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# On Injectivity of Maps between Grothendieck Groups Induced by Completion

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*Dedicated to Professor Melvin Hochster on the occasion of his sixty-fifth birthday*

## 1. Introduction

Let $(R, m, k)$ be a local ring and let $\hat{R}$ be the $m$-adic completion of $R$. Let $\mathcal{M}(R)$ be the category of finitely generated $R$-modules. The Grothendieck group of finitely generated modules over $R$ is defined as

$$\mathrm{G}(R) = \frac{\bigoplus_{M \in \mathcal{M}(R)} \mathbb{Z}[M]}{\langle [M_2] - [M_1] - [M_3] \mid 0 \to M_1 \to M_2 \to M_3 \to 0 \text{ is exact} \rangle}.$$

Kamoi and Kurano [KKu] studied injectivity of the map $\mathrm{G}(R) \to \mathrm{G}(\hat{R})$ induced by flat base-change. They showed that such a map is injective in the following cases: (i) $R$ is Hensenlian, (ii) $R$ is the localization at the irrelevant ideal of a positively graded ring over a field, or (iii) $R$ has only isolated singularity. Their results raise the question: Is the map between Grothendieck groups that is induced by completion always injective?

In [H1] Hochster announced a counterexample to this question as follows.

Theorem 1.1. *Let $k$ be a field, and let*

$$R = k[x_1, x_2, y_1, y_2]_{(x_1, x_2, y_1, y_2)}/(x_1x_2 - y_1x_1^2 - y_2x_2^2).$$

*Let $P = (x_1, x_2)$ and $M = R/P$. Then $[M]$ is in the kernel of the map $\mathrm{G}(R) \to \mathrm{G}(\hat{R})$. However, $[M] \neq 0$ in $\mathrm{G}(R)$.*

Hochster's example comes from the "direct summand hypersurface" in dimension 2 and is not normal. He claimed that there should also be an example that is normal. The main purpose of this note is to provide such an example by proving the following result.

Proposition 1.2. *Let $R = \mathbb{R}[x, y, z, w]_{(x,y,z,w)}/(x^2 + y^2 - (w+1)z^2)$, where $R$ is a normal domain. Let $P = (x, y, z)$ and $M = R/P$. Then $[M]$ is in the kernel of the map $\mathrm{G}(R) \to \mathrm{G}(\hat{R})$. However, $[M] \neq 0$ in $\mathrm{G}(R)$.*

This will be proved in Section 2. We note that Kurano and Srinivas [KuS] have recently constructed an example of a local ring $R$ such that the map $\mathrm{G}(R)_{\mathbb{Q}} \to$

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$G(\hat{R})_{\mathbb{Q}}$ is not injective. The ring in their example is not normal, and we do not know if a normal example exists in that context (i.e., with rational coefficients). In Section 3 we will discuss some questions on the kernel of the map $G(R) \to G(\hat{R})$.

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## 2. Our Example

We shall prove Proposition 1.2. First we need to recall some classical results.

COROLLARY 2.1 [Sw, Cor. 11.8]. *Let $k$ be a field of characteristic $\neq 2$, let $n > 1$ be an integer, and let $R = k[x_1, \dots, x_n]/(f)$ for $f$ a nondegenerate quadratic form in $k[x_1, \dots, x_n]$. Then $G(R) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if $C_0(f)$, the even part of the Clifford algebra of $f$, is simple.*

PROPOSITION 2.2 (Samuel; see [F, Prop. 11.5]). *Let $k$ be a field of characteristic $\neq 2$ and let $f$ be a nondegenerate quadratic form in $k[x_1, x_2, x_3]$. Let $R = k[x_1, x_2, x_3]/(f)$. If $f = 0$ has no nontrivial solution in $k$, then $\mathrm{Cl}(R) = 0$.*

PROPOSITION 2.3 [KKu]. *Let $S = \bigoplus_{n \geq 0} S_n$ be a graded ring over a field $S_0$ and let $S_+ = \bigoplus_{n > 0} S_n$. Let $A = S_{S_+}$. Then the map $G(S) \to G(A)$ induced by localization is an isomorphism.*

*Proof.* See the proof of [KKu, Thm. 1.5(ii)]. $\square$

Proposition 1.2 now follows from the next two propositions. (Clearly, $R$ is normal because the singular locus $V(x, y, z)$ has codimension 2.)

PROPOSITION 2.4. *$[\hat{M}] = 0$ in $G(\hat{R})$.*

*Proof.* Clearly $\hat{R} = \mathbb{R}[[x, y, z, w]]/(x^2 + y^2 - (w + 1)z^2)$. We want to show that $[\hat{R}/P\hat{R}] = 0$ in $G(\hat{R})$. Let $\alpha = \sqrt{w+1}$, which is a unit in $\hat{R}$, and let $Q = (x, y - \alpha z)\hat{R}$. Then clearly $Q$ is a height-1 prime in $\hat{R}$ and $P\hat{R} = Q + (y + \alpha z)\hat{R}$. The short exact sequence

$$0 \to \hat{R}/Q \to \hat{R}/Q \to \hat{R}/P\hat{R} \to 0,$$

where the second map is the multiplication by $y + \alpha z$, shows that $[\hat{R}/P\hat{R}] = 0$ in $G(\hat{R})$. $\square$

PROPOSITION 2.5. *$[M] \neq 0$ in $G(R)$.*

*Proof.* It is enough to show that $[M_P] \neq 0$ in $G(R_P)$. Let $K = \mathbb{R}(w)$; then $R_P \cong K[x, y, z]_{(x,y,z)}/(f)$, where $f = x^2 + y^2 - (w + 1)z^2$. Let $S = K[x, y, z]/(f)$. Clearly, $f$ is a nondegenerate quadratic form. Since the rank of $f$ is 3, an odd number, $C_0(f)$ is a simple algebra over $K$ (see e.g. [L, Chap. 5, Thm. 2.4]). By Corollary 2.1 and Proposition 2.3, $\mathrm{G}(R_P) = \mathrm{G}(S) = \mathbb{Z} \oplus \mathbb{Z}/(2)$. We claim that $f$ has no nontrivial solution in $K$. Suppose it has. Then, after clearing denominators, there exist polynomials $a(w), b(w), c(w) \in \mathbb{R}[w]$ such that

$$a(w)^2 + b(w)^2 = (w+1)c(w)^2.$$

The degree of $a(w)^2 + b(w)^2$ is always even; the degree of $(w+1)c(w)^2$ is odd unless $c(w) = 0$. But then $a(w)^2 + b(w)^2 = 0$, which forces $a(w) = b(w) = 0$—a contradiction. By the claim and Proposition 2.2, $\mathrm{Cl}(R_P) = \mathrm{Cl}(S) = 0$. Hence $[R_P]$ and $[R_P/PR_P]$ generate $\mathrm{G}(R_P) = \mathbb{Z} \oplus \mathbb{Z}/(2)$ (since the Grothendieck group is generated by $\{[R_P/Q], Q \in \operatorname{Spec} R_P\}$ and since $\dim R_P = 2$). Because $\mathbb{Z} \oplus \mathbb{Z}/(2)$ cannot be generated by one element, $[R_P/PR_P]$ must be nonzero (it is easy to see that $[R_P/PR_P]$ is 2-torsion). □

## 3. On the Kernel of the Map $\mathrm{G}(R) \to \mathrm{G}(\hat{R})$

In this section we raise some questions about the kernel of the map $\mathrm{G}(R) \to \mathrm{G}(\hat{R})$. First we fix some notation. Throughout this section we assume, for simplicity, that $R$ is excellent and is a homomorphic image of a regular local ring $T$. Let $d = \dim R$. Let $\mathrm{A}_i(R)$ be the $i$th Chow group of $R$; that is,

$$\mathrm{A}_i(R) = \frac{\bigoplus_{P \in \operatorname{Spec} R,\, \dim R/P = i} \mathbb{Z} \cdot [\operatorname{Spec} R/P]}{\langle \operatorname{div}(Q, x) \mid Q \in \operatorname{Spec} R,\, \dim R/Q = i+1,\, x \in R \backslash Q \rangle},$$

where

$$\operatorname{div}(Q, x) = \sum_{P \in \operatorname{Min}_R R/(Q,x)} l_{R_P}(R_P/(Q,x)R_P)[\operatorname{Spec} R/P].$$

For an abelian group $A$, we let $A_{\mathbb{Q}} = A \otimes_{\mathbb{Z}} \mathbb{Q}$. The Chow group of $R$ is defined to be $\mathrm{A}_*(R) = \bigoplus_{i=0}^{d} \mathrm{A}_i(R)$. It is well known that there is a $\mathbb{Q}$-vector space isomorphism

$$\tau_{R/T} : \mathrm{G}(R)_{\mathbb{Q}} \to \mathrm{A}_*(R)_{\mathbb{Q}}$$

(though it is unknown whether this is independent of $T$). We also remark that the Grothendieck group $\mathrm{G}(R)$ admits a filtration by $F_i\, \mathrm{G}(R) = \langle [M] \in \mathrm{G}(R) \mid \dim M \leq i \rangle$.

The existing examples on the failure of injectivity for the map $\mathrm{G}(R) \to \mathrm{G}(\hat{R})$ together with the affirmative results in [KKu] motivate the following question.

QUESTION 3.1. *Suppose that $R$ satisfies* $(\mathrm{R}_n)$ *(i.e., $R$ is regular in codimension $n$). Then is* $\ker(\mathrm{G}(R) \to \mathrm{G}(\hat{R}))$ *contained in* $F_{d-n-1}\, \mathrm{G}(R)$?

Question 3.1 is closely related to the following.

QUESTION 3.2. *Suppose that $R$ satisfies* $(\mathrm{R}_n)$. *Then is the map* $\mathrm{A}_i(R) \to \mathrm{A}_i(\hat{R})$ *injective for $i \geq d - n$?*

In fact, if we allow rational coefficients, then the previous questions are equivalent. Let $\mathrm{G}^i(R) = F_i\,\mathrm{G}(R)/F_{i-1}\,\mathrm{G}(R)$. Then clearly we have a decomposition:

$$\mathrm{G}(R)_{\mathbb{Q}} = \bigoplus_{i=0}^{d} \mathrm{G}^i(R)_{\mathbb{Q}}.$$

Also, the Riemann–Roch map decomposes into isomorphisms $\tau^i : \mathrm{G}^i(R)_{\mathbb{Q}} \to A_i(R)_{\mathbb{Q}}$ that make the diagram

$$\begin{tikzcd}
\mathrm{G}^i(R) \arrow[r, "\tau^i_{R/T}"] \arrow[d, "g_i"] & \mathrm{A}_i(R) \arrow[d, "f_i"] \\
\mathrm{G}^i(\hat{R}) \arrow[r, "\tau^i_{\hat{R}/\hat{T}}"] & \mathrm{A}_i(\hat{R})
\end{tikzcd}$$

commutative. It follows that

$$\ker(\mathrm{G}(R)_{\mathbb{Q}} \to \mathrm{G}(\hat{R})_{\mathbb{Q}}) \cong \bigoplus_{i}^{d} \ker(f_i) \cong \bigoplus_{i}^{d} \ker(g_i).$$

Hence we can state our next result.

PROPOSITION 3.3. *Let $R$ be an excellent local ring that is a homomorphic image of a regular local ring. Let* $\dim R = d$ *and let* $0 < l \leq d$ *be an integer. Then the maps* $\mathrm{A}_i(R)_{\mathbb{Q}} \to \mathrm{A}_i(\hat{R})_{\mathbb{Q}}$ *are injective for $i \geq l$ if and only if* $\ker(\mathrm{G}(R)_{\mathbb{Q}} \to \mathrm{G}(\hat{R})_{\mathbb{Q}}) \subseteq F_{l-1}\,\mathrm{G}(R)_{\mathbb{Q}}$.

Even if $l = 1$, we cannot answer Question 3.2. However, if $R$ is normal then Questions 3.1 and 3.2 can be answered in the affirmative when $l = 1$. In this case, $\mathrm{A}_1(R) \cong \mathrm{Cl}(R)$ and the map between class groups of $R$ and $\hat{R}$ is injective. Furthermore, it is well known that $\mathrm{G}(R)/F_{d-2}\,\mathrm{G}(R) \cong \mathrm{A}_d(R) \oplus \mathrm{A}_{d-1}(R)$ (see e.g. [C, Cor. 1]), so Question 3.1 is also true for $l = 1$.

Finally, one could formulate a stronger version of Question 3.1 as follows. Note that—both in Hochster's example and in the example presented here—the supports of the modules given are actually equal to the singular locus of $R$. So one could ask the following question.

QUESTION 3.4. *Let $R$ be an excellent local ring. Let $X =$* $\operatorname{Spec} R$, *$Y =$* $\operatorname{Sing}(X)$, *$\hat{X} =$* $\operatorname{Spec} \hat{R}$, *and $\hat{Y} =$* $\operatorname{Sing}(\hat{X})$. *One then has the commutative diagram*

$$\begin{tikzcd}
\mathrm{G}(Y) \arrow[r, "f"] \arrow[d] & \mathrm{G}(X) \arrow[d, "g"] \\
\mathrm{G}(\hat{Y}) \arrow[r] & \mathrm{G}(\hat{X}),
\end{tikzcd}$$

*where* $\mathrm{G}(X)$ *denotes the Grothendieck group of coherent $\mathcal{O}_X$-modules and where the maps are naturally induced by closed immersions or flat morphisms. Is* $\ker(g)$ *contained in* $\operatorname{im}(f)$?

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