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Singularities of Moduli Spaces of Vector Bundles over Curves in Characteristic 0 and p

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Melvin Hochster has made enormous contributions to commutative algebra, not only through his own work but also through the influence he has had on his students, colleagues, and co-authors; this paper is dedicated to him on the occasion of his 65th birthday

1. Introduction

Let X be a nonsingular projective curve over an algebraically closed field k of any characteristic. One has J, the *Jacobian* of X. This parameterizes the isomorphism classes of line bundles of degree 0 on X. This was constructed classically, so now one can use geometric invariant theory. Similiarly, one also considers J^d, the Jacobian of line bundles of degree d, for any integer d. If L is a line bundle of degree d on X, then $M \to M \otimes L$ is an isomorphism from J to J^d; hence J and J^d are isomorphic, but not canonically.

Suppose that k is the field $\mathbb{C}$. Viewing X as a complex manifold, we may consider $\operatorname{Hom}(\pi_1(X), S^1)$. This is isomorphic to $(S^1)^{2g}$, which is also a complex torus of complex dimension g and isomorphic to J. For bundles of rank > 1 (over any field), put $\mu(V) = \deg V/\operatorname{rank} V$. Then we say that V is *stable* (resp. *semistable*) if, for all proper subbundles W of V,

$$\mu(W) < \mu(V) \quad (\text{resp. } \mu(W) \le \mu(V)).$$

Over $\mathbb{C}$, if degree $V = 0$ then one has the following classical result.

Theorem 1.1. *V is stable (resp., a direct sum of stable bundles of degree 0) if and only if $V \simeq V_\sigma$ for an irreducible (arbitrary) $\sigma : \pi_1(X) \to U(r)$, where $U(r)$ is the unitary group on r variables, $r = \operatorname{rank} V$. This σ is unique up to conjugation.*

Now let k be arbitrary, and let $S = (V)$ be the set of semistable bundles of rank r and degree 0 on X. The set S is bounded; that is, there exists an $m \gg 0$ such that, for all V in S, one has $H^1(V(m)) = 0$ and that $H^0(V(m))$ generates $V(m)$. Let $n = H^0(V(m))$ and $G = \mathrm{GL}(n)$. Let H be the Hilbert scheme of quotients of $\mathcal{O}_X^n$,

$$0 \to K \to \mathcal{O}_X^n \to F \to 0.$$

Let R^s (resp. R^{ss}) be the open, G-invariant subset of H consisting of all those quotients F of $\mathcal{O}_X^n$ such that:

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(1) F is stable (resp. semistable);
(2) the canonical map $H^0(\mathcal{O}_X^n) \to H^0(F)$ is an isomorphism; and
(3) the Hilbert polynomial of F equals the Hilbert polynomial of $V(m)$ for some V in S, say P.

One takes the *good* quotient of R^{ss} by G, denoted by $U(r,0)$ or simply U. This has an open subset, the *geometric quotient* of R^s by G, denoted by $U^s(r,0)$ or simply U^s.

The points of U^s are the isomorphism classes of stable bundles of rank r and degree 0, and the points of U are the set of isomorphism classes of polystable bundles of rank r and degree 0. Here, a polystable bundle is the direct sum of stable bundles of degree 0.

One knows that R^s (resp. R^{ss}) is nonsingular and irreducible. Therefore, U is a normal projective variety and U^s is nonsingular. Observe that $R^s \to U^s$ is a principal $P\mathrm{GL}(n)$ bundle and that U is obtained by taking the ring of invariants of R^{ss} under $G = \mathrm{GL}(n)$.

We have constructed $U(r, rm)$ (resp. $U^s(r, rm)$), the space of semistable (resp. stable) bundles of rank r and degree rm. But $U(r,0)$ is isomorphic to $U(r, rm)$ by $V \to V \otimes \mathcal{O}_X^n(m)$, and similiarly $U^s(r,0)$ is isomorphic to $U^s(r, rm)$.

In this paper we consider a family of smooth curves $X_W \to \operatorname{Spec} W$ of genus $g \geq 2$. We construct the family of moduli spaces $U_W \to \operatorname{Spec} W$ (cf. [9]) and consider their specializations from characteristic 0 to characteristic p. We also include a brief discussion of their singularities.

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2. Main Theorem

Let W be the ring of Witt vectors over k, an algebraically closed field of characteristic p, with quotient field K and residue field k. Let $X_W \to \operatorname{Spec} W$ be a smooth projective curve. Take all the quotients F of $\mathcal{O}_{X_W}^n$ such that:

(1) both F_K and F_k are semistable vector bundles;
(2) the Hilbert polynomials of F_K and F_k are equal to the Hilbert polynomial of $V(m)$ for $V \in S$; and
(3) the canonical map $H^0(\mathcal{O}_{X_W}^n) \to H^0(F(m))$ is an isomorphism.

Again define R_W^{ss} and R_W^s, where both are open and invariant under $\mathrm{GL}(n, W)$. Taking quotients yields U_W and U_W^s, which are the *relative* moduli spaces of semistable and stable (respectively) bundles over Spec W [9]. Observe that $R_W^s \to U_W^s$ is a $P\mathrm{GL}(W)$ fibration.

We consider the following questions.

- When is $U_W \otimes K \simeq U_K$?
- When is $U_W \otimes k \simeq U_k$?

The main result of this paper is as follows.

THEOREM 2.1. *$U_W \otimes K \simeq U_K$ and $U_W \otimes k \simeq U_k$.*

Proof. We begin by remarking that the theorem is trivial for U_W^s, since it is a quotient by a fibration.

Moreover, $(R_W^{ss}/\mathrm{GL}(n, W)) \otimes K \simeq (R_W^{ss} \otimes K)/\mathrm{GL}(n, K)$ and so the theorem holds for K, since invariants commute with flat base-change. Hence we outline a proof of Theorem 2.1 for k (i.e., we prove that $U_W \otimes k \simeq U_k$).

We know that $U_W/(p)$ is integral and that there exists a morphism $U_k \to U_W/(p)$, which is bijective on closed points. This morphism is also *birational,* since $R_W^s \to U_W^s$ is a fibration and hence base-changes correctly. Thus we must prove that $U_W/(p)$ is *normal.*

The closed points of U_k or $U_W/(p)$ are the polystable bundles in characteristic p. Let $V \simeq A^a \oplus B^b \oplus C^c \cdots$ be one such. Then $\mathrm{Aut}\, V \simeq \mathrm{GL}(a) \times \mathrm{GL}(b) \times \mathrm{GL}(c) \times \cdots$.

Over any field—in particular, in characteristic p—the local ring of this polystable bundle V on U_k is given by $H^1(\mathcal{E}nd\, V)/\mathrm{Aut}\, V$ (see [8]). Here $H^1(\mathcal{E}nd\, V)$ is the *local moduli space* of V, which we denote by $M(V)$. One can lift V to a polystable bundle $V_W \simeq A_W^a \oplus B_W^b \oplus C_W^c \cdots$ over X_W. Consider the action of $\mathrm{Aut}\, V_W$ on $H^1(\mathcal{E}nd\, V_W)$. We show that the invariants of $\mathrm{Aut}\, V_W$ on $H^1(\mathcal{E}nd\, V_W)$, reduced modulo p, are the invariants of $\mathrm{Aut}\, V$ on $H^1(\mathcal{E}nd\, V)$. Then $U_W/(p)$ will be normal.

We begin by recalling the notion of a good filtration. Let G be a reductive group in characteristic p, and let V be a rational G module. We say that V has a *good filtration,* or that V is a *good G* module [1; 4], if there exists a filtration

$$V_0 \subset V_1 \subset \cdots \subset V_m = V$$

such that each V_i/V_{i-1} is isomorphic to a *dual Weyl module*—that is, isomorphic to $H^0(G/B, L(\mu))$, an induced module from a 1-dimensional B-module μ, where B is a Borel subgroup of G. If V is infinite-dimensional, then V is good if there is an increasing filtration V_n of V such that $\bigcup V_n = V$ and each V_i/V_{i-1} is a dual Weyl module. Now let $H_W \to \mathrm{Spec}\, W$ be a reductive group scheme over $\mathrm{Spec}\, W$ and V_W a representation of H_W over W. Assume that, for all n, $S^n(V_k^*)$ is good over H_k. Then we have the following statement (cf. [1, Rem. 4.9]).

THEOREM 2.2. *$(S(V_W^*)/H_W)/(p) \simeq S(V_k^*)/H_k$. In other words, under the previous assumption of good filtrations, invariants in characteristic* 0 *specialize to invariants in characteristic p.*

This theorem holds because, for each n, $\dim S^n(V_k^*)^G =$ the number of times k occurs as V_i/V_{i-1} in a good filtration of $S^n(V_k^*)$ [1, 4.7(1), p. 508]. This yields our next theorem.

THEOREM 2.3. *As an* Aut V *module,* $S^n(M(V)^*)$ *has a good filtration for all* n.

Proof. Let V be polystable of rank r. Assume that V has s isotypical factors, $A_1, \dots, A_s$, with multiplicities $a_1, \dots, a_s$. Then $\operatorname{Aut} V \simeq \mathrm{GL}(a_1) \times \mathrm{GL}(a_2) \times \cdots \times \mathrm{GL}(a_s)$. If $M_{p,q}$ denotes the space of $p \times q$ matrices, then the action of $\operatorname{Aut} V$ on $H^1(\mathcal{E}nd\, V)$ is isomorphic to the action of $\operatorname{Aut} V$ on $\bigoplus M_{a_i,a_j}$ for $1 \le i, j \le s$. This holds because, for stable bundles V and W of degree 0 on X, $\operatorname{Hom}_{\mathcal{O}_X}(V, W)$ equals k if V is isomorphic to W and equals 0 otherwise. Here, if $k \notin (i, j)$ then $\mathrm{GL}(a_k)$ acts trivially on M_{a_i,a_j}. That $H^1(\mathcal{E}nd\, V)$ has a good filtration as an $\operatorname{Aut} V$ module now follows from [4, Ex. 2, p. 621] and also from [1, 4.3(2), p. 504]. □

As a result, $U_W/(p)$ is normal because it is (locally) a ring of invariants of a smooth variety. Because $U_W/(p)$ is normal, the map $U_k \to U_W/(p)$ is an isomorphism. This concludes the proof of Theorem 2.1. □

One may also consider $SU(r)$, the variety of semistable bundles with trivial determinant. For V as defined previously, the local moduli space of V is $H^1(\mathcal{E}nd^0\, V)$, the bundle of trace-0 endomorphisms of V. Denoting this bundle by $M^0(V)$, we have that $S^n(M^0(V)^*)$ has a good filtration over $\operatorname{Aut} V$ for all n. This is seen as follows.

With the notation of Theorem 2.3, let $V \in SU(r)$. Then $\operatorname{Aut} V$ is $\mathrm{GL}(a_1) \times \cdots \times \mathrm{GL}(a_s)$. But $H^1(\mathcal{E}nd^0\, V) \simeq A \oplus M_{a_i,a_j}$ for $1 \le i \ne j \le s$, where A is given by

$$0 \to A \to \sum_{i=1}^{s} M(a_i) \to k \to 0.$$

Here $M(n)$ denotes the $n \times n$ matrix algebra, and the right arrow is

$$(X_1, X_2, \dots, X_s) \to \sum \operatorname{Tr}(X_i).$$

Now $S^\bullet(A^*)$ has a good filtration by [4, Lemma 3(1), p. 619] and [1, 4.3(2), p. 504]. Moreover, $S^\bullet\big(\big(\bigoplus M_{a_i,a_j},\ 1 \le i \ne j \le s\big)^*\big)$ has a good filtration by [4] and [1] again. So after constructing SU_W over $\operatorname{Spec} W$, one has that SU_W specializes to SU_k.

If A is a localization of a finitely generated k algebra with characteristic $k = p$, then we define A to be *strongly F-regular*: for any $c \in A$, there is a t and a map $R\colon A \to A$ such that $R(a+b) = R(a) + R(b)$, $R(x^{p^t} y) = xR(y)$, and $R(c) = 1$ [4; 5; 8]. This implies that A is Cohen–Macaulay, normal, F-split (or F-pure), and F-rational. This notion and result is due to Hochster and Huneke [5] and Smith [10]. The following theorem is due to Hashimoto [4].

THEOREM 2.4. *Let a reductive group* G *act on* V *in characteristic* p, *and assume that all the* $S^n(V^*)$ *have a good filtration. Then the ring of invariants* $S^\bullet(V^*)^G$ *is strongly F-regular.*

This implies that, in characteristic p, the moduli spaces U_k and SU_k are strongly F-regular. Instead of working over W, one can work over a Dedekind ring, say R.

If $X_R \to \operatorname{Spec} R$ is a smooth, projective relative curve then $U_R \to \operatorname{Spec} R$ and $SU_R \to \operatorname{Spec} R$, where both (by our previous results) specialize "correctly".

We need the following lemma.

LEMMA 2.5. *Both $U(r)$ and $SU(r)$ are Gorenstein in characteristic p.*

Proof. Let t be the worst point in $U(r)$—that is, the point corresponding to the trivial bundle T. Then, by looking at the action of $\operatorname{Aut} T$ on $H^1(\mathcal{E}nd\, T)$, we see that the local ring at t is Gorenstein as follows. Consider the action of $\operatorname{Aut} T$ on $H^1(\mathcal{E}nd\, T)$. We have $\operatorname{Aut} T \simeq \mathrm{GL}(r)$ and $H^1(\mathcal{E}nd\, T) \simeq \bigoplus_{i=1}^{g} M(r)$, where $M(r)$ is the space of $r \times r$ matrices. Denote $H^1(\mathcal{E}nd\, T)$ by W and $\operatorname{Aut} T$ by H. Then A, the completion of the local ring of $H \setminus W$ at the origin, is isomorphic to the completion of the local ring of $U(r)$ at t; we call this ring B. Observe that $H \setminus W$ is *factorial* because the representation of H on W factors through H/C, where C is the center of H. It is also Cohen–Macaulay, since $S^{\bullet}(M(r))$ has a good filtration as a $\mathrm{GL}(r)$ module [4, Ex.2, p. 621; 1, 4.3(2), p. 504]. Hence $H \setminus W$ is Gorenstein, and so is its localization at the origin. Therefore A is also Gorenstein and so is B. Thus the local ring of $U(r)$ at t is also Gorenstein. Here the main point is that the property of being factorial is not preserved by completions, in general, but the property of being Gorenstein is preserved.

Now let V and V_1 be two polystables bundles, $V \simeq A^a + B^b + \cdots$ and $V_1 \simeq A_1^a + B_1^b + \cdots$, with $\operatorname{rank} A = \operatorname{rank} A_1^a, \ldots$. Then $\operatorname{Aut} V$ is isomorphic to $\operatorname{Aut} V_1$, and the representations on $H^1(\mathcal{E}nd\, V)$ and $H^1(\mathcal{E}nd\, V_1)$ are isomorphic. The irreducibility of $U(r)$ now gives the result. An identical proof works for $SU(r)$ once we replace M_r with M_r^0. □

We next recall the definition of terminal and canonical singularities for normal Gorenstein varieties. Let Y be a normal Gorenstein variety in characteristic 0, and let $\pi : X \to Y$ be a resolution of singularities; in other words, X is nonsingular and π is an isomorphism over the smooth locus of Y. Then

$$\pi^* K_X \simeq \pi^* K_Y + \Sigma a_i E_i,$$

where E_i denotes the exceptional divisors. We say that Y has at most *canonical* (resp. *terminal*) singularities [3] if all the $a_i \geq 0$ (resp. $a_i > 0$).

Let X be a normal Gorenstein variety in characteristic 0. Then X has a model $X_R \to \operatorname{Spec} R$, where R is a finitely generated Z-algebra of characteristic 0 and quotient field K. Reduce X_R modulo p (denoted by X/p) for all $p \gg 0$. The following result is due to Hara and Watanabe [3].

THEOREM 2.6. *X_K has at most canonical singularities if and only if X/p, for all $p \gg 0$, is strongly F-regular.*

However, both U_k and SU_k are strongly F-regular in characteristic p. Hence both U_K and SU_K have at most canonical singularities in characteristic 0.

One may also consider $U(r,d)$, the moduli space of semistable bundles of rank r and degree d, and $SU(r,L)$, the space of semistable bundles of rank r

and determinant isomorphic to a fixed line bundle L. By considering the action of the automorphism group on the local moduli spaces, we can prove as before that:

(1) they are Gorenstein in every characteristic; and
(2) these varieties specialize correctly from characteristic 0 to characteristic p, in characteristic p they are strongly F-regular, and in characteristic 0 they have at most canonical singularities.

REMARK 2.7. In characteristic 0, Drezet–Narasimhan [2] have proved that both $U(r)$ and $SU(r)$ are locally factorial (and, of course, have rational singularities) and hence Gorenstein. This also proves that these varieties have canonical singularities.

REMARK 2.8. This work was completed in 2002. Soon after, Lakshmibai and Shukla [7] used the standard monomial theory to give another proof that $U(r)$ is Cohen–Macaulay.

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