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Asymptotic Expansion of the Heat Kernel for Orbifolds

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1. Introduction

Roughly speaking, a topological *orbifold* is a space locally homeomorphic to an orbit space of a finite group action on $\mathbb{R}^n$. A smooth orbifold consists of a Hausdorff second countable topological space together with an atlas of coordinate charts realizing such local homeomorphisms and satisfying compatibility conditions (see Section 2). Orbifolds were introduced by Satake and then studied by Thurston because of their utility in the investigation of 3-manifolds (e.g., a Seifert fibred 3-manifold is naturally a generalized circle bundle over a 2-orbifold); today, orbifolds arise naturally in diverse branches of mathematics and physics, including symplectic geometry, string theory, and vertex operator algebras.

We will be interested in orbifolds from a spectral-theoretic point of view. An orbifold endowed with a metric structure is a *Riemannian orbifold*. As in the manifold case, associated with every Riemannian metric is a Laplace operator acting on smooth functions on the orbifold. In the case of closed orbifolds, the Laplacian has a discrete spectrum. We study the relationship between the geometry and the Laplace spectrum of a closed orbifold via its heat kernel; as in the manifold case, the time-zero asymptotic expansion of the heat kernel furnishes geometric information about the orbifold.

Orbifolds began appearing sporadically in the spectral theory literature in the early 1990s and have received more concentrated attention in the last five years. Farsi [15] showed that the spectrum of an orbifold determines its volume by proving that Weyl's asymptotic formula holds for orbifolds. Dryden and Strohmaier [13] showed that, for a compact and negatively curved two-dimensional orbifold, the Laplace spectrum determines both the length spectrum and the orders of the singular points and vice versa; on the other hand, Doyle and Rossetti [12] gave (disconnected) examples of isospectral flat two-dimensional orbifolds with different length spectra and orders of singular points. Further investigations of the relationship between the lengths of closed geodesics and the spectrum were carried out by Stanhope and Uribe in [29]. It is natural to ask about the singularities that can appear in an isospectral family of orbifolds. Stanhope [28] showed that,

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in general, there can be at most finitely many isotropy types (up to isomorphism) in a set of isospectral Riemannian orbifolds that share a uniform lower bound on Ricci curvature. On the other hand, Shams, Stanhope, and Webb [27] constructed arbitrarily large (finite) isospectral sets of orbifolds that satisfy this curvature condition and whose isotropy types differ. Rossetti, Schueth, and Weilandt [25] recently constructed a pair of isospectral Riemannian orbifolds whose isotropy types have different orders.

For Riemannian manifolds, the asymptotic expansion of the heat kernel can be used to relate the geometry of the manifold to its spectrum. From the so-called *heat invariants* appearing in the asymptotic expansion, one can tell the dimension, the volume, and various quantities involving the curvature of the manifold. The heat kernel has been studied in various analogous or more general settings [4; 5; 6; 11; 17; 24].

In the case of a *good* Riemannian orbifold (i.e., an orbifold arising as the orbit space of a manifold under the action of a discrete group of isometries), Donnelly [10] proved the existence of the heat kernel and constructed the asymptotic expansion for the heat trace. We extend Donnelly's work to the case of general compact orbifolds. Moreover, in both the good case and the general case, we express the heat invariants in a form that clarifies the asymptotic contribution of each part of the singular set of the orbifold.

We calculate several terms in the asymptotic expansion explicitly in the case of two-dimensional orbifolds; we then use these terms to prove that the spectrum distinguishes elements within various classes of two-dimensional orbifolds. In particular, within the class of all two-dimensional orbifolds with nonnegative Euler characteristic, the spectrum is a complete topological invariant. Additional results are obtained for triangular pillow orbifolds endowed with a hyperbolic structure and for nonorientable two-dimensional orbifolds.

The paper is organized as follows. In Section 2 we give the background necessary for the rest of the paper, recalling several results that clarify the structure of the singular locus of an orbifold. Section 3 is devoted to the construction of the heat kernel on an arbitrary closed Riemannian orbifold by means of the construction of a parametrix. The existence of the heat kernel for closed orbifolds was shown previously by Chiang [8]; existence also follows from more general results for the heat kernel on Riemannian foliations (see [24]). However, we give a different construction in order to express the heat kernel in a convenient form that will allow us, in Section 4, to generalize Donnelly's asymptotic expansion. Section 5 is devoted to various applications of the heat expansion, including those mentioned in the previous paragraph.

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2. Orbifolds and Their Singular Sets

2.1. Definition. (i) An *orbifold chart* on a topological space X consists of a connected open subset $\tilde{U}$ of $\mathbb{R}^n$, a finite group G_U acting on $\tilde{U}$ by diffeomorphisms, and a mapping π_U from $\tilde{U}$ onto an open subset U of X inducing a homeomorphism from the orbit space $G_U\backslash\tilde{U}$ onto U. We will always assume that the group G_U acts effectively on $\tilde{U}$.

(ii) An *embedding* $\lambda\colon (\tilde{U}, G_U, \pi_U) \to (\tilde{U}', G_{U'}, \pi_{U'})$ between orbifold charts with $U \subseteq U'$ is a smooth embedding $\lambda\colon \tilde{U} \to \tilde{U}'$ such that the diagram

$$\begin{tikzcd}
\tilde{U} \arrow[r, tail, "\lambda"] \arrow[d, two heads, "\pi_U"'] & \tilde{U}' \arrow[d, two heads, "\pi_{U'}"] \\
G_U\backslash\tilde{U} \arrow[d, "\sim"'] & G_{U'}\backslash\tilde{U}' \arrow[d, "\sim"] \\
U \arrow[r, hook] & U'
\end{tikzcd}$$

commutes. Two charts $(\tilde{U}, G_U, \pi_U)$ and $(\tilde{U}', G_{U'}, \pi_{U'})$ on X are said to be *compatible* if, for each point $x \in U \cap U'$, there exists an orbifold chart $(\tilde{U}'', G_{U''}, \pi_{U''})$ with $U'' \subset U \cap U'$ and smooth embeddings $(\tilde{U}'', G_{U''}, \pi_{U''}) \to (\tilde{U}, G_U, \pi_U)$ and $(\tilde{U}'', G_{U''}, \pi_{U''}) \to (\tilde{U}', G_{U'}, \pi_{U'})$.

(iii) An n-dimensional *orbifold atlas* $\mathcal{A}$ on X is a compatible family of n-dimensional orbifold charts whose images form a covering of X. A *refinement* $\mathcal{A}'$ of an orbifold atlas $\mathcal{A}$ is an orbifold atlas each of whose charts embeds into a chart of $\mathcal{A}$. Two orbifold atlases are said to be *equivalent* if they have a common refinement. Every orbifold atlas is equivalent to a unique maximal one. An *orbifold* is a Hausdorff second-countable topological space together with a maximal orbifold atlas.

(iv) Let $\mathcal{O}$ be an orbifold. A point x of $\mathcal{O}$ is said to be *singular* if, for some (hence every) orbifold chart $(\tilde{U}, G_U, \pi_U)$ about x, the points in the inverse image of x in $\tilde{U}$ have nontrivial isotropy in G_U. The isomorphism class of the isotropy group, called the *abstract isotropy type* of x, is independent both of the choice of point in the inverse image of x in $\tilde{U}$ and of the choice of chart $(\tilde{U}, G_U, \pi_U)$ about x. Points that are not singular are called *regular.*

2.2. Remarks. (i) The notion of orbifold generalizes slightly the notion of V-manifold introduced by Satake [26]; V-manifolds are orbifolds for which the singular set has codimension at least 2.

(ii) Given an embedding $\lambda\colon (\tilde{U}, G_U, \pi_U) \to (\tilde{U}', G_{U'}, \pi_{U'})$ as in Definition 2.1, there exists a homomorphism $\tau\colon G_U \to G_{U'}$ such that $\lambda \circ \gamma = \tau(\gamma) \circ \lambda$ for all

$\gamma \in G_U$. This was proven by Satake for V-manifolds and was generalized to orbifolds by Moerdijk and Pronk [20]. From our convention that the group actions on each chart are effective, it follows that the homomorphism τ is injective.

(iii) There are some subtle differences among the definitions of *orbifold* in the literature; in particular, some authors do not require that the finite group actions in the orbifold charts be effective. The definition we use is that in [20]. Although these distinctions become significant when formulating the correct notion of the category of smooth orbifolds, they play no role in our computations.

An orbifold is said to be *good* if it is the orbit space of a manifold under the smooth action of a discrete group; otherwise, it is said to be *bad*. In particular, every point in an orbifold has a neighborhood that is a good orbifold. Even a bad orbifold can be expressed globally as the quotient of a manifold by a group action, although not a discrete group action. This is done by introducing a Riemannian structure and constructing the "bundle" of orthonormal frames. The orthonormal frame bundle is actually an *orbibundle,* the appropriate notion of vector bundle over an orbifold as described in [19] or [30]. However, its total space is a smooth manifold with an action of the orthogonal group, and one recovers the original orbifold as the orbit space of this orthogonal action. We shall review this construction after first establishing some notation in the setting of arbitrary group actions.

2.3. Definition (cf. [14, Chap. 2]). (i) Consider a smooth proper action of a Lie group H on a smooth manifold M. For $\tilde{x} \in M$, let $\mathrm{Iso}_H(\tilde{x})$ denote the subgroup of H that fixes $\tilde{x}$. Define an equivalence relation on M by $\tilde{x} \equiv \tilde{y}$ if $\mathrm{Iso}_H(\tilde{x})$ and $\mathrm{Iso}_H(\tilde{y})$ are conjugate. Each equivalence class is called an *H-orbit type*. Note that the equivalence classes are invariant under the action of H.

We will say that the H-orbit type of $\tilde{x}$ *dominates* that of $\tilde{y}$ if $\mathrm{Iso}_H(\tilde{x})$ is conjugate to a subgroup of $\mathrm{Iso}_H(\tilde{y})$.

(ii) Let $\pi : M \to H\backslash M$ be the projection onto the orbit space. Let $p \in H\backslash M$. As $\tilde{p}$ ranges over the H-orbit $\pi^{-1}(p)$ in M, the stabilizer $\mathrm{Iso}_H(\tilde{p})$ ranges over a conjugacy class of subgroups of H. We will denote this conjugacy class of subgroups by $\mathrm{Iso}_H(p)$ and refer to it as the *H-isotropy type* of p. Define an equivalence relation on $H\backslash M$ by $p \equiv q$ if $\mathrm{Iso}_H(p) = \mathrm{Iso}_H(q)$. The equivalence classes will be called *H-isotropy equivalence classes.* Note that π carries points of the same H-orbit type in M to points of the same H-isotropy equivalence class in $H\backslash M$.

We will say that the H-isotropy equivalence class of p *dominates* that of q if the groups making up the conjugacy class $\mathrm{Iso}_H(p)$ are conjugate to subgroups of those in $\mathrm{Iso}_H(q)$.

By an abuse of notation, we will write $|\mathrm{Iso}_H(p)|$ to mean the order of each of the groups making up the isotropy type $\mathrm{Iso}_H(p)$. We will refer to this quantity as the order of the H-isotropy at p.

2.4. Definition. (i) A *Riemannian structure* on an orbifold $\mathcal{O}$ is an assignment to each orbifold chart $(\tilde{U}, G_U, \pi_U)$ of a G_U-invariant Riemannian metric $g_{\tilde{U}}$ on $\tilde{U}$

satisfying the compatibility condition that each embedding λ appearing in Definition 2.1 be isometric. Every orbifold admits Riemannian structures.

(ii) We will say that an orbifold chart $(\tilde{U}, G_U, \pi_U)$ on a Riemannian orbifold $\mathcal{O}$ is a *distinguished* chart of radius r if $\tilde{U}$ is a convex geodesic ball of radius r. In this case, U is a convex geodesic ball in $\mathcal{O}$. The entire group G_U fixes the center $\tilde{p}$ of $\tilde{U}$, so the abstract isotropy type of $p := \pi_U(\tilde{p})$ is represented by G_U.

2.5. REMARK. Recall that, for a Riemannian manifold M and a point $p \in M$, the *convexity radius at* p is the largest positive real number $r(p)$ for which the geodesic ball of radius ε about p is geodesically convex for all $\varepsilon < r(p)$. If M is compact, the infimum r of $\{r(p) : p \in M\}$ is positive and is called the *convexity radius* of M. For a point p in an orbifold $\mathcal{O}$, we may define the convexity radius at p to be the largest real number $r(p)$ such that $\mathcal{O}$ admits a distinguished chart of radius ε centered at p for all $\varepsilon < r(p)$. It is immediate that $r(p)$ is positive. Moreover, if $\mathcal{O}$ is compact, then an elementary argument shows that the infimum r of $\{r(p) : p \in \mathcal{O}\}$ is positive; r is called the *convexity radius* of $\mathcal{O}$.

2.6. ORTHONORMAL FRAME BUNDLE. We give a brief description of the orthonormal frame bundle of a Riemannian orbifold. See [1] for more details. First consider a good Riemannian orbifold $\mathcal{O} = G\backslash M$, where M is a Riemannian manifold and G is a discrete group acting by isometries on M. Let $F(M) \to M$ be the orthonormal frame bundle of M. Each element $\gamma \in G$, as an isometry of M, induces a diffeomorphism γ_* of $F(M)$ carrying fibers to fibers; thus we obtain an action of G on $F(M)$ covering the action of G on M. The orthonormal frame bundle $F(\mathcal{O})$ of $\mathcal{O}$ is defined to be $G\backslash F(M) \to \mathcal{O}$. The *fiber* of $F(\mathcal{O}) \to \mathcal{O}$ over $x \in \mathcal{O}$ is the preimage of x in $G\backslash F(M)$. The right action of $O(n)$ on $F(M)$ commutes with the left action of G and hence descends to a right $O(n)$-action on $F(\mathcal{O})$. For a bad orbifold, the orthonormal frame bundle is defined in such a way that its restriction to any good neighborhood $U \cong G_U\backslash\tilde{U}$ is the orthonormal frame bundle of the good orbifold U.

The orthonormal frame bundle of $\mathcal{O}$ is a smooth manifold as well as an orbibundle on which the orthogonal group $O(n)$ acts smoothly on the right, preserving fibers. In particular, the orbifold $\mathcal{O}$ is the orbit space $F(\mathcal{O})/O(n)$ of the right action of $O(n)$ on the manifold $F(\mathcal{O})$.

2.7. NOTATION AND REMARKS. Let $\mathcal{O}$ be an orbifold. Endow $\mathcal{O}$ with a Riemannian metric and let $F(\mathcal{O})$ be the associated orthonormal frame bundle as in 2.6. The $O(n)$-action on the fiber of $F(\mathcal{O})$ over a point $x \in \mathcal{O}$ is free if and only if x is a regular point of $\mathcal{O}$. In particular, for each singular point x of $\mathcal{O}$ viewed as an element of $F(\mathcal{O})/O(n)$, the $O(n)$-isotropy type of x is a nontrivial conjugacy class $\mathrm{Iso}_{O(n)}(x)$ of subgroups of $O(n)$. (See Definition 2.3.)

Our realization of the orbifold $\mathcal{O}$ as a global quotient of a manifold (namely $F(\mathcal{O})$) by an action of $O(n)$ depends, of course, on the Riemannian metric. However, it is not difficult to show that the conjugacy class $\mathrm{Iso}_{O(n)}(x)$ of subgroups of $O(n)$ is actually independent of the choice of Riemannian metric used in the construction. This conjugacy class of subgroups of $O(n)$ will henceforth be denoted

$\mathrm{Iso}(x)$ (without the subscript $O(n)$ except when needed for clarity) and will be referred to as the *isotropy type* of the singular point x of $\mathcal{O}$. Its cardinality $|\mathrm{Iso}(x)|$ will be called the *order* of the isotropy at x. Similarly, the equivalence classes of elements of $\mathcal{O}$ with the same isotropy type will be called *isotropy equivalence classes* without mention of $O(n)$.

The subgroups of $O(n)$ in the conjugacy class $\mathrm{Iso}(x)$ lie in the isomorphism class defined by the abstract isotropy type of x given in Definition 2.1. Indeed, let $(\tilde{U}, G_U, \pi_U)$ be an orbifold chart with $x \in U$. Let $\tilde{x} \in \tilde{U}$ with $\pi_U(\tilde{x}) = x$. With respect to any choice of Riemannian metric on $\mathcal{O}$ (and associated Riemannian metric on $\tilde{U}$), the group G_U acts isometrically on $\tilde{U}$ and thus acts on the left on the orthonormal frame bundle $F(\tilde{U})$. The subgroup $\mathrm{Iso}_{G_U}(\tilde{x})$ leaves invariant the fiber of $F(\tilde{U})$ over $\tilde{x}$. For each q in this fiber, define a homomorphism $\sigma_q : \mathrm{Iso}_{G_U}(\tilde{x}) \to O(n)$ by the condition $\gamma(q) = (q)\sigma_q(\gamma)$, where $\gamma(\cdot)$ and $(\cdot)\sigma_q(\gamma)$ denote the left action of γ and right action of $\sigma_q(\gamma) \in O(n)$ on the fiber. By 2.6, the restriction of $F(\mathcal{O})$ to U is given by $G_U \backslash F(\tilde{U})$. Let $\rho\colon F(\tilde{U}) \to G_U \backslash F(\tilde{U})$ be the projection; then σ_q maps $\mathrm{Iso}_{G_U}(\tilde{x})$ isomorphically to the stabilizer of $\rho(q)$ in $O(n)$. This stabilizer is a representative of the conjugacy class $\mathrm{Iso}(x)$, and $\mathrm{Iso}_{G_U}(\tilde{x})$ represents the abstract isotropy type of x.

2.8. Definition. A smooth *stratification* of a manifold or orbifold M is a locally finite partition of M into locally closed submanifolds, called the *strata,* satisfying the following condition: For each stratum N, the closure of N is the union of N with a collection of lower-dimensional strata.

2.9. Remarks. (i) For any stratification of an orbifold (or manifold) $\mathcal{O}$, the strata of maximal dimension are open in $\mathcal{O}$ and their union has full measure in $\mathcal{O}$.

(ii) The stratifications that we shall discuss are Whitney stratifications. Because we do not explicitly use the additional properties of Whitney stratifications here, we omit the definition and refer the reader to [14]. The notion of Whitney stratification can be defined in the more general setting of spaces that can be at least locally embedded in a smooth manifold. As discussed in [14], the orbit space of a proper Lie group action on a smooth manifold has this property.

2.10. Proposition [14]. *Given a smooth action of a Lie group H on a manifold M, the following statements hold.*

(i) *The connected components of the H-orbit types form a Whitney stratification of M. The closure of a stratum $\tilde{N}$ is made up of the union of $\tilde{N}$ with a collection of lower-dimensional strata, each lying in an H-orbit type strictly dominated by that of $\tilde{N}$.*

(ii) *The connected components of the H-isotropy equivalence classes in $H\backslash M$ form a Whitney stratification of $H\backslash M$. The closure of a stratum N is made up of the union of N with a collection of lower-dimensional strata, each lying in an H-isotropy equivalence class strictly dominated by that of N. The map π carries each stratum in M onto a stratum in $H\backslash M$.*

(iii) *For $x \in H\backslash M$ and N the stratum through x, there exists a neighborhood U of x in $H\backslash M$ such that the isotropy equivalence class of each element of the complement of N in U strictly dominates that of x.*
(iv) *If M is compact, then the stratifications of M and of $H\backslash M$ are finite.*

2.11. Corollary. *Let $\mathcal{O}$ be an orbifold. Then the action of $O(n)$ on the frame bundle $F(\mathcal{O})$ gives rise to a (Whitney) stratification of $\mathcal{O}$. The strata are connected components of the isotropy equivalence classes in $\mathcal{O}$. The set of regular points of $\mathcal{O}$ intersects each connected component $\mathcal{O}_0$ of $\mathcal{O}$ in a single stratum that constitutes an open dense submanifold of $\mathcal{O}_0$.*

2.12. Notation. (i) We refer to the strata of $\mathcal{O}$ in Corollary 2.11 as $\mathcal{O}$*-strata*.

(ii) If $(\tilde{U}, G_U, \pi_U)$ is an orbifold chart on $\mathcal{O}$, then the action of G_U on $\tilde{U}$ gives rise to stratifications both of $\tilde{U}$ and of U as in Proposition 2.10. We will refer to these as $\tilde{U}$-strata and U-strata, respectively.

2.13. Proposition. *Let $\mathcal{O}$ be a Riemannian orbifold and $(\tilde{U}, G_U, \pi_U)$ an orbifold chart.*

(i) *The U-strata are precisely the connected components of the intersections of the $\mathcal{O}$-strata with U.*
(ii) *Any two elements of the same $\tilde{U}$-stratum have the same stabilizers in G_U (not just conjugate stabilizers).*
(iii) *If H is a subgroup of G_U, then each connected component W of the fixed point set* $\mathrm{Fix}(H)$ *of H in $\tilde{U}$ is a closed submanifold of $\tilde{U}$. Any $\tilde{U}$-stratum that intersects W nontrivially lies entirely in W. Thus the stratification of $\tilde{U}$ restricts to a stratification of W.*

Proof. (i) A consequence of Proposition 2.10 is that each U-stratum (resp. $\mathcal{O}$-stratum) is a connected component of the set of all points in U (resp. $\mathcal{O}$) having G_U-isotropy (resp. $O(n)$-isotropy) of a given order. Since by 2.7 the order of the G_U-isotropy of each $x \in U$ is equal to the order of the $O(n)$-isotropy of x, statement (i) follows.

(ii) This follows because G_U is discrete and the $\tilde{U}$-strata are connected.

(iii) The first statement is true for the fixed point set of any smooth proper action by a compact group on a manifold [14]; the second statement follows from (ii). □

2.14. Notation and Remarks. Let $\mathcal{O}$ be a Riemannian orbifold and $(\tilde{U}, G_U, \pi_U)$ an orbifold chart. Let $\tilde{N}$ be a $\tilde{U}$-stratum in $\tilde{U}$. By Proposition 2.13, all the points in $\tilde{N}$ have the same isotropy group in G_U; we will refer to this group as the isotropy group of $\tilde{N}$, denoted $\mathrm{Iso}(\tilde{N})$.

Given a $\tilde{U}$-stratum $\tilde{N}$, denote by $\mathrm{Iso}^{\max}(\tilde{N})$ the set of all $\gamma \in \mathrm{Iso}(\tilde{N})$ such that $\tilde{N}$ is open in the fixed point set $\mathrm{Fix}(\gamma)$ of γ.

For $\gamma \in G_U$, Proposition 2.13 tells us that each component W of the fixed point set $\mathrm{Fix}(\gamma)$ of γ (equivalently, the fixed point set of the cyclic group generated by

γ) is a manifold stratified by a collection of $\tilde{U}$-strata. By Remark 2.9(i), the strata in W of maximal dimension are open and their union has full measure in W. In particular, the union of those $\tilde{U}$-strata $\tilde{N}$ for which $\gamma \in \mathrm{Iso}^{\max}(\tilde{N})$ has full measure in $\mathrm{Fix}(\gamma)$.

2.15. Example. On $\mathbb{R}^2$, let r_x and r_y denote the reflections across the x-axis and y-axis (respectively) and let r_0 denote the rotation through angle π about the origin 0. Then $G := \{r_x, r_y, r_0, \mathrm{Id}\}$ is a Klein 4-group acting isometrically on $\mathbb{R}^2$. The quotient of $\mathbb{R}^2$ by the semi-direct product of G with the lattice $\mathbb{Z}^2$ of translations is a closed orbifold $\mathcal{O}$ whose underlying space is a square of side length $\frac{1}{2}$. The points on the boundary of the square are singular points (not boundary points) of the orbifold and constitute eight strata: each corner point forms a single stratum with isotropy of order 4, while each open edge forms a stratum with isotropy of order 2. The strata of codimension 1 are called *reflectors* or *mirrors,* and the single-point strata are called *corner reflectors* (or dihedral points). The intersection U of the square with a disk of radius less than $\frac{1}{2}$ centered at one of the corners is the image of an orbifold chart $(\tilde{U}, G, \pi_U)$, where $\tilde{U}$ is a disk in $\mathbb{R}^2$ centered at the origin and G is the Klein 4-group described previously. The $\tilde{U}$-strata of this action consist of the single point 0 and the intersections of the disk $\tilde{U}$ with the positive and negative x-axis and the positive and negative y-axis. If $\tilde{N}$ is the intersection of $\tilde{U}$ with one of the half-axes, then $\mathrm{Iso}(\tilde{N})$ consists of a reflection and the identity while $\mathrm{Iso}^{\max}(\tilde{N})$ contains only the reflection. For $\tilde{N} = \{0\}$, we have $\mathrm{Iso}(\tilde{N}) = G$ but $\mathrm{Iso}^{\max}(\tilde{N}) = \{r_0\}$.

3. Construction of the Heat Expansion

In this section, we address the heat kernel on closed Riemannian orbifolds.

3.1. Proposition. *Let $\mathcal{O}$ be a closed Riemannian orbifold. The Laplacian Δ of $\mathcal{O}$ has a discrete spectrum $\lambda_1 \le \lambda_2 \le \cdots$, with each eigenvalue having finite multiplicity. The normalized eigenfunctions φ_j are C^∞ and form an orthonormal basis of $L^2(\mathcal{O})$.*

This proposition was proved by Chiang [8] in the case of V-manifolds (as defined in Remark 2.2). For an orbifold $\mathcal{O}$ that is not a V-manifold, those strata of the singular set of codimension 1 are called *reflectors.* Doubling along all reflectors yields a V-manifold X that doubly covers $\mathcal{O}$. Thus $\mathcal{O}$ is the quotient of a V-manifold X by a $\mathbb{Z}_2$ action. Since the eigenfunctions on $\mathcal{O}$ are then the $\mathbb{Z}_2$-invariant eigenfunctions on X, Proposition 3.1 follows immediately.

3.2. Definition. Set

$$\mathbb{R}_+ = [0, \infty) \quad \text{and} \quad \mathbb{R}_+^* = (0, \infty).$$

We say that $K : \mathbb{R}_+^* \times \mathcal{O} \times \mathcal{O} \to \mathbb{R}$ is a *fundamental solution* of the heat equation, or *heat kernel,* if it satisfies:

(i) K is C^0 in the three variables, C^1 in the first, and C^2 in the second;
(ii) $\left(\frac{\partial}{\partial t} + \Delta_x\right)K(t,x,y) = 0$, where Δ_x is the Laplacian with respect to the second variable; and
(iii) $\lim_{t\to 0^+} K(t,x,\cdot) = \delta_x$ for all $x \in \mathcal{O}$.

By the same argument as in the manifold case (see [2, III.E.2]), Proposition 3.1 implies the following statement.

3.3. Corollary. *If a heat kernel exists, then it is unique and is given by*

$$K(t,x,y) = \sum_{j=1}^{\infty} e^{-\lambda_j t}\varphi_j(x)\varphi_j(y).$$

Chiang proved the existence of the heat kernel on a compact V-manifold (from which existence on an arbitrary closed orbifold trivially follows) by proving the convergence of $\sum_{j=1}^{\infty} e^{-\lambda_j t}\varphi_j(x)\varphi_j(y)$. She also showed that the heat kernel can be approximated on good neighborhoods by the Dirichlet heat kernel on the local manifold covering. Existence follows also from more general results on existence of the heat kernel for the basic Laplacian on Riemannian foliations [22]. However, in order to apply Donnelly's results on the heat trace for good orbifolds—and so obtain the terms in the asymptotic expansion for arbitrary orbifolds in an applicable form—we will not assume the existence results for the heat kernel or heat trace. Instead we construct a parametrix and then follow the standard construction of the heat kernel from the parametrix as in [2]. Our construction of the parametrix (and consequently the heat kernel) will use directly the local structure of orbifolds as quotients of manifolds by finite group actions.

3.4. Definition. A *parametrix* for the heat operator on $\mathcal{O}$ is a function H: $\mathbb{R}_+^* \times \mathcal{O} \times \mathcal{O} \to \mathbb{R}$ satisfying:

(i) $H \in C^\infty(\mathbb{R}_+^* \times \mathcal{O} \times \mathcal{O})$;
(ii) $\left(\frac{\partial}{\partial t} + \Delta_x\right)H(t,x,y)$ extends to a function in $C^0(\mathbb{R}_+ \times \mathcal{O} \times \mathcal{O})$; and
(iii) $\lim_{t\to 0^+} H(t,x,\cdot) = \delta_x$ for all $x \in \mathcal{O}$.

Recall that the heat kernel on a closed n-dimensional Riemannian manifold M has an asymptotic expansion along the diagonal in $M \times M$ as $t \to 0^+$ of the form

$$K(t,x,x) \sim (4\pi t)^{-n/2}(u_0(x,x) + tu_1(x,x) + t^2u_2(x,x) + \cdots), \qquad (3.5)$$

where the u_i are local Riemannian invariants defined in a neighborhood of the diagonal in $M \times M$. Let ζ be a cut-off function that is identically 1 near the diagonal; then, for $m > n/2$, the function

$$K^{(m)}(t,x,y) = \zeta(x,y)(4\pi t)^{-n/2}e^{-d(x,y)^2/4t}(u_0(x,y) + \cdots + t^m u_m(x,y)) \qquad (3.6)$$

is a parametrix for the heat operator on M.

3.7. REMARK. In what follows, we shall take a local covering of our orbifold $\mathcal{O}$ by distinguished charts and piece together a parametrix for the heat operator on $\mathcal{O}$ from the expressions in (3.6). The key to piecing together the parametrix on $\mathcal{O}$ is to note that, whereas the parametrix $K^{(m)}$ in (3.6) is defined globally on M, the three defining conditions of a parametrix in Definition 3.4 are satisfied locally as well as globally by $K^{(m)}$. Indeed, the first two conditions are trivially local. The third condition is local in the following sense: the expression $e^{-d(x,y)^2/4t}$ goes to zero uniformly as $t \to 0^+$ when $d(x,y)$ is bounded away from zero. Thus for $f \in C^0(M)$ we have

$$f(x) = \lim_{t\to 0^+} \int_M K^{(m)}(t,x,y) f(y)\,dy = \lim_{t\to 0^+} \int_W K^{(m)}(t,x,y) f(y)\,dy,$$

where W is any neighborhood of x.

We will also use the fact (see [2]) that if $m > n/2+2l$ then $\left(\frac{\partial}{\partial t}+\Delta_x\right)K^{(m)}(t,x,y)$ extends to a function in $C^l(\mathbb{R}_+ \times M \times M)$. Moreover, the extension is of class C^{2l} in the last two variables.

3.8. NOTATION. Let $\mathcal{O}$ be an orbifold of dimension n. Fix $\varepsilon > 0$ such that, for each $x \in \mathcal{O}$, there exists a distinguished coordinate chart of radius ε centered at x. Cover $\mathcal{O}$ with finitely many such charts $(\tilde{W}_\alpha, G_\alpha, \pi_\alpha)$, $\alpha = 1, \dots, s$. (Here we write G_α for G_{W_α} and π_α for π_{W_α}.) Let p_α be the center of W_α and $\tilde{p}_\alpha$ the center of $\tilde{W}_\alpha$. Let U_α (resp. V_α) be the geodesic ball of radius $\varepsilon/4$ (resp. $\varepsilon/2$) centered at p_α, and let $\tilde{U}_\alpha$ and $\tilde{V}_\alpha$ be the corresponding balls centered at $\tilde{p}_\alpha$ in $\tilde{W}_\alpha$. We may assume that the family of balls $\{U_\alpha\}_{1\le\alpha\le s}$ still covers $\mathcal{O}$.

For each α and each nonnegative integer m, define $\tilde{H}_\alpha^{(m)} \colon \mathbb{R}_+^* \times \tilde{W}_\alpha \times \tilde{W}_\alpha$ by

$$\tilde{H}_\alpha^{(m)}(t,\tilde{x},\tilde{y}) = (4\pi t)^{-n/2} e^{-d(\tilde{x},\tilde{y})^2/4t}(u_0(\tilde{x},\tilde{y}) + \cdots + t^m u_m(\tilde{x},\tilde{y})),$$

where the u_i are the invariants in (3.5). Since each $\gamma \in G_\alpha$ is an isometry of $\tilde{W}_\alpha$, it follows that $u_i(\gamma\tilde{x},\gamma\tilde{y}) = u_i(\tilde{x},\tilde{y})$ for all $\tilde{x},\tilde{y} \in \tilde{W}_\alpha$. Therefore, the function

$$(t,\tilde{x},\tilde{y}) \mapsto \sum_{\gamma\in G_\alpha} \tilde{H}_\alpha^{(m)}(t,\tilde{x},\gamma\tilde{y})$$

is G_α-invariant in both $\tilde{x}$ and $\tilde{y}$ and thus descends to a well-defined function, which we denote by $H_\alpha^{(m)}$, on $\mathbb{R}_+^* \times W_\alpha \times W_\alpha$.

Let $\psi_\alpha \colon \mathcal{O} \to \mathbb{R}$ be a C^∞ cut-off function that is identically 1 on V_α and is supported in W_α. Let $\{\eta_\alpha : \alpha = 1, \dots, s\}$ be a partition of unity on $\mathcal{O}$ with the support of η_α contained in $\bar{U}_\alpha$. Define $H^{(m)} \colon \mathbb{R}_+^* \times \mathcal{O} \times \mathcal{O} \to \mathbb{R}$ by

$$H^{(m)}(t,x,y) = \sum_{\alpha=1}^{s} \psi_\alpha(x)\eta_\alpha(y) H_\alpha^{(m)}(t,x,y). \tag{3.9}$$

We will show that $H^{(m)}$ is a parametrix for the heat kernel on $\mathcal{O}$ when $m > n/2$.

3.10. LEMMA. $H^{(m)} \in C^\infty(\mathbb{R}_+^* \times \mathcal{O} \times \mathcal{O})$.

The proof is immediate.

3.11. LEMMA. *Let l be a nonnegative integer. Then*:

(i) $\left(\frac{\partial}{\partial t}+\Delta_x\right)H^{(m)}(t,x,y)$ *extends to a function in* $C^l(\mathbb{R}_+\times\mathcal{O}\times\mathcal{O})$ *if* $m>n/2+2l$, *and it is of class* C^{2l} *in the last two variables*;

(ii) *for any given* $T>0$ *and for each* $m>n/2$, *there exists a constant* A *such that* $\left|\left(\frac{\partial}{\partial t}+\Delta_x\right)H^{(m)}(t,x,y)\right|<At^{m-n/2}$ *when* $0<t<T$.

Proof. (i) Let $l\geq 0$ and suppose $m>n/2+2l$. By Remark 3.7, the function $\left(\frac{\partial}{\partial t}+\tilde{\Delta}_{\tilde{x}}\right)\tilde{H}_\alpha^{(m)}(t,\tilde{x},\tilde{y})$ on $\mathbb{R}_+^*\times\tilde{W}_\alpha\times\tilde{W}_\alpha$ extends to $C^l(\mathbb{R}_+\times\tilde{W}_\alpha\times\tilde{W}_\alpha)$. (We use $\tilde{\Delta}$ to denote the Laplacian on $\tilde{W}_\alpha$ for all choices of α.) Thus the same is true for $\sum_{\gamma\in G_\alpha}\left(\frac{\partial}{\partial t}+\tilde{\Delta}_{\tilde{x}}\right)\tilde{H}_\alpha^{(m)}(t,\tilde{x},\gamma\tilde{y})$, and hence it follows that $\left(\frac{\partial}{\partial t}+\Delta_x\right)H_\alpha^{(m)}(t,x,y)$ extends to $C^l(\mathbb{R}_+\times W_\alpha\times W_\alpha)$.

Now consider the function

$$f_\alpha(t,x,y):=\left(\frac{\partial}{\partial t}+\Delta_x\right)(\psi_\alpha(x)\eta_\alpha(y)H_\alpha^{(m)}(t,x,y)).$$

Noting that ψ_α and η_α are compactly supported inside W_α, we may view f_α as a function on $\mathbb{R}_+^*\times\mathcal{O}\times\mathcal{O}$ that is 0 whenever x or y lies outside of W_α. We show that f_α extends to $C^l(\mathbb{R}_+\times\mathcal{O}\times\mathcal{O})$. Since $\psi_\alpha\equiv 1$ on V_α, it follows immediately from the previous paragraph that f_α extends to $C^l(\mathbb{R}_+\times V_\alpha\times\mathcal{O})$. Moreover, $f_\alpha\equiv 0$ on $\mathbb{R}_+^*\times\mathcal{O}\times(\mathcal{O}\setminus U_\alpha)$ and so f_α also extends smoothly (to zero) on $\mathbb{R}_+\times\mathcal{O}\times(\mathcal{O}\setminus U_\alpha)$. Finally, for $(x,y)\in(\mathcal{O}\setminus V_\alpha)\times\bar{U}_\alpha$, we have $d(x,y)\geq\varepsilon/4$. Thus, as $t\to 0^+$, $f_\alpha(t,x,y)$ and all its derivatives converge to 0 uniformly for $(x,y)\in(\mathcal{O}\setminus V_\alpha)\times\bar{U}_\alpha$. Hence $H^{(m)}$ extends to a function in $C^l(\mathbb{R}_+\times\mathcal{O}\times\mathcal{O})$.

The claim regarding class C^{2l} follows from Remark 3.7.

(ii) The u_i are constructed so that

$$\left(\frac{\partial}{\partial t}+\tilde{\Delta}_{\tilde{x}}\right)\tilde{H}_\alpha^{(m)}(t,\tilde{x},\tilde{y})=(4\pi t)^{-n/2}e^{-d(\tilde{x},\tilde{y})^2/4t}t^m\tilde{\Delta}_{\tilde{x}}u_m(t,\tilde{x},\tilde{y})$$

(see [2]). Because the u_i are C^∞ functions, there must exist a constant B_α such that

$$\left(\frac{\partial}{\partial t}+\tilde{\Delta}_{\tilde{x}}\right)\tilde{H}_\alpha^{(m)}(t,\tilde{x},\tilde{y})<B_\alpha t^{m-n/2}$$

on $(0,T]\times\tilde{W}_\alpha\times\tilde{W}_\alpha$. Consequently,

$$\left(\frac{\partial}{\partial t}+\Delta_x\right)H_\alpha^{(m)}(t,x,y)<|G_\alpha|B_\alpha t^{m-n/2}$$

on $(0,T]\times W_\alpha\times W_\alpha$ and

$$\left(\frac{\partial}{\partial t}+\Delta_x\right)(\psi_\alpha(x)\eta_\alpha(y)H_\alpha^{(m)}(t,x,y))<|G_\alpha|B_\alpha t^{m-n/2}$$

on $(0,T]\times V_\alpha\times\mathcal{O}$, since $\psi_\alpha\equiv 1$ on V_α. Once again, for x outside of V_α and y in the support of η_α, we have $d(x,y)\geq\varepsilon/4$; hence $\left(\frac{\partial}{\partial t}+\Delta_x\right)(\psi_\alpha(x)\eta_\alpha(y)H_\alpha^{(m)}(t,x,y))$ can be bounded in terms of any power of t on $(0,T)\times(\mathcal{O}\setminus V_\alpha)\times\mathcal{O}$.

Statement (ii) now follows from equation (3.9). $\square$

3.12. REMARK. If $m > n/2+2l$, then we can obtain bounds on the partial derivatives of order at most l of $\left(\frac{\partial}{\partial t} + \Delta_x\right)H^{(m)}(t,x,y)$ by an argument analogous to that used in the proof of Lemma 3.11(ii).

3.13. LEMMA. *Let m be any nonnegative integer. For $f \in C^\infty(\mathcal{O})$ and $x \in \mathcal{O}$, we have*

$$\lim_{t\to 0^+} \int_{\mathcal{O}} H^{(m)}(t,x,y) f(y)\, dy = f(x);$$

in other words, $\lim_{t\to 0^+} H^{(m)}(t,x,y) = \delta_x(y)$. Moreover, if N is a topological space and if f is a continuous function on $N \times \mathcal{O}$, then the convergence of $\int_{\mathcal{O}} H^{(m)}(t,x,y) f(p,y)\, dy$ to $f(p,x)$ is locally uniform on $N \times \mathcal{O}$.

Proof. Let $\tilde{\psi}_\alpha$ and $\tilde{\eta}_\alpha$ be the lifts of ψ_α and η_α to $\tilde{W}_\alpha$.

Since $\operatorname{supp}(\eta_\alpha) \subset \bar{U}_\alpha \subset W_\alpha$, we have

$$\int_{\mathcal{O}} \psi_\alpha(x)\eta_\alpha(y) H_\alpha^{(m)}(t,x,y) f(p,y)\, dy = \frac{\psi_\alpha(x)}{|G_\alpha|} \sum_{\gamma\in G_\alpha} \int_{\tilde{W}_\alpha} \tilde{H}_\alpha^{(m)}(t,\tilde{x},\gamma\tilde{y})\tilde{\eta}_\alpha(\tilde{y})\tilde{f}_\alpha(p,\tilde{y})\, d\tilde{y}, \quad (3.14)$$

where $\tilde{f}_\alpha$ is a lift of $f|_{N\times W_\alpha}$ to $N \times \tilde{W}_\alpha$ and $\tilde{x}$ is an arbitrarily chosen point in the preimage of x under the map $\tilde{W}_\alpha \to W_\alpha$.

We change variables in each of the integrals in the right side of equation (3.14), letting $\tilde{u} = \gamma(\tilde{y})$. Since γ is an isometry and since $\tilde{\eta}_\alpha$ and $\tilde{f}_\alpha(p,\cdot)$ are γ-invariant, each integral in the summand is equal to

$$\int_{\tilde{W}_\alpha} \tilde{H}_\alpha^{(m)}(t,\tilde{x},\tilde{u})\tilde{\eta}_\alpha(\tilde{u})\tilde{f}_\alpha(p,\tilde{u})\, d\tilde{u}. \quad (3.15)$$

As $t \to 0^+$, the integral (3.15) converges to $\tilde{\eta}_\alpha(\tilde{x})\tilde{f}_\alpha(p,\tilde{x}) = \eta_\alpha(x) f(p,x)$ (see Remark 3.7). Moreover, this convergence is locally uniform on $N \times \tilde{W}_\alpha$ (see [2]). Once we note that $\psi_\alpha \equiv 1$ on the support of η_α, it follows that both sides of equation (3.14) converge to $\eta_\alpha(x) f(p,x)$ as $t \to 0^+$, and the convergence is locally uniform on $N \times W_\alpha$. Since both sides of (3.14) are identically zero when x lies outside of $\operatorname{supp}(\psi_\alpha) \subset W_\alpha$, we thus have locally uniform convergence to $\eta_\alpha(x) f(p,x)$ on all of $N \times \mathcal{O}$.

Finally, it follows from equation (3.9) that

$$\lim_{t\to 0^+} \int_{\mathcal{O}} H^{(m)}(t,x,y) f(p,y)\, dy = \sum_\alpha \eta_\alpha(x) f(p,x) = f(p,x)$$

locally uniformly. □

3.16. PROPOSITION. *$H^{(m)}$ is a parametrix for the heat operator on $\mathcal{O}$ if $m > n/2$.*

Proof. The proof is immediate from Lemmas 3.10, 3.11(i), and 3.13. □

The construction of the heat kernel from the parametrix $H^{(m)}$ follows exactly as in [2]. Here we give only a brief summary.

3.17. NOTATION. For $A, B \in C^0(\mathbb{R}_+ \times \mathcal{O} \times \mathcal{O})$, define the convolution $A * B \in C^0(\mathbb{R}_+^* \times \mathcal{O} \times \mathcal{O})$ by

$$A * B(t, x, z) = \int_0^t d\theta \int_{\mathcal{O}} A(t - \theta, x, y) B(\theta, y, z)\, dy.$$

Note that the convolution operator $*$ is associative.

3.18. LEMMA. *Let l be any nonnegative integer, and let $m > n/2 + 2l$. Define $F_m(t, x, y) = \left(\frac{\partial}{\partial t} + \Delta_x\right) H^{(m)}(t, x, y)$. (See Lemma 3.11 for regularity properties of F_m.) Then, for each $T > 0$, the series $\sum_{j=1}^{\infty} (-1)^{j+1} F_m^{*j}(t, x, y)$ converges uniformly on $[0, T] \times \mathcal{O} \times \mathcal{O}$. Let $Q_m : \mathbb{R}_+ \times \mathcal{O} \times \mathcal{O} \to \mathbb{R}$ be the sum of this series. Then $Q_m \in C^l(\mathbb{R}_+ \times \mathcal{O} \times \mathcal{O})$. Moreover, for any $T > 0$, there exists a constant C such that*

$$|Q_m(t, x, y)| \le C t^{m - n/2}$$

on $[0, T] \times \mathcal{O} \times \mathcal{O}$.

The proof of Lemma 3.18 is identical to that of [2, Lemma E.III.6] and uses only Lemma 3.11(ii) and Remark 3.12.

3.19. LEMMA. *Let $m > n/2$. For $P \in C^0(\mathbb{R}_+ \times \mathcal{O} \times \mathcal{O})$, the function $H^{(m)} * P$, defined formally by the expression in Notation 3.17, exists and is in $C^0(\mathbb{R}_+^* \times \mathcal{O} \times \mathcal{O})$. Moreover, if $m > n/2 + l$, then $H^{(m)} * P$ is of class C^l in the second variable. For $m > n/2 + 2$, we have that $\left(\frac{\partial}{\partial t} + \Delta_x\right)(H^{(m)} * P(t, x, y))$ exists and equals $(P + H^{(m)} * P)(t, x, y)$.*

Again the proof is identical to that of [2, Lemma E.III.7] and is based on Lemma 3.13.

Using Lemmas 3.18 and 3.19, we obtain (as in [2, Prop. EIII.8]) the following result.

3.20. PROPOSITION. *Let $m > n/2 + 2$ and define Q_m as in Lemma 3.18. Then $K := H^{(m)} - H^{(m)} * Q_m$ is a fundamental solution of the heat equation on $\mathcal{O}$.*

Uniqueness of the heat kernel implies that $H^{(m)} - H^{(m)} * Q_m$ is independent of the choice of $m > n/2 + 2$.

3.21. NOTATION. Let

$$\tilde{H}_\alpha(t, \tilde{x}, \tilde{y}) = \sum_{\gamma \in G_\alpha} (4\pi t)^{-n/2} e^{-d(\tilde{x}, \gamma(\tilde{y}))^2/4t} (u_0(\tilde{x}, \gamma(\tilde{y})) + t u_1(\tilde{x}, \gamma(\tilde{y})) + \cdots).$$

Observe that $\tilde{H}_\alpha$ is G_α-invariant in both $\tilde{x}$ and $\tilde{y}$ and thus descends to a well-defined function, which we denote by H_α, on $\mathbb{R}_+^* \times W_\alpha \times W_\alpha$.

3.22. THEOREM. *In the notation of Proposition 3.1 and 3.21, the trace of the heat kernel has an asymptotic expansion as $t \to 0^+$ given by*

$$\sum_{j=1}^{\infty} e^{-\lambda_j t} \sim_{t \to 0^+} \sum_{\alpha=1}^{s} \int_{\mathcal{O}} \eta_\alpha(x) H_\alpha(t, x, x)\, dx.$$

Proof. By Corollary 3.3 and Proposition 3.20, we have

$$\sum_{j=1}^{\infty} e^{-\lambda_j t} = \int_{\mathcal{O}} (H^{(m)} - H^{(m)} * Q_m)(t, x, x)\, dx$$

for $m > n/2+2$. Because $(4\pi t)^{n/2} H^{(m)}(t, x, x)$ is uniformly bounded for $(t, x) \in (0, T] \times \mathcal{O}$ for any given $T > 0$, it follows from Lemma 3.18 that

$$\sum_{j=1}^{\infty} e^{-\lambda_j t} = \int_{\mathcal{O}} H^{(m)}(t, x, x)\, dx + O(t^{m-n}) \tag{3.23}$$

on any interval $(0, T]$. (*Note:* If $\mathcal{O}$ is a manifold, then $\int_{\mathcal{O}} H^{(m)}(t, x, x)\, dx = (4\pi t)^{-n/2}(a_0 + a_1 t + \cdots + a_m t^m)$. For general orbifolds, the arguments in the next section will show that $\int_{\mathcal{O}} H^{(m)}(t, x, x)\, dx$ is of the form $(4\pi t)^{-n/2} \sum_{j=0}^{2m} c_j t^{j/2}$. Thus the error term can be improved to $O(t^{m-(n-1)/2})$, since $\int_{\mathcal{O}} H^{(m)}(t, x, x)\, dx = \int_{\mathcal{O}} H^{(m+n)}(t, x, x)\, dx + O(t^{m-(n-1)/2})$.)

Since ψ_α is identically 1 on the support of η_α, Notation 3.8 yields

$$H^{(m)}(t, x, x) = \sum_{\alpha} \eta_\alpha(x) H_\alpha^{(m)}(t, x, x). \tag{3.24}$$

Substituting (3.24) into (3.23) then yields the theorem. □

4. Computation of the Heat Asymptotics

4.1. Notation and Remarks. Let γ be an isometry of a Riemannian manifold M and let $\Omega(\gamma)$ denote the set of components of the fixed point set of γ. Each element of $\Omega(\gamma)$ is a submanifold of M. For each nonnegative integer k, Donnelly [10] defined a real-valued function, which we temporarily denote $b_k((M, \gamma), \cdot)$, on the fixed point set of γ. For each $W \in \Omega(\gamma)$, the restriction of $b_k((M, \gamma), \cdot)$ to W is smooth. Two key properties of the b_k are:

(i) *Locality.* For $a \in W$, $b_k((M, \gamma), a)$ depends only on the germs at a of the Riemannian metric of M and of the isometry γ. In particular, if U is a γ-invariant neighborhood of a in M, then $b_k((M, \gamma), a) = b_k((U, \gamma), a)$.

(ii) *Universality.* If M and M' are Riemannian manifolds admitting the respective isometries γ and γ', and if $\sigma\colon M \to M'$ is an isometry satisfying $\sigma \circ \gamma = \gamma' \circ \sigma$, then $b_k((M, \gamma), x) = b_k((M', \gamma'), \sigma(x))$ for all $x \in \mathrm{Fix}(\gamma)$.

In view of the locality property, we will usually delete the explicit reference to M and rewrite these functions as $b_k(\gamma, \cdot)$, as they are written in [10].

4.2. Computation of the b_k [10]. In the notation of 4.1, let $W \in \Omega(\gamma)$ and let $n = \dim(M)$ and $m = \dim(W)$. For $x \in W$, the orthogonal complement $T_x(W)^\perp$ of $T_x(W)$ in the tangent space $T_x(M)$ is invariant under γ_*. Define $A_\gamma(x) = \gamma_*\colon T_x(W)^\perp \to T_x(W)^\perp$, and observe that $A_\gamma(x)$ is nonsingular. Set

$$B_\gamma(x) = (I - A_\gamma(x))^{-1}.$$

Donnelly showed that

$$b_k(\gamma, x) = |\det(B_\gamma(x))|\tilde{b}_k(\gamma, x),$$

where $\tilde{b}_k(\gamma, \cdot)$ is an $O(m) \times O(n-m)$ universal invariant polynomial in the components of B_γ and in the curvature tensor R of M and its covariant derivatives.

Explicit formulas for b_0 and b_1 are given in [10, Thm. 5.1] using these indexing conventions: $1 \leq \alpha, \beta \leq m$, $m+1 \leq i, j, k \leq n$, and $1 \leq a, b, c \leq n$. At each point $x \in W$, choose an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_x(M)$ so that the first m vectors are tangent to W. The sign convention on the curvature tensor R of M is chosen so that R_{abab} is the sectional curvature of the plane spanned by e_a and e_b. Set

$$\tau = \sum_{a,b=1}^{n} R_{abab} \quad \text{and} \quad \rho_{ab} = \sum_{c=1}^{n} R_{acbc}.$$

Thus τ is the scalar curvature and ρ the Ricci tensor of M. Then

$$b_0(\gamma, x) = |\det(B_\gamma(x))|, \tag{4.3}$$

and summing over repeated indices yields

$$\begin{aligned} &b_1(\gamma, x) \\ &\quad = |\det(B_\gamma(x))|\big(\tfrac{1}{6}\tau + \tfrac{1}{6}\rho_{kk} + \tfrac{1}{3}R_{iksh}B_{ki}B_{hs} + \tfrac{1}{3}R_{ikth}B_{kt}B_{hi} - R_{kaha}B_{ks}B_{hs}\big). \end{aligned} \tag{4.4}$$

4.5. NOTATION. Let $\mathcal{O}$ be an orbifold and let $(\tilde{U}, G_U, \pi_U)$ be an orbifold chart.

In the notation of 2.12 and 2.14, let $\tilde{N}$ be a $\tilde{U}$-stratum and let $\gamma \in \mathrm{Iso}^{\max}(\tilde{N})$. Then $\tilde{N}$ is an open subset of a component of $\mathrm{Fix}(\gamma)$ and thus, by 4.1, $b_k(\gamma, \cdot)$ $(= b_k((\tilde{U}, \gamma), \cdot))$ is smooth on $\tilde{N}$ for each nonnegative integer k. Define a function $b_k(\tilde{N}, \cdot)$ on $\tilde{N}$ by

$$b_k(\tilde{N}, x) = \sum_{\gamma \in \mathrm{Iso}^{\max}(\tilde{N})} b_k(\gamma, x).$$

4.6. LEMMA. *Let $\mathcal{O}$ be a Riemannian orbifold, let N be an $\mathcal{O}$-stratum, and let $p \in N$. Let $(\tilde{U}, G_U, \pi_U)$ and $(\tilde{U}', G_{U'}, \pi_{U'})$ be two orbifold charts with $p \in U \cap U'$. Let $\tilde{p} \in \tilde{U}$ and $\tilde{p}' \in \tilde{U}'$ with $\pi_U(\tilde{p}) = p = \pi_{U'}(\tilde{p}')$, and let $\tilde{N}$ (resp. $\tilde{N}'$) be the $\tilde{U}$-stratum through $\tilde{p}$ (resp. $\tilde{U}'$-stratum through $\tilde{p}'$). Then, for each k, we have $b_k(\tilde{N}, \tilde{p}) = b_k(\tilde{N}', \tilde{p}')$.*

Proof. By Definition 2.1, it suffices to consider the case where one chart embeds in the other—say, $\lambda: (\tilde{U}, G_U, \pi_U) \to (\tilde{U}', G_{U'}, \pi_{U'})$ is an isometric embedding with $\lambda(\tilde{p}) = \tilde{p}'$. The associated homomorphism $\tau: G_U \to G_{U'}$ carries $\mathrm{Iso}_{G_U}(\tilde{p})$ isomorphically onto $\mathrm{Iso}_{G_{U'}}(\tilde{p}')$ and $\mathrm{Iso}^{\max}(\tilde{p})$ to $\mathrm{Iso}^{\max}(\tilde{p}')$. The $\tilde{U}$-stratum $\tilde{N}$ is carried to an open subset of the $\tilde{U}'$-stratum $\tilde{N}'$. The lemma is thus an immediate consequence of the universality of the b_k, as discussed in 4.1. □

4.7. DEFINITION. Let $\mathcal{O}$ be a Riemannian orbifold and let N be an $\mathcal{O}$-stratum.

(i) For each nonnegative integer k, define a real-valued function $b_k(N, \cdot)$ by setting $b_k(N, p) = b_k(\tilde{N}, \tilde{p})$ where $(\tilde{U}, G_U, \pi_U)$ is any orbifold chart about p,

$\tilde{p} \in \pi_U^{-1}(p)$, and $\tilde{N}$ is the $\tilde{U}$-stratum through $\tilde{p}$. By Lemma 4.6, the function $b_k(N, \cdot)$ is well-defined.

(ii) The Riemannian metric on $\mathcal{O}$ induces a Riemannian metric—and thus a volume element—on the manifold N. Set

$$I_N := (4\pi t)^{-\dim(N)/2} \sum_{k=0}^{\infty} t^k \int_N b_k(N, x)\, d\operatorname{vol}_N(x),$$

where $d\operatorname{vol}_N$ is the Riemannian volume element.

(iii) Also set

$$I_0 = (4\pi t)^{-\dim(\mathcal{O})/2} \sum_{k=0}^{\infty} a_k(\mathcal{O}) t^k,$$

where the $a_k(\mathcal{O})$ (which we will usually write simply as a_k) are the familiar heat invariants. More precisely, the invariants u_i in (3.5), which are defined in terms of the curvature and its covariant derivatives on any Riemannian manifold, also make sense on any Riemannian orbifold. The invariants $a_k(\mathcal{O})$ are given by $a_k = \int_{\mathcal{O}} u_k(x, x)\, d\operatorname{vol}_{\mathcal{O}}(x)$. In particular, $a_0 = \operatorname{vol}(\mathcal{O})$, $a_1 = \frac{1}{6} \int_{\mathcal{O}} \tau(x)\, d\operatorname{vol}_{\mathcal{O}}(x)$, and so forth. Observe that if $\mathcal{O}$ is finitely covered by a Riemannian manifold M (say, $\mathcal{O} = G \backslash M$) then $a_k(\mathcal{O}) = \frac{1}{|G|} a_k(M)$.

4.8. THEOREM. *Let $\mathcal{O}$ be a Riemannian orbifold and let $\lambda_1 \le \lambda_2 \le \cdots$ be the spectrum of the associated Laplacian acting on smooth functions on $\mathcal{O}$. The heat trace $\sum_{j=1}^{\infty} e^{-\lambda_j t}$ of $\mathcal{O}$ is asymptotic as $t \to 0^+$ to*

$$I_0 + \sum_{N \in S(\mathcal{O})} \frac{I_N}{|\operatorname{Iso}(N)|},$$

where $S(\mathcal{O})$ is the set of all $\mathcal{O}$-strata and where $|\operatorname{Iso}(N)|$ is the order of the isotropy at each $p \in N$ as defined in Remark 2.7. This asymptotic expansion is of the form

$$(4\pi t)^{-\dim(\mathcal{O})/2} \sum_{j=0}^{\infty} c_j t^{j/2} \tag{4.9}$$

for some constants c_j.

4.10. REMARK. Suppose $\mathcal{O} = G \backslash M$ is a good closed orbifold. Note that M may be noncompact and G may be an infinite group, although the isotropy group at any point of M must be a finite subgroup of G. In this setting, Donnelly [11] proved the existence and uniqueness of the heat kernel K^M on M and of an asymptotic expansion for K^M. He then obtained an asymptotic expansion for the heat trace on $\mathcal{O}$. Theorem 4.8, in the case of good orbifolds, organizes the information in [11] in a way that clarifies the contribution of each $\mathcal{O}$-stratum to the asymptotics.

The expression for the heat asymptotics of good orbifolds in [11] differs from that in (4.9) in that the half-powers are missing. However, the absence of these powers is apparently a typographical error in transcribing a result of Donnelly's earlier paper [10], to be stated here as Proposition 4.11.

Richardson [24] obtained an asymptotic expansion for the heat trace associated with the basic Laplacian on a Riemannian foliation. Also referring to Donnelly's work on good orbifolds, he showed that the expansion is of the form given in (4.9).

The remainder of this section is devoted to proving Theorem 4.8.

4.11. PROPOSITION [10]. *Let M be a closed Riemannian manifold, let $K(t,x,y)$ be the heat kernel of M, and let γ be a nontrivial isometry of M. Then, in the notation of 4.1, $\int_M K(t,x,\gamma(x))\,d\,\mathrm{vol}_M(x)$ is asymptotic as $t \to 0^+$ to*

$$\sum_{W\in\Omega(\gamma)} (4\pi t)^{-\dim(W)/2} \sum_{k=0}^{\infty} t^k \int_W b_k(\gamma,a)\,d\,\mathrm{vol}_W(a),$$

where $d\,\mathrm{vol}_W$ is the volume form on W defined by the Riemannian metric induced from M.

4.12. PROOF OF THEOREM 4.8 IN SPECIAL CASE. We prove Theorem 4.8 for $\mathcal{O} = G\backslash M$ a good closed orbifold with G finite (and thus M compact). In particular, (M,G,π) is a global orbifold chart where $\pi : M \to \mathcal{O}$ is the projection. In this special case, the theorem is an easy consequence of Proposition 4.11. Indeed, if we let K denote the heat kernel of M and let $\pi : M \to \mathcal{O}$ be the projection, then the heat kernel $K^{\mathcal{O}}$ of $\mathcal{O}$ is given by

$$K^{\mathcal{O}}(t,x,y) = \sum_{\gamma\in G} K(t,\tilde{x},\gamma(\tilde{y})),$$

where $\tilde{x}$ (resp. $\tilde{y}$) are any elements of $\pi^{-1}(x)$ (resp. $\pi^{-1}(y)$). Thus

$$\int_{\mathcal{O}} K^{\mathcal{O}}(t,x,x)\,d\,\mathrm{vol}_{\mathcal{O}}(x) = \frac{1}{|G|}\sum_{\gamma\in G}\int_M K(t,\tilde{x},\gamma(\tilde{x}))\,d\,\mathrm{vol}_M(\tilde{x})$$

and so Proposition 4.11 implies that

$$\begin{aligned}\int_{\mathcal{O}} & K^{\mathcal{O}}(t,x,x)\,d\,\mathrm{vol}_{\mathcal{O}}(x)\\ &\sim_{t\to 0^+} \frac{1}{|G|}\int_M K(t,\tilde{x},\tilde{x})\,d\,\mathrm{vol}_M(\tilde{x})\\ &\quad + \frac{1}{|G|}\sum_{1\neq\gamma\in G}\sum_{W\in\Omega(\gamma)} (4\pi t)^{-\dim(W)/2}\sum_{k=0}^{\infty} t^k\int_W b_k(\gamma,a)\,d\,\mathrm{vol}_W(a). \qquad (4.13)\end{aligned}$$

The first term on the right-hand side of (4.13) is given by

$$\frac{1}{|G|}\int_M K(t,\tilde{x},\tilde{x})\,d\,\mathrm{vol}_M(\tilde{x}) = \frac{1}{|G|}(4\pi t)^{-\dim(M)/2}\sum_{k=0}^{\infty} a_k(M)t^k = I_0. \qquad (4.14)$$

(See the final comment in Definition 4.7(iii).)

Next let $1 \neq \gamma \in G$, let $W \in \Omega(\gamma)$, and let $\tilde{N}$ be an M-stratum contained in W. Then either $\tilde{N}$ has measure zero in W (in which case $\gamma \notin \mathrm{Iso}^{\max}(\tilde{N})$) or $\tilde{N}$ is open in W and so $\gamma \in \mathrm{Iso}^{\max}(\tilde{N})$. Thus—by replacing the integral over W with the integrals over the M-strata that are open in W, reordering the summations in (4.13), and taking note of (4.14)—we obtain

$$\int_{\mathcal{O}} K^{\mathcal{O}}(t,x,x)\,d\,\mathrm{vol}_{\mathcal{O}}(x) \sim_{t\to 0^+} I_0 + \frac{1}{|G|}\sum_{\tilde{N}\in\tilde{S}(M)} \tilde{I}_{\tilde{N}}, \tag{4.15}$$

where $\tilde{S}(M)$ denotes the set of all M-strata and where

$$\tilde{I}_{\tilde{N}} = (4\pi t)^{-\dim(\tilde{N})/2}\sum_{k=0}^{\infty} t^k \int_{\tilde{N}} b_k(\tilde{N},a)\,d\,\mathrm{vol}_{\tilde{N}}(a).$$

Let N be an $\mathcal{O}$-stratum. Then $\pi^{-1}(N)$ is a union of finitely many mutually isometric M-stratum $\tilde{N}_1,\dots,\tilde{N}_k$ and $\pi:\pi^{-1}(N)\to N$ is a covering map of degree $\frac{|G|}{|\mathrm{Iso}(N)|}$. We have

$$\tilde{I}_{\tilde{N}_1}+\cdots+\tilde{I}_{\tilde{N}_k} = \frac{|G|}{|\mathrm{Iso}(N)|} I_N$$

and then Theorem 4.8, in the case of orbifolds finitely covered by manifolds, follows from (4.15).

The proof in the general case will apply the argument in 4.12 to orbifold charts and then piece the computations together via a partition of unity. We first generalize Proposition 4.11 slightly. The manifolds in the two lemmas that follow do not have boundaries but could, for example, be bounded domains in a larger manifold.

4.16. Lemma. *We use the notation of 4.1. Let M be an n-dimensional Riemannian manifold* (*without boundary*) *of finite volume and let $\gamma: M\to M$ be a nontrivial isometry. Assume that the distance $d(\tilde{x},\gamma(\tilde{x}))$ remains bounded away from* 0 *off arbitrarily small tubular neighborhoods of the fixed point set of γ and that each component of the fixed point set of γ has finite volume. Then, as $t\to 0^+$,*

$$\int_M (4\pi t)^{-n/2} e^{-d(\tilde{x},\gamma(\tilde{x}))^2/4t}(u_0(\tilde{x},\gamma(\tilde{x})) + t u_1(\tilde{x},\gamma(\tilde{x}))+\cdots)\,d\tilde{x}$$
$$\sim \sum_{W\in\Omega(\gamma)} (4\pi t)^{-\dim(W)/2}\sum_{k=0}^{\infty} t^k \int_W b_k(\gamma,\tilde{x})\,d\,\mathrm{vol}_W(\tilde{x}).$$

This result is proven in Donnelly [10, Thm. 4.1] for M closed. In that case, of course, the hypotheses on the distance function and on the fixed point set of γ are automatic and so the lemma is a restatement of Proposition 4.11. The proof goes through verbatim in the more general setting of Lemma 4.16.

4.17. Lemma. *With the notation and hypotheses of the previous lemma, let $\tilde{\eta}$ be a smooth bounded γ-invariant function on M. Then, for $k=0,1,2,\dots$, there exists a family of functions $c_k(\gamma,\tilde{\eta},\cdot)$ defined on the fixed point set of γ, smooth on each component $W\in\Omega(\gamma)$, and such that, as $t\to 0^+$,*

$$\int_M \tilde{\eta}(\tilde{x})(4\pi t)^{-n/2} e^{-d(\tilde{x},\gamma(\tilde{x}))^2/4t}(u_0(\tilde{x},\gamma(\tilde{x})) + t u_1(\tilde{x},\gamma(\tilde{x}))+\cdots)\,d\tilde{x}$$
$$\sim \sum_{W\in\Omega(\gamma)} (4\pi t)^{-\dim(W)/2}\sum_{k=0}^{\infty} t^k \int_W c_k(\gamma,\tilde{\eta},\tilde{x})\,d\,\mathrm{vol}_W(\tilde{x}).$$

Moreover, $c_k(\gamma, \tilde{\eta}, \cdot)$ satisfies the following properties.

(i) *Locality: $c_k(\gamma, \tilde{\eta}, \tilde{x})$ depends only on the germs of γ, $\tilde{\eta}$ and the Riemannian metric of M at $\tilde{x} \in W$.*
(ii) *$c_k(\gamma, \tilde{\eta}, \cdot)$ is* 0 *off* $\operatorname{supp}(\tilde{\eta}) \cap W$.
(iii) *The dependence of $c_k(\gamma, \tilde{\eta}, \cdot)$ on $\tilde{\eta}$ is linear.*
(iv) *$c_k(\gamma, 1, \cdot) = b_k(\gamma, \cdot)$, where* 1 *denotes the constant function $\tilde{\eta} \equiv 1$.*
(v) *Universality: If M' is another Riemannian manifold, γ' is an isometry of M', and $\sigma : M \to M'$ is an isometry satisfying $\sigma \circ \gamma = \gamma' \circ \sigma$, then $c_k(\gamma', \tilde{\eta} \circ \sigma^{-1}, \sigma(\tilde{x})) = c_k(\gamma, \tilde{\eta}, \tilde{x})$ for all $\tilde{x}$ in the fixed point set of γ.*

The proof requires only minor changes in the proof of [10, Thm. 4.1]. In the proof of that theorem, the functions $b_k(\gamma, \cdot)$ are expressed as linear combinations of certain derivatives of explicitly defined functions h_j, $j = 0, \dots, k$. To obtain the functions $c_k(\gamma, \tilde{\eta}, \cdot)$, we replace the functions h_j by the functions $\tilde{\eta} h_j$.

4.18. Notation and Remarks. Let $\mathcal{O}$ be a closed orbifold, and consider the charts $\tilde{U}_\alpha, \tilde{V}_\alpha, \tilde{W}_\alpha$ and partition of unity $\{\eta_\alpha\}$ given in Notation 3.8. Let $\tilde{V}_\alpha$ play the role of M in Lemmas 4.16 and 4.17, and let $\tilde{\eta}_\alpha = \eta_\alpha \circ \pi_\alpha$ play the role of $\tilde{\eta}$. Since $\tilde{V}_\alpha$ has compact closure inside the larger Riemannian manifold $\tilde{W}_\alpha$ on which G_α acts by isometries and since the fixed point set of γ in $\tilde{W}_\alpha$ is connected (in fact, it is the union of a collection of geodesics radiating from the center point $\tilde{p}_\alpha$), one easily verifies for each $\gamma \in G_\alpha$ that the hypothesis concerning the distance function in the two lemmas holds.

(i) For each $\tilde{V}_\alpha$-stratum $\tilde{N}$, define a smooth function $c_{k,\alpha}(\tilde{N}, \cdot)$ on $\tilde{N}$ by
$$c_{k,\alpha}(\tilde{N}, \tilde{x}) = \sum_{\gamma \in \operatorname{Iso}^{\max}(\tilde{N})} c_k(\gamma, \tilde{\eta}_\alpha, \tilde{x}).$$
(ii) Let N be an $\mathcal{O}$-stratum. For each nonnegative integer k and each $\alpha = 1, \dots, s$, define a continuous (in fact, smooth) function $c_{k,\alpha}(N, \cdot)$ on N as follows. First, for $x \in N \cap V_\alpha$, set $c_{k,\alpha}(N, x) = c_k(\tilde{N}, \tilde{x})$, where $\tilde{x}$ is any element of $\pi_\alpha^{-1}(x)$ and $\tilde{N}$ is the $\tilde{V}_\alpha$-stratum through $\tilde{x}$. By an argument analogous to that of Lemma 4.6, this definition is independent of the choice of $\tilde{x}$ in $\pi_\alpha^{-1}(x)$. Since $\tilde{\eta}_\alpha$ is supported in $\bar{\tilde{U}}_\alpha$, Lemma 4.17(ii) implies that $c_{k,\alpha}(N, \cdot)$ is 0 off $N \cap \bar{U}_\alpha$ and thus extends to a continuous function on N that is 0 off $N \cap U_\alpha$.
(iii) Set
$$I_{N,\alpha} := (4\pi t)^{-\dim(N)/2} \sum_{k=0}^{\infty} t^k \int_N c_{k,\alpha}(N, x)\, d\operatorname{vol}_N(x).$$

4.19. Lemma. *Let $\mathcal{O}$ be a closed Riemannian orbifold and let N be an $\mathcal{O}$-stratum. Then, for each nonnegative integer k,*
$$\sum_{\alpha=1}^{s} c_{k,\alpha}(N, \cdot) = b_k(N, \cdot) \quad \textit{and} \quad \sum_{\alpha=1}^{s} I_{N,\alpha} = I_N.$$

Proof. Let $x \in N$, and let $\alpha_1, \cdots, \alpha_r$ be those $\alpha \in \{1, \dots, s\}$ for which $x \in V_{\alpha_i}$. Then we can find a coordinate chart $(\tilde{U}, G_U, \pi_U)$ such that $U \subset V_{\alpha_1} \cap \cdots \cap V_{\alpha_r}$ and such that the chart $(\tilde{U}, G_U, \pi_U)$ embeds in each of the charts $(\tilde{V}_\alpha, G_\alpha, \pi_\alpha)$. Let $\lambda_i : \tilde{U} \to \tilde{V}_{\alpha_i}$ be the embedding. Let $\tilde{x} \in \pi_U^{-1}(x)$, let $\tilde{N}$ be the $\tilde{U}$-stratum through $\tilde{x}$, and let $\tilde{x}_i = \lambda_i(\tilde{x})$. As in the proof of Lemma 4.6, $\lambda_i(\tilde{N})$ is an open subset of the $\tilde{V}_{\alpha_i}$-stratum $\tilde{N}_i$ through $\tilde{x}_i$. Using the universality property (v) of Lemma 4.17, Notation 4.18, and an argument analogous to that of Lemma 4.6, we see that

$$c_{k,\alpha_i}(N, x) := c_{k,\alpha_i}(\tilde{N}_i, \tilde{x}_i) = \sum_{\gamma \in \mathrm{Iso}^{\max}(\tilde{N})} c_k(\gamma, \eta_{\alpha_i} \circ \pi_U, \tilde{x}).$$

Thus, since $c_{k,\alpha}(N, x) = 0$ when α is not one of $\alpha_1, \dots, \alpha_r$, we have

$$\sum_{\alpha=1}^{s} c_{k,\alpha}(N, x) = \sum_{\gamma \in \mathrm{Iso}^{\max}(\tilde{N})} \sum_{i=1}^{r} c_k(\gamma, \eta_{\alpha_i} \circ \pi_U, \tilde{x}). \tag{4.20}$$

Given (iii) and (iv) of Lemma 4.17 and since $\sum_{i=1}^{r} \eta_{\alpha_i} \equiv 1$ on U, we see that

$$\sum_{i=1}^{r} c_k(\gamma, \eta_{\alpha_i} \circ \pi_U, \tilde{x}) = b_k(\gamma, \tilde{x}) \tag{4.21}$$

on $\tilde{N}$. By Definition 4.7,

$$\sum_{\gamma \in \mathrm{Iso}^{\max}(\tilde{N})} b_k(\gamma, \tilde{x}) = b_k(N, x)$$

and thus the first equation in the lemma follows from (4.20) and (4.21). The second equation is then immediate. □

Proof of Theorem 4.8. Let $n = \dim(\mathcal{O})$. Because the support of η_α is contained in V_α, by Theorem 3.22 we have

$$\sum_{j=1}^{\infty} e^{-\lambda_j t} \sim_{t \to 0^+} \sum_{\alpha=1}^{s} \int_{V_\alpha} \eta_\alpha(x) H_\alpha(t, x, x)\, d\operatorname{vol}(\mathcal{O}), \tag{4.22}$$

where $H_\alpha(t, x, x)$ is defined in Notation 3.21. By Notation 3.21,

$$\begin{aligned}
&\sum_{\alpha=1}^{s} \int_{V_\alpha} \eta_\alpha(x) H_\alpha(t, x, x)\, d\operatorname{vol}_{\mathcal{O}}(x) \\
&\quad = \sum_{\alpha=1}^{s} \frac{1}{|G_\alpha|} \int_{\tilde{V}_\alpha} \tilde{\eta}_\alpha(\tilde{x})(4\pi t)^{-n/2}(u_0(\tilde{x}, \tilde{x}) + t u_1(\tilde{x}, \tilde{x}) + \cdots)\, d\operatorname{vol}_{\tilde{V}_\alpha}(\tilde{x}) \\
&\qquad + \sum_{\alpha=1}^{s} \frac{1}{|G_\alpha|} \sum_{1 \neq \gamma \in G_\alpha} \int_{\tilde{V}_\alpha} \tilde{\eta}_\alpha(\tilde{x})(4\pi t)^{-n/2} e^{-d(\tilde{x}, \gamma(\tilde{x}))^2/4t}(u_0(\tilde{x}, \gamma(\tilde{x})) \\
&\qquad + t u_1(\tilde{x}, \gamma(\tilde{x})) + \cdots)\, d\operatorname{vol}_{\tilde{V}_\alpha}(\tilde{x}),
\end{aligned} \tag{4.23}$$

where $\tilde{\eta}_\alpha = \eta_\alpha \circ \pi_\alpha$.

Consider the first sum on the right-hand side of equation (4.23). Since η_α is supported in V_α, we have

$$\begin{aligned}\sum_{\alpha=1}^{s}\frac{1}{|G_\alpha|}(4\pi t)^{-n/2}&\int_{\tilde V_\alpha}\tilde\eta_\alpha(\tilde x)(u_0(\tilde x,\tilde x)+tu_1(\tilde x,\tilde x)+\cdots)\,d\,\mathrm{vol}_{\tilde V_\alpha}(\tilde x)\\ &=\sum_{\alpha=1}^{s}(4\pi t)^{-n/2}\int_{\mathcal{O}}\eta_\alpha(x)(u_0(x,x)+tu_1(x,x)+\cdots)\,d\,\mathrm{vol}_{\mathcal{O}}(x)\\ &=(4\pi t)^{-n/2}\int_{\mathcal{O}}(u_0(x,x)+tu_1(x,x)+\cdots)\,d\,\mathrm{vol}_{\mathcal{O}}(x)=I_0.\end{aligned}\tag{4.24}$$

Next, by Lemma 4.17 and the remarks in 4.18, for each $1\neq\gamma\in G_\alpha$ we have

$$\begin{aligned}\frac{1}{|G_\alpha|}\sum_{1\neq\gamma\in G_\alpha}\int_{\tilde V_\alpha}\tilde\eta_\alpha(\tilde x)(4\pi t)^{-n/2}e^{-d(\tilde x,\gamma(\tilde x))^2/4t}&(u_0(\tilde x,\gamma(\tilde x))\\ &+tu_1(\tilde x,\gamma(\tilde x))+\cdots)\,d\,\mathrm{vol}_{\tilde V_\alpha}(\tilde x)\\ \sim\frac{1}{|G_\alpha|}\sum_{1\neq\gamma\in G_\alpha}\sum_{W\in\Omega(\gamma)}(4\pi t)^{-\dim(W)/2}\sum_{k=0}^{\infty}t^k\int_W c_k(\gamma,\tilde\eta_\alpha,\tilde x)\,d\,\mathrm{vol}_W(\tilde x).&\end{aligned}\tag{4.25}$$

By 4.18 and an argument identical to that in 4.12, the right-hand side of (4.25) is equal to

$$\sum_{N\in S(\mathcal{O})}\frac{1}{|\mathrm{Iso}(N)|}I_{N,\alpha}.$$

Consequently, Lemma 4.19 and (4.25) together imply that the second sum on the right-hand side of equation (4.23) is equal to

$$\sum_{N\in S(\mathcal{O})}\frac{1}{|\mathrm{Iso}(N)|}I_N.$$

Thus, in view of (4.22)–(4.24), the theorem is proved. □

5. Applications

5.1. THEOREM. *Let $\mathcal{O}$ be a Riemannian orbifold with singularities. If $\mathcal{O}$ is even dimensional* (*resp., odd dimensional*) *and if some $\mathcal{O}$-stratum of the singular set is odd dimensional* (*resp., even dimensional*), *then $\mathcal{O}$ cannot be isospectral to a Riemannian manifold.*

Proof. It is clear from equation (4.3) that, if N is any $\mathcal{O}$-stratum of the singular set, then the function $b_0(\gamma,\cdot)$ is strictly positive on N for each $\gamma\in\mathrm{Iso}^{\max}(N)$. Thus, in the two cases, that $\mathcal{O}$ is an orbifold can be gleaned from the presence of half-integer (rep. integer) powers of t in the asymptotic expansion in Theorem 4.8. □

5.2. REMARK. In [18] this theorem was stated for good orbifolds, but here we also include bad orbifolds.

We now restrict our attention to closed two-dimensional orbifolds (2-orbifolds). The singularities that may occur in 2-orbifolds are cone points, dihedral corner reflectors, and mirror reflectors (recall Example 2.15). A cone point p of order n is an isolated singularity; an orbifold chart for a neighborhood of p is $(\mathbb{D}^2, \mathbb{Z}_n, \pi)$, where $\mathbb{D}^2$ is an open 2-disk in $\mathbb{R}^2$ and $\mathbb{Z}_n$ is the cyclic group of order n. The Euler characteristic of a 2-orbifold is 2 minus the sum of the related values: each cone point of order n has value $(n-1)/n$; each dihedral corner reflector has value $(n-1)/2n$; each handle has value 2; each cross-cap has value 1; and each mirror reflector has value 1. Every good 2-orbifold admits a (metrically) spherical, Euclidean or hyperbolic structure depending on whether the Euler characteristic is (respectively) positive, zero, or negative [30]. In addition, all bad 2-orbifolds have positive Euler characteristic.

5.3. EXAMPLE. Let $\mathcal{O}$ be a 2-orbifold and let p be a cone point of order m. If $N = \{p\}$, then $\operatorname{Iso}(N)$ is a cyclic group of order m and $\operatorname{Iso}^{\max}(N)$ contains all of the nontrivial elements. If we let γ be the generator then, for $j = 1, \dots, m-1$,

$$A_{\gamma^j} = \gamma_*^j = \begin{bmatrix} \cos\left(\frac{2j\pi}{m}\right) & -\sin\left(\frac{2j\pi}{m}\right) \\ \sin\left(\frac{2j\pi}{m}\right) & \cos\left(\frac{2j\pi}{m}\right) \end{bmatrix},$$

where A_{γ^j} is as defined in 4.2. Thus

$$b_0(\gamma^j) = |\det((I - A_{\gamma^j})^{-1})| = \frac{1}{2 - 2\cos\left(\frac{2j\pi}{m}\right)} = \frac{1}{4\sin^2\left(\frac{j\pi}{m}\right)}.$$

(We write $b_0(\gamma^j)$, rather than using the function notation $b_0(\gamma^j, \cdot)$, because N consists of a single point.)

5.4. LEMMA.

$$\sum_{j=1}^{m-1} \frac{1}{\sin^2\left(\frac{j\pi}{m}\right)} = \frac{m^2-1}{3}.$$

Proof. A well-known formula (see e.g. [23, Ex. 7.9.1]), proven by the calculus of residues, states that

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{k=-\infty}^{\infty} \frac{1}{(k-z)^2}.$$

Thus

$$\sum_{j=1}^{m-1} \frac{1}{\sin^2\left(\frac{j\pi}{m}\right)} = \frac{m^2}{\pi^2} \sum_{j=1}^{m-1} \sum_{k=-\infty}^{\infty} \frac{1}{(mk-j)^2}.$$

Since

$$\sum_{j=1}^{m-1} \sum_{k=-\infty}^{\infty} \frac{1}{(mk-j)^2} = 2\sum_{n=1}^{\infty} \frac{1}{n^2} - 2\sum_{n=1}^{\infty} \frac{1}{m^2 n^2}$$

and since

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

the lemma follows. □

5.5. PROPOSITION. *Let $\mathcal{O}$ be a 2-orbifold, let p be a cone point in $\mathcal{O}$ of order m, and let $N = \{p\}$. Then, in the notation of Theorem 4.8,*

$$I_N = \frac{m^2-1}{12} + O(t).$$

Proof. By Definition 4.7(ii) and Example 5.3,

$$I_N = \sum_{j=1}^{m-1} \frac{1}{4\sin^2\left(\frac{j\pi}{m}\right)} + O(t).$$

Thus, Proposition 5.5 follows from Lemma 5.4. □

5.6. EXAMPLE (Calculating heat invariants for 2-orbifolds).

Degree 0 term for orientable 2-orbifolds. An orientable 2-orbifold $\mathcal{O}$ can have only isolated singularities (i.e., cone points). Suppose $\mathcal{O}$ has k cone points of orders $m_1, \dots, m_k$. In the notation of 4.7(iii),

$$I_0 = \frac{1}{4\pi}(a_0 t^{-1} + a_1 + O(t)).$$

Thus, by Theorem 4.8 and Proposition 5.5, the term of degree 0 in the asymptotic expansion in Theorem 4.8 is given by

$$\frac{a_1}{4\pi} + \sum_{i=1}^{k} \frac{1}{m_i} \frac{m_i^2-1}{12}.$$

By the Gauss–Bonnet theorem (valid also for orbifolds; see [26; 30]), we have

$$a_1 = \frac{2\pi}{3}\chi(\mathcal{O}).$$

Hence the degree 0 term is

$$\frac{\chi(\mathcal{O})}{6} + \sum_{i=1}^{k} \frac{m_i^2-1}{12 m_i}. \tag{5.7}$$

Degree 0 term for nonorientable 2-orbifolds. For a 2-orbifold $\mathcal{O}$, the dimension 0 singular locus is the only portion (aside from the $a_1/4\pi = \chi(\mathcal{O})/6$ component) that contributes to the degree 0 term of the asymptotic expansion in Theorem 4.8. A nonorientable 2-orbifold $\mathcal{O}$ can have cone points and/or dihedral corner reflector points that contribute in the following ways. As in the computation of the degree 0 term for orientable 2-orbifolds, here a simple cone point of order m contributes $\frac{1}{m}\frac{m^2-1}{12}$. Let $N = \{p\}$, where p is a corner reflector point created by a rotation of order n and a reflection. Then $|\mathrm{Iso}(N)| = 2n$. By Notation 2.14,

$\mathrm{Iso}^{\max}(N)$ contains only the nontrivial elements of the rotation group, since the reflection fixes one-dimensional strata of the mirror locus, a higher-dimensional stratum of the singular set. Hence the computations in Example 5.3 and Proposition 5.5 remain the same, but the difference in $|\mathrm{Iso}(N)|$ is seen as an extra $\frac{1}{2}$ factor in Theorem 4.8. The point contributes $\frac{1}{2n}\frac{n^2-1}{12}$ to the degree 0 term.

Thus, for a 2-orbifold $\mathcal{O}$ with cone points $p_1, \dots, p_k$ of orders $m_1, \dots, m_k$ and dihedral corner reflector points $q_1, \dots, q_r$ of orders $n_1, \dots, n_r$, the term of degree 0 in the asymptotic expansion of the heat trace is

$$\frac{\chi(\mathcal{O})}{6} + \sum_{i=1}^{k} \frac{1}{m_i}\frac{m_i^2 - 1}{12} + \sum_{j=1}^{r} \frac{1}{2n_j}\frac{n_j^2 - 1}{12}. \tag{5.8}$$

Degree 1 term for 2-orbifolds. Aside from a_2, only dimension 0 strata of the singular set contribute to the t term of the asymptotic expansion in Theorem 4.8. We first note that the last term in (4.4) is 0; this follows from the summation convention and the fact that our singular set is zero dimensional. Using symmetry properties of the curvature, we can further simplify (4.4) as

$$b_1(\gamma^j) = \frac{1}{4\sin^2\left(\frac{j\pi}{m}\right)}\left(\frac{\tau}{6} + \frac{\rho_{kk}}{6} + \frac{2}{3}(R_{1212}(B_{21}^2 + B_{12}^2 - B_{12}B_{21} - B_{22}B_{11}))\right),$$

where R_{1212} is evaluated in the local covering manifold. By the definitions of scalar and Ricci curvature, the preceding equation becomes

$$b_1(\gamma^j) = \frac{R_{1212}(1 + B_{21}^2 + B_{12}^2 - B_{12}B_{21} - B_{22}B_{11})}{6\sin^2\left(\frac{j\pi}{m}\right)}.$$

In general, straightforward calculations show that

$$B_{\gamma^j} = (I - A_{\gamma^j})^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{\sin\left(\frac{2j\pi}{m}\right)}{2 - 2\cos\left(\frac{2j\pi}{m}\right)} \\ \frac{\sin\left(\frac{2j\pi}{m}\right)}{2 - 2\cos\left(\frac{2j\pi}{m}\right)} & \frac{1}{2} \end{bmatrix},$$

which implies

$$b_1(\gamma^j) = \frac{R_{1212}}{8\sin^4\left(\frac{j\pi}{m}\right)}$$

for $j = 1, \dots, m-1$.

Thus, for a 2-orbifold $\mathcal{O}$ with cone points $p_1, \dots, p_k$ of orders $m_1, \dots, m_k$ and dihedral corner reflector points $q_1, \dots, q_r$ of orders $n_1, \dots, n_r$, the coefficient of the term of degree 1 in the asymptotic expansion of the heat trace is

$$\frac{a_2}{4\pi} + \sum_{i=1}^{k} \frac{1}{m_i}\left(\sum_{j=1}^{m_i-1} \frac{R_{1212}}{8\sin^4\left(\frac{j\pi}{m_i}\right)}\right) + \sum_{i=1}^{r} \frac{1}{2n_i}\left(\sum_{j=1}^{n_i-1} \frac{R_{1212}}{8\sin^4\left(\frac{j\pi}{n_i}\right)}\right). \tag{5.9}$$

Recall (see e.g. [2]) that $a_2(\mathcal{O}) = \frac{1}{360}\int_{\mathcal{O}}(2|R|^2 - 2|\rho|^2 + 5\tau^2)\, d\,\mathrm{vol}_{\mathcal{O}}(g)$, where R is the curvature, ρ is the Ricci curvature, and τ is the scalar curvature of $\mathcal{O}$.

We can further simplify (5.9) by making the substitution

$$\sum_{j=1}^{m_i-1} \frac{1}{\sin^4\left(\frac{j\pi}{m_i}\right)} = \frac{m_i^4 + 10m_i^2 - 11}{45}.$$

See [3] and the references therein or [7; 16] for evaluations of this and similar finite trigonometric sums.

With this substitution, (5.9) becomes

$$\frac{a_2}{4\pi} + \sum_{i=1}^{k} \frac{R_{1212}(m_i^4 + 10m_i^2 - 11)}{360m_i} + \sum_{i=1}^{r} \frac{R_{1212}(n_i^4 + 10n_i^2 - 11)}{720n_i}. \tag{5.10}$$

Degree $-\frac{1}{2}$ term for 2-orbifolds. The only $\mathcal{O}$-strata that contribute to the degree $1/\sqrt{t}$ term are those of codimension 1 in $\mathcal{O}$. To obtain these $\mathcal{O}$-strata, remove any dihedral points from the mirror locus and then take the connected components of the remaining set. Let $x \in N$, an $\mathcal{O}$-stratum of the mirror locus, and note that $\gamma \in \mathrm{Iso}^{\max}(N)$ must act as a reflection.

To compute $b_0(\gamma, x) = |\det((I - A_\gamma)^{-1})|$, observe that $\gamma_* = [-1]$ on the normal bundle to N and so

$$b_0(\gamma, x) = |\det((I - A_\gamma)^{-1})| = |[2]^{-1}| = \frac{1}{2}.$$

Applying 4.7(ii) yields

$$\begin{aligned} I_N &= (4\pi t)^{-1/2} \sum_{\gamma \in \mathrm{Iso}^{\max}(N)} \sum_{k=0}^{\infty} t^k \int_N \frac{1}{2}\, d\,\mathrm{vol}_N(x) + O(t) \\ &= \frac{\mathrm{length}(N)}{4\sqrt{\pi}} \frac{1}{\sqrt{t}} + O(\sqrt{t}\,). \end{aligned}$$

We sum over all $\mathcal{O}$-strata of the mirror locus to obtain the coefficient of the $1/\sqrt{t}$ term in Theorem 4.8:

$$\sum_N \frac{I_N}{|\mathrm{Iso}(N)|} = \sum_N \frac{1}{2} \frac{\mathrm{length}(N)}{4\sqrt{\pi}} = \frac{\mathrm{length}(\mathrm{MirrorLocus}(\mathcal{O}))}{8\sqrt{\pi}}. \tag{5.11}$$

Degree $\frac{1}{2}$ term for 2-orbifolds. For a 2-orbifold $\mathcal{O}$, the dimension 1 singular locus gives the sole contribution to the $\sqrt{t}$ term of the asymptotic expansion in Theorem 4.8.

We have

$$\begin{aligned} b_1(\gamma, x) = |\det(B_\gamma(x))| \bigg(&\frac{\tau}{6} + \frac{\rho_{kk}}{6} + \frac{1}{3} R_{iksh} B_{ki} B_{hs} \\ &+ \frac{1}{3} R_{ikth} B_{kt} B_{hi} - R_{k\alpha h\alpha} B_{ks} B_{hs} \bigg)(x), \end{aligned}$$

where B_{ij} denotes the i, j entry of $B_\gamma(x)$, τ is the scalar curvature of M at x, and ρ is the Ricci curvature.

In the case of a 2-orbifold with a point x in its mirror locus (but not a dihedral corner reflector point), the matrix $B_\gamma(x)$ is one dimensional and thus the third and fourth terms in the sum vanish. Since $B_\gamma(x) = \frac{1}{2}$, the last term is $-\frac{1}{4}R_{1212}$. We also have $\rho_{kk} = R_{1212}$ and $\tau = 2R_{1212}$. Thus $b_1(\gamma, x) = \frac{1}{2}\left(\frac{1}{4}R_{1212}\right)(x) = \frac{1}{8}R_{1212}(x) = \frac{1}{16}\tau(x)$. Hence, for N a stratum in the mirror locus, the term of degree $\frac{1}{2}$ in I_N is given by

$$\frac{\sqrt{t}}{\sqrt{4\pi}}\int_N \frac{1}{16}\tau(x)\,d\,\mathrm{vol}_N(x) = \frac{\sqrt{t}}{32\sqrt{\pi}}\int_N \tau(x)\,d\,\mathrm{vol}_N(x).$$

The coefficient of the degree $\frac{1}{2}$ term in the asymptotic expansion for the heat trace on $\mathcal{O}$ is thus

$$\frac{1}{64\sqrt{\pi}}\int_{\mathrm{MirrorLocus}(\mathcal{O})}\tau, \tag{5.12}$$

where the scalar curvature is the scalar curvature of $\mathcal{O}$ computed at points in the mirror locus and the integral is with respect to the induced Riemannian metric on the one-dimensional mirror locus. See Table 1 for the asymptotic expansions of the heat kernel for orbifolds $\mathcal{O}$ with $\chi(\mathcal{O}) \geq 0$.

Our notation for orbifolds is adapted from Conway's convention [9], with commas added for readability. Namely, $\mathcal{O}(a, b, *c, d)$ denotes an orbifold with simple cone points of orders a and b and with dihedral corner reflectors of orders $2c$ and $2d$. In addition, $\mathcal{O}(n\times)$ is a disk with a simple interior cone point and the edge identified via the antipodal map, while $\mathcal{O}(n*)$ is a disk with a simple interior cone point and a mirror edge corresponding to a reflection. A detailed explanation of the orbifold notation can be found in [9] (cf. the figures in [21, pp. 80–90]). See [30] for a proof that these are all of the closed 2-orbifolds with $\chi(\mathcal{O}) \geq 0$ and that, for $1 < m \neq n$, the only bad 2-orbifolds are $\mathcal{O}(m)$, $\mathcal{O}(*m)$, $\mathcal{O}(m, n)$, and $\mathcal{O}(*m, n)$.

Let $\mathcal{O}$ be an orientable 2-orbifold with k cone points of orders $m_1, \dots, m_k$, denoted $\mathcal{O}(m_1, \dots, m_k)$, and consider the quantity c defined as twelve times the degree 0 term:

$$c = 2\chi(\mathcal{O}) + \sum_{i=1}^{k}\left(m_i - \frac{1}{m_i}\right). \tag{5.13}$$

This quantity is a spectral invariant; it depends only on the topology, not on the Riemannian metric. For $\mathcal{O}(m_1, \dots, m_k)$, we denote by $c(m_1, \dots, m_k)$ the associated spectral invariant. We now investigate classes of orientable 2-orbifolds for which c is a complete topological invariant. Although Theorem 5.14 is a special case of Theorem 5.15, we begin with the more restricted class in order to give the reader intuition for the proof techniques used.

5.14. THEOREM. *Within the class of all footballs (good or bad) and all teardrops, the spectral invariant c is a complete topological invariant. That is, c distinguishes footballs from teardrops and determines the orders of the cone points.*

Table 1 2-Orbifold Expansions with $\chi(\mathcal{O}) \geq 0$

$\mathcal{O}$ with $\chi(\mathcal{O}) > 0$	Asymptotic Expansion
$\mathcal{O}(m)$	$V\frac{1}{t} + \frac{1}{12}\left(2 + m + \frac{1}{m}\right) + O(t)$
$\mathcal{O}(*m)$	$V\frac{1}{t} + ML\frac{1}{\sqrt{t}} + \frac{1}{24}\left(2 + m + \frac{1}{m}\right) + O\left(\sqrt{t}\right)$
$\mathcal{O}(m,n)$	$V\frac{1}{t} + \frac{1}{12}\left(m + n + \frac{1}{m} + \frac{1}{n}\right) + O(t)$
$\mathcal{O}(*m,n)$	$V\frac{1}{t} + ML\frac{1}{\sqrt{t}} + \frac{1}{24}\left(m + n + \frac{1}{m} + \frac{1}{m}\right) + O\left(\sqrt{t}\right)$
$\mathcal{O}(m\times)$	$V\frac{1}{t} + \frac{1}{12}\left(m + \frac{1}{m}\right) + O(t)$
$\mathcal{O}(m*)$	$V\frac{1}{t} + ML\frac{1}{\sqrt{t}} + \frac{1}{12}\left(m + \frac{1}{m}\right) + O\left(\sqrt{t}\right)$
$\mathcal{O}(2,2,m)$	$V\frac{1}{t} + \frac{1}{12}\left(3 + m + \frac{1}{m}\right) + O(t)$
$\mathcal{O}(*2,2,m), \mathcal{O}(2,*m)$	$V\frac{1}{t} + ML\frac{1}{\sqrt{t}} + \frac{1}{24}\left(3 + m + \frac{1}{m}\right) + O\left(\sqrt{t}\right)$
$\mathcal{O}(2,3,3)$	$V\frac{1}{t} + \frac{43}{72} + O(t)$
$\mathcal{O}(*2,3,3), \mathcal{O}(3,*2)$	$V\frac{1}{t} + ML\frac{1}{\sqrt{t}} + \frac{43}{144} + O\left(\sqrt{t}\right)$
$\mathcal{O}(2,3,4)$	$V\frac{1}{t} + \frac{97}{144} + O(t)$
$\mathcal{O}(*2,3,4)$	$V\frac{1}{t} + ML\frac{1}{\sqrt{t}} + \frac{97}{288} + O\left(\sqrt{t}\right)$
$\mathcal{O}(2,3,5)$	$V\frac{1}{t} + \frac{271}{360} + O(t)$
$\mathcal{O}(*2,3,5)$	$V\frac{1}{t} + ML\frac{1}{\sqrt{t}} + \frac{271}{720} + O\left(\sqrt{t}\right)$
$\mathcal{O}$ with $\chi(\mathcal{O}) = 0$	Asymptotic Expansion
torus, Klein bottle	$V\frac{1}{t} + O(t)$
*torus, *Klein bottle	$V\frac{1}{t} + ML\frac{1}{\sqrt{t}} + O\left(\sqrt{t}\right)$
$\mathcal{O}(2,2,2,2)$	$V\frac{1}{t} + \frac{1}{2} + O(t)$
$\mathcal{O}(*2,2,2,2), \mathcal{O}(2,*2,2), \mathcal{O}(2,2*)$	$V\frac{1}{t} + ML\frac{1}{\sqrt{t}} + \frac{1}{4} + O\left(\sqrt{t}\right)$
$\mathcal{O}(2,2\times)$	$V\frac{1}{t} + \frac{1}{4} + O(t)$
$\mathcal{O}(2,4,4)$	$V\frac{1}{t} + \frac{3}{4} + O(t)$
$\mathcal{O}(*2,4,4), \mathcal{O}(4,*2)$	$V\frac{1}{t} + ML\frac{1}{\sqrt{t}} + \frac{3}{8} + O\left(\sqrt{t}\right)$
$\mathcal{O}(3,3,3)$	$V\frac{1}{t} + \frac{2}{3} + O(t)$
$\mathcal{O}(*3,3,3), \mathcal{O}(3,*3)$	$V\frac{1}{t} + ML\frac{1}{\sqrt{t}} + \frac{1}{3} + O\left(\sqrt{t}\right)$
$\mathcal{O}(2,3,6)$	$V\frac{1}{t} + \frac{5}{6} + O(t)$
$\mathcal{O}(*2,3,6)$	$V\frac{1}{t} + ML\frac{1}{\sqrt{t}} + \frac{5}{12} + O\left(\sqrt{t}\right)$

Here $m, n \geq 1$, $V = \mathrm{vol}(\mathcal{O})/4\pi$ and $ML = \mathrm{length}(\mathrm{MirrorLocus}(\mathcal{O}))/8\sqrt{\pi}$. Note that *torus is an annulus with two mirror reflector edges and *Klein bottle is a Möbius band with one mirror reflector edge.

Proof. Denote by $\mathcal{O}(m)$ the teardrop with cone point of order m and by $\mathcal{O}(r,s)$ the football with cone points of orders r and s. Let $c(m)$ and $c(r,s)$ denote the invariant defined in equation (5.13) in the two cases. Then $\mathcal{O}(m)$ has Euler characteristic $1 + 1/m$ and thus

$$c(m) = 2 + m + \frac{1}{m}.$$

The football $\mathcal{O}(r,s)$ has Euler characteristic $1/r + 1/s$, so

$$c(r,s) = r + s + \frac{1}{r} + \frac{1}{s}.$$

The invariant is an integer only in the case of $\mathcal{O}(2,2)$, so the football $\mathcal{O}(2,2)$ is spectrally distinguishable from the other footballs and teardrops. Hence, for the remainder of the proof, all footballs will be assumed to have at least one cone point of order strictly greater than 2.

For teardrops, $c(m)$ trivially determines m. We next claim that footballs are distinguishable from teardrops. Indeed, suppose that $c(m) = c(r,s)$. Then $m + 2 = r + s$ and $1/m = 1/r + 1/s$. The latter equation implies that $m < \min(r,s)$. Since also $r, s \geq 2$, we have $2 + m < r + s$—a contradiction, thus proving the claim.

It remains only to show that, for footballs, $c(r,s)$ determines r and s. From $c(r,s)$ one can read off the quantities $r + s$ and

$$\frac{1}{r} + \frac{1}{s} = \frac{r+s}{rs}.$$

Therefore, $c(r,s)$ determines both $r+s$ and rs and thus also $|r-s|$, since $(r-s)^2 = (r+s)^2 - 4rs$. Hence (r,s) is determined up to order, completing the proof. □

5.15. Theorem. *Let C be the class consisting of all closed orientable 2-orbifolds with $\chi(\mathcal{O}) \geq 0$. The spectral invariant c is a complete topological invariant within C and, moreover, it distinguishes the elements of C from smooth oriented closed surfaces.*

Proof. We first consider the 2-orbifolds for which c is an integer. Note that—among teardrops, footballs, and triangular pillows—the only integer values are $c(2,2) = 5$, $c(2,3,6) = 10$, $c(2,4,4) = 9$, and $c(3,3,3) = 8$ (cf. Tables 1 and 2). In addition, $c(2,2,2,2) = 6$, $c(S^2) = 4$, and $c(T^2) = 0$. Let S_g be a Riemann surface of genus $g \geq 2$. Then we also have $c(S_g) = 4 - 4g$. It is clear that the values of c are distinct in each case, so that the spectrum distinguishes these 2-orbifolds.

For the rest of the proof it suffices to consider orbifolds in C with $\chi(\mathcal{O}) > 0$, since these include all orbifolds within C for which c is not an integer. Table 2 lists the values of c for these triangular pillows. Setting $c(2,3,3) = c(2,2,m)$ and solving for m gives $m = (25 \pm \sqrt{481})/12$, which contradicts the assumption that m is an integer. Similar calculations for $c(2,3,4)$ and $c(2,3,5)$ show that the spectrum distinguishes among triangular pillows with $\chi(\mathcal{O}) > 0$.

By Theorem 5.14, c distinguishes among teardrops and footballs. We next show that c distinguishes both teardrops and footballs from triangular pillows with $\chi(\mathcal{O}) > 0$. We have $c(m) = 2 + m + 1/m$ and $c(p,q,r) = -2 + p + q + r + 1/p + 1/q + 1/r$; setting the integer and fractional parts of these equations equal gives

Table 2 Triangular Pillow Orbifolds

	$\mathcal{O}$	$\chi(\mathcal{O})$	$c(\mathcal{O})$
$\chi(\mathcal{O}) > 0$	$\mathcal{O}(2,2,2)$	$\frac{1}{2}$	$5\frac{1}{2}$
	$\mathcal{O}(2,2,m)$	$\frac{1}{m}$	$3+m+\frac{1}{m}$
	$\mathcal{O}(2,3,3)$	$\frac{1}{6}$	$7\frac{1}{6}$
	$\mathcal{O}(2,3,4)$	$\frac{1}{12}$	$8\frac{1}{12}$
	$\mathcal{O}(2,3,5)$	$\frac{1}{30}$	$9\frac{1}{30}$
$\chi(\mathcal{O}) = 0$	$\mathcal{O}(3,3,3)$	0	8
	$\mathcal{O}(2,4,4)$	0	9
	$\mathcal{O}(2,3,6)$	0	10
$\chi(\mathcal{O}) < 0$	$\mathcal{O}(3,3,4)$	$-\frac{1}{12}$	$8\frac{11}{12}$
	$\mathcal{O}(3,4,4)$	$-\frac{1}{6}$	$9\frac{5}{6}$
	$\mathcal{O}(3,3,5)$	$-\frac{2}{15}$	$9\frac{13}{15}$
	$\mathcal{O}(2,4,5)$	$-\frac{1}{20}$	$9\frac{19}{20}$
	$\vdots$	$-1 < \chi(\mathcal{O}) < 0$	$c(\mathcal{O}) > 10$

$$2+m = -1+p+q+r,$$
$$\frac{1}{m} = \frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1.$$

Solving the first equation for m and plugging the result into the second equation yields

$$0 = pr(p+r-3)+pq(p+q-3)+qr(q+r-3)+pqr(5-p-q-r).$$

None of the possible triples (p,q,r) satisfy this equation, showing that teardrops are distinguished from triangular pillows with $\chi(\mathcal{O}) > 0$. The argument that c distinguishes good footballs from these triangular pillows is analogous.

To see that c distinguishes triangular pillows with $\chi(\mathcal{O}) > 0$ from bad footballs, we compare the respective integer and fractional parts of c in each case. For example, comparing $c(2,3,3)$ and $c(r,s)$ with $r \neq s$ gives

$$7 = r+s,$$
$$\frac{1}{6} = \frac{1}{r}+\frac{1}{s},$$

which implies $s^2-7s+42 = 0$. Thus $s = (7 \pm \sqrt{-119})/2$, which contradicts that s is an integer. The calculations are similar for $\mathcal{O}(2,3,4)$ and $\mathcal{O}(2,3,5)$, while for $\mathcal{O}(2,2,m)$ we have

$$3 + m = r + s,$$
$$\frac{1}{m} = \frac{1}{r} + \frac{1}{s},$$

which implies $r(r-3) + s(s-3) + rs = 0$. We can assume that one of r and s (say, r) is strictly greater than 2. Thus $r - 3 \geq 0$ and $s - 3 \geq -1$, which implies

$$r(r-3) + s(s-3) + rs \geq s(r-1) > 0.$$

Thus triangular pillows with $\chi(\mathcal{O}) > 0$ are distinguished from bad footballs. □

5.16. REMARK. Notably absent from the class C are triangular pillows with $\chi(\mathcal{O}) < 0$. The invariant c does not seem sufficiently strong to distinguish among these triangular pillows. However, as a special case of a result in [13], the spectrum does determine the orders of the cone points in such a 2-orbifold *provided* that it is endowed with a metric of constant curvature -1. In contrast, we do not need such a metric assumption to distinguish, say, triangular pillows with $\chi(\mathcal{O}) < 0$ from triangular pillows with $\chi(\mathcal{O}) > 0$.

If $\mathcal{O}(p,q,r)$ is any triangular pillow with $\chi(\mathcal{O}) < 0$ and if $\operatorname{spec}(\mathcal{O}(p,q,r)) = \operatorname{spec}(\mathcal{O}(2,2,m))$, then by setting the integer and fractional parts of the respective values of c equal we obtain

$$m + 3 = p + q + r - 2,$$
$$\frac{1}{m} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}.$$

Solving the first equation for m and plugging the resulting value into the second equation yields

$$2pqr + pr(p+r-5) + pq(p+q-5) + qr(q+r-5) = 0. \tag{5.17}$$

Observe that, for a triangular pillow with $\chi(\mathcal{O}) < 0$, we must have $1/p + 1/q + 1/r < 1$, which implies that the sum of any two of p, q, r is at least 5. Thus, each term on the left-hand side of (5.17) is nonnegative. This contradiction implies in turn that $\mathcal{O}(2,2,m)$ cannot be isospectral to such a triangular pillow. For the remaining triples where $\chi(\mathcal{O}) > 0$, we first set the integer and fractional parts of the respective value of c equal to those of a triangular pillow with $\chi(\mathcal{O}) < 0$ and then note that there are no $\chi(\mathcal{O}) < 0$ triples satisfying these equations. Thus c distinguishes between triangular pillows with $\chi(\mathcal{O}) < 0$ and $\chi(\mathcal{O}) > 0$. A similar argument shows that c distinguishes triangular pillows with $\chi(\mathcal{O}) < 0$ from teardrops; it is clear that c also distinguishes triangular pillows with $\chi(\mathcal{O}) < 0$ from the smooth surfaces and the remaining elements of C, with the exception of footballs. In this last case, it seems that metric assumptions are again necessary.

5.18. REMARK. Other difficulties arise during the consideration of the expanded class that includes nonorientable orbifolds. Metric assumptions are necessary in

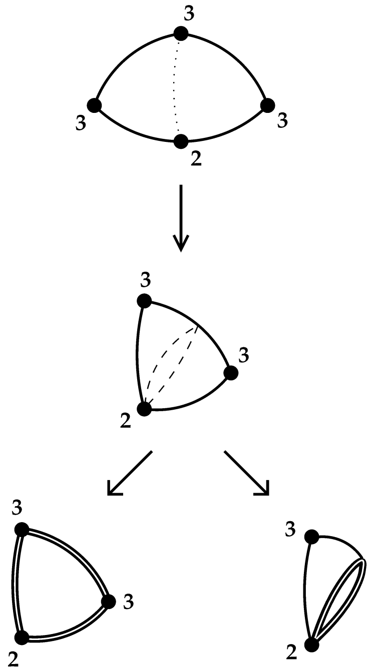

Figure 1 The uppermost object is a fundamental domain for $\mathcal{O}(2,3,3)$, with its quotient $\mathcal{O}(2,3,3)$ below (here the vertices are cone points labeled with their orders and the double edges represent reflector edges; on the bottom left we show $\mathcal{O}(*2,3,3)$, which is obtained by reflecting $\mathcal{O}(2,3,3)$ in the plane of the paper, and on the right is $\mathcal{O}(3,*2)$, obtained by reflecting $\mathcal{O}(2,3,3)$ in the plane containing the dashed loop)

numerous cases, such as distinguishing nonorientable orbifolds with the same orientable double cover (e.g., $\mathcal{O}(*2,3,3)$ and $\mathcal{O}(3,*2)$; see Table 1 and Figure 1).

We now examine classes that include nonorientable orbifolds.

5.19. Proposition. *Within the class of all closed 2-orbifolds with $\chi(\mathcal{O}) \geq 0$, the spectrum distinguishes whether the orbifold has zero or positive Euler characteristic.*

Proof. Note that, for 2-orbifolds $\mathcal{O}$ with $\chi(\mathcal{O}) = 0$, c is either an integer or equal to 4.5. Hence c distinguishes all but the following cases: S^2 from the orbifolds $\mathcal{O}(*3,3,3)$ and $\mathcal{O}(3,*3)$ (with $c = 4$), the good football $\mathcal{O}(2,2)$ from $\mathcal{O}(*2,3,6)$ (with $c = 5$), and the bad teardrop $\mathcal{O}(2)$ from the orbifolds $\mathcal{O}(*2,4,4)$ and $\mathcal{O}(4,*2)$ (with $c = 4.5$). The lack of a mirror locus in the $\chi(\mathcal{O}) > 0$ cases and the presence of a mirror locus in the corresponding $\chi(\mathcal{O}) = 0$ orbifolds can be gleaned from the degree $-\frac{1}{2}$ term, and so they are distinguished. □

5.20. Proposition. *Within the class of closed 2-orbifolds of constant nonzero curvature R or $-R$, the spectrum determines the sign of the curvature—that is, whether the orbifold is spherical or hyperbolic.*

Proof. Assume that $\mathcal{O}$ has cone points $p_1, \ldots, p_k$ of orders $m_1, \ldots, m_k$ and/or dihedral corner reflector points $q_1, \ldots, q_r$ of orders $n_1, \ldots, n_r$. Now look at the coefficient of the t term in the expansion, as in (5.10), which reduces to

$$\frac{a_2}{4\pi} \pm R\left(\sum_{i=1}^{k} \frac{(m_i^2+11)(m_i^2-1)}{360m_i} + \sum_{i=1}^{r} \frac{(n_i^2+11)(n_i^2-1)}{720n_i}\right)$$

in the presence of constant curvature. The $1/t$ term in the expansion gives us $\mathrm{vol}(\mathcal{O})$ and we know the size of the curvature, so we know the a_2 component and may subtract it off. Observe that the summands are nonnegative; hence we can read the sign of the curvature unless there were no cone points and no dihedral corner reflector points. In that case, examine the degree 0 term, which now has no point contributions and reduces to $a_1/4\pi = \chi(\mathcal{O})/6$, and read off the Euler characteristic. □

5.21. REMARK. In the case of closed 2-orbifolds with a nontrivial mirror locus, (5.12) enables us to make a stronger statement. In particular: among such orbifolds that are endowed with a metric of strictly positive, strictly negative, or zero curvature, the spectrum determines the sign of the curvature. This class includes the bad orbifolds $\mathcal{O}(*m)$ and $\mathcal{O}(*p,q)$ with $p \neq q$, since they admit a metric of strictly positive (but variable) curvature.

5.22. PROPOSITION. *Within the class of spherical* 2*-orbifolds of constant curvature* $R > 0$, *the spectrum determines the orbifold.*

Proof. Notice that the metric requirement eliminates teardrops and bad footballs (and their quotients) from this class. In Table 1, c distinguishes among the remaining spherical orbifolds with the exception that c is unable to distinguish between orbifolds that are nonorientable but have the same orientable double cover: $\mathcal{O}(*m,m)$, $\mathcal{O}(m\times)$, and $\mathcal{O}(m*)$ with double cover $\mathcal{O}(m,m)$; $\mathcal{O}(*2,2,m)$ and $\mathcal{O}(2,*m)$ with double cover $\mathcal{O}(2,2,m)$; and $\mathcal{O}(*2,3,3)$ and $\mathcal{O}(3,*2)$ with double cover $\mathcal{O}(3,3,2)$ (see Figure 1). However, c is able to distinguish each nonorientable class from the remaining orbifolds.

Consider such a class of nonorientable spherical orbifolds with a common orientable double cover. Equation (5.11), the coefficient of the degree $-\frac{1}{2}$ term, distinguishes nonorientable orbifolds with mirror loci from those without—that is, in this class it distinguishes orbifolds with only cross-caps from those with mirror loci. In the presence of constant curvature, (5.11) also distinguishes among the remaining spherical cases: among $\mathcal{O}(*m,m)$ and $\mathcal{O}(m*)$, the length of the mirror locus of $\mathcal{O}(*m,m)$ is larger in the constant curvature metric; among $\mathcal{O}(*2,2,m)$ and $\mathcal{O}(2,*m)$, the length of the mirror locus of $\mathcal{O}(*2,2,m)$ is larger; and among $\mathcal{O}(*2,3,3)$ and $\mathcal{O}(3,*2)$, the length of the mirror locus of $\mathcal{O}(*2,3,3)$ is larger (see Figure 1). □

5.23. REMARK. Metric assumptions are needed only to distinguish within each nonorientable class. We cannot make a similar statement for flat 2-orbifolds. For example, it is possible to endow $\mathcal{O}(2,*2,2)$ and $\mathcal{O}(2,2*)$ with a metric of zero curvature so that they have the same area and also have mirror loci of the same length. Then they cannot be distinguished by the asymptotic expansion of the heat trace.

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