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Duality and Tameness

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Introduction

The purpose of this paper is to construct examples of strange behavior of local cohomology. In these constructions we follow a strategy that was already used in [CH] and that relates, via a spectral sequence introduced in [HRa], the local cohomology for the two distinguished bigraded prime ideals in a standard bigraded algebra.

In the first part we consider algebras with rather general gradings and deduce a similar spectral sequence in this more general situation. A typical example of such an algebra is the Rees algebra of a graded ideal. The proof for the spectral sequence given here is simpler than that of the corresponding spectral sequence in [HRa].

In the second part of this paper we construct examples of standard graded rings A, which are algebras over a field K, such that the function

$$j \mapsto \dim_K(H^i_{A_+}(A)_{-j}) \tag{1}$$

is an interesting function for $j \gg 0$. In our examples, this dimension will be finite for all j.

Suppose that A_0 is a Noetherian local ring and that $A = \bigoplus_{j\ge 0} A_j$ is a standard graded ring, and set $A_+ := \bigoplus_{j>0} A_j$. Let M be a finitely generated graded A-module and let $\mathcal{F} := \tilde{M}$ be the sheafification of M on $Y = \operatorname{Proj}(A)$. We then have graded A-module isomorphisms

$$H^{i+1}_{A_+}(M) \cong \bigoplus_{n\in\mathbf{Z}} H^i(Y, \mathcal{F}(n))$$

for $i \ge 1$ as well as a similar expression for $i = 0$ and 1.

By Serre vanishing, $H^i_{A_+}(M)_j = 0$ for all i and $j \gg 0$. However, the asymptotic behavior of $H^i_{A_+}(M)_{-j}$ for $j \gg 0$ is much more mysterious.

In the case when $A_0 = K$ is a field, the function (1) is in fact a polynomial for large enough j. The proof is a consequence of graded local duality ([BrS, 13.4.6] or [BH, 3.6.19]) and follows also from Serre duality on a projective variety.

For more general A_0, the $H^i_{A_+}(M)_{-j}$ are finitely generated A_0 modules but need not have finite length.

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The following problem was proposed by Brodmann and Hellus [BrHe].

Tameness Problem. Are the local cohomology modules $H^i_{A_+}(M)$ tame? That is, is it true that either

$$\{H^i_{A_+}(M)_j \neq 0 \ \forall j \ll 0\} \quad \text{or} \quad \{H^i_{A+}(M)_j = 0 \ \forall j \ll 0\}?$$

The problem has a positive solution for A_0 of small dimension (see e.g. [Br; BrHe; L; RSe]).

Theorem 0.1 [BrHe]. *If* $\dim(A_0) \leq 2$, *then M is tame.*

However, it has recently been shown by two of the authors that tameness can fail if $\dim(A_0) = 3$.

Theorem 0.2 [CH]. *There are examples with* $\dim(A_0) = 3$ *where M is not tame.*

The statement of this example is reproduced in Theorem 3.1 of this paper. The function (1) is periodic for large j. Specifically, the function (1) is 2 for large even j and is 0 for large odd j.

In Theorem 3.3 we construct an example of failure of tameness of local cohomology that is not periodic and is not even a quasi-polynomial (in $-j$) for large j. Specifically, for $j > 0$ we have

$$\dim_K(H^2_{A_+}(A)_{-j}) = \begin{cases} 1 & \text{if } j \equiv 0 \text{ (modulo } p+1), \\ 1 & \text{if } j = p^t \text{ for some odd } t \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where the characteristic of K is p. We have $p^t \equiv -1$ (modulo $p+1$) for all odd $t \geq 0$.

We also give an example (Theorem 3.5) of failure of tameness where (1) is a quasi-polynomial with linear growth in even degree and is 0 in odd degree.

In Theorem 3.6 we give a tame example, but we have

$$\lim_{j\to\infty} \frac{\dim_K(H^2_{A_+}(A)_{-j})}{j^3} = 54\sqrt{2}$$

and so (1) is far from being a quasi-polynomial in $-j$ for large j.

Whereas the example of [CH] is for $M = \omega_A$ with ω_A the canonical module of A, the examples in this paper are all for $M = A$. This allows us to easily reinterpret our examples as Rees algebras in Section 4; thus we have examples of Rees algebras over local rings for which the preceding failure of tameness holds.

Finally, in Section 5 we give an analysis of the explicit and implicit roles of bigraded duality in the construction of the examples and, in addition, some comments on how it affects the geometry of the constructions.

1. Duality for Polynomial Rings in Two Sets of Variables

Let K be any commutative ring (with unit). In later applications K will be mostly a field. Furthermore, let $S = K[x_1, \dots, x_m, y_1, \dots, y_n]$, $P = (x_1, \dots, x_m)$, and $Q = (y_1, \dots, y_n)$.

The homology of the Čech complex $\mathcal{C}_P(\cdot)$ (resp. $\mathcal{C}_Q(\cdot)$) will be denoted by $H_P(\cdot)$ (resp. $H_Q(\cdot)$). Observe that, for any commutative ring K, this homology is the local cohomology supported in P (resp. Q) because P and Q are generated by regular sequences.

Assume that S is Γ-graded for some abelian group Γ and that $\deg(a) = 0$ for $a \in K$. If $x^s y^p \in R$, then $\deg(x^s y^p) = l(s) + l'(p)$ with $l(s) := \sum_i s_i \deg(x_i)$ and $l'(p) := \sum_j p_j \deg(y_j)$.

Definition 1.1. Let $I \subset S$ be a Γ-graded ideal. For every i and $\gamma \in \Gamma$, the Γ-grading of S is *I-sharp* if $H_I^i(S)_\gamma$ is a finitely generated K-module.

Lemma 1.2. *The following conditions are equivalent*:

(i) *the Γ-grading of S is P-sharp*;
(ii) *the Γ-grading of S is Q-sharp*;
(iii) *for all $\gamma \in \Gamma$, $|\{(\alpha, \beta) : \alpha \ge 0, \beta \ge 0, l(\alpha) = \gamma + l'(\beta)\}| < \infty$.*

Note that if K is Noetherian, M is a finitely generated Γ-graded S-module, and the Γ-grading of S is I-sharp, then $H_I^i(M)_\gamma$ is a finite K-module for every i and $\gamma \in \Gamma$. This follows from the converging Γ-graded spectral sequence $H_{p-q}(H_I^p(\mathbb{F})) \Rightarrow H_I^q(M)$, where $\mathbb{F}$ is a Γ-graded free S-resolution of M with $\mathbb{F}_i$ finite for every i.

We will assume from now on that the Γ-grading of S is P-sharp (equivalently, Q-sharp). Set $\sigma = \deg(x_1 \cdots x_m y_1 \cdots y_n)$, and for N a Γ-graded module let $N^\vee = \operatorname{Hom}_S(N, S(-\sigma))$ and $N^* = {}^*\operatorname{Hom}_K(N, K)$, where the Γ-grading of N^* is given by $(N^*)_\gamma = \operatorname{Hom}_K(N_{-\gamma}, K)$. More generally, we always denote the graded K-dual of a graded module N (over any graded K-algebra) by N^*. Finally, we denote by $\varphi_{\alpha\beta}$ the map $S(-a) \to S(-b)$ induced by multiplication by $x^\alpha y^\beta$, where $a = \deg x^\alpha$ and $b = -\deg y^\beta$.

Lemma 1.3. $H_P^m(\varphi_{\alpha\beta})_\gamma \cong H_Q^n((\varphi_{\alpha\beta}^\vee)_{-\gamma})^*$.

Proof. The free K-module $H_P^m(S)_\gamma$ is generated by the elements $x^{-s-1}y^p$ with $s, p \ge 0$ and $-l(s) - l(1) + l'(p) = \gamma$, and $H_Q^n(S)_{\gamma'}$ is generated by the elements $x^t y^{-q-1}$ with $t, q \ge 0$ and $l(t) - l'(q) - l'(1) = \gamma'$.

Let $d_\gamma \colon H_P^m(S)_\gamma \to (H_Q^n(S^\vee)^*)_\gamma = \operatorname{Hom}_K(H_Q^n(S)_{-\gamma-\sigma}, K)$ be the K-linear map defined by

$$d_\gamma(x^{-s-1}y^p)(x^t y^{-q-1}) = \begin{cases} 1 & \text{if } s = t \text{ and } p = q, \\ 0 & \text{otherwise.} \end{cases}$$

Then d_γ is an isomorphism (because the Γ-grading of S is Q-sharp) and there is a commutative diagram

$$\begin{array}{ccc} H^m_P(S)_{\gamma-a} & \xrightarrow{H^m_P(\varphi_{\alpha\beta})_\gamma} & H^m_P(S)_{\gamma-b} \\ {\scriptstyle d_{\gamma-a}}\downarrow & & \downarrow{\scriptstyle d_{\gamma-b}} \\ (H^n_Q(S)_{-\gamma+a-\sigma})^* & \xrightarrow{H^n_Q((\varphi^\vee_{\alpha\beta})_{-\gamma})^*} & (H^n_Q(S)_{-\gamma+b-\sigma})^*. \end{array}$$

The assertion follows. □

As an immediate consequence we obtain the following statement.

COROLLARY 1.4. (a) *Let $f \in S$ be an homogeneous element of degree $a - b$, and let $\varphi\colon S(-a) \to S(-b)$ be the graded degree-0 map induced by multiplication with f. Then*

$$H^m_P(\varphi) \simeq H^n_Q(\varphi^\vee)^*.$$

(b) *Let $\mathbb{F}$ be a Γ-graded complex of finitely generated free S-modules. Then*:

(i) $H^i_P(\mathbb{F}) = 0$ *for* $i \neq m$ *and* $H^j_Q(\mathbb{F}) = 0$ *for* $j \neq n$;
(ii) $H^m_P(\mathbb{F}) \simeq H^n_Q((\mathbb{F})^\vee)^*$.

The main result of this section is as follows.

THEOREM 1.5. *Assume that K is Noetherian, that the Γ-grading of S is P-sharp (equivalently, Q-sharp), and that M is a finitely generated Γ-graded S-module. Set $\omega_{S/K} := S(-\sigma)$, and let $\mathbb{F}$ be a minimal Γ-graded S-resolution of M. Then the following statements hold.*

(a) *For all i, there is a functorial isomorphism*

$$H^i_P(M) \simeq H_{m-i}(H^m_P(\mathbb{F})).$$

(b) *There is a convergent Γ-graded spectral sequence*

$$H^i_Q(\mathrm{Ext}^j_S(M, \omega_{S/K})) \Rightarrow H^{i+j-n}(H^m_P(\mathbb{F})^*).$$

In particular, if K is a field then there is a convergent Γ-graded spectral sequence

$$H^i_Q(\mathrm{Ext}^j_S(M, \omega_S)) \Rightarrow H_P^{\dim S-(i+j)}(M)^*.$$

Proof. Claim (a) is an immediate consequence of Corollary 1.4 via the Γ-graded spectral sequence $H_{p-i}(H^p_P(\mathbb{F})) \Rightarrow H^i_P(M)$. For (b), the two spectral sequences arising from the double complex $C^{\bullet\bullet} := \mathcal{C}_Q\mathbb{F}^\vee$ with $C^{ij} = \mathcal{C}^i_Q(\mathrm{Hom}_S(\mathbb{F}_j, S(-\sigma)))$ have as respective second terms $'E^{ij}_2 = H^i_Q(\mathrm{Ext}^j_S(M, \omega_{S/K}))$ and $''E^{ij}_2 = 0$ (for $i \neq n$) and $''E^{nj}_2 = H^j(H^n_Q(\mathbb{F}^\vee)) \simeq H^j(H^m_P(\mathbb{F})^*)$. If K is also a field, then $H^j(H^m_P(\mathbb{F})^*) \simeq (H_j(H^m_P(\mathbb{F})))^* \simeq H^{m-j}_P(M)^*$. □

COROLLARY 1.6. *Under the hypotheses of the theorem, if K is a field then, for any $\gamma \in \Gamma$, there exist convergent spectral sequences of finite-dimensional K-vector spaces*

$$H^i_Q(\mathrm{Ext}^j_S(M, \omega_S))_\gamma \Rightarrow H_P^{\dim S-(i+j)}(M)_{-\gamma},$$

$$H^i_P(\mathrm{Ext}^j_S(M, \omega_S))_\gamma \Rightarrow H_Q^{\dim S-(i+j)}(M)_{-\gamma}.$$

We now consider the special case where $\Gamma = \mathbb{Z}^2$ and $S := K[x_1, \dots, x_m, y_1, \dots, y_n]$ with $\deg(x_i) = (1, 0)$ and $\deg(y_j) = (d_j, 1)$ for $d_j \geq 0$. Set $T := K[x_1, \dots, x_m]$ and let M be a Γ-graded S-module. We view M as a $\mathbb{Z}$-graded module by defining $M_k = \bigoplus_j M_{(j,k)}$. Observe that each M_k is itself a graded T-module with $(M_k)_j = M_{(j,k)}$ for all j. We also note that $H^i_P(M)_k \cong H^i_{P_0}(M_k)$, as can been seen from the definition of local cohomology using the Čech complex. Here $P_0 = (x_1, \dots, x_m)$ is the graded maximal ideal of T.

Corollary 1.7. *With notation as before, let* $s := \dim S = m+n$ *and* $d := \dim M$.
(a) $H^0_P(\operatorname{Ext}^{s-d}_S(M, \omega_S)) \cong H^d_Q(M)^*$ *for any* k.
(b) *There is an exact sequence*

$$0 \to H^1_P(\operatorname{Ext}^{s-d}_S(M, \omega_S)) \to H^{d-1}_Q(M)^* \to H^0_P(\operatorname{Ext}^{s-d+1}_S(M, \omega_S)).$$

(c) *Let* $i \geq 2$. *If* $\operatorname{Ext}^j_S(M, \omega_S)$ *is annihilated by a power of* P *for all* $s-d < j < s-d+i$, *then there is an exact sequence*

$$\begin{aligned}\operatorname{Ext}^{s-d+i-1}_S(M, \omega_S) &\to H^i_P(\operatorname{Ext}^{s-d}_S(M, \omega_S)) \\ &\to H^{d-i}_Q(M)^* \to H^0_P(\operatorname{Ext}^{s-d+i}_S(M, \omega_S)).\end{aligned}$$

In particular, if $\operatorname{Ext}^j_S(M, \omega_S)$ *has finite length for all* $s-d < j \leq s-d+i_0$ *then, for some integer* i_0,

$$H^i_{P_0}(\operatorname{Ext}^{s-d}_S(M, \omega_S)_k) \cong (H^{d-i}_Q(M)_{-k})^* \quad \text{for all } i \leq i_0 \text{ and } k \gg 0.$$

Consequently, if M *is a generalized Cohen–Macaulay module* (*i.e.*, $\operatorname{Ext}^{s-i}_S(M, \omega_S)$ *has finite length for all* $i \neq d$) *and if we set* $N = \operatorname{Ext}^{s-d}_S(M, \omega_S)$, *then*

$$H^i_{P_0}(N_k) \cong (H^{d-i}_Q(M)_{-k})^* \quad \text{for all } i \text{ and all } k \gg 0.$$

Proof. (a), (b), and (c) are direct consequences of Corollary 1.6. For the application, notice that if $\gamma = (\ell, k) \in \Gamma$ with $k \gg 0$ then one has $\operatorname{Ext}^j_S(M, \omega_S)_\gamma = 0$ for all $s-d < j \leq s-d+i_0$. Therefore, for such γ, the desired conclusion follows. □

A typical example to which this situation applies is the Rees algebra of a graded ideal I in the standard graded polynomial ring $T = K[x_1, \dots, x_m]$. Suppose I is generated by the homogeneous polynomials $f_1, \dots, f_n$ with $\deg f_j = d_j$ for $j = 1, \dots, n$. Then the Rees algebra $\mathcal{R}(I) \subset T[t]$ is generated by the elements $f_j t$. If we set $\deg f_j t = (d_j, 1)$ for all j and $\deg x_i = (1, 0)$ for all i, then $\mathcal{R}(I)$ becomes a Γ-graded S-module via the K-algebra homomorphism $S \to \mathcal{R}(I)$ with $x_i \mapsto x_i$ and $y_j \mapsto f_j t$. According to this definition, we have $\mathcal{R}(I)_k = I^k$ for all k.

Since $\dim \mathcal{R}(I) = m+1$, the module $\omega_{\mathcal{R}(I)} = \operatorname{Ext}^{n-1}_S(\mathcal{R}(I), \omega_S)$ is the canonical module of $\mathcal{R}(I)$ (in the sense of [HK, 5. Vortrag]). Recall that if a ring R is a finite S-module of dimension $m+1$, then the natural finite map $R \to \operatorname{Hom}(\omega_R, \omega_R) \cong \operatorname{Ext}^{n-1}_S(\omega_R, \omega_S)$ is an isomorphism if and only if R is S_2. Together with Corollary1.7, this yields the following result.

COROLLARY 1.8. *Let $R := \mathcal{R}(I)$. Suppose that $R_{\mathfrak{p}}$ is Cohen–Macaulay for all $\mathfrak{p} \neq (\mathfrak{m}, R_+)$, where $\mathfrak{m} = (x_1, \dots, x_m)$ and $R_+ = \bigoplus_{k>0} I^k t^k$. Then*

$$H^i_{\mathfrak{m}}(I^k) \cong (H^{m+1-i}_{R_+}(\omega_R)_{-k})^* \quad \textit{for all } i \textit{ and all } k \gg 0.$$

Proof. Because ω_R localizes, the conditions imply that $(\omega_R)_{\mathfrak{p}}$ is Cohen–Macaulay for all $\mathfrak{p} \neq (\mathfrak{m}, R_+)$. Hence the natural "into" map $R \to R' := \operatorname{Ext}^{n-1}_S(\omega_R, \omega_S)$ has a cokernel of finite length. In particular, $R'_k = R_k = I^k$ for $k \gg 0$. Thus Corollary 1.7 applied to $M = \omega_R$ gives the desired conclusion. □

REMARK 1.9. Let $R := \mathcal{R}(I)$. If the cokernel of $R \to \operatorname{Hom}(\omega_R, \omega_R)$ is annihilated by a power of R_+ (in other words, if the blow-up is S_2 as a projective scheme over $\operatorname{Spec}(T)$), then $R'_k = I^k$ for $k \gg 0$ and one therefore has an exact sequence

$$\begin{aligned} 0 &\to H^0_{\mathfrak{m}}(T/I^k) \to (H^m_{R_+}(\omega_R)_{-k})^* \\ &\to H^0_{\mathfrak{m}}(\operatorname{Ext}^n_S(\omega_R, \omega_S)_k) \to H^1_{\mathfrak{m}}(T/I^k) \to (H^{m-1}_{R_+}(\omega_R)_{-k})^* \end{aligned}$$

for such a k.

2. A Method of Constructing Examples

Suppose that $R = \bigoplus_{i,j\geq 0} R_{ij}$ is a standard bigraded algebra over a ring $K = R_{00}$. Define $R^i = \bigoplus_{j\geq 0} R_{ij}$ and $R_j = \bigoplus_{i\geq 0} R_{ij}$. Define ideals $P = \bigoplus_{i>0} R^i$ and $Q = \bigoplus_{j>0} R_j$ in R. Suppose that $M = \bigoplus_{ij\in \mathbf{Z}} M_{ij}$ is a finitely generated, bigraded R-module. Define $M^i = \bigoplus_{j\in\mathbf{Z}} M_{ij}$ and $M_j = \bigoplus_{i\in\mathbf{Z}} M_{ij}$. Note that M^i is a graded R^0-module and M_j is a graded R_0-module. Let $Q_0 = R_{01}R^0$, so that $Q = Q_0 R$; let $P_0 = R_{10}R_0$, so that $P = R_{10}R$. Then we have K-module isomorphisms

$$H^l_Q(M)_{m,n} \cong H^l_{Q_0}(M^m)_n$$

for $m, n \in \mathbf{Z}$. Let $\widetilde{M^m}$ be the sheafification of the graded R^0-module M^m on $\operatorname{Proj}(R^0)$. Then we have K-module isomorphisms

$$H^l_{Q_0}(M^m)_n \cong H^{l-1}(\operatorname{Proj}(R^0), \widetilde{M^m}(n))$$

for $l \geq 2$ as well as exact sequences

$$\begin{aligned} 0 \to H^0_{Q_0}(M^m)_n &\to (R^m)_n \\ &= R_{m,n} \to H^0(\operatorname{Proj}(R^0), \widetilde{M^m}(n)) \to H^1_{Q_0}(M^m)_n \to 0. \end{aligned}$$

We have similar formulas for the calculation of $H^l_P(M)$.

Now assume that X is a projective scheme over K and that $\mathcal{F}_1$ and $\mathcal{F}_2$ are very ample line bundles on X. Let

$$R_{m,n} = \Gamma(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n}).$$

We require that $R = \bigoplus_{m,n\geq 0} R_{m,n}$ be a standard bigraded K-algebra. We have

$$X \cong \operatorname{Proj}(R_0) \cong \operatorname{Proj}(R^0).$$

The sheafification of the graded R^0-module R^m on X is $\widetilde{R^m} = \mathcal{F}_1^{\otimes m}$, and the sheafification of the graded R_0-module R_n on X is $\widetilde{R_n} \cong \mathcal{F}_2^{\otimes n}$ [Ha, Exer. II.5.9].

For $l \geq 2$ we have bigraded isomorphisms

$$H^l_Q(R) \cong \bigoplus_{m\geq 0,\, n\in\mathbf{Z}} H^l_{Q_0}(R^m)_n \cong \bigoplus_{m\geq 0,\, n\in\mathbf{Z}} H^{l-1}(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n}).$$

Viewing R as a graded R_0 algebra, we thus have graded isomorphisms

$$H^l_Q(R)_n \cong \bigoplus_{m\geq 0} H^{l-1}(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n}) \tag{2}$$

for $l \geq 2$ and $n \in \mathbf{Z}$. Let $d = \dim(R) = \dim(X) + 2$.

We now further assume that K is an algebraically closed field and that X is a nonsingular K-variety. Let

$$V = \mathbf{P}(\mathcal{F}_1 \oplus \mathcal{F}_2),$$

a projective space bundle over X with projection $\pi : V \to X$. Since $\mathcal{F}_1 \oplus \mathcal{F}_2$ is an ample bundle on X, it follows that $\mathcal{O}_V(1)$ is ample on V. Since

$$R \cong \bigoplus_{t\geq 0} \Gamma(V, \mathcal{O}_V(t))$$

with

$$\Gamma(V, \mathcal{O}_V(t)) \cong \Gamma(X, S^t(\mathcal{F}_1 \oplus \mathcal{F}_2)) \cong \bigoplus_{i+j=t} R_{ij}$$

and since R is generated in degree 1 with respect to this grading, it follows that $\mathcal{O}_V(1)$ is very ample on V and that R is the homogeneous coordinate ring of the nonsingular projective variety V. Hence R is generalized Cohen–Macaulay, since all local cohomology modules $H^i_{R_+}(R)$ of R with respect to the maximal bigraded ideal R_+ of R have finite length for $i < d$. Moreover, V is projectively normal by this embedding [Ha, Exer. II.5.14] and so R is normal.

3. Strange Behavior of Local Cohomology

In [CH] we constructed the following example of failure of tameness of local cohomology. In the example, R_0 has dimension 3, which is the lowest possible for failure of tameness [Br].

THEOREM 3.1. *Suppose that K is an algebraically closed field. Then there exist a normal standard graded K-algebra R_0 with* $\dim(R_0) = 3$ *and a normal standard graded R_0-algebra R with* $\dim(R) = 4$ *such that, for $j \gg 0$,*

$$\dim_K(H^2_Q(\omega_R)_{-j}) = \begin{cases} 2 & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd,} \end{cases}$$

where ω_R is the canonical module of R and $Q = \bigoplus_{n>0} R_n$.

We first show that this theorem is also true for the local cohomology of R.

THEOREM 3.2. *Suppose that K is an algebraically closed field. Then there exist a normal standard graded K-algebra R_0 with* $\dim(R_0) = 3$ *and a normal standard graded R_0-algebra R with* $\dim(R) = 4$ *such that, for $j > 0$,*

$$\dim_K(H_Q^2(R)_{-j}) = \begin{cases} 2 & \textit{if } j \textit{ is even,} \\ 0 & \textit{if } j \textit{ is odd,} \end{cases}$$

where $Q = \bigoplus_{n>0} R_n$.

Proof. We compute this directly for the R of Theorem 3.1 from (2) and the calculations of [CH]. Translating from the notation of this paper to the notation of [CH], we have $X = S$ is an Abelian surface, $\mathcal{F}_1 = \mathcal{O}_S(r_2 laH)$, and $\mathcal{F}_2 = \mathcal{O}_S(r_2(D + alH))$.

By (2), for $n \in \mathbf{N}$ we have

$$\begin{aligned} \dim_K(H_Q^2(R)_n) &= \sum_{m \geq 0} h^1(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n}) \\ &= \sum_{m \geq 0} h^1(S, \mathcal{O}_S((m+n)r_2 alH + nr_2 D)). \end{aligned}$$

Formula (1) of [CH] tells us that, for $m, n \in \mathbf{Z}$,

$$h^1(S, \mathcal{O}_S(mH + nD)) = \begin{cases} 2 & \text{if } m = 0 \text{ and } n \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Thus, for $n < 0$ we have

$$\dim_K(H_Q^2(R)_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

giving the conclusions of the theorem. □

The following example shows nonperiodic failure of tameness.

Theorem 3.3. *Suppose that* p *is a prime number such that* $p \equiv 2 \bmod 3$ *and* $p \geq 11$. *Then there exist a normal standard graded* K*-algebra* R_0 *over a field* K *of characteristic* p *with* $\dim(R_0) = 4$ *as well as a normal standard graded* R_0*-algebra* R *with* $\dim(R) = 5$ *such that, for* $j > 0$,

$$\dim_K(H_Q^2(R)_{-j}) = \begin{cases} 1 & \textit{if } j \equiv 0 \ (\textit{modulo } p+1), \\ 1 & \textit{if } j = p^t \textit{ for some odd } t \geq 0, \\ 0 & \textit{otherwise,} \end{cases}$$

where $Q = \bigoplus_{n>0} R_n$. *We have* $p^t \equiv -1$ $(\textit{modulo } p+1)$ *for all odd* $t \geq 0$.

To establish this result, we need the following simple lemma.

Lemma 3.4. *Let* C *be a nonsingular curve of genus* g *over an algebraically closed field* K, *and let* $\mathcal{M}, \mathcal{N}$ *be line bundles on* C. *If* $\deg(\mathcal{M}) \geq 2(2g+1)$ *and* $\deg(\mathcal{N}) \geq 2(2g+1)$, *then the natural map*

$$\Gamma(C, \mathcal{M}) \otimes \Gamma(C, \mathcal{N}) \to \Gamma(C, \mathcal{M} \otimes \mathcal{N})$$

is a surjection.

Proof. If $\mathcal{L}$ is a line bundle on C, then (a) $H^1(C, \mathcal{L}) = 0$ if $\deg(\mathcal{L}) > 2g - 2$ and (b) $\mathcal{L}$ is very ample if $\deg(\mathcal{L}) \geq 2g + 1$ [Ha, Chap. IV, Sec. 3].

Suppose that $\mathcal{L}$ is very ample and that $\mathcal{G}$ is another line bundle on C. If $\deg(\mathcal{G}) > 2g-2-\deg(\mathcal{L})$, then $\mathcal{G}$ is 2-regular for $\mathcal{L}$ [M1, Lec. 14]. Therefore, if $\deg(\mathcal{G}) > 2g-2+\deg(\mathcal{L})$, then

$$\Gamma(C,\mathcal{G}) \otimes \Gamma(C,\mathcal{L}) \to \Gamma(C,\mathcal{G}\otimes\mathcal{L})$$

is a surjection by Castelnuovo's proposition [M1, Lec. 14, p. 99].

We now apply these remarks to prove the lemma. Write $\mathcal{M} \cong \mathcal{A}^{\otimes q} \otimes \mathcal{B}$, where $\mathcal{A}$ is a line bundle such that $\deg(\mathcal{A}) = 2g+1 \le \deg(\mathcal{B}) < 2(2g+1)$. Observe that $\deg(\mathcal{N}) > 2g-2+\deg(\mathcal{A})$. Hence there exists a surjection

$$\Gamma(C,\mathcal{N}) \otimes \Gamma(C,\mathcal{A}) \to \Gamma(C,\mathcal{A}\otimes\mathcal{N}).$$

We iterate to obtain the surjections

$$\Gamma(C,\mathcal{A}^{\otimes i}\otimes\mathcal{N}) \otimes \Gamma(C,\mathcal{A}) \to \Gamma(C,\mathcal{A}^{\otimes(i+1)}\otimes\mathcal{N})$$

for $i \le q$ as well as the surjection

$$\Gamma(C,\mathcal{A}^{\otimes q}\otimes\mathcal{N}) \otimes \Gamma(C,\mathcal{B}) \to \Gamma(C,\mathcal{M}\otimes\mathcal{N}). \qquad \square$$

Proof of Theorem 3.3. For the construction, we start with an example from [CSr, Sec. 6]. There is an algebraically closed field K of characteristic p, a curve C of genus 2 over K, a point $q \in C$, and a line bundle $\mathcal{M}$ on C of degree 0 such that, for $n \ge 0$,

$$H^1(C,\mathcal{O}_C(q)\otimes\mathcal{M}^{\otimes n}) = \begin{cases} 1 & \text{if } n = p^t \text{ for some } t \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, $H^1(C,\mathcal{O}_C(2q)\otimes\mathcal{M}^{\otimes n}) = 0$ for all $n > 0$.

Let $a = p+1$. Let E be an elliptic curve over K and $T = E \times E$, with projections $\pi_i\colon T \to E$. Let $b \in E$ be a point and let $\mathcal{A} = \pi_1^*(\mathcal{O}_E(b)) \otimes \pi_2^*(\mathcal{O}_E(b))$. Let $X = T \times C$, with projections $\varphi_1\colon X \to T$ and $\varphi_2\colon X \to C$. Let $\mathcal{L} = \mathcal{O}_C(q)$. Let

$$\mathcal{F}_1 = \varphi_1^*(\mathcal{A})^{\otimes a} \otimes \varphi_2^*(\mathcal{L})^{\otimes a} \quad \text{and}$$
$$\mathcal{F}_2 = \varphi_1^*(\mathcal{A})^{\otimes(1+a)} \otimes \varphi_2^*(\mathcal{L}^{\otimes(1+a)} \otimes \mathcal{M}^{-1}).$$

For $m,n \ge 0$, we have the natural surjections

$$\begin{aligned}&\Gamma(X,\mathcal{F}_1)^{\otimes m} \otimes \Gamma(X,\mathcal{F}_2)^{\otimes n} \\ &\quad = \Gamma(T,\mathcal{A}^{\otimes a})^{\otimes m} \otimes \Gamma(T,\mathcal{A}^{\otimes(1+a)})^{\otimes n} \otimes \Gamma(C,\mathcal{L}^{a})^{\otimes m} \otimes \Gamma(C,\mathcal{L}^{\otimes(1+a)}\otimes\mathcal{M}^{-1})^{\otimes n} \\ &\quad \to \Gamma(T,\mathcal{A}^{\otimes(ma+n(1+a))}) \otimes \Gamma(C,\mathcal{L}^{\otimes(ma+n(1+a))}\otimes\mathcal{M}^{-\otimes n}) = \Gamma(X,\mathcal{F}_1^{\otimes m}\otimes\mathcal{F}_2^{\otimes n})\end{aligned} \tag{4}$$

by the Künneth formula (see [M1, Lec. 11, IV]) and Lemma 3.4.

Let $R_{m,n} = \Gamma(X,\mathcal{F}_1^{\otimes m}\otimes\mathcal{F}_2^{\otimes n})$. Since (by (4)) $R = \bigoplus_{m,n\ge 0} R_{m,n}$ is a standard bigraded K-algebra, it follows that (2) holds.

By the Riemann–Roch theorem we compute

$$h^0(C,\mathcal{L}^{\otimes r}\otimes\mathcal{M}^{-\otimes s}) = h^1(C,\mathcal{L}^{\otimes r}\otimes\mathcal{M}^{-\otimes s}) + r - 1, \tag{5}$$

and for $s < 0$ we obtain

$$h^1(C, \mathcal{L}^{\otimes r} \otimes \mathcal{M}^{-\otimes s}) = \begin{cases} 1-r & \text{if } r < 0, \\ 1 & \text{if } r = 0 \text{ and } s < 0, \\ 1 & \text{if } r = 1 \text{ and } s = -p^t \text{ for some } t \in \mathbf{N}, \\ 0 & \text{if } r = 1 \text{ and } s \neq -p^t \text{ for some } t \in \mathbf{N}, \\ 0 & \text{if } r = 2 \text{ and } s < 0, \\ 0 & \text{if } r \geq 3. \end{cases} \tag{6}$$

We also have

$$h^1(T, \mathcal{A}^{\otimes r}) = \begin{cases} 0 & \text{if } r \neq 0, \\ 2 & \text{if } r = 0, \end{cases} \tag{7}$$

and

$$h^0(T, \mathcal{A}^{\otimes r}) = \begin{cases} 0 & \text{if } r < 0, \\ 1 & \text{if } r = 0, \\ r^2 & \text{if } r > 0. \end{cases} \tag{8}$$

By (2), for $n \in \mathbf{Z}$ we have

$$\dim_K(H^2_Q(R)_n) = \sum_{m \geq 0} h^1(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n}).$$

By the Künneth formula,

$$\begin{aligned} &H^1(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n}) \\ &\quad \cong H^0(T, \mathcal{A}^{\otimes(ma+n(1+a))}) \otimes H^1(C, \mathcal{L}^{\otimes(ma+n(1+a))} \otimes \mathcal{M}^{-\otimes n}) \\ &\quad\quad \oplus H^1(T, \mathcal{A}^{\otimes(ma+n(1+a))}) \otimes H^0(C, \mathcal{L}^{\otimes(ma+n(1+a))} \otimes \mathcal{M}^{-\otimes n}). \end{aligned}$$

Thus, from (5)–(8) it follows that, for $j > 0$,

$$\dim_K(H^2_Q(R)_{-j}) = \begin{cases} 1 & \text{if } j \equiv 0 \bmod a, \\ 1 & \text{if } j = p^t \text{ for some odd } t \in \mathbf{N}, \\ 0 & \text{otherwise,} \end{cases}$$

which confirms the conclusions of Theorem 3.3. □

Theorem 3.5 gives an example of failure of tameness of local cohomology with larger growth.

Theorem 3.5. *Suppose that K is an algebraically closed field. Then there exist a normal standard graded K-algebra R_0 over K with* $\dim(R_0) = 4$ *and a normal standard graded R_0-algebra R with* $\dim(R) = 5$ *such that, for* $j > 0$,

$$\dim_K(H^3_Q(R)_{-j}) = \begin{cases} 6j & \textit{if } j \textit{ is even}, \\ 0 & \textit{if } j \textit{ is odd}, \end{cases}$$

where $Q = \bigoplus_{n>0} R_n$.

Proof. Let E be an elliptic curve over K and let $q \in E$ be a point. Let $\mathcal{L} = \mathcal{O}_E(3q)$. By [Ha, Prop. IV.4.6], $\mathcal{L}$ is very ample on E and

$$\bigoplus_{n \geq 0} \Gamma(E, \mathcal{L}^{\otimes n}) \tag{9}$$

is generated in degree 1 as a K-algebra. For $n \in \mathbf{N}$,

$$h^0(C, \mathcal{L}^{\otimes n}) = \begin{cases} 0 & \text{if } n < 0, \\ 1 & \text{if } n = 0, \\ 3n & \text{if } n > 0. \end{cases} \tag{10}$$

Moreover,

$$h^1(C, \mathcal{L}^{\otimes n}) = \begin{cases} -3n & \text{if } n < 0, \\ 1 & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases} \tag{11}$$

Let $X = E^3$ with the three canonical projections $\pi_i \colon X \to E$. Define

$$\mathcal{F}_1 = \pi_1^*(\mathcal{L}^{\otimes 2}) \otimes \pi_2^*(\mathcal{L}^{\otimes 2}) \otimes \pi_3^*(\mathcal{L}^{\otimes 2})$$

and

$$\mathcal{F}_2 = \pi_1^*(\mathcal{L}) \otimes \pi_2^*(\mathcal{L}) \otimes \pi_3^*(\mathcal{L}^{\otimes 2}).$$

Let

$$R_{m,n} = \Gamma(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n}),$$
$$R = \bigoplus_{m,n \geq 0} R_{m,n}.$$

By (9) and the Künneth formula, R is standard bigraded. Given (2) and that $\omega_X \cong \mathcal{O}_X$, by Serre duality we have

$$\dim_K(H_Q^3(R)_{-j}) = \sum_{m \geq 0} h^2(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{-\otimes j}) = \sum_{m \leq 0} h^1(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes j})$$

for $j \in \mathbf{Z}$.

Now (10), (11), and the Künneth formula yield, for $n > 0$,

$$h^1(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n}) = \begin{cases} 0 & \text{if } 2m + n \neq 0, \\ 2h^0(X, \mathcal{L}^{\otimes n}) & \text{if } 2m + n = 0. \end{cases}$$

Thus the conclusions of Theorem 3.5 hold. $\square$

The following theorem gives an example of tame but still rather strange local cohomology. Let $[x]$ denote the greatest integer in a real number x.

THEOREM 3.6. *Suppose that K is an algebraically closed field. Then there exist a normal standard graded K-algebra R_0 with* $\dim(R_0) = 3$ *and a normal standard graded R_0-algebra R with* $\dim(R) = 4$ *such that, for $j > 0$,*

$$\dim_K(H_Q^2(R)_{-j})$$
$$= 162\left(j^2\left(\left[\frac{j}{\sqrt{2}}\right] + \frac{1}{2}\right) - \frac{1}{3}\left[\frac{j}{\sqrt{2}}\right]\left(\left[\frac{j}{\sqrt{2}}\right] + 1\right)\left(2\left[\frac{j}{\sqrt{2}}\right] + 1\right)\right)$$

and

$$\lim_{j \to \infty} \frac{\dim_K(H_Q^2(R)_{-j})}{j^3} = 54\sqrt{2},$$

where $Q = \bigoplus_{n>0} R_n$.

Proof. We use the method of [C, Ex. 1.6]. Let E be an elliptic curve over an algebraically closed field K and let $p \in E$ be a point. Let $X = E \times E$ with projections $\pi_i : X \to E$. Let $C_1 = \pi_1^*(p)$ and $C_2 = \pi_2^*(p)$, and let

$$\Delta = \{(q,q) \mid q \in E\}$$

be the diagonal of X. We compute (as in [C]) that

$$(C_1^2) = (C_2^2) = (\Delta^2) = 0 \tag{12}$$

and

$$(\Delta \cdot C_1) = (\Delta \cdot C_2) = (C_1 \cdot C_2) = 1. \tag{13}$$

If $\mathcal{N}$ is an ample line bundle on X, then

$$H^i(X, \mathcal{N}) = 0 \quad \text{for } i > 0 \tag{14}$$

by the vanishing theorem of [M2, Sec. 16].

Suppose that $\mathcal{L}$ is a very ample line bundle on X and that $\mathcal{M}$ is a numerically effective (nef) line bundle. Then $\mathcal{M}$ is 3-regular for $\mathcal{L}$, so that

$$\Gamma(X, \mathcal{M} \otimes \mathcal{L}^{\otimes n}) \otimes \Gamma(X, \mathcal{L}) \to \Gamma(X, \mathcal{M} \otimes \mathcal{L}^{\otimes(n+1)})$$

is a surjection if $n \geq 3$. Since $C_1 + 2C_2$ is an ample divisor (by the Moishezon–Nakai criterion; see [Ha, Thm. V.1.10]), it follows by the Lefschetz theorem [M2, Sec. 17, Thm.] that $3(C_1 + 2C_2)$ is very ample. Let

$$\mathcal{F}_1 = \mathcal{O}_X(9(C_1 + 2C_2)).$$

Then $\mathcal{O}_X$ is 3-regular for $\mathcal{O}_X(3(C_1 + 2C_2))$, so we have surjections

$$\Gamma(X, \mathcal{F}_1^{\otimes n}) \otimes \Gamma(X, \mathcal{F}_1) \to \Gamma(X, \mathcal{F}_1^{\otimes(n+1)})$$

for all $n \geq 1$.

By the Moishezon–Nakai criterion, $\Delta + C_2$ is ample. Let $D = 3(\Delta + C_2)$. By the Lefschetz theorem, D is very ample and so $\mathcal{O}_X(D) \otimes \mathcal{F}_1$ is very ample. Let

$$\mathcal{F}_2 = \mathcal{O}_X(3D) \otimes \mathcal{F}_1^{\otimes 3}.$$

Because $\mathcal{O}_X$ is 3-regular for $\mathcal{O}_X(D) \otimes \mathcal{F}_1$, we have surjections

$$\Gamma(X, \mathcal{F}_2^{\otimes n}) \otimes \Gamma(X, \mathcal{F}_2) \to \Gamma(X, \mathcal{F}_2^{\otimes(n+1)})$$

for all $n \geq 1$.

For $n > 0$ and $m \geq 0$,

$$\mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n} \cong \mathcal{O}_X(3nD) \otimes \mathcal{F}_1^{\otimes(m+3n)}.$$

Since D is nef, it is 3-regular for $\mathcal{F}_1$ and we have a surjection for all $m \geq 0$ and $n > 0$:

$$\Gamma(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n}) \otimes \Gamma(X, \mathcal{F}_1) \to \Gamma(X, \mathcal{F}_1^{\otimes(m+1)} \otimes \mathcal{F}_2^{\otimes n}).$$

Let

$$R_{m,n} = \Gamma(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n}).$$

We have shown that $\bigoplus_{m,n \geq 0} R_{m,n}$ is a standard bigraded K-algebra. Thus (2) holds.

For $m, n \in \mathbf{Z}$, let $\mathcal{G} = \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n}$. By (14) and Serre duality ($\omega_X \cong \mathcal{O}_X$, since X is an abelian variety), we deduce (as in [C, Ex. 1.6]) that:

1. $(\mathcal{G}^2) > 0$ and $(\mathcal{G} \cdot \mathcal{F}_1) > 0$ imply that $\mathcal{G}$ is ample and that $h^1(X, \mathcal{G}) = h^2(X, \mathcal{G}) = 0$;
2. $(\mathcal{G}^2) < 0$ implies $h^0(X, \mathcal{G}) = h^2(X, \mathcal{G}) = 0$; and
3. $(\mathcal{G}^2) > 0$ and $(\mathcal{G} \cdot \mathcal{F}_1) < 0$ imply that $\mathcal{G}^{-1}$ is ample and that $h^0(X, \mathcal{G}) = h^1(X, \mathcal{G}) = 0$.

Let $\tau_2 = -4 - \sqrt{2}/2$ and $\tau_1 = -4 + \sqrt{2}/2$. Using (12) and (13), we compute

$$(\mathcal{F}_1^2) = 2 \cdot 162, \quad (\mathcal{F}_2)^2 = 31 \cdot 162, \quad (\mathcal{F}_1 \cdot \mathcal{F}_2) = 8 \cdot 162.$$

Then

$$\begin{aligned}(\mathcal{G}^2) &= 324(m^2 + 8mn + \tfrac{31}{2}n^2) \\ &= 324(m - \tau_1 n)(m - \tau_2 n)\end{aligned}$$

and

$$(\mathcal{G} \cdot \mathcal{F}_1) = 324(m + 4n).$$

Since $\tau_2 < -4 < \tau_1 < 0$, for $n < 0$ and $m \in \mathbf{Z}$ it follows that:

1. $m > \tau_2 n$ if and only if $\mathcal{G}^2 > 0$ and $\mathcal{G} \cdot \mathcal{F}_1 > 0$;
2. $\tau_1 n < m < \tau_2 n$ if and only if $(\mathcal{G}^2) < 0$; and
3. $m < \tau_1 n$ if and only if $(\mathcal{G}^2) > 0$ and $(\mathcal{G} \cdot \mathcal{F}_1) < 0$.

By the Riemann–Roch theorem for an abelian surface [M2, Sec. 16],

$$\chi(\mathcal{G}) = \frac{1}{2}(\mathcal{G}^2).$$

Thus, for $m \in \mathbf{Z}$ and $n < 0$,

$$h^1(X, \mathcal{G}) = \begin{cases} -\frac{1}{2}(\mathcal{G}^2) = -162(m^2 + 8mn + \frac{31}{2}n^2) & \text{if } \tau_1 n < m < \tau_2 n, \\ 0 & \text{otherwise.} \end{cases}$$

For $n \in \mathbf{Z}$, let $\sigma(n) = \dim_K(H_Q^2(R_n))$. By (2),

$$\sigma(n) = \sum_{m \geq 0} h^1(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n}).$$

For $n < 0$, we have

$$\sigma(n) = -162\Bigg(\sum_{\tau_1 n < m < \tau_2 n} \bigg(m^2 + 8mn + \frac{31}{2}n^2\bigg)\Bigg).$$

Setting $r = m + 4n$, we obtain

$$\begin{aligned}
&\sigma(n) \\
&= -162\Bigg(\sum_{\sqrt{2}/2n < r < -\sqrt{2}/2n} \bigg(r^2 - \frac{1}{2}n^2\bigg)\Bigg) \\
&= -324 \sum_{r=1}^{[-n/\sqrt{2}]} \bigg(r^2 - \frac{1}{2}n^2\bigg) + 81n^2 \\
&= -324\bigg(\frac{1}{6}\Big[-\frac{n}{\sqrt{2}}\Big]\Big(\Big[-\frac{n}{\sqrt{2}}\Big] + 1\Big)\Big(2\Big[-\frac{n}{\sqrt{2}}\Big] + 1\Big) - \frac{1}{2}n^2\Big[-\frac{n}{\sqrt{2}}\Big]\bigg) + 81n^2 \\
&= 162\bigg(n^2\Big(\Big[-\frac{n}{\sqrt{2}}\Big] + \frac{1}{2}\Big) - \frac{1}{3}\Big[-\frac{n}{\sqrt{2}}\Big]\Big(\Big[-\frac{n}{\sqrt{2}}\Big] + 1\Big)\Big(2\Big[-\frac{n}{\sqrt{2}}\Big] + 1\Big)\bigg).
\end{aligned}$$

We thus have the conclusions of the theorem. □

4. Strange Examples of Rees Algebras

Let notation and assumptions be as in Section 2. Since $\mathcal{F}_1$ is ample, there exists an $l > 0$ such that $\Gamma(X, \mathcal{F}_1^{\otimes l} \otimes \mathcal{F}_2^{-1}) \neq 0$. Thus we have an embedding $\mathcal{F}_2 \otimes \mathcal{F}_1^{-l} \subset \mathcal{O}_X$. Let $\mathcal{A} = \mathcal{F}_2 \otimes \mathcal{F}_1^{-l}$, which we have embedded as an ideal sheaf of X. For $j \geq 0$ and $i \geq jl$, let

$$T_{ij} = \Gamma(X, \mathcal{F}_1^{\otimes i} \otimes \mathcal{A}^{\otimes j}) = R_{i-jl,j}.$$

For $j \geq 0$, let $T_j = \bigoplus_{i \geq jl} T_{ij}$ and $T = \bigoplus_{j \geq 0} T_j$. Let $B = \bigoplus_{j>0} T_j$. Observe that $R \cong T$ as graded rings over $R_0 \cong T_0$, although they have different bigraded structures. Hence for all i, j we have

$$H_B^i(T)_j \cong H_Q^i(R)_j. \tag{15}$$

We know that T_1 is a homogeneous ideal of T_0 and that T is the Rees algebra of T_1. Thus all of the examples of Section 3 can be interpreted as Rees algebras over normal rings T_0 with isolated singularities.

We thus obtain the following theorems from Theorems 3.2–3.6. Theorems 4.1, 4.2, and 4.3 give examples of Rees algebras with nontame local cohomology.

THEOREM 4.1. *Suppose that K is an algebraically closed field. Then there exist a normal, standard graded K-algebra T_0 with* $\dim(T_0) = 3$ *and a graded ideal $A \subset T_0$ such that the Rees algebra $T = T_0[At]$ of A is normal and, for $j > 0$,*

$$\dim_K(H_B^2(T)_{-j}) = \begin{cases} 2 & \textit{if } j \textit{ is even,} \\ 0 & \textit{if } j \textit{ is odd,} \end{cases}$$

where B is the graded ideal AtT of T.

THEOREM 4.2. *Suppose that p is a prime number such that $p \equiv 2 \bmod 3$ and $p \geq 11$. Then there exist a normal standard graded K-algebra T_0 over a field K of characteristic p with* $\dim(T_0) = 4$ *as well as a graded ideal $A \subset T_0$ such that the Rees algebra $T = T_0[At]$ of A is normal and, for $j > 0$,*

$$\dim_K(H_Q^2(T)_{-j}) = \begin{cases} 1 & \textit{if } j \equiv 0 \ (\textit{modulo } p+1), \\ 1 & \textit{if } j = p^t \textit{ for some odd } t \geq 0, \\ 0 & \textit{otherwise,} \end{cases}$$

where B is the graded ideal AtT of T. Then $p^t \equiv -1$ (modulo $p+1$) for all odd $t \geq 0$.

THEOREM 4.3. *Suppose that K is an algebraically closed field. Then there exist a normal, standard graded K-algebra T_0 with* $\dim(T_0) = 4$ *and a graded ideal $A \subset T_0$ such that the Rees algebra $T = R_0[At]$ of A is normal and, for $j > 0$,*

$$\dim_K(H_B^3(T)_{-j}) = \begin{cases} 6j & \textit{if } j \textit{ is even,} \\ 0 & \textit{if } j \textit{ is odd,} \end{cases}$$

where B is the graded ideal AtT of T.

THEOREM 4.4. *Suppose that K is an algebraically closed field. Then there exist a normal standard graded K-algebra T_0 with* $\dim(T_0) = 3$ *and a graded ideal $A \subset T_0$ such that the Rees algebra $T = T_0[At]$ of A is normal and, for $j > 0$,*

$$\dim_K(H_B^2(T)_{-j}) = 162\left(j^2\left(\left[\frac{j}{\sqrt{2}}\right]+\frac{1}{2}\right)-\frac{1}{3}\left[\frac{j}{\sqrt{2}}\right]\left(\left[\frac{j}{\sqrt{2}}\right]+1\right)\left(2\left[\frac{j}{\sqrt{2}}\right]+1\right)\right)$$

and

$$\lim_{j\to\infty}\frac{\dim_K(H_B^2(T)_{-j})}{j^3} = 54\sqrt{2},$$

where B is the graded ideal AtT of T.

By localizing at the graded maximal ideal of T_0, we obtain examples of Rees algebras of local rings with strange local cohomology. In all of these examples, T_0 is generalized Cohen–Macaulay but is not Cohen–Macaulay. This follows because, in all of these examples,

$$H^2_{P_0}(R_0)_0 \cong H^1(X, \mathcal{O}_X) \neq 0.$$

5. Local Duality in the Examples

The example of [CH], giving failure of tameness of local cohomology, is stated in Theorem 3.1 of this paper. The proof of [CH] uses the bigraded local duality theorem of [HRa], which now follows from the much more general bigraded local duality theorem (Theorem 1.5 and Corollary 1.7 of this paper) to conclude that in our situation, where R is generalized Cohen–Macaulay,

$$(H_Q^{d-i}(\omega_R)_{-j})^* \cong H_P^i(R)_j \tag{16}$$

for $j \gg 0$.

In [CH], the formula

$$\begin{aligned} H_P^i(R)_j &\cong H_{P_0}^i(R_j) \\ &\cong \bigoplus_{m\in\mathbf{Z}} H^{i-1}(X, \widetilde{R_j}(m)) \\ &\cong \bigoplus_{m\in\mathbf{Z}} H^{i-1}(X, \mathcal{F}_1^{\otimes m}\otimes\mathcal{F}_2^{\otimes j}) \end{aligned} \tag{17}$$

for $i \geq 2$ and $j \geq 0$ is then used with formula (1) of [CH] ((3) of this paper) to prove Theorem 3.1.

In Section 2 we derived (2), from which we directly computed the local cohomology in the examples of this paper. We made essential use of Serre duality on X in computing the examples. In this section, we show how (16) can be obtained directly from the geometry of X and V and how this formula can be directly interpreted as Serre duality on X.

Let notation be as in Section 2, so that K is an algebraically closed field and $\mathcal{F}_1, \mathcal{F}_2$ are very ample line bundles on the nonsingular variety X. Let ω_R be the dualizing module of R, and let ω_X be the canonical bundle of X (which is a dualizing sheaf on X). For a K-module W, let $W' = \mathrm{Hom}_K(W, K)$.

Lemma 5.1. *We have that*

$$(\omega_R)_{ij} = \begin{cases} \Gamma(X, \mathcal{F}_1^{\otimes i} \otimes \mathcal{F}_2^{\otimes j} \otimes \omega_X) & \text{if } i \geq 1 \text{ and } j \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Set $(\omega_R)^i = \bigoplus_{j \in \mathbf{Z}} (\omega_R)_{i,j}$, *a graded* R^0 *module. Then the sheafification of* $(\omega_R)^i$ *on* X *is*

$$\widetilde{(\omega_R)^i} = \begin{cases} \mathcal{F}_1^{\otimes i} \otimes \omega_X & \text{if } i \geq 1, \\ 0 & \text{if } i \leq 0. \end{cases} \tag{18}$$

Set $(\omega_R)_j = \bigoplus_{i \in \mathbf{Z}} (\omega_R)_{i,j}$, *a graded* R_0 *module. Then the sheafification of* $(\omega_R)_j$ *on* X *is*

$$\widetilde{(\omega_R)_j} = \begin{cases} \mathcal{F}_2^{\otimes j} \otimes \omega_X & \text{if } j \geq 1, \\ 0 & \text{if } j \leq 0. \end{cases} \tag{19}$$

Proof. Give R the grading where the elements of degree e in R are $[R]_e = \sum_{i+j=e} R_{ij}$. We have realized R (with this grading) as the coordinate ring of the projective embedding of $V = \mathbf{P}(\mathcal{F}_1 \oplus \mathcal{F}_2)$ by the very ample divisor $\mathcal{O}_V(1)$ with projection $\pi : V \to X$.

Let ω_V be the canonical line bundle on V. We first calculate ω_V. Let f be a fiber of the map $\pi : V \to X$. By adjunction, we have that $(f \cdot \omega_V) = -2$. Since

$$\mathrm{Pic}(V) \cong \mathbf{Z}\mathcal{O}_V(1) \oplus \pi^*(\mathrm{Pic}(X)),$$

there exists a line bundle $\mathcal{G}$ on X such that

$$\omega_V \cong \mathcal{O}_V(-2) \otimes \pi^*(\mathcal{G}).$$

The natural split exact sequence

$$0 \to \mathcal{F}_2 \to \mathcal{F}_1 \oplus \mathcal{F}_2 \to \mathcal{F}_1 \to 0 \tag{20}$$

determines a section X_0 of X such that π_* of the exact sequence

$$0 \to \mathcal{O}_V(1) \otimes \mathcal{O}_V(-X_0) \to \mathcal{O}_V(1) \to \mathcal{O}_V(1) \otimes \mathcal{O}_{X_0} \to 0$$

is (20) [Ha, Prop. II.7.12]. Therefore,

$$\mathcal{O}_V(1) \otimes \mathcal{O}_V(-X_0) \cong \pi^*(\mathcal{F}_2)$$

and

$$\mathcal{O}_V(1) \otimes \mathcal{O}_{X_0} \cong \mathcal{F}_1.$$

By adjunction, we have that the canonical line bundle of X_0 is

$$\omega_{X_0} \cong \omega_V \otimes \mathcal{O}_V(X_0) \otimes \mathcal{O}_{X_0}.$$

Putting these expressions together, we see that

$$\mathcal{G} \cong \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \omega_X.$$

Hence

$$\omega_V \cong \mathcal{O}_V(-2) \otimes \pi^*(\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \omega_X).$$

We realize R as a bigraded quotient of a bigraded polynomial ring

$$S = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$$

with $\deg(x_i) = (1,0)$ for all i and $\deg(y_j) = (0,1)$ for all j. Viewing S as a graded K-algebra with the grading determined by $d(x_i) = d(y_j) = 1$ for all i, j, we have a projective embedding $V \subset \mathbf{P} = \mathrm{Proj}(S)$. Since V is nonsingular, we see from [Ha, Sec. III.7] that

$$\omega_V \cong \mathcal{E}xt^r_{\mathbf{P}}(\mathcal{O}_V, \mathcal{O}_{\mathbf{p}}(-e)),$$

where $e = m+n$ is the dimension of S and $r = e - \dim(R)$. Here ω_R is defined as

$$\omega_R = {}^*\mathrm{Ext}^r_S(R, S(-e)) \cong \bigoplus_{m \in \mathbf{Z}} \mathrm{Ext}^r_{\mathbf{P}}(\mathcal{O}_V, \mathcal{O}_{\mathbf{P}}(m-e)).$$

For $m \gg 0$,

$$\Gamma(\mathbf{P}, \mathcal{E}xt^r_{\mathbf{P}}(\mathcal{O}_V, \mathcal{O}_{\mathbf{p}}(m-e))) \cong \mathrm{Ext}^r_{\mathbf{P}}(\mathcal{O}_V, \mathcal{O}_{\mathbf{P}}(m-e))$$

by [Ha, Prop. III.6.9]. Thus ω_R and

$$\Gamma_*(\omega_V) = \bigoplus_{m \in \mathbf{Z}} \Gamma(V, \omega_V(m))$$

are isomorphic in high degree. Since both modules have depth ≥ 2 at the maximal bigraded ideal of R, we see that

$$\omega_R \cong \Gamma_*(\omega_V).$$

Thus

$$\begin{aligned}\omega_R &= \bigoplus_{m \in \mathbf{Z}} \Gamma(V, \omega_V(m)) \\ &= \bigoplus_{m \in \mathbf{Z}} \Gamma(V, \mathcal{O}_V(m-2) \otimes \pi^*(\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \omega_X)).\end{aligned}$$

Since a fiber f of π satisfies $(f \cdot \mathcal{O}_V(m-2) \otimes \pi^*(\mathcal{F}_1 \otimes \mathcal{F}_2)) < 0$ if $m < 2$, we see that (with this grading) $[\omega_R]_m = 0$ if $m < 2$ and, for $m \geq 2$,

$$\begin{aligned}[][\omega_R]_m &= \Gamma(X, S^{m-2}(\mathcal{F}_1 \oplus \mathcal{F}_2) \otimes \mathcal{F}_1 \otimes \mathcal{F}_2) \\ &= \bigoplus_{i+j=m-2} \Gamma(X, \mathcal{F}_1^{\otimes(i+1)} \otimes \mathcal{F}_2^{\otimes(j+1)} \otimes \omega_X).\end{aligned}$$

The conclusions of the lemma now follow. □

Suppose that $2 \leq i \leq d-2$. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ are ample and since $d-(i+1) > 0$, there exists a natural number n_0 such that

$$H^{d-(i+1)}(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{n} \otimes \omega_X) = 0 \tag{21}$$

for $n \geq n_0$ and all $m \geq 0$.

By (18), we have graded isomorphisms

$$H^i_Q(\omega_R)_n \cong \bigoplus_{m \geq 1} H^{i-1}(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n} \otimes \omega_X) \tag{22}$$

for $n \in \mathbf{Z}$.

By Serre duality,

$$H_Q^i(\omega_R)_n \cong \bigoplus_{m\ge 1}(H^{d-i-1}(X,\mathcal{F}_1^{-\otimes m}\otimes\mathcal{F}_2^{-\otimes n}))'. \tag{23}$$

By (21), there exists an n_0 such that

$$H_Q^i(\omega_R)_{-n} \cong \bigoplus_{m\in\mathbf{Z}}(H^{d-i-1}(X,\mathcal{F}_1^{-\otimes m}\otimes\mathcal{F}_2^{\otimes n}))' \tag{24}$$

for $n \ge n_0$.

Now apply to (24) the functor $L^* = \operatorname{Hom}_K(L,K)$ on graded R_0-modules with the grading

$$(L^*)_i = \operatorname{Hom}_K(L_{-i},K),$$

and compare with (17) to obtain

$$H_P^{d-i}(R)_n \cong (H_Q^i(\omega_R)_{-n})^* \tag{25}$$

for $n \ge n_0$, from which (16) immediately follows.

We can now use (22) and (3) to verify that Theorem 3.1 is, in fact, true for all $j > 0$.

We finally comment that an alternate proof of Theorem 3.2 for $j \gg 0$ is obtained from Theorem 3.1, formulas (2) and (22), the fact that X is an abelian variety (so that $\omega_X \cong \mathcal{O}_X$), and the observation that

$$h^1(X,\mathcal{F}_2^{-\otimes n}) = h^1(X,\mathcal{F}_2^{\otimes n}) = 0$$

for $n > 0$.

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