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Canonical Hilbert–Burch Matrices for Ideals of $k[x, y]$

Aldo Conca & Giuseppe Valla

1. Introduction

Let k be a field of arbitrary characteristic and R the polynomial ring $k[x_1, \dots, x_n]$. Let τ be a term order on R. Given a nonzero $f \in R$, we denote by $\mathrm{Lt}_\tau(f)$ the largest term with respect to τ appearing in f and by $\mathrm{Lc}_\tau(f)$ the coefficient of $\mathrm{Lt}_\tau(f)$ in f. For an ideal I of R we denote by $\mathrm{Lt}_\tau(I)$ the (monomial) ideal generated by $\mathrm{Lt}_\tau(f)$ with $f \in I \setminus \{0\}$. Let E be a monomial ideal of R. Consider the set $V(E)$ of the homogeneous ideals I of R such that $\mathrm{Lt}_\tau(I) = E$. The set $V(E)$ has a natural structure of affine variety. Namely, given I in $V(E)$, we can consider I as a point in an affine space $\mathbf{A}^N$ with coordinates given by the coefficients of the nonleading terms in the reduced Gröbner basis of I; see Section 2 for details. The equations defining (at least set-theoretically) $V(E)$ can be obtained from Buchberger's Gröbner basis criterion. Provided $\dim_k R/E$ is finite, one can give the structure of affine variety also to the set $V_0(E)$ of the ideals I (homogeneous or not) such that $\mathrm{Lt}_\tau(I) = E$.

These and similarly defined varieties play important roles in many contexts, such as the study of various types of Hilbert schemes and the problem of deforming nonradical to radical or prime ideals (see [ASt; Br; CRV; ES1; ES2; G1; G2; Ha; H1; H2; I1; I2; IY; KRo2; MSt]).

Many of the equations defining $V(E)$ or $V_0(E)$ contain parameters that appear in degree 1 and that can be eliminated. It happens quite often that, after dispensing with the superfluous parameters, one is left with no equations—that is, the variety is an affine space. But it is well known that, in general, $V(E)$ can be reducible and can have irreducible components that are not affine spaces; see Examples 2.1–2.3.

On the other hand, for $n = 2$ and $d = \dim_k R/E < \infty$ it is known that $V_0(E)$ and $V(E)$ are affine spaces. This has been proved (for some term orders) by Briançon [Br] and Iarrobino [I1]; for general τ, it follows from a general result of Bialynicki-Birula [Bi1; Bi2] on smooth varieties with k^*-actions. Here it is important to note that $V_0(E)$ coincides with the set of points of the Hilbert scheme $\mathrm{Hilb}^d(\mathbf{A}^2)$ that degenerate to E under a suitable k^*-action associated to a weight vector representing the term order on monomials of degree $\le d+1$. By the analogy with Schubert cells for Grassmannians, we call $V_0(E)$ and $V(E)$ *Gröbner cells*.

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Our goal here is to show that, for $n = 2$ and τ the lexicographic order induced by $x > y$, both $V(E)$ and $V_0(E)$ can be described as affine spaces in a very explicit way (see Theorem 3.3). To achieve this goal, we identify canonical Hilbert–Burch matrices of the ideals involved. The main point is to introduce (redundant) systems of generators for the ideals in $V_0(E)$ that, instead of being themselves "simple", have "simple" syzygies.

We can then easily deduce formulas for the dimensions of $V(E)$ and $V_0(E)$ and of two other subvarieties of $V_0(E)$; see Corollary 3.1. Dimension formulas for these varieties were originally obtained in [Br; ES1; ES2; G2; I1; IY]. In Section 4 we re-prove and generalize some results of Iarrobino [I2] concerning the Betti strata of $V(E)$.

For standard facts on Gröbner bases we refer the reader to [KRo1] or [Ei]. The results of this paper were discovered, suggested, and double-checked by extensive computer algebra experiments performed with CoCoA [Co]. We thank Anthony Iarrobino and Lorenzo Robbiano for useful discussions regarding the material of this paper.

2. $V(E)$ as an Affine Variety

With notation as just described, we first recall how $V(E)$ and $V_0(E)$ can be given the structure of affine varieties. For every minimal monomial generator m of E, consider the polynomial

$$f_m = m - \sum \lambda(m, m')m',$$

where the sum is extended to the monomials $m' \notin E$ such that $\deg m = \deg m'$ and $m' < m$ with respect to τ. Denote by N the total number of parameters $\lambda(m, m')$. The property of being a Gröbner basis for the f_m is turned into the vanishing of polynomials (say, $B_1, \dots, B_r$) on the parameters $\lambda(m, m')$. Because an ideal has a unique reduced Gröbner basis, the points of the affine variety of $\mathbf{A}^N$ defined by the vanishing of the B_i are in bijection with the elements of $V(E)$. The polynomials B_i can be explicitly computed through Buchberger's criterion for Gröbner basis. There are many degrees of freedom in the application of Buchberger's criterion (e.g., one can use all the S-pairs or carefully chosen subsets of them, the reduction process can be performed in various ways, and so on). So the actual nature of the polynomials B_i depends on these choices but not, of course, on the variety that they define. Similarly, if $\dim_k R/E$ is finite then one can give the structure of an affine variety to $V_0(E)$ by dropping the assumption that $\deg m = \deg m'$ in the definition of f_m.

We note in passing that $V_0(E)$ is contained, usually properly, in the variety

$$U_E = \{I \subset R : \text{the monomials not in } E \text{ form a } K\text{-basis of } R/I\}.$$

For $R = k[x, y]$, the variety U_E is studied in detail in [Ha; H1; KRo2; MSt].

As stated in the Introduction, the varieties $V(E)$ and $V_0(E)$ quite often are affine spaces. Roughly speaking, what happens is as follows. Let m, n be monomial generators of E with $m' < m$ and let $t = m'n/\mathrm{GCD}(m,n)$ satisfy $t \notin E$. Then the coefficient of t in the S-polynomial associated to f_m and f_n is just $\lambda(m,m')$ or $\lambda(m,m') - \lambda(n,n')$ depending on whether or not there exists an $n' < n$ such that $t = n'm/\mathrm{GCD}(m,n)$. Performing the reduction procedure, $\lambda(m,m')$ cannot be cancelled because at each iteration the degree of the coefficients involved increases by 1. At the end of the reduction procedure, the coefficient of t in the polynomial we are left with must vanish. Hence we have equations of the form

$$\lambda(m,m') + B = 0 \quad \text{or} \quad \lambda(m,m') - \lambda(n,n') + B = 0, \tag{2.1}$$

where B is a polynomial in the $\lambda(\cdot,\cdot)$ not involving monomials of degree 1. Of course, if B does not involve $\lambda(m,m')$ at all then we can use (2.1) to remove parameter $\lambda(m,m')$ from the equations. This elimination process can be iterated. In many cases, at the end of the elimination process the equations vanish completely, and this shows that the associated variety is an affine space. We have implemented this rough algorithm in CoCoA [Co]. For example, we have verified that $V(E)$ is an affine space for $\tau = \mathrm{Lex}$, $n = 3$, and E any ideal generated by monomials of degree 3.

The following examples show that in general the variety $V(E)$ has a more complicated structure. For simplicity, the coordinates of the ambient affine spaces $\lambda(m,m')$ are denoted by a_i. All the computations are done over a field of characteristic 0.

EXAMPLE 2.1. Set $n = 3$, $E = (x_3^4, x_2^4, x_1x_2^2x_3, x_1^3x_3)$, and $\tau = \mathrm{Lex}$. Then $V(E)$ is a subvariety of $\mathbf{A}^{17}$, where the inclusion is given by the parameterization

$$\begin{aligned}
&x_1^3x_3 - x_1^2x_2^2a_1 - x_1^2x_2x_3a_2 - x_1^2x_3^2a_3 - x_1x_2^3a_4 - x_1x_2x_3^2a_5 - x_1x_3^3a_6\\
&\qquad - x_2^3x_3a_7 - x_2^2x_3^2a_8 - x_2x_3^3a_9,\\
&x_1x_2^2x_3 - x_1x_2x_3^2a_{10} - x_1x_3^3a_{11} - x_2^3x_3a_{12} - x_2^2x_3^2a_{13} - x_2x_3^3a_{14},\\
&x_2^4 - x_2^3x_3a_{15} - x_2^2x_3^2a_{16} - x_2x_3^3a_{17},\\
&x_3^4.
\end{aligned}$$

Buchberger's criterion yields three equations, two of which can be written as

$$\begin{aligned}
&a_{14} = -a_{10}^2a_{12} - a_{10}a_{13} - a_{11}a_{12} \quad \text{and}\\
&a_9 = 2a_1a_{10}^2a_{12}^2a_{15} + \text{other 46 terms in the } a_i \textit{ not} \text{ involving } a_9 \text{ and } a_{14}.
\end{aligned}$$

Setting $b_{17} = a_{10}^3 - a_{10}^2a_{15} + 2a_{10}a_{11} - a_{10}a_{16} - a_{11}a_{15} - a_{17}$, the third equation is $a_1b_{17} = 0$. Hence $V(E)$ has two irreducible components, each of which is isomorphic to $\mathbf{A}^{14}$.

EXAMPLE 2.2. Let $n = 4$, $E = (x_4^2, x_2x_4, x_2^2, x_1x_4)$, and $\tau = \mathrm{Lex}$. Then $V(E)$ is a subvariety of $\mathbf{A}^8$, where the inclusion is given by the parameterization

$$\begin{aligned} &x_4^2, \\ &x_2x_4 - x_3^2a_1 - x_3x_4a_2, \\ &x_2^2 - x_2x_3a_3 - x_3^2a_4 - x_3x_4a_5, \\ &x_1x_4 - x_2x_3a_6 - x_3^2a_7 - x_3x_4a_8. \end{aligned}$$

The parameters a_1, a_7, a_4 can be eliminated, so $V(E)$ is indeed contained in $\mathbf{A}^5$. After renaming $b_3 = 2a_2 - a_3$, the defining ideal of $V(E)$ in $\mathbf{A}^5$ takes the form b_3a_6, a_5a_6. Hence $V(E)$ has two irreducible components, one isomorphic to $\mathbf{A}^3$ and the other to $\mathbf{A}^4$.

EXAMPLE 2.3. Let $n = 4$, $E = (x_4^2, x_2x_4, x_1x_4, x_1x_2, x_1^2)$, and $\tau =$ Lex. Then $V(E)$ is a subvariety of $\mathbf{A}^{16}$, where the inclusion is given by the parameterization

$$\begin{aligned} &x_4^2, \\ &x_2x_4 - x_3^2a_1 - x_3x_4a_2, \\ &x_1x_4 - x_2^2a_3 - x_2x_3a_4 - x_3^2a_5 - x_3x_4a_6, \\ &x_1x_2 - x_1x_3a_7 - x_2^2a_8 - x_2x_3a_9 - x_3^2a_{10} - x_3x_4a_{11}, \\ &x_1^2 - x_1x_3a_{12} - x_2^2a_{13} - x_2x_3a_{14} - x_3^2a_{15} - x_3x_4a_{16}. \end{aligned}$$

The parameters $a_1, a_3, a_4, a_5, a_{10}, a_{13}, a_{14}, a_{15}$ can be eliminated, so $V(E)$ is indeed contained in $\mathbf{A}^8$. After renaming

$$b_9 = a_2a_8 + a_7a_8 - a_6 + a_9, \quad b_{12} = 2a_6 + b_9 - a_{12}, \quad b_7 = a_2 - a_7,$$

the defining ideal of $V(E)$ in $\mathbf{A}^8$ takes the form $(b_7b_9, b_9b_{12}, a_{11}b_{12} - b_7a_{16})$. Thus $V(E)$ has two components: one is isomorphic to $\mathbf{A}^6$, and the other is a quadric hypersurface of rank 4 in $\mathbf{A}^7$.

3. Ideals in $k[x, y]$

From now on, let k be a field and let $R = k[x, y]$ be the polynomial ring over k. We equip R with the lexicographic term order $>$ induced by $x > y$. In what follows, $\mathrm{Lt}_\tau(f)$, $\mathrm{Lt}_\tau(I)$, and $\mathrm{Lc}_\tau(f)$ will be more simply denoted as $\mathrm{Lt}(f)$, $\mathrm{Lt}_(I)$, and $\mathrm{Lc}(f)$.

Given a monomial ideal $E \subset R$ with $\dim_k R/E < \infty$, we want to describe the set of ideals

$$V_0(E) = \{I \text{ such that } \mathrm{Lt}(I) = E\}$$

and its subsets

$$\begin{aligned} V_1(E) &= \{I : \mathrm{Lt}(I) = E \text{ and } y \in \sqrt{I}\}, \\ V_2(E) &= \{I : \mathrm{Lt}(I) = E \text{ and } \sqrt{I} = (x, y)\}, \\ V(E) = V_3(E) &= \{I : \mathrm{Lt}(I) = E; \text{ and } I \text{ is homogeneous}\}. \end{aligned}$$

Our goal is to prove Theorem 3.3. As a corollary, we have the following result.

COROLLARY 3.1. *The set $V_0(E)$ is an affine space. The subsets $V_1(E)$, $V_2(E)$, and $V_3(E)$ are also affine spaces; indeed, they are coordinate subspaces of $V_0(E)$. Furthermore,*

$$\dim V_i(E) = \begin{cases} \dim_k R/E + \min\{j : y^j \in E\} & \text{if } i = 0, \\ \dim_k R/E & \text{if } i = 1, \\ \dim_k R/E - \min\{j : x^j \in E\} & \text{if } i = 2, \\ \#\mathcal{S}(E) & \text{if } i = 3, \end{cases}$$

where $\mathcal{S}(E)$ is the set described just before Definition 3.2 and $\#\mathcal{S}(E)$ denotes its cardinality.

As stated in the Introduction, dimension formulas for the $V_i(E)$—as well as the fact that they are affine spaces—have been proved in [Br; ES1; ES2; G2; I1; IY].

To prove Corollary 3.1 we could try to analyze the equations coming from Buchberger's criterion. Yet this turns out to be quite difficult, so instead we parameterize the syzygies and identify canonical Hilbert–Burch matrices.

Next we introduce some notation. Given a monomial ideal E such that $\dim_k R/E$ is finite, we set $t = \min\{j : x^j \in E\}$, $m_0 = 0$, and $m_i = \min\{j : x^{t-i}y^j \in E\}$ for every $1 \le i \le t$. It is clear that $m_0 = 0 < m_1 \le m_2 \le \cdots \le m_t$,

$$E = (x^t, x^{t-1}y^{m_1}, \dots, xy^{m_{t-1}}, y^{m_t}),$$

and $\dim_k R/E = \sum_{i=0}^{t} m_i$. These generators of E are not minimal in general. They minimally generate E if and only if $m_0 < m_1 < m_2 < \cdots < m_t$—that is, iff E is a lex-segment ideal. By construction, the correspondence

$$E \longleftrightarrow (m_0, \dots, m_t)$$

is a bijection between monomial ideals of R with radical equal to (x, y) and sequences of integers $0 = m_0 < m_1 \le m_2 \le \cdots \le m_t$.

Given E or, equivalently, $(m_0, \dots, m_t)$, we set

$$d_i = m_i - m_{i-1}$$

for $i = 1, \dots, t$. Here $d_1 > 0$ and $d_i \ge 0$ for every $i = 2, \dots, t$. Clearly, E can be as well described via the vector $(d_1, \dots, d_t)$. Furthermore, the lex-segment corresponds exactly to the vectors with $d_i > 0$ for $i = 1, \dots, t$.

The matrix

$$M_0(E) = \begin{pmatrix} y^{d_1} & 0 & 0 & \cdots & 0 & 0 \\ -x & y^{d_2} & 0 & \cdots & 0 & 0 \\ 0 & -x & y^{d_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & -x & y^{d_{t-1}} & 0 \\ 0 & 0 & 0 & 0 & -x & y^{d_t} \\ 0 & 0 & 0 & 0 & 0 & -x \end{pmatrix}$$

has size $(t+1) \times t$ and is a Hilbert–Burch matrix of E in the sense that the (signed) t-minors of $M_0(E)$ are the monomials $x^{t-i}y^{m_i}$ and the columns generate their syzygy module.

The matrix $M_0(E)$ represents a map from $F_1 = \bigoplus_{i=1}^{t} R(-t+i-m_i-1)$ to $F_0 = \bigoplus_{i=1}^{t+1} R(-t+i-1-m_{i-1})$. It is useful to consider also the corresponding degree matrix $U(E) = (u_{ij})$. The entries of $U(E)$ are the degrees of the (homogeneous) entries of every matrix representing a map of degree 0 from F_1 to F_0. We have

$$u_{ij} = m_j - m_{i-1} + i - j \quad \text{for } i = 1, \dots, t+1 \text{ and } j = 1, \dots, t. \tag{3.1}$$

We remark that $u_{ii} = m_i - m_{i-1} = d_i$ and $u_{i+1,i} = 1$ for every $i = 1, \dots, t$. Define

$$\mathcal{S}(E) = \{(i,j) : 1 \le j < i \le t+1 \text{ and } 0 \le u_{ij} < d_j\}.$$

DEFINITION 3.2. Let $T_0(E)$ be the set of $(t+1) \times t$ matrices $N = (n_{i,j})$, where

$$n_{i,j} = \begin{cases} 0 & \text{if } i < j, \\ \text{a polynomial in } k[y] \text{ of degree} < d_j & \text{if } i \ge j. \end{cases}$$

Consider also the following conditions.

(1) $n_{i,i} = 0$ for every $i = 1, \dots, t$.
(2) For every j such that $d_j > 0$, the polynomial $n_{i,j}$ has no constant term for every $i = j+1, \dots, k+1$, where $k = \max\{v : j \le v \le t \text{ and } m_v = m_j\}$.
(3) The polynomial

$$n_{i,j} = \begin{cases} 0 & \text{if } (i,j) \notin \mathcal{S}(E), \\ p_{ij}y^{u_{ij}} & \text{if } (i,j) \in \mathcal{S}(E), \end{cases}$$

with $p_{ij} \in k$.

Accordingly, we define:

$$T_1(E) = \{N \in T_0(E) : N \text{ satisfies (1)}\};$$
$$T_2(E) = \{N \in T_0(E) : N \text{ satisfies (1) and (2)}\};$$
$$T_3(E) = \{N \in T_0(E) : N \text{ satisfies (3)}\}.$$

THEOREM 3.3. *For every monomial ideal E, the map $\phi\colon T_0(E) \to V_0(E)$ defined by sending $N \in T_0(E)$ to the ideal of t-minors of the matrix $M_0(E) + N$ is a bijection. Furthermore, the restriction of ϕ induces bijections between $T_i(E)$ and $V_i(E)$ for $i = 1, 2, 3$.*

By construction, the sets $T_i(E)$ are affine spaces and their dimension can be easily computed from their defining conditions. Therefore, Corollary 3.1 is an immediate consequence of Theorem 3.3.

Before embarking in the proof of this theorem, we consider an example.

EXAMPLE 3.4. Let $E = (x^3, xy^3, y^5) = (x^3, x^2y^3, xy^3, y^5)$. Then $m = (0, 3, 3, 5)$, $d = (3, 0, 2)$, and

$$M_0(E) = \begin{pmatrix} y^3 & 0 & 0 \\ -x & 1 & 0 \\ 0 & -x & y^2 \\ 0 & 0 & -x \end{pmatrix}, \qquad U(E) = \begin{pmatrix} 3 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}.$$

We have $t = 3$, $\min\{i : y^i \in E\} = 5$, $\dim R/E = 11$, and $\#\mathcal{S}(E) = 4$.

The matrices in $T_0(E)$ have the form

$$\begin{pmatrix} n_{1,1} & 0 & 0 \\ n_{2,1} & 0 & 0 \\ n_{3,1} & 0 & n_{3,3} \\ n_{4,1} & 0 & n_{4,3} \end{pmatrix},$$

where the $n_{i,1}$ are polynomials in y of degree < 3 and the $n_{i,3}$ are polynomials in y of degree < 2. The matrices in $T_1(E)$ are those of $T_0(E)$ such that $n_{1,1} = 0$ and $n_{3,3} = 0$. The matrices in $T_2(E)$ are those of $T_1(E)$ such that $n_{2,1}, n_{3,1}, n_{4,3}$ have no constant term. Finally, the matrices in $T_3(E)$ have the form

$$\begin{pmatrix} 0 & 0 & 0 \\ p_{21}y & 0 & 0 \\ p_{31}y^2 & 0 & 0 \\ p_{41}y & 0 & p_{43}y \end{pmatrix},$$

where the $p_{ij} \in k$.

As predicted by Theorem 3.1, we obtain $\dim V_0(E) = 16$, $\dim V_1(E) = 11$, $\dim V_2(E) = 8$, and $\dim V_3(E) = 4$.

Proof of Theorem 3.3. The proof proceeds in three steps, which show in turn that the map ϕ is well-defined (Step 1); that the map ϕ is bijective (Step 2); and that, for $i = 1, 2, 3$, we have $\phi(N) \in V_i(E)$ iff $N \in T_i(E)$ (Step 3).

Step 1. *Proof that ϕ is well-defined*

For $N \in T_0(E)$ set $I = \phi(N)$. We show that $\mathrm{Lt}(I) = E$. For $i = 0, \ldots, t$, let f_i be $(-1)^{t-i}$ times the determinant of the submatrix of $M_0(E) + N$ obtained by deleting the $(i + 1)$th row. By construction, $\mathrm{Lt}(f_i) = x^{t-i}y^{m_i}$ and $\mathrm{Lc}(f_i) = 1$. We show that $f_0, \ldots, f_t$ form a Gröbner basis of I. The syzygy module of leading terms of the f_i is generated by the syzygies

$$y^{d_i}(x^{t-i+1}y^{m_{i-1}}) - x(x^{t-i}y^{m_i}) = 0 \tag{3.2}$$

with $i = 1, \ldots, t$. To prove that the f_i form a Gröbner basis it is enough to show that the S-polynomials associated to these syzygies reduce to 0. Because

$$y^{d_i}f_{i-1} - xf_i + \sum_{j=i-1}^{t} n_{j+1,i}f_j = 0,$$

it is enough to show that if $y^{d_i}f_{i-1} - xf_i \neq 0$ then $\mathrm{Lt}(n_{j+1,i}f_j) \leq \mathrm{Lt}(y^{d_i}f_{i-1} - xf_i)$ for every $n_{j+1,i} \neq 0$. Observe that the nonzero factors $n_{j+1,i}f_j$ have leading terms that involve different powers of x. Therefore, $\max(\mathrm{Lt}(n_{j+1,i}f_j) : n_{j+1,i} \neq 0) = \mathrm{Lt}(y^{d_i}f_{i-1} - xf_i)$.

Step 2 will be a corollary of the following two lemmas. Indeed, Lemma 3.6 is the main technical result of the paper and is similar, in spirit, to analogous results appearing in [Br] and [IY].

LEMMA 3.5. *Let I be an ideal of R such that* $\mathrm{Lt}(I) = E$, *and let* $f_0, \dots, f_t \in I$ *such that* $\mathrm{Lt}(f_i) = x^{t-i}y^{m_i}$ *and* $\mathrm{Lc}(f_i) = 1$. *Then, for every $f \in I$ such that* $\mathrm{Lt}(f) = x^{t-i}y^b$ *for some* $0 \le i \le t$, *there exist polynomials* $g_j \in k[y]$ *with* $j = i, \dots, t$ *and of* $\deg g_i = b - m_i$ *such that* $f + g_i f_i + \dots + g_t f_t = 0$.

Proof. By assumption, $f_0, \dots, f_t$ is a Gröbner basis of I. Hence $x^{t-i}y^b$ is divisible by some $x^{t-j}y^{m_j}$, so $t - j \le t - i$ and $m_j \le b$. It follows that $i \le j$ and $m_i \le m_j \le b$. Therefore, $f - \mathrm{Lc}(f)y^{b-m_i}f_i$ is still in I and has a smaller leading term (if it is nonzero). We desire the desired representation by iterating the procedure. □

LEMMA 3.6. *Let I be an ideal of R such that* $\mathrm{Lt}(I) = E$. *Then there exist* $f_0, \dots, f_t \in I$ *such that*:

(1) $\mathrm{Lt}(f_i) = x^{t-i}y^{m_i}$ *and* $\mathrm{Lc}(f_i) = 1$ *for every* $i = 0, \dots, t$; *and*
(2) *for every* $i = 1, \dots, t$ *there exist* $n_{j+1,i} \in k[y]$ *with* $i - 1 \le j \le t$ *and* $\deg n_{j+1,i} < d_i$ *such that*

$$y^{d_i}f_{i-1} - xf_i + \sum_{j=i-1}^{t} n_{j+1,i}f_j = 0. \tag{3.3}$$

Moreover, the polynomials f_i and $n_{j+1,i}$ with these two properties are uniquely determined by I.

Proof. We prove the existence first. A set of polynomials $f_0, \dots, f_t \in I$ satisfying (1) clearly exists. We show how to modify those polynomials in order to satisfy (2). For a given k ($1 \le k \le t$), suppose that we have already modified $f_k, \dots, f_t$ so that (1) is still fulfilled and (2) is fulfilled for $i = k+1, \dots, t$. We show how to modify f_{k-1} in order to fulfill (2) for $i = k$. Note that (a) $y^{d_k}f_{k-1} - xf_k$ is in I and involves only terms with x-exponent $\le t-(k-1)$ and (b) if $x^{t-(k-1)}y^b$ is indeed present then $b < m_k$. By Lemma 3.5 we have that there exist $g_{k-1}, \dots, g_t \in k[y]$ such that g_{k-1} is either 0 or of degree $< d_k$ and

$$y^{d_k}f_{k-1} - xf_k + g_{k-1}f_{k-1} + g_kf_k + \dots + g_tf_t = 0. \tag{3.4}$$

Set $h = y^{d_k} + g_{k-1}$ and perform, for $j = k, \dots, t$, division with remainder: $g_j = hq_j + r_j$ with $q_j, r_j \in k[y]$ and r_j either 0 or of degree $< d_k$. Then

$$y^{d_k}f'_{k-1} - xf_k + g_{k-1}f'_{k-1} + r_kf_k + \dots + r_tf_t = 0 \tag{3.5}$$

with $f'_{k-1} = f_{k-1} + q_kf_k + \dots + q_tf_t$. Note that f'_{k-1} is in I and that $\mathrm{Lt}(f'_{k-1}) = \mathrm{Lt}(f_{k-1})$ and $\mathrm{Lc}(f_{k-1}) = \mathrm{Lc}(f'_{k-1})$. We may replace f_{i-1} with f'_{i-1} and then (3.5) is the desired relation.

We prove now the uniqueness of the f_i and $n_{j+1,i}$ fulfilling (1) and (2). Suppose we have other polynomials f'_i and $n'_{j+1,i}$ fulfilling (1) and (2). Observe that $f_t = f'_t$ because they are both the monic generator of $I \cap k[y]$. Hence we may assume that $f_j = f_j'$ for $j = k, \dots, t$ in order to show that $f_{k-1} = f_{k-1}'$. By assumption,

$$y^{d_k} f_{k-1} - x f_k + n_{k,k} f_{k-1} + \sum_{j=k}^{t} n_{j+1,k} f_j = 0 \tag{3.6}$$

and

$$y^{d_k} f_{k-1}' - x f_k' + n_{k,k}' f_{k-1}' + \sum_{j=k}^{t} n_{j+1,k}' f_j' = 0, \tag{3.7}$$

where the $n_{j+1,k}$ and $n_{j+1,k}'$ are polynomials in $k[y]$ of degree $< d_k$.

Lemma 3.5 applied with $f = f_{k-1}'$ yields the equation

$$f_{k-1}' = f_{k-1} + g_k f_k + \cdots + g_t f_t \tag{3.8}$$

with $g_j \in k[y]$. Set $h = y^{d_i} + n_{k,k}$ and $h' = y^{d_i} + n_{k,k}'$. After replacing f_{k-1}' in (3.7) with the right-hand side of (3.8) and then subtracting (3.6), we obtain

$$(h' - h) f_{k-1} + \sum_{j=k}^{t} (h' g_j + n_{k,j+1}' - n_{k,j+1}) f_j = 0. \tag{3.9}$$

Since the leading terms of the f_i involve distinct powers of x, the f_i are linearly independent over $k[y]$. Hence the coefficients $h'g_j + n_{k,j+1}' - n_{k,j+1}$ of (3.9) must be 0. Therefore, $h'g_j = -n_{k,j+1}' + n_{k,j+1}$. But $n_{k,j+1}' + n_{k,j+1}$ has degree $<$ d_k and h' has degree d_k. Hence $g_j = 0$ for every j and so $f_{k-1} = f_{k-1}'$. Having shown that the f_i fulfilling (1) and (2) are uniquely determined by I, it remains to show that the coefficients $n_{j+1,i}$ are also uniquely determined. This is easy: given other coefficients $n_{j+1,i}'$ satisfying (2), say

$$y^{d_i} f_{i-1} - x f_i + \sum_{j=i-1}^{t} n_{j+1,i}' f_j = 0, \tag{3.10}$$

we may subtract (3.3) from (3.10) and obtain

$$\sum_{j=i-1}^{t} (n_{j+1,i}' - n_{j+1,i}) f_j = 0.$$

This implies that $n_{j+1,i}' = n_{j+1,i}$ by the linear independence of the f_i over $k[y]$.

Step 2. *Proof that ϕ is bijective*
We first prove that ϕ is injective. Suppose $I = \phi(N) = \phi(N')$ for matrices $N, N' \in T_0(E)$. We saw in Step 1 that the signed t-minors $f_0, \dots, f_t$ of $M_0(E) + N$ fulfill (1) and (2) of Lemma 3.6. The same is true for the signed t-minors $f_0', \dots, f_t'$ of $M_0(E) + N'$. By the uniqueness of the f_i in Lemma 3.6, $f_i = f_i'$ for every i. By the uniqueness of the coefficients in (3.3), we conclude that $N = N'$.

We now show that ϕ is surjective. Let $I \in V_0(E)$. We may find $f_0, \dots, f_t \in I$ satisfying (1) and (2) of Lemma 3.6. Equation (3.3) is the reduction to 0 of the S-polynomial corresponding to the syzygy (3.2) among the leading terms. We know that these syzygies generate the syzygy module of the leading term of the f_i, so Schreyer's theorem implies that (3.3) gives a system of generators for the syzygy module of the f_i. The corresponding matrix is of the form $M_0(E) + N$ with $N \in T_0(E)$, and the Hilbert–Burch theorem (see [BHe]) then implies that $\phi(N) = I$.

Step 3. *Proof that* $\phi(N) \in V_i(E)$ *iff* $N \in T_i(E)$
Throughout the proof, N denotes a matrix in $T_0(E)$, $I = \phi(N)$, and $f_0, \dots, f_t$ are the signed t-minors of $M_0(E) + N$.

Because the f_i form a Gröbner basis with respect to the lexicographic order, $f_t = \prod_{i=1}^{t}(y^{d_i} + n_{i,i})$ generates $I \cap k[y]$. We have that $y \in \sqrt{I}$ iff f_t divides some power of y. But this is clearly equivalent to the vanishing of $n_{i,i}$ for $i = 1, \dots, t$, which proves that $N \in T_1(E)$ iff $\phi(N) \in V_1(E)$.

In order to prove that $N \in T_2(E)$ iff $\phi(N) \in V_2(E)$, we may assume that $N \in T_1(E)$ and then show that $\sqrt{I} = (x, y)$ iff N fulfills condition (2) of Definition 3.2. Since we already know $y \in \sqrt{I}$, it follows that $\sqrt{I} = \sqrt{I + (y)}$. Replace y with 0 in $M_0(E) + N$, and denote by W_1 the resulting matrix. The first row of W_1 is 0 (since $d_1 > 0$). Denote by W the submatrix of W_1 obtained by deleting the first row. By construction, $I + (y) = (\det W, y)$. We must show that $\det W$ is a power of x iff N fulfills condition (2) of Definition 3.2.

Let $C = \{i : i = 1, \dots, t \text{ and } d_i > 0\}$, say $C = \{i_1, \dots, i_p\}$ with $i_1 < \cdots < i_p$. By assumption, $i_1 = 1$ and we set $i_{p+1} = t + 1$ by convention. The matrix W has a block decomposition

$$W = \begin{pmatrix} J_1 & 0 & 0 & \cdots & 0 \\ * & J_2 & 0 & \cdots & 0 \\ * & * & J_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ * & * & \cdots & * & J_p \end{pmatrix},$$

where each J_v is a square block of size, say, $u = i_{v+1} - i_v$ and has the form

$$\begin{pmatrix} -x + a_1 & 1 & 0 & \cdots & \cdots & 0 \\ a_2 & -x & 1 & 0 & \cdots & 0 \\ a_3 & 0 & -x & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{u-1} & 0 & \cdots & 0 & -x & 1 \\ a_u & 0 & \cdots & 0 & 0 & -x \end{pmatrix};$$

here $a_j = n_{i_v+j, i_v}(0)$ for $j = 1, \dots, u$. Now $\det W = \prod_v \det J_v$. The determinant of the matrix J_v is, up to sign, $x^u - a_1 x^{u-1} - a_2 x^{u-2} - \cdots - a_u$. Hence $\det W$ is a power of x if and only if the coefficients a_j in each J_v are 0. This is condition (2) of Definition 3.2.

Finally, we have to show that $N \in T_3(E)$ iff I is homogeneous. The "only if" direction follows immediately because the matrix $M_0(E) + N$ is homogeneous for every $N \in T_3(N)$. The "if" direction follows from the observation that the polynomials f_i and $n_{i,j}$ of Lemma 3.6 are homogeneous if we start with a homogeneous ideal I.

This completes Step 3 and thus the proof of Theorem 3.3. □

4. Betti Strata of $V(E)$

Let $h = h(z)$ be the Hilbert series of a graded Artinian quotient of R. It is known that $h(z)$ is of the form

$$h(z) = 1 + 2z + \cdots + cz^{c-1} + \sum_{j=c}^{s} h_j z^j$$

with $s + 1 \geq c \geq h_c \geq \cdots \geq h_s > 0$. Denote by $\mathbb{G}(h)$ the variety that parameterizes graded ideals I in R such that the Hilbert series $h_{R/I}(z) = h(z)$. Iarrobino proved in [I1] that $\mathbb{G}(h)$ is a smooth projective variety whose dimension is given by the beautiful formula

$$\dim \mathbb{G}(h) = h_c + \sum_{j=c}^{s} p_j p_{j+1}, \tag{4.1}$$

where $p(z) = \sum_0^{s+1} p_i z^i = (1 - z)h(z)$ is the first difference of $h(z)$.

Among the ideals with Hilbert series $h(z)$, the lex-segment plays a special role. We denote it by $L(h)$ or just L if $h(z)$ is clear from the context. If char $k = 0$, then $V(L)$ is dense in $\mathbb{G}(h)$ so that $\dim V(L) = \dim \mathbb{G}(h)$. Therefore, by Corollary 3.1 we have

$$\dim \mathbb{G}(h) = \#\mathcal{S}(L). \tag{4.2}$$

To double-check, the curious reader can show directly that the right-hand sides of (4.1) and (4.2) indeed coincide. It is a simple, but not obvious, exercise.

We come now to study the Betti strata of $V(E)$. For a homogeneous ideal I in $k[x, y]$, denote by $\beta_{i,j}(I)$ the (i, j)th Betti number. In particular, $\beta_{0,j}(I)$ is the number of minimal generators of I of degree j. It is well known that each pair of the three sets of invariants $\{\beta_{0,j}(I)\}_j$, $\{\beta_{1,j}(I)\}_j$, and $\{\dim I_j\}_j$ determine the third. Given integers j and u, we define

$$V(E, j, u) = \{I \in V(E) : \beta_{0,j}(I) = u\},$$
$$V(E, j, \geq u) = \{I \in V(E) : \beta_{0,j}(I) \geq u\}.$$

For $b = (b_1, \ldots, b_j, \ldots)$ a vector with integral entries, we define

$$V(E, b) = \bigcap_j V(E, j, b_j)$$

and

$$V(E, \geq b) = \bigcap_j V(E, j, \geq b_j). \tag{4.3}$$

We consider a monomial ideal E and its associated sequence $m_0, \ldots, m_t$. The ideals in $V(E)$ are parameterized by the affine space $\mathbf{A}^n$, where $n = \#\mathcal{S}(E)$. We denote by p_{ij} with $(i, j) \in \mathcal{S}(E)$ (or simply by $p_1, \ldots, p_n$) the coordinates of $\mathbf{A}^n$.

Given $p \in \mathbf{A}^n$, we consider the matrix $N \in T_3(E)$, defined in condition (3) of Definition 3.2. Set $M(p) = M_0(E) + N$. By the Hilbert–Burch theorem, the ideal I of maximal minors of $M(p)$ has the free resolution

$$0 \longrightarrow \bigoplus_{i=1}^{t} R(-b_i) \xrightarrow{M(p)} \bigoplus_{i=1}^{t+1} R(-a_i) \longrightarrow 0, \tag{4.4}$$

where $a_i = t+1-i+m_{i-1}$ for $i = 1, \dots, t+1$ and $b_i = a_{i+1}+1$ for $i = 1, \dots, t$.

For every j, we set

$$w_j = \{i : a_i = j\} \quad \text{and} \quad v_j = \{i : b_i = j\}.$$

Tensoring (4.4) with k and then taking the degree-j component yields the complex of vector spaces

$$0 \longrightarrow k^{\#v_j} \xrightarrow{M(p)_j} k^{\#w_j} \longrightarrow 0,$$

whose homology gives the Betti numbers of I. Here $M(p)_j$ is the submatrix of $M(p)$ with rows indices w_j and column indices v_j. It follows that

$$\beta_{0,j}(I) = \#w_j - \operatorname{rank} M(p)_j \tag{4.5}$$

and hence $V(E, j, \geq u)$ is the determinantal variety defined by the condition

$$\operatorname{rank} M(p)_j \leq \#w_j - u.$$

If $i_1 \in w_j$ and $i_2 \in v_j$, then the (i_1, i_2)th entry of $M(p)$ is:

$$\begin{array}{ll} p_{i_1 i_2} & \text{if } i_1 > i_2 \text{ and } d_{i_2} > 0; \\ 0 & \text{if } i_1 > i_2 \text{ and } d_{i_2} = 0; \\ 1 & \text{if } i_1 = i_2; \\ 0 & \text{if } i_1 < i_2. \end{array}$$

Hence the matrices $M(p)_j$ have entries that are either variables or 0 or 1. Furthermore, the sets of the variables involved in $M(p)_j$ and in $M(p)_i$ are disjoint if $i \neq j$. We may summarize as follows.

LEMMA 4.1. *The variety $V(E, \geq b)$ is the transversal intersection of the determinantal varieties $V(E, j, \geq b_j)$. In particular, the codimension of $V(E, \geq b)$ is the sum of the codimensions of the $V(E, j, \geq b_j)$, and $V(E, \geq b)$ is irreducible iff $V(E, j, \geq b_j)$ is irreducible for every j.*

From now on we concentrate our attention on the variety $V(E, j, \geq u)$. If $i \in w_j \cap v_j$, then the (i,i)th entry of $M(p)_j$ is 1 and all the other entries in that column are 0; hence we can simply eliminate the column and the row containing the 1s. Denote by $M(p)_j^*$ the submatrix obtained from $M(p)_j$ by removing the 1s together with their columns and rows. Since the 1s are in different rows and columns, it follows that

$$\operatorname{rank} M(p)_j = \operatorname{rank} M(p)_j^* + \#(w_j \cap v_j).$$

After noticing that $\#(w_j \setminus w_j \cap v_j)$ is exactly $\beta_{0,j}(E)$, we may conclude as follows.

LEMMA 4.2. *The variety* $V(E,j,\geq u)$ *is defined by the condition*

$$\operatorname{rank} M(p)_j^* \leq \beta_{0,j}(E) - u.$$

The matrices $M(p)_j^*$ have entries that are either 0 or distinct variables, and if the (i_1,i_2)th entry is 0 then the same is true for the (h_1,h_2)th entry also when $h_1 \leq i_1$ and $h_2 \geq i_2$; that is, they look like

$$\begin{pmatrix} \bullet & \bullet & 0 & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix}, \tag{4.6}$$

where each $\bullet$ is a distinct variable.

REMARK 4.3. The ideals of minors of a given size of the matrices of type (4.6) are radical (to prove this one can use Gröbner bases) but obviously are not prime in general. They can have clearly minimal primes of different codimension.

The following example shows that $V(E,\geq u)$ is not irreducible in general.

EXAMPLE 4.4. Let $E = (x^6, x^5y, x^4y^3, x^3y^4, x^2y^4, xy^5, y^7)$. Here $d = (1,2,1,0,1,2)$ and $a = (6,6,7,7,6,6,7)$ and $b = (7,8,8,7,7,8)$. We have that $V(E)$ is an 8-dimensional affine space parameterized by the following matrix.

$$M(p) = \begin{array}{c|cccccc} & 7 & 8 & 8 & 7 & 7 & 8 \\ \hline 6 & y & 0 & 0 & 0 & 0 & 0 \\ 6 & -x & y^2 & 0 & 0 & 0 & 0 \\ 7 & p_1 & -x+p_4y & y & 0 & 0 & 0 \\ 7 & p_2 & p_5y & -x & 1 & 0 & 0 \\ 6 & 0 & 0 & 0 & -x & y & 0 \\ 6 & 0 & 0 & 0 & 0 & -x & y^2 \\ 7 & p_3 & p_6y & 0 & 0 & p_7 & -x+p_8y \end{array}$$

The numbers on the boundary are the degree of the syzygies (the first row) and the degree of the generators (the first column).

Here the only interesting variety is $V(E,7,\geq u)$. We have $w_7 = \{3,4,7\}$ and $v_7 = \{1,4,5\}$. The matrix $M(p)_7$ is obtained via $M(p)$, selecting the rows and columns marked with 7:

$$M(p)_7 = \begin{array}{c|ccc} & 7 & 7 & 7 \\ \hline 7 & p_1 & 0 & 0 \\ 7 & p_2 & 1 & 0 \\ 7 & p_3 & 0 & p_7 \end{array}$$

In order to get $M(p)_7^*$ we must cancel rows and columns containing 1s:

$$M(p)_7^* = \begin{pmatrix} p_1 & 0 \\ p_3 & p_7 \end{pmatrix}.$$

Hence $V(E, 7, \geq u)$ is defined by the condition

$$\operatorname{rank} M(p)_7^* \leq 2 - u.$$

Therefore, $V(E, 7, \geq 1)$ is defined by $p_1 p_7 = 0$ and has two irreducible components of codimension 1. The variety $V(E, 7, \geq 2)$ is defined by $p_1 = p_3 = p_7 = 0$ and is irreducible of codimension 3.

This example can be generalized to show that every matrix of type (4.6) can arise as $M(p)_j^*$ for some E and some j. Instead of giving complicated and cumbersome details, we just give the following (it is hoped, illuminating) example and leave the details to the interested reader.

EXAMPLE 4.5. Starting with E associated to the sequence

$$d = (1, 1, 2, 1, 0, 1, 1, 1, 2, 1, 1, 0, 1, 1, 2, 1, 1, 1),$$

the matrix $M(p)_{19}^*$ is

$$\begin{pmatrix} \bullet & \bullet & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

and $V(E, 19, \geq u)$ is defined by the condition $\operatorname{rank} M(p)_{19}^* \leq 7 - u$.

If E is a lex-segment then $d_i > 0$ for every i. As a consequence, the matrices $M(p)_j$ are matrices of indeterminates. Thus we obtain the following results of Iarrobino [I2].

COROLLARY 4.6. *Let L be a lex-segment. Then the variety $V(L, j, \geq u)$ is defined by the condition* $\operatorname{rank} M(p)_j \leq \beta_{0,j}(L) - u$, *where $M(p)_j$ is a matrix of distinct variables of size $\beta_{0,j}(L) \times \beta_{1,j}(L)$. In particular*:

(1) *$V(L, j, \geq u)$ is irreducible; it coincides with the closure of $V(L, j, u)$ provided $V(L, j, u)$ is not empty—that is, provided $\beta_{0,j}(L) - \beta_{1,j}(L) \leq u \leq \beta_{0,j}(L)$.*

(2) *If* $\beta_{0,j}(L) - \beta_{1,j}(L) \leq u \leq \beta_{0,j}(L)$, *then the codimension of* $V(L, j, \geq u)$ *is* $(\beta_{1,j}(L) - \beta_{0,j}(L) + u)u$.

If I is an ideal with the same Hilbert function as the lex-segment L and if $\beta_{0,j}(I) = u$, then $\beta_{1,j}(L) - \beta_{0,j}(L) + u$ is exactly $\beta_{1,j}(I)$. Hence the formula for the codimension of $V(L, j, \geq u)$ can be written as $\beta_{1,j}(I)\beta_{0,j}(I)$.

As a result, we have the following statement.

COROLLARY 4.7. *Let L be a lex-segment ideal and let I be a homogeneous ideal with the Hilbert function of L. Set $b = \{\beta_{0,j}(I)\}$. Then the variety $V(L, \geq b)$ is irreducible; it is the closure of $V(L, b)$ and has codimension $\sum_j \beta_{1,j}(I)\beta_{0,j}(I)$.*

We conclude the paper with an example.

EXAMPLE 4.8. Let $L = (x^8, x^7y, x^6y^2, x^5y^4, x^4y^5, x^3y^6, x^2y^7, xy^9, y^{10})$. Then $V(L)$ is $\mathbf{A}^{22}$, whose parameterization is given via the following matrix $M(p)$.

$$\begin{array}{c|cccccccc}
 & 9 & 9 & 10 & 10 & 10 & 10 & 11 & 11 \\ \hline
8 & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & -x & y & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & 0 & -x & y^2 & 0 & 0 & 0 & 0 & 0 \\
9 & p_1 & p_5 & -x + yp_9 & y & 0 & 0 & 0 & 0 \\
9 & p_2 & p_6 & yp_{10} & -x & y & 0 & 0 & 0 \\
9 & p_3 & p_7 & yp_{11} & 0 & -x & y & 0 & 0 \\
9 & p_4 & p_8 & yp_{12} & 0 & 0 & -x & y^2 & 0 \\
10 & 0 & 0 & p_{13} & p_{15} & p_{17} & p_{19} & -x + yp_{21} & y \\
10 & 0 & 0 & p_{14} & p_{16} & p_{18} & p_{20} & yp_{22} & -x
\end{array}$$

The matrices whose ranks describe the Betti strata are

$$M(p)_9 = \begin{pmatrix} p_1 & p_5 \\ p_2 & p_6 \\ p_3 & p_7 \\ p_4 & p_8 \end{pmatrix} \quad \text{and} \quad M(p)_{10} = \begin{pmatrix} p_{13} & p_{15} & p_{17} & p_{19} \\ p_{14} & p_{16} & p_{18} & p_{20} \end{pmatrix}.$$

For instance, with $b = (b_j)$ defined by $b_9 = 3$, $b_{10} = 1$, and $b_j = \beta_{0,j}(L)$ for $j \neq 9, 10$, the Betti strata $V(L, \geq b)$ is described by rank $M(p)_9 \leq 1$ and rank $M(p)_{10} \leq 1$.

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A. Conca
Dipartimento di Matematica
Universitá di Genova
Via Dodecaneso 35
I-16146 Genova
Italy

conca@dima.unige.it

G. Valla
Dipartimento di Matematica
Universitá di Genova
Via Dodecaneso 35
I-16146 Genova
Italy

valla@dima.unige.it