \nabla_Y^{[m]}$ (as in Lemma 3.9). Then we have, finally,

$$D_Y f_h = \sigma_h(B_{X\setminus Y} *' \tilde{Q}_{X\setminus Y} h^{-|Y|}\nabla_Y^{[m]}\sigma_{1/h} f). \tag{3.13}$$

Now compare (3.13) with the approximants to $D_Y f$:

$$(D_Y f)_h = \sigma_h(B_{X\setminus Y} *' Q_{X\setminus Y}\sigma_{1/h} D_Y f).$$

By construction, $\tilde{Q}_{X\setminus Y} - Q_{X\setminus Y}$ is a sum of terms of order $\ge m(X\setminus Y)$ and so these approximants differ (by Lemma 3.6) from the approximants

$$(\widetilde{D_Y f})_h := \sigma_h(B_{X\setminus Y} *' \tilde{Q}_{X\setminus Y}\sigma_{1/h} D_Y f)$$

by order $O(h^{m(X\setminus Y)})$.

Given the estimate

$$|D_Y f_h - (D_Y f)_h| \le |D_Y f_h - (\widetilde{D_Y f})_h| + |(\widetilde{D_Y f})_h - (D_Y f)_h|,$$

we need only show that $|D_Y f_h - (\widetilde{D_Y f})_h| = O(h^{m(X)-|Y|+1})$. First,

$$D_Y f_h - (\widetilde{D_Y f})_h = \sigma_h(B_{X\setminus Y} *' \tilde{Q}_{X\setminus Y}[h^{-|Y|}\nabla_Y^{[m]}\sigma_{1/h} f - \sigma_{1/h} D_Y f]).$$

Then clearly we have

$$|\sigma_h(B_{X\setminus Y} *' \tilde{Q}_{X\setminus Y}[h^{-|Y|}\nabla_Y^{[m]}\sigma_{1/h} f - \sigma_{1/h} D_Y f])|_\infty$$
$$= |\tilde{Q}_{X\setminus Y} B_{X\setminus Y} *' [h^{-|Y|}(\nabla_Y^{[m]} - D_Y)\sigma_{1/h} f]|_\infty.$$

The norm $|f * a|_\infty$ of the convolution $f * a$ of a function with compact support f by a sequence a is clearly bounded by $C|a|_\infty$, where $C = |f|_\infty d$ for d the maximum number of points in Λ that lie in the support of a translate $\tau_a f$ of f as a varies in Λ. We can now apply the estimate to the sequence $(\nabla_Y^{[m]} - D_Y)\sigma_{1/h} f$ (computed on points of Λ) given by (3.9). This yields the estimate $O(h^{m(X)+1})$, proving formula (3.12).

Now let $m \leq m(X) - 1$. Since for any sublist Y of X with m elements we have that $X \setminus Y$ still generates V, it follows by a simple induction that the operators D_Y span linearly the same space spanned by the partial derivatives ∂^α for $\alpha \in \mathbb{N}^s$ with $|\alpha| = m$. Thus, the claim of the theorem follows. □

4. Super Functions and Nil Functions

4.1. Super Functions

There is an alternative way of presenting algorithm (3.3) that makes use of certain special functions known as super functions.

DEFINITIONS 4. A *super function* is an element $F \in \mathcal{S}_X$ with compact support such that $F *' q = q$ for every polynomial q of degree $\leq m(X)$. A *nil function* is an element $F \in \mathcal{S}_X$ with compact support such that $F *' q = 0$ for every polynomial $q \in D(X)$. A function $F \in \mathcal{S}_X^f$ is *nilpotent of size r* if, for every polynomial q of degree $< r$, we have $F *' q = 0$.

The following statement is an obvious consequence.

PROPOSITION 4.1. *If Q is any difference operator inverting $B_X *' -$ on $D(X)$, then QB_X is a super function. In particular, since $Q_X B_X *' q = q$ for every $q \in D(X)$, it follows that $Q_X B_X$ is a super function.*

If F_1 is a super function and if $F_2 \in \mathcal{S}_X$ has compact support, then F_2 is a super function if and only if $F_1 - F_2$ is a nil function.

We can therefore write the basic algorithm of formula (3.4) as *convolution by a super function QB_X*:

$$f \mapsto B_X *' Qf = (QB_X) *' f. \tag{4.1}$$

Once we have chosen a super function, we may describe them all using Proposition 4.1 and a description of nil functions. Such functions can be described, but this falls beyond the scope of this paper (see [11]).

With notation as in the paragraph preceding Example 3.3, we denote by $s_m(x)$ the super function associated to $b_m(x)$:

$$s_m(x) = \left(\sum_{i=0}^{m} \frac{\nabla^i}{i+1} \right)^{m+1} b_m(x).$$

Denote by $n_{a,m} = \nabla^a b_m$ the nil function (assume $a > 0$).

Figures 1 and 2 show examples of super functions associated to $B_{1,1,1}, B_{1^6}$. In Figures 3 and 4 we add some nil functions.

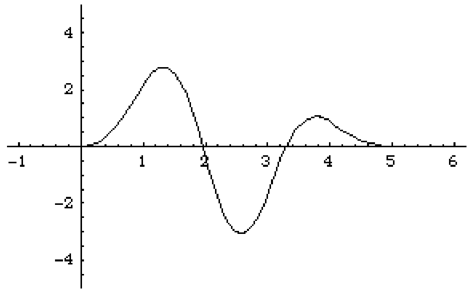

Figure 1 $s_2(x)$

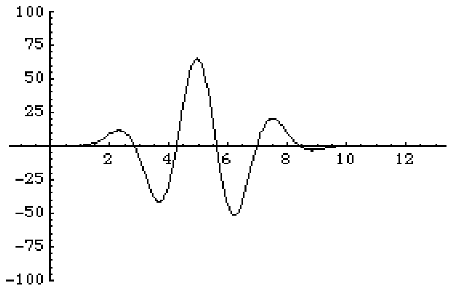

Figure 2 $s_5(x)$

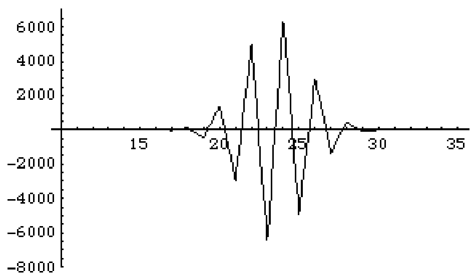

Figure 3 $n_{2,14}(x)$

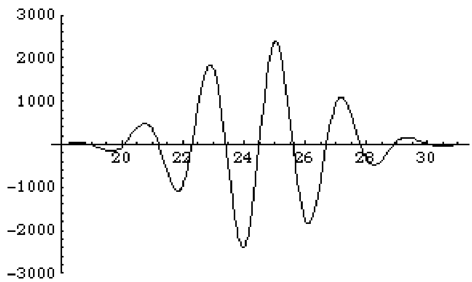

Figure 4 $n_{4,14}(x)$

REMARK 4.2. For any integrable function f we have $\int_V \nabla_a f\, d\mu = 0$, where $d\mu$ is a translation-invariant measure.

Thus, given any polynomial $p(x_1, \dots, x_s)$ and elements $a_i \in X$,

$$\int_V p(\nabla_{a_1}, \dots, \nabla_{a_s}) B_X\, d\mu = p(0).$$

In particular, by our previous remarks and the expression (3.2) of Q_X we deduce that $\int_V F\, d\mu = 1$ for any super function F.

5. Quasi-interpolants

The algorithm proposed in Section 3.1 is certainly not the only possible one. Let us analyze some properties of possible algorithms of interpolation. We follow closely the two papers [17] and [18].

The starting point of any such algorithm is a linear functional L mapping functions on splines in the cardinal space $\mathcal{S}$. Once such a functional is given, one can repeat the approximation scheme associating to a function f and a parameter h the approximant $f_h := \sigma_h L \sigma_{1/h} f$. Of course, for such an algorithm to be a real approximation algorithm requires suitable hypotheses on L. Let us review some of the principal requirements that we may ask of L.

- *Translation invariance*. By this we mean that L commutes with the translations τ_a for all $a \in \Lambda$.
- *Locality.* In this case one cannot impose a strong form of locality. However, one may ask whether there is a bounded open set Ω such that, if $f = 0$ on Ω, then also Lf equals 0 on Ω.

  If we assume translation invariance then we also have locality for all the open sets $a + \Omega$ with $a \in \Lambda$.
- *Projection property.* One may ask whether L is the identity on suitable subspaces of $\mathcal{S}_X$. For instance, the algorithm treated in Section 3.1 is the identity on the space $D(X)$ of polynomials contained in $\mathcal{S}_X$.

We shall commence analyzing these conditions in the *unimodular case*—that is, when all the bases extracted from $X \subset \Lambda$ are in fact integral bases of the lattice Λ. In this case one knows the linear independence of translates of the box spline (see [7]).

For this unimodular case, the functional L can be expressed through its *coordinates*:

$$Lf = \sum_{a \in \Lambda} L_a(f) B_X(x - a).$$

If we assume translation invariance, then

$$L\tau_b f = \sum_{a \in \Lambda} L_a(f) B_X(x - a - b) = \sum_{a \in \Lambda} L_{a+b}(f) B_X(x - a).$$

On the other hand, $L\tau_b f = \sum_{a \in \Lambda} L_a(\tau_b f) B_X(x - a)$ and so, by linear independence, we obtain the identities $L_{a+b}(f) = L_a \tau_b f$ for all $a, b \in \Lambda$. In particular, L is determined by the unique linear functional L_0 by setting $L_a f := L_0 \tau_a f$.

A question studied in [17; 18] is whether we can construct L_0 and hence L so that L is the identity on the entire cardinal space $\mathcal{S}_X$. From the previous discussion we need only construct a linear functional L_0 such that, on a spline $g = \sum_{a\in\Lambda} c_a B_X(x-a)$, we have $L_0 g = c_0$.

Here is one way to construct such a linear functional. Consider a bounded open set Ω with $0 \in \Omega$. The set

$$A = \{\alpha_1, \dots, \alpha_k\} := \{\alpha \in \Lambda \mid (\alpha + B(X)) \cap \Omega \neq \emptyset\}$$

is finite. Since B_X is supported on $B(X)$, only the translates $B_X(x-\alpha)$, $\alpha \in A$, do not vanish on Ω. Using the local polynomiality of B_X, one can show (cf. [7]) that the restrictions of the functions $B_X(x-\alpha)$, $\alpha \in A$, to Ω are still linearly independent. Let us denote by $\mathcal{S}_X(A)$ the k-dimensional space with basis the elements $B_X(x-\alpha)$ for $\alpha \in A$.

Observe that if L is local on Ω then

$$L_0\Biggl(\sum_{a\in\Lambda} c_a B_X(x-a)\Biggr) = c_0 = L_0\Biggl(\sum_{\alpha\in A} c_\alpha B_X(x-\alpha)\Biggr) = \sum_{\alpha\in A} c_\alpha L(B_X(x-\alpha));$$

hence it suffices to evaluate L_0 on the space $\mathcal{S}_X(A)$.

Suppose that we show the existence of k linear functionals T_i $(i = 1, \dots, k)$ local on Ω and such that the $k \times k$ matrix with entries $T_i B_X(x-\alpha_j)$ is invertible. By inverting the matrix one can easily write c_0 on this space as a linear combination of the functionals T_i. So we take this formula as the definition of L_0.

There are several possible approaches to the construction of functionals T_i with the required properties. One is to consider the Hilbert space structure and define, for $\alpha_i \in A$, the functional $T_i f := \int_\Omega f(x) B_X(x-\alpha_i)\,dx$. By the local linear independence, these functionals are clearly linearly independent on the space $\mathcal{S}_X(A)$.

A second method consists of observing that, by the linear independence, one can find k points $p_1, \dots, p_k \in \Omega$ such that the evaluation of functions in $\mathcal{S}_X(A)$ at these points establishes a linear isomorphism between $\mathcal{S}_X(A)$ and $\mathbb{R}^k$. In other words, the $k \times k$ matrix with entries $B_X(p_i - a)$ $(i = 1, \dots, k, a \in A)$ is invertible and we can define T_i as the evaluation at the point p_i. In general it seems difficult to exhibit explicit points with the required property (although most k-tuples of points satisfy the property).

We offer some remarks for when $\Omega = \mathring{B}(X)$, the interior of the zonotope. Let $X = \{a_1, \dots, a_m\}$ be a list of integral vectors spanning the ambient space $\mathbb{R}^s$, and denote by $\Omega := \bigl\{\sum_{i=1}^m t_i a_i,\ 0 < t_i < 1\bigr\}$ the *open zonotope*. Let $\sigma_X := \sum_{i=1}^m a_i$ and note the symmetry condition $a \in \Omega$ if and only if $\sigma_X - a \in \Omega$.

Let $A := \Omega \cap 2^{-1}\Lambda$ be the set of half-integral points contained in Ω, and let $B := \{j \in \Lambda \mid [\Omega - j] \cap \Omega \neq \emptyset\}$.

Proposition 5.1. *The mapping* $\phi\colon a \mapsto 2a - \sigma_X$, *with inverse* $\phi^{-1}\colon b \mapsto 1/2(b + \sigma_X)$, *is a bijiection between A and B.*

Proof. First we show that, if $a \in A$, then $2a - \sigma_X \in B$. In fact we have

$$\sigma_X - a = a - (2a - \sigma_X) \in \Omega \cap (\Omega - (2a - \sigma_X)).$$

Conversely, assume $b \in B$ and let $a := 1/2(b + \sigma_X)$; clearly $a \in 2^{-1}\Lambda$, and we claim also that $a \in \Omega$. By assumption there exists an element $c \in \Omega$ with $c - b \in \Omega$, and by symmetry $\sigma_X - c + b \in \Omega$. Since Ω is convex, we conclude that $1/2(\sigma_X - c + b + c) = a \in \Omega$. □

CONJECTURE 5.1. *The half-integral points in Ω give rise to an invertible matrix C with entries $c_{a,b} := B_X(a - 2b + \sigma_X)$.*

This can be verified by simple examples.

If Conjecture 5.1 is satisfied then one can use this matrix and the method previously sketched in order to construct an explicit projection to the spline space, where the operator L_0 is an explicit linear combination of the operators $f \mapsto f(a)$, $a \in A$, of evaluation at the half-integral points in $\mathring{B}(X)$.

When $f = \sum_{b \in B} c_b B(x - b)$ we have

$$f(c) = \sum_{a \in A} c_{2a-\sigma_X} B_X(c - 2a + \sigma_X) = \sum_{a \in A} c_{2a-\sigma_X} c_{c,a},$$

so if $c^{a,b}$ are the entries of the matrix C^{-1} then

$$c_0 = \sum_{b \in A} c^{\sigma_X/2,b} f(b),$$

and the projection operator L is given by

$$Lf = \sum_{a \in \Lambda} \left[\sum_{b \in A} c^{\sigma_X/2,b} f(b - a) \right] B_X(x - a).$$

Appendix

A. BJÖRNER

In this appendix we review some facts about face rings of matroid complexes that are relevant to the foregoing text.

A.1. Shellability and Face Rings

An abstract simplicial complex Δ is *pure* if all its *facets* (maximal faces) are of the same dimension. A linear order $F_1, F_2, \ldots, F_t$ of its facets is called a *shelling* if each facet F_i $(i > 1)$ intersects the complex $\Delta_{i-1} = 2^{F_1} \cup \cdots \cup 2^{F_{i-1}}$ generated by the preceding facets in a pure $(\dim \Delta - 1)$-dimensional subcomplex. Equivalently, each facet F_i has a subface $\hat{F}_i$, called its *restriction,* such that $\Delta_i \setminus \Delta_{i-1} = \{G : \hat{F}_i \subseteq G \subseteq F_i\}$ for all $i > 1$. We put $\hat{F}_1 = \emptyset$. A pure complex Δ is said to be *shellable* if it admits a shelling order of its facets. See [1, pp. 228–232] or [22, pp. 78–83] for motivation and more details about the notion of shellability.

Let $V = \{x_1, \ldots, x_n\}$ be the set of vertices of a simplicial complex Δ. To each subset $F \subseteq V$ we associate a squarefree monomial $x(F) = \prod_{x_i \in F} x_i$. The commutative ring

$$\mathbf{k}[\Delta] = \mathbf{k}[x_1, \ldots, x_n]/(x(F) \mid F \notin \Delta)$$

($\mathbf{k}$ a field) is called the *face ring* (or Stanley–Reisner ring, or discrete Hodge algebra) of Δ.

The relevance of the concept of shellability for commutative algebra is the following theorem, which shows that shellability induces a combinatorial decomposition of face rings from which Hilbert series can be read.

Theorem A1. *Let $F_1, F_2, \dots, F_t$ be a shelling of the simplicial complex Δ, and let $(\theta) = (\theta_1, \theta_2, \dots, \theta_d)$ be a linear system of parameters for the ring $\mathbf{k}[\Delta]$. Then $\mathbf{k}[\Delta]$ is Cohen–Macaulay, and*

$$\{x(\hat{F}_i) : i = 1, 2, \dots, t\}$$

is a $\mathbf{k}$-basis of the quotient ring $\mathbf{k}[\Delta]/(\theta)$. It follows that

$$\mathrm{Hilb}_{\mathbf{k}[\Delta]}(z) = \frac{h(z)}{(1-z)^d},$$

where $h(z) = \sum_{i=1}^{t} z^{\|\hat{F}_i\|}$.

The realization that shellability implies Cohen–Macaulayness can be traced back to the seminal work of Hochster [14] and Stanley [22]. The more detailed form of the theorem is due to Garsia [13], Kind and Kleinschmidt [16], and Stanley [22, Thm. 2.5, p. 82].

A.2. Matroids

A *matroid* $M = (E, IN)$ consists of a family IN of subsets of a finite set E such that the following properties hold.

M1 *Closure:* $A \subseteq B \in IN$ implies $A \in IN$.
M2 *Exchange:* $A, B \in IN$ and $|A| > |B|$ implies that $B \cup \{x\} \in IN$ for some $x \in A \setminus B$.

The sets in IN are called *independent,* and maximal independent sets are *bases.* All bases have the same cardinality, called the *rank* of M. The minimal dependent sets are called *circuits.* A premier example of a matroid is given by linear independence among a finite set of vectors. See for example the book series [26] for an exposition on matroids.

Matroid theory contains a pleasant duality operation, defined as follows. If M is a matroid on the ground set E, then there is a *dual matroid* M^* (having the same ground set) whose bases are given by the set complements $E \setminus B$ of bases B of M. The circuits of M^* are called *cocircuits* of M, and $\mathrm{rank}(M^*) = |E| - \mathrm{rank}(M)$.

If B is a basis of M and if $p \notin B$, then there is a unique circuit $\mathrm{cir}(B, p)$ contained in $B \cup p$ (we here dispense with set brackets for singletons). Dually, if $q \in B$ then there is a unique cocircuit $\mathrm{cocir}(B, q)$ contained in $(E \setminus B) \cup q$. These *basic* circuits and cocircuits are related in the following way:

$$q \in \mathrm{cir}(B, p) \iff (B \setminus q) \cup p \text{ is a basis} \iff p \in \mathrm{cocir}(B, q).$$

From now on, assume that the ground set E is linearly ordered. Given a basis B and an element $p \notin B$, we say that p is *externally active* in B if p is the least element of $\mathrm{cir}(B, p)$. Likewise, an element $q \in B$ is said to be *internally active* in B if q is the least element of $\operatorname{cocir}(B, p)$. Note that these concepts are dual: p is externally active in the basis B of M if and only if p is internally active in the basis $E \setminus B$ of M^*.

Let $\operatorname{ext}(B)$ denote the number of elements that are externally active in the basis B and $\operatorname{int}(B)$ the number of internally active elements. The two-variable polynomial

$$T_M(x, y) = \sum_{\text{bases } B \text{ of } M} x^{\operatorname{int}(B)} y^{\operatorname{ext}(B)}$$

is called the *Tutte polynomial* of M. It depends only on the matroid M, and not on the chosen linear order on the ground set needed for its computation in terms of basis activities.

It follows from the axioms that the independent sets $IN(M)$ of a matroid M form a pure simplicial complex whose facets are the bases of M. The following result appears in [1, pp. 233–236].

THEOREM A2. *The lexicographic order of the bases of M, induced by the linear order of the ground set E, is a shelling of $IN(M)$. The corresponding restriction operator sends a basis B to the subset $\hat{B} = \{p \in B \mid p$ is not internally active in $B\}$.*

A.3. Face Ring of a Matroid

Combining the information from Sections A.1 and A.2 allows us to draw the following conclusions about the face ring of a matroid complex. Let M be a matroid of rank r on the set $E = \{x_1, \dots, x_n\}$ and consider the polynomial ring $\mathbf{k}[M] := \mathbf{k}[x_1, \dots, x_n]/J$, where J is the ideal generated by the squarefree monomials $x(C)$ corresponding to the cocircuits C of M. Then $\mathbf{k}[M]$ is the face ring of the simplicial complex $IN(M^*)$ of independent sets of the dual matroid M^*, a complex that is pure $(n - r - 1)$-dimensional.

According to Theorem A2, the complex $IN(M^*)$ is shellable and

$$\sum_B z^{\|\widehat{E \setminus B}\|} = \sum_B z^{n-r-\operatorname{ext}(B)} = z^{n-r} T_M(1, 1/z),$$

with summation running over all bases B of M. Then, by Theorem A1, the face ring $\mathbf{k}[M]$ is Cohen–Macaulay with Hilbert series

$$\operatorname{Hilb}_{\mathbf{k}[M]}(z) = \frac{z^{n-r} T_M(1, 1/z)}{(1 - z)^{n-r}}.$$

Hence, modding out by a linear system of parameters (l.s.o.p.) yields the following conclusion.

THEOREM A3. *Let (θ) be a linear system of parameters for $\mathbf{k}[M]$. Then $\mathbf{k}[M]/(\theta)$ is a finite-dimensional algebra with Hilbert series*

$$\operatorname{Hilb}_{\mathbf{k}[M]/(\theta)}(z) = z^{n-r} T_M(1, 1/z).$$

Furthermore, for each basis B of M, let $\operatorname{Ext}(B)$ *be the set of its externally active elements. Then the system of squarefree monomials*

$$\{x((E \setminus B) \setminus \mathrm{Ext}(B)) \mid B \textit{ is a matroid basis of } M\}$$

gives a $\mathbf{k}$*-basis of* $\mathbf{k}[M]/(\theta)$.

A.4. Remark

The following question, asked by C. Procesi, is to our knowledge open.

Given a matroid M, is there a (naturally defined) bigraded finite-dimensional algebra $A[M]$ whose bigraded Hilbert polynomial is either the Tutte polynomial $T_M(x, y)$ or a closely related evaluation such as $x^r y^{n-r} T_M(1/x, 1/y)$?

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