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Proper Pseudoholomorphic Maps between Strictly Pseudoconvex Regions

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1. Introduction

The regularity up to the boundary of a proper holomorphic map between strictly pseudoconvex bounded domains D and D' of $\mathbb{C}^n$ has been widely studied when $n \geq 2$ and is now as well understood as the one-dimensional case. A continuous map $F : D \to D'$ is said to be *proper* if $F^{-1}(K)$ is compact for every compact set K in D'. If D and D' have $\mathcal{C}^r$-boundaries ($r \geq 2$) then such a map F has a $\mathcal{C}^{r-1/2}$-extension to the boundary, and this is the maximal regularity [4; 20] that can be expected. Various authors have contributed to this result. We just mention Fefferman, who proved in 1974 that if D and D' have smooth boundaries and if F is a biholomorphism, then F extends smoothly to the boundary [9]. We refer to the survey of Forstnerič [10] for a thorough history.

Our aim here is to study the boundary behavior of proper pseudoholomorphic maps between strictly pseudoconvex regions in almost complex manifolds. In order to establish an analogue of the known result in the complex situation, we will need to adapt objects and tools specific to the integrable case. Note, for example, that there is no longer a notion of an analytic set and that the Jacobian of a pseudoholomorphic map is not pseudoholomorphic. Our method uses pseudoholomorphic discs. Originally introduced by E. Bishop, this method has provided geometric proofs of various versions of Fefferman's theorem [17; 25] even in the almost complex case [5; 12].

We consider the following situation. Let D be a bounded domain in some smooth (real) manifold, and let J be an almost complex structure of class $\mathcal{C}^1$ on $\bar{D}$ that is smooth in D. Throughout this paper, we will say that (D, J) is a *strictly pseudoconvex region* if D is defined by $\{\rho < 0\}$, where ρ is a $\mathcal{C}^2$-regular defining function that is strictly J-plurisubharmonic on $\bar{D}$. We say (D, J) is a *strictly pseudoconvex region of class* $\mathcal{C}^r$ when ρ and J are at least of class $\mathcal{C}^r$. In the complex situation, the regularity is thus the regularity of the boundary.

The first step of our proof is to derive the Hölder $\frac{1}{2}$-continuous extension, which comes from a sized estimate of the set of regular values and from estimates of the Kobayashi metric. To obtain more regularity, the main obstacle compared with the biholomorphic case is obviously the existence of critical points. Thus we begin with studying the locus of all these points. For the complex case, Pinchuk [19; 20]

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used the scaling method to show that every proper holomorphic map between two bounded strictly pseudoconvex domains in $\mathbb{C}^n$ having $\mathcal{C}^2$-boundaries is locally biholomorphic. We prove that this statement is always true in the almost complex case, at least near the boundary, as follows.

THEOREM 1. *Let (D, J) and (D', J') be some strictly pseudoconvex regions of the same dimension, and let F be a proper pseudoholomorphic map from D to D'. Then the Jacobian of F is far from zero near the boundary of D.*

In particular, the critical points of F form a compact set in D. Our proof is based on an almost complex adaptation of the scaling method (see also [5]). The idea of this method consists in anisotropically dilating the domains in order to construct a limit map between simple model domains. When the manifolds are almost complex, the transformations operating on the domains are not pseudoholomorphic and so we simultaneously dilate the almost complex structures. Note that if $n = 2$ then one can normalize the initial structures to obtain the standard structure as a limit. In the general case, the limit almost complex structures are not necessarily integrable.

The essential tool for gaining more regularity for the extension to the boundary is the study of a family of pseudoholomorphic discs attached to the boundary of the domain, which is invariant under the action of pseudobiholomorphisms. We obtain that the regularity of the extension depends on the regularity of the almost complex structures at the boundary, and we also give explicit estimates of the Hölderian norms via results of [3]. More precisely, we have the following theorem.

THEOREM 2. *Let (D, J) and (D', J') be strictly pseudoconvex regions of the same dimension, respectively $\mathcal{C}^r$ and $\mathcal{C}^{r'}$, where $r, r' \geq 2$ are not integers. Then every proper pseudoholomorphic map $F\colon D \to D'$ has a $\mathcal{C}^s$-extension to $\bar{D}$, where*

$$s = \begin{cases} \min(r, r') & \text{if } |r' - r| \geq 1, \\ \max(r-1, r'-1) & \text{if } |r' - r| < 1. \end{cases}$$

Moreover, for $s' = \min(r-1, r')$,

$$\|F\|_{\mathcal{C}^{s'-1}(\bar{D})} \leq c(s') \|(F, {}^t(dF)^{-1})\|_\infty \left(1 + \frac{c'}{\sqrt{\lambda^{J'}_{N^*(\partial D')}}}\right).$$

Here $\lambda^{J'}_{N^*(\partial D')}$ denotes the smallest eigenvalue of the Levi form of the square root of the distance to the conormal bundle $N^*(\partial D')$. We will call it the *minimal J-curvature of $N^*(\partial D')$*.

When (D, J) and (D', J') are smooth, we thus obtain that F has a smooth extension. Note that $J' = F_* J$ near the boundary by Theorem 1. Hence, as in the biholomorphic case [5], we have the following necessary and sufficient condition on J' for the smooth extension of F.

COROLLARY 1. *Let (D, J) and (D', J') be strictly pseudoconvex regions of the same dimension. Assume that (D, J) is of class $\mathcal{C}^\infty$. Then a proper pseudoholomorphic map $F\colon D \to D'$ extends smoothly to the boundary if and only if $\partial D'$ is smooth and J' extends smoothly on $\overline{D'}$.*

The paper is organized as follows. Section 2 consists of some preliminaries about almost complex manifolds. In Section 3 we prove the boundary distance preserving property for a proper pseudoholomorphic map, which leads to the existence of the Hölder $\frac{1}{2}$-continuous extension. Section 4 covers application of the scaling method, after which we study the properties of the limit map. Finally, in Section 5 we prove Theorems 1 and 2 and Corollary 1.

2. Preliminaries

2.1. Strictly Pseudoconvex Domains in Almost Complex Manifolds

We begin by recalling some definitions.

Definition 1. An *almost complex structure* on a smooth (real) manifold M^{2n} is a $(1,1)$-tensor J—that is, a section from M to $\mathrm{End}(TM)$ such that $J^2 = -\mathrm{Id}$.

Every almost complex structure admits a Hermitian metric and also provides an orientation on the manifold.

Definition 2. A map $F\colon (M, J) \to (M', J')$ of class $\mathcal{C}^1$ between two almost complex manifolds is said to be (J, J')*-holomorphic* if $J' \circ dF = dF \circ J$.

If (M, J) is the unit disc of $\mathbb{C}$ (i.e., $\Delta \subset \mathbb{R}^2$ equipped with the standard complex structure), we say that F is a J*-holomorphic disc*. Nijenhuis and Woolf [18] proved that such maps exist. Moreover, one can prescribe $F(0)$ and $dF_0(\partial/\partial x)$.

As in the complex case, the maps verifying the pseudoholomorphy's equation inherit their smoothness from the smoothness of the almost complex structures: if J and J' are of class $\mathcal{C}^r$, then every (J, J')-holomorphic map is of class $\mathcal{C}^{r+1}$. We also have the following lemma.

Lemma 1. *Let F be a pseudoholomorphic map between almost complex manifolds of the same dimension. Then F either preserves or inverts the orientation provided by the almost complex structures.*

Proof. After fixing local coordinates, we must show that the sign of the Jacobian of F is constant on D. The almost complex structure J verifies $J_p = PJ_{st}P^{-1}$ for some matrix P, where $J_{st} = \left(\begin{smallmatrix} 0 & -I_n \\ I_n & 0 \end{smallmatrix}\right)$ is the standard complex structure on $\mathbb{R}^{2n}$. Such a factorization is not unique, but if $J_p = PJ_{st}P^{-1} = QJ_{st}Q^{-1}$ then the matrix $Q^{-1}P$ commutes with J_{st} and hence $\det(Q^{-1}P) > 0$. Thus, the sign of $\det P$ depends only on p and J; we denote it by $\delta^J(p)$.

Given $p_0 \in D$, let $(e_1^{(p_0)}, \dots, e_{2n}^{(p_0)})$ be a basis such that, for all $k \geq 1$, $e_{n+k}^{(p_0)} = J_{p_0} e_k^{(p_0)}$. Denote by P_p the matrix of $(e_1^{(p_0)}, \dots, e_n^{(p_0)}, J_p e_1^{(p_0)}, \dots, J_p e_n^{(p_0)})$, and let V be a neighborhood of p_0 in which the matrix P_p remains invertible. In particular, for every $p \in V$, $\det P_p$ and $\det P_{p_0}$ have the same sign. It follows that δ^J is locally constant on D and hence is constant.

For any $p \in D$ and $q \in D'$, we write $J_p = P_p J_{st} P_p^{-1}$ and $J'_q = P'_q J_{st} P_q'^{-1}$. By the (J, J')-holomorphy of F, for all $p \in D$ we have $dF_p J_p = J'_{F(p)} dF_p$, so $(P_{F(p)}'^{-1} dF_p P_p)$ commutes with the complex standard structure and

$$\det(P'^{-1}_{F(p)} dF_p P_p) \geq 0.$$

Finally, the sign of the Jacobian of F is equal to $\delta^J \times \delta^{J'}$ at any noncritical point. □

DEFINITION 3. Let (M, J) be an almost complex manifold and let ρ be a $\mathcal{C}^2$-smooth function from M to $\mathbb{R}$. For all $X \in TM$, define $d^c_J\rho(X) = -d\rho(JX)$ and $\mathcal{L}^J\rho = d(d^c_J\rho)(X, JX)$. The quadratic form $\mathcal{L}^J\rho$ is called the *Levi form* of ρ. The function ρ is said to be *strictly J-plurisubharmonic* (resp. *J-plurisubharmonic*) if its Levi form is positive definite (resp. positive).

REMARK 1. One may show that a $\mathcal{C}^2$-regular map $\rho\colon \Omega \to \mathbb{R}$ is J-plurisubharmonic if and only if all the compositions $\rho \circ h$, for any J-holomorphic disc h valued in Ω, are subharmonic on the unit disc Δ. Indeed, there is the following link between the Laplacian of $\rho \circ h$ and the Levi form of ρ [7; 13]:

$$\forall \zeta \in \Delta, \quad \Delta(\rho \circ h)_\zeta = \mathcal{L}^J_{h(\zeta)}(\rho)\left(\frac{\partial h}{\partial x}(\zeta)\right).$$

2.2. *Holomorphic Mappings between Simple Model Domains*

Model almost complex structures naturally appear as limits of rescaled almost complex structures. We refer to [12] for a detailed treatment of model structures.

Throughout this paper, we denote by $(x_0, y_0, \dots, x_n, y_n)$ the coordinates in $\mathbb{R}^{2n+2}$ and by $z = (z_0, \dots, z_n) = (z_0, {}'z)$ the associated complex coordinates. Thus $\mathbb{R}^{2n+2}$ may be identified with $\mathbb{C}^{n+1}$ by means of the standard complex structure

$$\mathcal{J}^{(n+1)}_{st} = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & (0) & \\ & & \ddots & & \\ & (0) & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}.$$

DEFINITION 4. An almost complex structure J on $\mathbb{R}^{2n+2}$ is called a *model structure* if it is defined by

$$J(z) = \begin{pmatrix} \mathcal{J}^{(1)}_{st} & B^J({}'z) \\ 0 & \mathcal{J}^{(n)}_{st} \end{pmatrix}, \tag{1}$$

where $B^J({}'z) \in \mathcal{M}_{2,2n}(\mathbb{R})$ is $\mathbb{R}$-linear in $x_1, \dots, x_n, y_1, \dots, y_n$. A pair (Σ, J) is called a *model domain* if $\Sigma = \{z \in \mathbb{C}^{n+1} \mid \operatorname{Re} z_0 + P({}'z, {}'\bar{z}) < 0\}$, where P is some real homogeneous polynomial of degree 2 and J is a model structure such that Σ is strictly J-pseudoconvex at 0.

For such a matrix B^J, the corresponding complex matrix is

$$\begin{aligned} B_{\mathbb{C}}({}'z) &= (B^{2j-1}_{2k-1} + iB^{2j}_{2k-1})_{1\leq j,k\leq n} \\ &= \left(\sum_{k=1}^n (a^J_{1,k} z_k + b^J_{1,k}\bar{z}_k) \cdots \sum_{k=1}^n (a^J_{n,k} z_k + b^J_{n,k}\bar{z}_k)\right), \end{aligned}$$

where $a^J_{j,k}$ and $b^J_{j,k}$ are complex constants.

DEFINITION 5. The model structure J given by (1) is *simple* if $a^J_{j,k} = 0$ for all j, k.

For any model domain (Σ, J) there exists a simple model structure $\mathcal{J}$ and a $(J, \mathcal{J})$-biholomorphism between Σ and $\mathbb{H}$ fixing $(-1, '0)$, where $\mathbb{H} = \{z \in \mathbb{C}^{n+1} \mid \operatorname{Re} z_0 + \|'z\|^2 < 0\}$ is the Siegel half-plane [16].

The proof shows that the map constructed is in fact a global diffeomorphism of $\mathbb{C}^{n+1}$ given by

$$\Psi(z) = (z_0 + \psi('z), '\Psi('z)),$$

where ψ is a polynomial of degree at most 2 in $'z$ and $'\bar{z}$ and where $'\Psi : \mathbb{C}^n \to \mathbb{C}^n$ is $\mathbb{C}$-linear in $'z$. Consequently:

(1) for all $t < 0$, $\Psi(t, '0) = (t, '0)$;
(2) if the model domain Σ is given by the equation $0 = \tilde{r}(z) = \operatorname{Re} z_0 + P('z, '\bar{z}) < 0$, with P some real homogeneous polynomial of degree 2 on $\mathbb{C}^n$, then

$$\tilde{r} \circ \Psi^{-1}(z) = \operatorname{Re} z_0 + \|'z\|^2.$$

The proof also implies that, if the model structure $\mathcal{J}$ is integrable, then we can in fact prescribe $\mathcal{J} = \mathcal{J}_{st}$.

Because of the special form of simple model structures, a pseudoholomorphic map $F = (F_0, 'F)$ between simple model domains has an interesting behavior. We can suppose that the domains are both the Siegel half-plane $\mathbb{H}$.

PROPOSITION 1 [16]. *Assume that $\mathcal{J}$ and $\mathcal{J}'$ are* non-integrable *simple model structures on $\mathbb{H}$. If $F : (\mathbb{H}, \mathcal{J}) \to (\mathbb{H}, \mathcal{J}')$ is a pseudoholomorphic map, then there exists a real constant c such that*

$$\forall z = (z_0, 'z) \in \mathbb{H}, \quad F(z) = (cz_0 + f('z), 'F('z)), \tag{2}$$

where $f : \mathbb{C}^n \to \mathbb{C}$ is antiholomorphic and $'F : \mathbb{C}^n \to \mathbb{C}^n$ is holomorphic (*with respect to the standard complex structure*).

Proof. The proof is given in [16] in the case of a pseudobiholomorphism. We sketch it here for the general case using the notation of Definition 4.

Computation of the coefficients of the Nijenhuis tensor $N_{\mathcal{J}}$ together with the hypothesis that $\mathcal{J}$ is nonintegrable yields some j, k such that $b^{\mathcal{J}}_{j,k} - b^{\mathcal{J}}_{k,j} \neq 0$. Let us fix such a pair (j, k). We may identify the coefficients in $dF\big(N_{\mathcal{J}}\big(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\big)\big) = N_{\mathcal{J}'}\big(dF\big(\frac{\partial}{\partial z_j}\big), dF\big(\frac{\partial}{\partial z_k}\big)\big)$ and obtain, equivalently,

$$\forall l \geq 1, \quad \frac{\partial F_l}{\partial z_0} = \frac{\partial F_l}{\partial \bar{z}_0} = 0, \tag{3}$$

$$\frac{\partial F_0}{\partial z_0} = \frac{1}{b^{\mathcal{J}}_{j,k} - b^{\mathcal{J}}_{k,j}} \sum_{l,m=1}^{n} \frac{\partial \bar{F}_l}{\partial \bar{z}_j} \frac{\partial \bar{F}_m}{\partial \bar{z}_k} (b^{\mathcal{J}'}_{l,m} - b^{\mathcal{J}'}_{m,l}). \tag{4}$$

The system obtained in (3) means that $F_1, \ldots, F_n$ do not depend on z_0 and $\bar{z}_0$ (precisely, of x_0 and y_0). Moreover, F is $(\mathcal{J}, \mathcal{J}')$-holomorphic, and since $\mathcal{J}$ and $\mathcal{J}'$ are simple model structures it follows that

$$dF \circ \mathcal{J} = \mathcal{J}' \circ dF \implies \begin{cases} \mathcal{J}^{(1)}_{st} \circ d_{z_0} F = d_{z_0} F \circ \mathcal{J}^{(n)}_{st}, \\ \mathcal{J}^{(n)}_{st} \circ d('F) = d('F) \circ \mathcal{J}^{(n)}_{st}. \end{cases}$$

Hence $'F\colon {}'z \mapsto (F_1('z), \dots, F_n('z))$ and $z_0 \mapsto F_0(z_0, 'z)$ are holomorphic (with respect to the standard complex structure). This implies that $\frac{\partial F_0}{\partial z_0}$ is independent of z_0 and antiholomorphic in $'z$:

$$F_0(z) = c('z)z_0 + f('z),$$

where c and f are antiholomorphic. For any $'z \in \mathbb{C}^n$, the map $z_0 \mapsto F_0(z_0, 'z)$ is defined on $\{\zeta \in \mathbb{C} \mid \operatorname{Re}\zeta < -\|'z\|^2\}$, and it takes its values in $\{\zeta \in \mathbb{C} \mid \operatorname{Re}\zeta < -\|'F('z)\|^2\}$. This requires that c be real-valued and antiholomorphic, hence constant. $\square$

3. Hölder $\frac{1}{2}$-Continuous Extension

3.1. Regular Values of Proper Maps

We recall that a *critical point* of a $\mathcal{C}^1$ map F from D to D' is a point $p \in D$ such that the Jacobian of F vanishes at p. In the sequel, we will denote by C the set of all critical points. A *critical value* is the image by F of some critical point. Every point of D' that is not a critical value, even if it is not in $F(D)$, is called a *regular value*.

REMARK 2. If $F\colon (D, J) \to (D', J')$ is pseudoholomorphic then, for any critical point $p \in D$, the subspace $\operatorname{Ker} dF_p$ contains a subspace of dimension 2 because it is preserved by J_p. By [8, Thm. 3.4.3] we then have that the Hausdorff dimension of the set $F(C)$ of all critical values is less than $2n - 2$.

We also show the following property of proper pseudoholomorphic maps.

LEMMA 2. *Let (D, J) and (D', J') be strictly pseudoconvex regions of dimension $2n$, and let F be a proper pseudoholomorphic map from D to D'. Then there is at least one point $p \in D$ such that $F(p)$ is a regular value.*

Proof. It suffices to prove that the open set $D \setminus C$ is not empty. Indeed, this will imply, by means of the rank theorem, that the (Hausdorff) dimension of $F(D \setminus C)$ is equal to $2n$. According to Remark 2, we will have $F(C) \neq F(D \setminus C)$.

Assume by way of contradiction that $C = D$. Then the maximal rank r_0 of dF in D is less than $2n - 2$, and it is nonzero because F is necessarily nonconstant. The locus C_0 where the rank of dF is r_0 is an open set, so $F(C_0)$ is a submanifold of (Hausdorff) dimension r_0.

Since the rank of the map dF is less than $r_0 - 1$ (in fact, $r_0 - 2$) on $D \setminus C_0$, [8, Thm. 3.4.3] yields $\dim_H F(D \setminus C_0) \le r_0 - 1$. Thus $F(C_0) \neq F(D \setminus C_0)$. This allows us to pick $q \in F(C_0) \setminus F(D \setminus C_0)$. The set $N = F^{-1}(\{q\})$ included in the open set C_0 is then a submanifold of D of dimension $2n - r_0$. Moreover, N is compact and its tangent bundle $\operatorname{Ker} dF$ is preserved by J. Hence J induces an almost complex structure on N.

Let ρ be some $\mathcal{C}^2$-regular and strictly J-plurisubharmonic function on D. The map ρ reaches its maximum on N at some point p. According to [18], there exists a pseudoholomorphic disc h that takes its values in N, is centered at p, and verifies $\frac{\partial h}{\partial x}(0) \neq 0$. By the maximum principle, the strictly subharmonic function $\rho \circ h$ is constant. This leads to a contradiction because $0 = \triangle(\rho \circ h)_p = \mathcal{L}^J_p(\rho)\big(\frac{\partial h}{\partial x}(0)\big) > 0$ in view of Remark 1. □

The next statement is well known in the complex case.

PROPOSITION 2. *Let (D, J) and (D', J') be strictly pseudoconvex regions of the same dimension. If F is proper and pseudoholomorphic from D to D', then $F(D) = D'$. Moreover, all the regular values of F have the same (finite) number of antecedents, and they form a path-connected open set that is dense in D'.*

Proof. Recall (see e.g. [24]) that if X and Y are two oriented connected manifolds of the same dimension and if $F\colon X \to Y$ is a smooth proper map, then the degree of F is equal to $\sum_{p\in F^{-1}(q)} \operatorname{sgn}(\det dF_p)$ for any regular value $q \in Y$. In particular, if q is not in the image of F, then the degree of F is 0. Let $q \in F(D) \setminus F(C)$ be a regular value of F. The degree of F is either positive or negative by Lemma 1, so $F(D) = D'$ and the preimage of any regular value of F has exactly $\deg F$ elements.

The set of all critical points of F is closed in D. Indeed, F is proper and hence closed, and $D' \setminus F(C)$ is an open set in D'. According to [22, Prop. 14.4.2] and Proposition 2, $F(C)$ has no interior. That $D' \setminus F(C)$ is path-connected follows from classical geometric arguments, as in the complex case. □

3.2. Boundary Distance Preserving Property

The aim of this section is to prove the following statement, which will give as a corollary the Hölder $\frac{1}{2}$-continuous extension of a proper pseudoholomorphic map.

PROPOSITION 3. *Let (D, J) and (D', J') be strictly pseudoconvex regions of dimension $2n$, and let F be a proper pseudoholomorphic map from D to D'. Then there exist $c_1, c_2 > 0$ such that*

$$\forall p \in D, \quad c_1 \leq \frac{\operatorname{dist}(F(p), \partial D')}{\operatorname{dist}(p, \partial D)} \leq c_2.$$

Proof. The Hopf lemma in the almost complex situation [5] applies to the map $\rho' \circ F$ on D and gives some $c' > 0$ such that

$$\forall p \in D, \quad |\rho'(F(p))| \geq c' \operatorname{dist}(p, \partial D).$$

The idea of the proof of Proposition 3 is to construct a J-plurisubharmonic map on D', say $\rho' \circ F^{-1}$, in order to apply again the Hopf lemma. More precisely, we define

$$u\colon D' \ni q \mapsto \operatorname*{Max}_{p\in F^{-1}(\{q\})} \{\rho(p)\}.$$

Note that u takes negative values and is continuous on $D' \setminus F(C)$. Indeed, for any q in $D' \setminus F(C)$, the compact set $K = F^{-1}(\{q\})$ consists of noncritical, hence isolated, points; thus K is finite. Set $K = \{p_1, \dots, p_k\}$, where $k = |\deg F|$. According to the inverse function theorem, for each $j = 1, \dots, k$ one can construct a neighborhood V_j of p_j and a neighborhood W of q included in the open set $D' \setminus F(C)$ such that (a) F induces a $\mathcal{C}^1$-diffeomorphism from V_j to W and (b) $F^{-1}(W) = \bigsqcup V_j$. For $j = 1, \dots, k$, set $F_j = F|_{V_j} : V_j \to W$ and $u_j = \rho \circ F_j^{-1}$. The maps u_j are continuous and J-plurisubharmonic. Consequently, $u = \operatorname{Max}_{1 \le j \le k} u_j$ is continuous.

We also have that u is J-plurisubharmonic on W' in the following sense: the composition $u \circ h$ is subharmonic on the unit disc $\Delta \subset \mathbb{C}$ for every J-holomorphic disc $h : \Delta \to W$ (this extends Definition 3 to the case of an upper semicontinuous function). Thus u is locally J-plurisubharmonic on $D' \setminus F(C)$, which is equivalent to being globally J-plurisubharmonic.

Since $\dim_H(F(C)) \le 2n - 2$, we obtain that $\limsup(u)$ is plurisubharmonic on the whole D' (see [6]). By the Hopf lemma, there is some constant $c > 0$ such that $|\limsup(u)(q)| \ge c \operatorname{dist}(q, \partial D')$ for any $q \in D'$. The map ρ is continuous and so,

$$\forall p \in D, \quad |\rho(p)| \ge |\limsup(u)(F(p))| \ge c \operatorname{dist}(F(p), \partial D');$$

thus we obtain the desired inequalities. □

COROLLARY 2. *Let (D, J) and (D', J') be strictly pseudoconvex regions of dimension $2n$, and let F be a proper pseudoholomorphic map from D to D'. Then F has a continuous extension $\bar F : \bar D \to \overline{D'}$ such that $\bar F(\partial D) \subset \partial D'$. Moreover, $\bar F$ is Hölder continuous with exponent $\frac{1}{2}$.*

Proof. Once F is known to preserve the distance to the boundary, Corollary 2 follows from estimates of the infinitesimal Kobayashi pseudometric (as in the proof of [5, Prop. 3.3]). We sketch the arguments here for the sake of completeness.

For every $p \in D$ and for v a tangent vector at point p, set

$$K_{(D,J)}(p, v) = \inf\{\alpha > 0 \mid \exists h \in \mathcal{O}^J(\Delta, D) \text{ with } h(0) = p \text{ and } (\partial h/\partial x)(0) = v/\alpha\},$$

which is well-defined according to [18]. Let us now recall the following result [5; 11]: under our hypotheses, there exists a constant $C > 0$ such that

$$\forall p \in D, \quad \forall v \in T_pM, \quad \frac{1}{C}\frac{\|v\|}{\operatorname{dist}(p, \partial D)^{1/2}} \le K_{(D,J)}(p, v) \le C\frac{\|v\|}{\operatorname{dist}(p, \partial D)}.$$

Then, by the decreasing property of the infinitesimal Kobayashi pseudometric, for any $p \in D$ and $v \in T_pM$ we have

$$C_1\frac{\|dF_p(v)\|}{\operatorname{dist}(F(p), \partial D')^{1/2}} \le K_{(D',J')}(F(p), dF_p(v)) \le K_{(D,J)}(p, v) \le C_2\frac{\|v\|}{\operatorname{dist}(p, \partial D)},$$

which implies the estimate

$$|||dF_p||| \le C\frac{\|v\|}{\operatorname{dist}(p, \partial D)^{1/2}}.$$

This gives the statement by Hardy and Littlewood's theorem. □

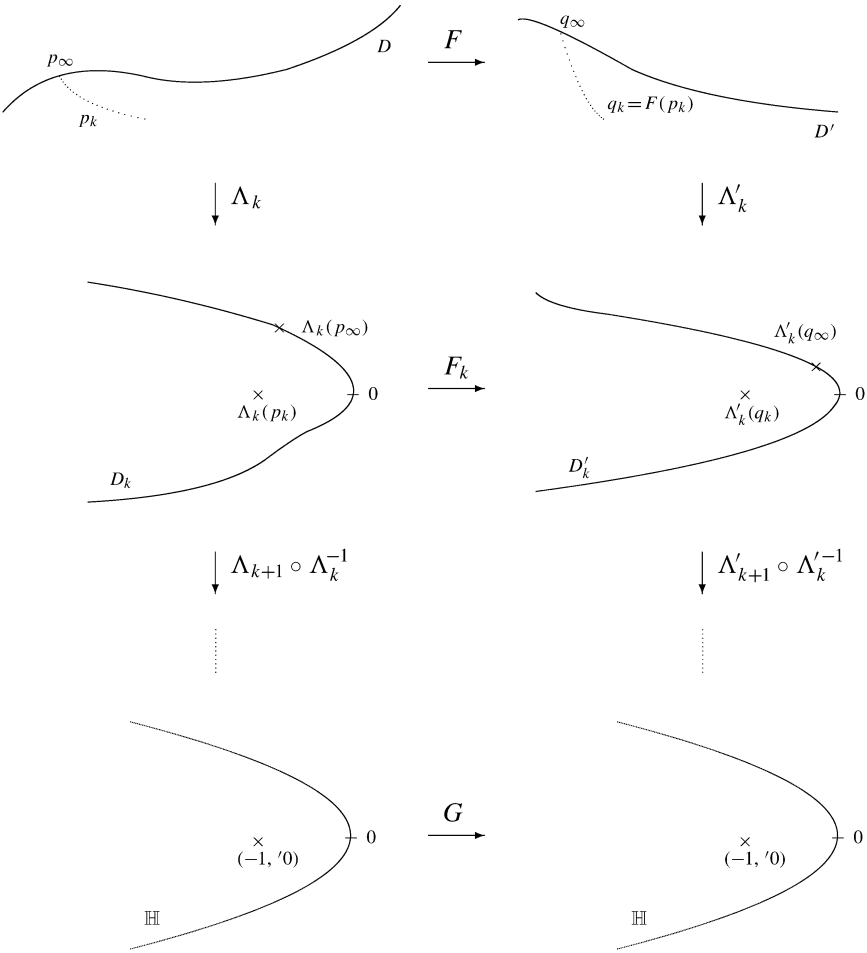

Figure 1

4. The Scaling Method

The proof of Theorem 1 is based on the scaling method. The idea is to rectify ∂D and $\partial D'$ via successive changes of variable in order to obtain $\partial \mathbb{H}$. To make the transformations holomorphic, we also rescale the almost complex structures as in [12] (see also [16]). See Figure 1.

4.1. Dilations

Let (p_k) be a sequence of points of D converging to $p_\infty \in \partial D$, and set $q_k = F(p_k)$. According to the boundary distance preserving property, (q_k) converges to $q_\infty = F(p_\infty) \in \partial D'$.

4.1.1. *Choice of Local Coordinates*

In an adapted local coordinate system $\Phi\colon U \to \mathbb{R}^{2n+2}$ about p_∞ with $\Phi(p_\infty) = 0$, we identify p_∞ with 0 and U with $\mathbb{R}^{2n+2}$. Moreover, we may assume that $\rho \circ \Phi^{-1}$ is bounded for the $\mathcal{C}^1$ norm and that the following statements hold.

- $J(0) = J'(0) = \mathcal{J}_{st}$.
- $D = \{p \in \mathbb{R}^{2n+2} \mid \rho(p) < 0\}$ and $T_0(\partial D) = \{x_0 = 0\}$, where the defining function ρ can be expressed by

$$\rho(z) = \operatorname{Re} z_0 + \operatorname{Re}\Bigg(z_0 \sum_{j\geq 1}(\rho_j z_j + \rho_{\bar{j}}\bar{z}_j)\Bigg) + P('z, '\bar{z}) + \rho_\varepsilon(z)$$

  with P a real homogeneous polynomial of degree 2 and $\rho_\varepsilon(z) = o(\|z\|^2)$.
- $D' = \{p \in \mathbb{R}^{2n+2} \mid \rho'(p) < 0\}$ and $T_0(\partial D') = \{x_0 = 0\}$, where the defining function ρ can be expressed by

$$\rho'(z) = \operatorname{Re} z_0 + \operatorname{Re}\Bigg(z_0 \sum_{j\geq 1}(\rho'_j z_j + \rho'_{\bar{j}}\bar{z}_j)\Bigg) + Q('z, '\bar{z}) + \rho'_\varepsilon(z)$$

  with Q a real homogeneous polynomial of degree 2 and $\rho'_\varepsilon(z) = o(\|z\|^2)$.

4.1.2. *Centering*

Recall that for every neighborhood V of 0 one can find some constant $\delta > 0$ such that, for all $p \in V \cap \partial D$, the closed ball of radius δ centered at $p - \delta\vec{n}_p$ is in $D \cup \{p\}$ (here $\vec{n}_p$ denotes the outer normal to D at p). Hence $\dot{p}_k \in \partial D$ and $\dot{q}_k \in \partial D'$ such that

$$\operatorname{dist}(p_k, \partial D) = \|p_k - \dot{p}_k\| = d_k \quad \text{and} \quad \operatorname{dist}(q_k, \partial D') = \|q_k - \dot{q}_k\| = d'_k$$

are uniquely defined for some sufficiently large k. Then there exists a rigid motion $\phi_k\colon \mathbb{R}^{2n+2} \to \mathbb{R}^{2n+2}$, with $\phi_k(\dot{p}_k) = 0$ and $\phi_k(p_k) = (-d_k, 0, \ldots, 0)$, verifying:

- the tangent space to $\partial(\phi_k(D))$ at 0 is $\{\operatorname{Re} z_0 = 0\}$ and the complex tangent space to $\partial(\phi_k(D))$ at 0 (for the induced almost complex structure $(\phi_k)_* J$) is $\{0\} \times \mathbb{C}^n$;
- ϕ_k converges to the identity mapping on any compact subset of $\mathbb{R}^{2n+2}$ with respect to the $\mathcal{C}^2$-topology.

Consequently, $\dot{J}^k = (\phi_k)_* J$ converges to J in the $\mathcal{C}^1$ sense on any compact subset and is expressed by

$$\dot{J}^k(0) = \begin{pmatrix} \dot{J}^k_{(1,1)}(0) & 0_{2,2n} \\ \dot{J}^k_{(2,1)}(0) & \dot{J}^k_{(2,2)}(0) \end{pmatrix}. \tag{5}$$

The sequence $\rho_k = \rho \circ \phi_k^{-1}$ also converges to ρ at second order with respect to the compact-open topology:

$$\begin{aligned}\rho_k(z) &= \rho \circ \phi_k^{-1}(z) \\ &= \tau_k\Bigg(\operatorname{Re} z_0 + \operatorname{Re}\Bigg(z_0 \sum_{j\geq 1}(\rho^k_j z_j + \rho^k_{\bar{j}}\bar{z}_j)\Bigg) + P^k('z, '\bar{z}) + \rho^k_\varepsilon(z)\Bigg),\end{aligned}$$

where P^k is some real homogeneous polynomial of degree 2 and $\rho_\varepsilon^k(z) = o(\|z\|^2)$ uniformly in k. Let us also define the inhomogeneous dilation

$$\delta_k\colon (z_1,\dots,z_n) \mapsto \left(\frac{z_1}{d_k}, \frac{z_2}{\sqrt{d_k}}, \dots, \frac{z_n}{\sqrt{d_k}}\right)$$

and set

$$\Lambda_k = \delta_k \circ \phi_k \circ \Phi, \qquad D_k = \Lambda_k(D),$$

$$r_k = \frac{1}{d_k\tau_k}\rho \circ \Lambda_k^{-1}, \qquad J^k = (\Lambda_k)_* J.$$

We construct in the same way, $\phi'_k\colon \mathbb{R}^{2n+2} \to \mathbb{R}^{2n+2}$, $\dot{J}'^k = (\phi'_k)_* J$, $\rho'_k = \rho' \circ \phi_k'^{-1}$, and the dilation δ'_k. We also define Λ'_k, D'_k, r'_k, and J'^k. Finally, we set

$$F_k = \Lambda'_k \circ F \circ \Lambda_k^{-1}\colon D_k \to D'_k.$$

4.1.3. *Convergence*

Dilation yields $d_k r_k(z) = d_k(\operatorname{Re} z_0 + P^k('z, '\bar{z})) + O\big(d_k\sqrt{d_k}\,\big)$, from which our next lemma follows.

LEMMA 3. *The sequence (r_k) converges at second order to $\tilde{r}$ with respect to the compact-open topology, and D_k converges in the sense of local Hausdorff set convergence to $\tilde{D} = \{z \in \mathbb{R}^{2n+2} \mid \tilde{r}(z) < 0\}$, where*

$$\tilde{r}(z) = \operatorname{Re} z_0 + P('z, '\bar{z}).$$

There is a similar statement for (r'_k) and D'_k.

LEMMA 4. *The sequence of almost complex structures (J^k), respectively (J'^k), converges on any compact subset to a model structure $\tilde{J}$, respectively $\tilde{J}'$, in the $\mathcal{C}^1$ sense.*

Proof. We follow [16]. Writing almost complex structures as matrices, we have

$$J(z) = J(0) + \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} = \begin{pmatrix} \mathcal{J}_{st}^{(1)} + A(z) & B(z) \\ C(z) & \mathcal{J}_{st}^{(n-1)} + D(z) \end{pmatrix}$$

and

$$\begin{aligned} \dot{J}^k(z) &= \dot{J}^k(0) + \begin{pmatrix} \dot{A}^k(z) & \dot{B}^k(z) \\ \dot{C}^k(z) & \dot{D}^k(z) \end{pmatrix} \\ &= \begin{pmatrix} \dot{J}^k_{(1,1)}(0) + \dot{A}^k(z) & \dot{B}^k(z) \\ \dot{J}^k_{(2,1)}(0) + \dot{C}^k(z) & \dot{J}^k_{(2,2)}(0) + \dot{D}^k(z) \end{pmatrix}, \end{aligned}$$

where $\dot{A}^k \to A$, $\dot{B}^k \to B$, $\dot{C}^k \to C$, and $\dot{D}^k \to D$ with respect to the $\mathcal{C}^1$-topology on any compact subset. Let us define

$$\begin{aligned} J^k(z) &= \begin{pmatrix} \frac{1}{d_k} I_2 & 0 \\ 0 & \frac{1}{\sqrt{d_k}} I_{2n} \end{pmatrix} \dot{J}^k(\delta_k^{-1}(z)) \begin{pmatrix} d_k I_2 & 0 \\ 0 & \sqrt{d_k} I_{2n} \end{pmatrix} \\ &= \begin{pmatrix} \dot{J}^k_{(1,1)} + \dot{A}^k(\delta_k^{-1}(z)) & \frac{1}{\sqrt{d_k}} \dot{B}^k(\delta_k^{-1}(z)) \\ \sqrt{d_k}\,\dot{J}^k_{(2,1)} + \sqrt{d_k}\,\dot{C}^k(\delta_k^{-1}(z)) & \dot{J}^k_{(2,2)} + \dot{D}^k(\delta_k^{-1}(z)) \end{pmatrix}. \end{aligned}$$

Since δ_k^{-1} converges uniformly to 0 and $\dot{J}^k$ converges uniformly to J on any compact subset, it follows that

$$\begin{aligned} \dot{J}^k_{(1,1)} + \dot{A}^k(\delta_k^{-1}(z)) &\to \mathcal{J}_{st}^{(1)}, \\ \sqrt{d_k}\,\dot{J}^k_{(2,1)} + \sqrt{d_k}\,\dot{C}^k(\delta_k^{-1}(z)) &\to 0, \\ \dot{J}^k_{(2,2)} + \dot{D}^k(\delta_k^{-1}(z)) &\to \mathcal{J}_{st}^{(n)} \end{aligned}$$

on any compact subset with respect to the $\mathcal{C}^1$ topology. Thus $\dot{B}^k(z)$ and $B(z)$ may be expressed as

$$\dot{B}^k(z) = \sum_{j=1}^{n} (B^k_{2j-1}x_j + B^k_{2j}y_j) + B^k_\varepsilon(z),$$

$$B(z) = \sum_{j=1}^{n} (B_{2j-1}x_j + B_{2j}y_j) + B_\varepsilon(z),$$

where B^k_j is a sequence of constant matrices that converges to B_j as $k \to +\infty$, $B^k_\varepsilon \to B_\varepsilon$ in the $\mathcal{C}^1$ sense on any compact subset, and $B^k_\varepsilon = o(\|z\|)$ uniformly in k. Therefore,

$$\begin{aligned} \frac{1}{\sqrt{d_k}}\dot{B}^k(\delta_k^{-1}(z)) &= \sqrt{d_k}(B^k_1 x_1 + B^k_2 y_1) \\ &\quad + \sum_{j=2}^{n} (B^k_{2j-1}x_j + B^k_{2j}y_j) + \frac{1}{\sqrt{d_k}} B^k_\varepsilon(d_k z_1, \sqrt{d_k}\,'z) \\ &\to \sum_{j=2}^{n} (B_{2j-1}x_j + B_{2j}y_j) \quad \text{as } k \to +\infty. \end{aligned}$$

We obtain that J^k converges on any compact subset of $\mathbb{R}^{2n+2}$ with respect to the $\mathcal{C}^1$ topology to $\tilde{J}$ defined as

$$\tilde{J}(z) = \begin{pmatrix} \mathcal{J}_{st}^{(1)} & \tilde{B}('z) \\ 0 & \mathcal{J}_{st}^{(n)} \end{pmatrix}, \quad \tilde{B}('z) = \sum_{j=2}^{n} (B_{2j-1}x_j + B_{2j}y_j). \qquad \square$$

LEMMA 5. *$(\tilde{D}, \tilde{J})$ and $(\tilde{D}', \tilde{J}')$ are model domains.*

Proof. We recall the proof given in [12] for the sake of completeness. Define $\tilde{r}_k = \rho \circ \delta_k^{-1}$ and $\tilde{J}^k = \delta_{k*}J$. As for $\dot{r}_k$ and $\dot{J}^k$, one can show that $\tilde{r}_k/d_k$ converges to $\tilde{r}$ at second order with respect to the compact-open topology and that $\tilde{J}^k$ converges to $\tilde{J}$ in the $\mathcal{C}^1$ sense on any compact subset. Consequently, for any v,

$$\mathcal{L}_0^{\tilde{J}^k}\left(\frac{\tilde{r}_k}{d_k}\right)(v) \xrightarrow[k\to+\infty]{} \mathcal{L}_0^{\tilde{J}}\tilde{r}(v).$$

By the invariance of the Levi form we obtain $\mathcal{L}_0^J\rho(v) = \mathcal{L}_0^{\tilde{J}^k}\tilde{r}_k(d\delta_k(v))$. Since $\tilde{J}^k(0) = \mathcal{J}_{st}$ it follows that any complex tangent vector to the domain defined by $\tilde{r}_k$ is of the form $(0, v')$, and $d\delta_k(v) = v/\sqrt{d_k}$. For such a v,

$$\mathcal{L}_0^J \rho(v) = \mathcal{L}_0^{\tilde{J}^k} \tilde{r}_k(d\delta_k(v)) = \mathcal{L}_0^{\tilde{J}^k} \tilde{r}_k\left(\frac{v}{\sqrt{d_k}}\right) = \mathcal{L}_0^{\tilde{J}^k}\left(\frac{\tilde{r}_k}{d_k}\right)(v).$$

Passing to the limit, we have that $\mathcal{L}_0^{\tilde{J}} \tilde{r}(v) > 0$ for every v in the complex tangent space to $\tilde{D}$ at 0. □

LEMMA 6. *The sequence (F_k) admits a subsequence that converges at first order with respect to the compact-open topology to a map $\tilde{F}$ defined on $\tilde{D}$ and valued in $\overline{\tilde{D}'}$. The map $\tilde{F}$ is $(\tilde{J}, \tilde{J}')$-holomorphic and verifies $\tilde{F}(-1,'0) = (-1,'0)$.*

Proof. Suppose that K is some compact subset in $\tilde{D}$. Observe that, in order to show the existence of the desired subsequence, we need only prove that (F_k) is bounded in the $\mathcal{C}^0$-norm on K. Indeed, cover K by small bidiscs and consider two transversal foliations by J-holomorphic curves on every bidisc. Such a foliation is a small perturbation of the foliation by complex lines because J is a small perturbation of the standard structure (see [18]). The restriction of F_k on every such curve is bounded in the $\mathcal{C}^0$-norm; hence it is bounded in the $\mathcal{C}^1$-norm by the elliptic estimates [23]. Since the bounds are uniform with respect to curves, the sequence (F_k) is bounded in the $\mathcal{C}^1$-norm on K.

Now we follow [16]. First, recall that there exists an $\alpha > 0$ such that, for any sufficiently large k and $r \in [0; 1[$ and for any J^k-holomorphic disc $h\colon \Delta \to D_k \cap U$ such that $h(0) \in Q(0,\alpha)$, there is a positive constant C_r such that $h(\Delta_r) \subset Q(0, C_r\alpha)$.

For every $p \in \tilde{D}$, there is a neighborhood U_p of p and a family $\mathcal{H}_p$ of pseudoholomorphic discs centered at p such that $U_p \subset \bigcup_{h\in\mathcal{H}_p} h(\Delta_{r(p)})$ (see [7; 14; 15]). Hence one can find a finite covering $\{U_{t_j}\}_{j=0,\dots,m}$ of K such that $t_0 = (-1,'0)$ and $U_{t_j} \cap U_{t_{j+1}} \neq \emptyset$. Set $r = \max\{r(t_j)\}$. Since $\delta_k'^{-1} \circ F_k(-1,'0) = (-d_k','0) \in Q(0, d_k')$, we have $\delta_k'^{-1} \circ F_k \circ h(\Delta_r) \subset Q(0, C_r d_k')$ for all $h \in \mathcal{H}_{t_0}$, and

$$\delta_k'^{-1} \circ F_k(U_{t_0}) \subset Q(0, C_r d_k').$$

For all $h \in \mathcal{H}_{t_1}$, there exists an $\omega \in \Delta_r$ such that $h(\omega) \in U_{t_0} \cap U_{t_1}$. The pseudoholomorphic disc $g\colon \zeta \mapsto h\left(\frac{\zeta+\omega}{1+\bar{\omega}\zeta}\right)$ verifies $g(0) \in Q(0, C_r d_k')$ and $g(\omega) = h(0)$. Thus $\delta_k'^{-1} \circ F_k(t_1) \in Q(0, C_r^2 d_k')$ and $\delta_k'^{-1} \circ F_k(U_{t_1}) \subset Q(0, C_r^2 d_k')$. Iterating this process yields $\delta_k'^{-1} \circ F_k(U_{t_m}) \subset Q(0, C_r^{2m+1} d_k')$, whence

$$F_k(K) \subset \delta_k'(Q(0, C_K d_k')) = Q(0, C_K).$$

Let $(F_{k'})$ be a subsequence converging at first order with respect to the compact-open topology to some map $\tilde{F}\colon \tilde{D} \to \overline{\tilde{D}'}$. Passing to the limit in the pseudoholomorphy condition, we get that $\tilde{F}$ is $(\tilde{J}, \tilde{J}')$-holomorphic. □

In the particular case $n = 2$, one can also find a proof of this statement in [5] that is based on the method developed in [1; 2].

4.2. Properties of the Limit Map G

There exist simple model structures $\mathcal{J}, \mathcal{J}'$ on $\mathbb{H}$ and pseudobiholomorphisms $\Psi\colon \tilde{D} \to \mathbb{H}$ and $\Psi'\colon \tilde{D}' \to \mathbb{H}$ fixing $(-1,'0)$ and both continuous and one-to-one to the boundary. Let us define $G = \Psi' \circ \tilde{F} \circ \Psi^{-1}$. By construction, $G\colon \mathbb{H} \to \bar{\mathbb{H}}$ is $(\mathcal{J}, \mathcal{J}')$-holomorphic and fixes $(-1, '0)$.

If the almost complex structure $\mathcal{J}$ (resp. $\mathcal{J}'$) is integrable, then we can prescribe $\mathcal{J} = \mathcal{J}_{st}$ (resp. $\mathcal{J}' = \mathcal{J}_{st}$).

4.2.1. *Boundary Distance Preserving Property*

LEMMA 7. *For any bounded subset K in $\mathbb{H}$, there exist some constants $C_K, C'_K > 0$ such that, for all $p \in K$,*

$$C_K \leq \frac{\operatorname{dist}(G(p), \partial\mathbb{H})}{\operatorname{dist}(p, \partial\mathbb{H})} \leq C'_K.$$

In particular, G takes its values in $\mathbb{H}$ (and not only in $\bar{\mathbb{H}}$) and admits a locally Hölder $\frac{1}{2}$-continuous extension to $\bar{\mathbb{H}}$ verifying $G(\partial\mathbb{H}) \subset \partial\mathbb{H}$.

Proof. The proof of Proposition 3 gives two constants $c, c' > 0$ such that, for any $p \in D$,

$$|\rho'(F(p))| \geq c \operatorname{dist}(p, \partial D) \quad \text{and} \quad |\rho(p)| \geq c' \operatorname{dist}(F(p), \partial D').$$

Since $F_k = \Lambda'_k \circ F \circ \Lambda_k^{-1}$, for all $p \in D_k = \Lambda_k(D)$ it follows that

$$\begin{aligned} c \operatorname{dist}(\Lambda_k^{-1}(p), \partial D) &\leq |\rho' \circ \Lambda_k'^{-1}(F_k(p))| = d'_k \tau'_k |r'_k(F_k(p))|, \\ c' \operatorname{dist}(F \circ \Lambda_k^{-1}(p), \partial D') &\leq |\rho \circ \Lambda_k^{-1}(p)| = d_k \tau_k |r_k(p)|. \end{aligned} \tag{6}$$

Let q be a point of the boundary ∂D. Then

$$\|\Lambda_k^{-1}(p) - q\| \geq \frac{1}{\operatorname{Max}_{\bar{D}} \|d\phi_k\|} \|\delta_k^{-1}(p - \Lambda_k(q))\| \geq \frac{d_k}{\operatorname{Max}_{\bar{D}} \|d\phi_k\|} \|p - \Lambda_k(q)\|.$$

Therefore, by (6) we obtain

$$c_k \operatorname{dist}(p, \partial \tilde{D}_k) \leq |\tilde{r}'_k(F_k(p))| \quad \text{where} \quad c_k = c \, \frac{d_k}{d'_k \tau'_k \operatorname{Max}_{\bar{D}} \|d\phi_k\|};$$

$$c'_k \operatorname{dist}(F_k(p), \partial \tilde{D}'_k) \leq |\tilde{r}_k(p)| \quad \text{where} \quad c'_k = c' \, \frac{d'_k}{d_k \tau_k \operatorname{Max}_{\bar{D}} \|d\phi'_k\|}.$$

According to Proposition 3,

$$\frac{d'_k}{d_k} = \frac{\operatorname{dist}(F(p_k), \partial D')}{\operatorname{dist}(p_k, \partial D)}$$

is bounded between two positive constants. Passing to the limit, we obtain constants $C, C' > 0$ such that, for any $p \in \tilde{D}$ (and hence $p \in \tilde{D}_k$ for some sufficiently large k),

$$\begin{aligned} c \operatorname{dist}(\tilde{F}(p), \partial \tilde{D}') &\leq |\tilde{r}(p)|, \\ c' \operatorname{dist}(p, \partial \tilde{D}) &\leq |\tilde{r}'(\tilde{F}(p))|. \end{aligned} \tag{7}$$

Applying the diffeomorphisms Ψ' and Ψ^{-1}, we obtain the two desired inequalities. The same arguments as in the proof of Corollary 2 give the locally Hölder $\frac{1}{2}$-continuous extension to $\bar{\mathbb{H}}$. $\square$

COROLLARY 3. *Writing* $G = (G_0, 'G)$, *one has*

$$\operatorname{Re}(G_0(t, '0)) \xrightarrow[t\in\mathbb{R},\ t\to-\infty]{} -\infty.$$

Proof. Since $r(z) = \operatorname{Re} z_0 + \|'z\|^2 \geq \operatorname{Re} z_0$, we need only show that $r(G(t, '0)) \to -\infty$. Moreover, for any $t \in \mathbb{R}^-$,

$$r(G(t, '0)) = \tilde{r}'(\tilde{F}(\Psi^{-1}(t, '0))) = \tilde{r}'(\tilde{F}(t, '0)) \geq c \operatorname{dist}((t, '0), \partial\tilde{D})$$

according to (7). So it suffices to derive $\operatorname{dist}((t, '0), \partial\tilde{D}) \xrightarrow[t\in\mathbb{R},\ t\to-\infty]{} \infty$. The domain $\tilde{D}$ is defined by $0 = \operatorname{Re} z_0 + P('z, '\bar{z})$, where P is a real homogeneous polynomial of degree 2. Let $\gamma > 0$ be such that, for any $'z \in \mathbb{C}^n$, $|P('z, '\bar{z})| \leq \gamma \|'z\|^2$. One may easily verify that, for any $z = (z_0, 'z) \in \mathbb{C}^{n+1}$ such that $\|(t, '0) - z\| < \sqrt{|t|/(1+\gamma)}$, we have $\tilde{r}(z) < 0$ as soon as $|t|/(1+\gamma) \geq 1$. Whence, for t sufficiently large, $\operatorname{dist}((t, '0), \partial\tilde{D}) \geq \sqrt{|t|/(1+\gamma)}$. □

4.2.2. *Studying the Jacobian*

In order to simplify the notation, we assume hereafter that $\tilde{D} = \tilde{D}' = \mathbb{H}$ and $\Psi = \Psi' = \mathrm{Id}$.

LEMMA 8. *There exist some constants* $0 < \alpha \leq \beta < \infty$ *such that, for all* $p \in \mathbb{H}$,

$$\alpha|\operatorname{Jac}_p G| \leq \liminf|\operatorname{Jac}_{\Lambda_k^{-1}\circ\delta_k^{-1}(p)} F| \leq \limsup|\operatorname{Jac}_{\Lambda_k^{-1}\circ\delta_k^{-1}(p)} F| \leq \beta|\operatorname{Jac}_p G|.$$

Proof. Pick $p \in \mathbb{H}$. Then $p \in D_k$ for some sufficiently large k, and

$$d(F_k)_p = d(\delta_k') \circ d(\Lambda_k') \circ dF_{\Lambda_k^{-1}\circ\delta_k^{-1}(p)} \circ d\Lambda_k^{-1} \circ d\delta_k^{-1}.$$

The rigid motion Λ_k converges to the identity mapping. Hence, taking the determinant in the previous equality, we have

$$\operatorname{Jac}_{\Lambda_k^{-1}\circ\delta_k^{-1}(p)} F = \mu_k \operatorname{Jac}_p F_k, \tag{8}$$

where μ_k depends only on k. Moreover, $\mu_k \underset{k\to+\infty}{\sim} (d_k'/d_k)^{n+1}$ remains bounded between two positive constants by the boundary distance preserving property of F. Since $\operatorname{Jac}_p F_k = \det(d(F_k)_p) \xrightarrow[k\to+\infty]{} \operatorname{Jac}_p G$, we conclude the proof by passing to the limit in (8). □

LEMMA 9. *For any sequence* $\mathcal{P} = (p_k)_k$ *of points of* D *converging to* $p_\infty \in \partial D$, *the sequence* $(\operatorname{Jac}_{p_k} F)_k$ *is bounded. If* $\operatorname{Jac}_{p_k} F \xrightarrow[+\infty]{} 0$ *then, for any other sequence* $(p_k')_k$ *of points of* D *converging to* p_∞, *we have* $\operatorname{Jac}_{p_k'} F \xrightarrow[+\infty]{} 0$.

Proof. Denote by $G^{\mathcal{P}}$ the limit map obtained by applying the scaling method to the sequence $\mathcal{P}$. Lemma 8 taken at $p = (-1, '0) \in \mathbb{H}$ gives

$$\alpha|\operatorname{Jac}_{(-1,'0)} G^{\mathcal{P}}| \leq \liminf|\operatorname{Jac}_{p_k} F| \leq \limsup|\operatorname{Jac}_{p_k} F| \leq \beta|\operatorname{Jac}_{(-1,'0)} G^{\mathcal{P}}|.$$

This implies that the sequence $(\operatorname{Jac}_{p_k} F)_k$ is bounded.

Assume that $\mathrm{Jac}_{p_k} F \to 0$, and let $(p'_k)_k$ be another sequence converging to p_∞. Set $p''_{2k} = p_k$ and $p''_{2k+1} = p'_k$. For any cluster point λ of $(\mathrm{Jac}_{p'_k} F)$, the sequence $(\mathrm{Jac}_{p''_k} F)$ has at least 0 and λ as cluster points. The scaling method applied to the sequence $\mathcal{P}'' = (p''_k)$ and Lemma 8 taken at $p = (-1, '0) \in \mathbb{H}$ together show that

$$\alpha''|\mathrm{Jac}_{(-1,'0)}\, G^{\mathcal{P}''}| \leq 0 \leq |\lambda| \leq \beta''|\mathrm{Jac}_{(-1,'0)}\, G^{\mathcal{P}''}|.$$

Hence $\mathrm{Jac}_{(-1,'0)}\, G^{\mathcal{P}''} = 0$ and $\lambda = 0$. □

LEMMA 10. *Let $\mathcal{P} = (p_k)$ be a sequence of points of D converging to $p_\infty \in \partial D$. Then the Jacobian of $G = G^{\mathcal{P}}$ does not vanish in $\mathbb{H}$.*

Proof. We may assume $p_\infty = 0$.

First, we show that if the Jacobian of G vanishes at some point $p \in \mathbb{H}$ then it vanishes identically in $\mathbb{H}$. The scaling method applyed to $\mathcal{P}$ and Lemma 8 taken at p give a sequence (p''_k), where $p''_k = \Lambda_k^{-1} \circ \delta_k^{-1}(p)$, such that $\mathrm{Jac}_{p''_k} F \to 0$. Then, for any $p' \in \mathbb{H}$, we have

$$\begin{aligned}\alpha|\mathrm{Jac}_{p'}\, G| &\leq \liminf|\mathrm{Jac}_{\Lambda_k^{-1}\circ\delta_k^{-1}(p')}\, F| \\ &\leq \limsup|\mathrm{Jac}_{\Lambda_k^{-1}\circ\delta_k^{-1}(p')}\, F| \leq \beta|\mathrm{Jac}_{p'}\, G|. \qquad (9)\end{aligned}$$

Hence $\mathrm{Jac}_{p'}\, G = 0$ if and only if $\mathrm{Jac}_{\Lambda_k^{-1}\circ\delta_k^{-1}(p')}\, F \to 0$. According to Lemma 9, it only remains to prove that the sequence (p'_k) converges to 0 with $p'_k = \Lambda_k^{-1} \circ \delta_k^{-1}(p')$. But

$$\Lambda_k^{-1} \circ \delta_k^{-1}(p') \xrightarrow[k\to+\infty]{} 0,$$

which gives the statement.

Suppose by contradiction that the Jacobian of G is identically zero in $\mathbb{H}$. There exist a neighborhood U of 0, a constant $\delta > 0$, and a function φ that is continuous on $\overline{U \cap \mathbb{H}}$ and strictly J-plurisubharmonic on $U \cap \mathbb{H}$ such that,

$$\forall z \in U \cap \mathbb{H}, \quad \varphi(z) < -\delta\|z\|^2. \qquad (10)$$

Let us fix $\varepsilon > 0$ such that $\overline{\mathbb{B}(0, \sqrt{\varepsilon/\delta})} \subset U$ and set $H^\varepsilon = \{z \in U \cap \mathbb{H} \mid \varphi(z) > -\varepsilon\}$. Then $\overline{H^\varepsilon} \subset U$ by (10). By hypothesis, the maximal rank r_0 of dG on $U \cap \mathbb{H}$ is less than $2n+1$. Moreover, according to Lemma 7, $G(U \cap \mathbb{H}) \subset U \cap \mathbb{H}$ and its continuous extension verifies $G(\partial\mathbb{H}) \subset \partial\mathbb{H}$. Hence G is nonconstant and $r_0 > 0$.

As in the proof of Lemma 2, we obtain the existence of some $q \in G(U \cap \mathbb{H})$ such that $N = G^{-1}(\{q\})$ is an almost complex submanifold of (real) dimension $2n + 2 - r_0$ in $U \cap \mathbb{H}$. The continuous function φ reaches its maximum on the compact set $N \cap \overline{H^\varepsilon}$ at some point p_0. If $p_0 \in N \cap H^\varepsilon$ then there exists a pseudo-holomorphic disc h in the open subset $N \cap H^\varepsilon$ of N, centered at p_0, such that $\frac{\partial h}{\partial x}(0) \neq 0$ [18]—but this is impossible by the maximum principle. Thus $p_0 \in N \cap \partial H^\varepsilon$. Because of the continuity of φ,

$$\partial H^\varepsilon = (U \cap \partial\mathbb{H}) \cup \{z \in U \cap \mathbb{H} \mid \varphi(z) = -\varepsilon\}.$$

The boundary distance preserving property of G implies that N does not intersect $\partial\mathbb{H}$. Consequently,

$$\varphi(p_0) = -\varepsilon \quad \text{and} \quad \underset{N\cap\overline{H^\varepsilon}}{\mathrm{Max}}\, \varphi = -\varepsilon.$$

Hence φ is constant equal to $-\varepsilon$ on $N \cap \overline{H^\varepsilon}$, which contradicts the strict plurisubharmonicity of φ. $\square$

4.2.3. *Computation of* $\frac{\partial G_0}{\partial z_0}$

LEMMA 11. *For all* $z \in \mathbb{H}$,

$$\frac{\partial G_0}{\partial z_0}(z) = 1.$$

Proof. First notice that, by Lemma 10, the almost complex structures $\mathcal{J}$ and $\mathcal{J}'$ are either both integrable or both nonintegrable.

Suppose that $\mathcal{J}$ and $\mathcal{J}'$ are both integrable. In this case, we have seen that $\mathcal{J} = \mathcal{J}' = \mathcal{J}_{st}$. The map $G\colon \mathbb{H} \to \mathbb{H}$ is thus holomorphic for the standard structure and admits a continuous extension to the boundary (Lemma 7). Denote by Φ the biholomorphism (for the standard structure) from $\mathbb{H}$ to the unit ball $\mathbb{B}$ of $\mathbb{C}^{n+1}$ defined by

$$\Phi(z_0, {}'z) \mapsto \left(\frac{z_0+1}{z_0-1}, \frac{1}{1-z_0}{}'z\right);$$

Φ extends to a homeomorphism from $\bar{\mathbb{H}}$ to $\bar{\mathbb{B}}$ by defining $\Phi(\infty) = (1, {}'0)$ and $\Phi^{-1}(1, {}'0) = \infty$.

The map $\tilde{G} = \Phi \circ G \circ \Phi^{-1}$ from $\mathbb{B}$ to $\mathbb{B}$ is holomorphic and continuous up to $S^* = \partial\mathbb{B} \setminus \{(1, {}'0)\}$. Moreover, $\tilde{G}(S^*) \subset \partial\mathbb{B}$. Such a map is an automorphism of the ball (see [21, Prop. 2.3]). We have $\tilde{G}(0) = 0$ and, for all $u \in [0; 1[$,

$$\tilde{G}(u, {}'0) = \Phi \circ G\left(\frac{u+1}{u-1}, {}'0\right) = \left(\frac{Z_0+1}{Z_0-1}, \frac{\sqrt{1}}{1-Z_0}{}'Z\right),$$

where $G\left(\frac{u+1}{u-1}, {}'0\right) = (Z_0, {}'Z)$. Corollary 3 implies that, as u tends to 1^-, the real part of Z_0 tends to $-\infty$ and so $\mathrm{Re}\, \frac{Z_0+1}{Z_0-1} \to 1$. Since the image of Φ is the unit ball, necessarily

$$\tilde{G}(u, {}'0) \xrightarrow[u\in[0;1[,\ u\to 1]{} (1, {}'0).$$

By [4, p. 467], we obtain $\tilde{G}_0 \equiv \mathrm{Id}$. Hence $G_0(z_0, {}'z) = z_0$ for every $z \in \mathbb{H}$.

Suppose that $\mathcal{J}$ and $\mathcal{J}'$ are both nonintegrable. In this case, by (2) we have

$$G(z_0, {}'z) = (cz_0 + f_1({}'z) + if_2({}'z), {}'G({}'z)),$$

where $c \neq 0$ is a real constant and where f_1 and f_2 are real-valued. Since the map G is continuous to the boudary and verifies $G(\partial\mathbb{H}) \subset \partial\mathbb{H}$, for any ${}'z \in \mathbb{C}^n$ we obtain

$$\mathrm{Re}\, z_0 + \|{}'z\|^2 = 0 \implies c\,\mathrm{Re}(z_0) + f_1({}'z) + \|{}'G({}'z)\|^2 = 0.$$

As a consequence, $f_1('z) = c\|'z\|^2 - \|'G('z)\|^2$, $f_1('0) = \|'G('0)\|^2$, and

$$(-1,'0) = G(-1,'0) = (-c + f_1('0) + if_2('0),'G('0)) \implies \begin{cases} 'G('0) = '0, \\ c = 1. \end{cases}$$

Therefore, $\frac{\partial G}{\partial z_0}(z) = 1$ for all $z \in \mathbb{H}$. □

5. Proofs of the Theorems

5.1. Behavior Near the Boundary

The different properties of G proved in the previous section give us some information on F near the boundary. In particular, Lemma 10 implies that there is no sequence (p_k) of points of D converging to some point of the boundary such that $\mathrm{Jac}_{p_k} F \to 0$. This proves Theorem 1.

As a consequence, the map F is locally pseudobiholomorphic out of some compact set. Whence, in order to study the regularity near the boundary, it suffices to understand the pseudobiholomorphic case. We also know (by [5, Prop. 3.5]) some precise estimates of the Kobayashi metric, which give the asymptotic behavior of the differential according to the directions. We begin by fixing notation. Consider the vector fields

$$v^0 = \frac{\partial\rho}{\partial x_0}\frac{\partial}{\partial y_0} - \frac{\partial\rho}{\partial y_0}\frac{\partial}{\partial x_0}$$

and

$$v^j = \frac{\partial\rho}{\partial x_0}\frac{\partial}{\partial x_j} - \frac{\partial\rho}{\partial x_j}\frac{\partial}{\partial x_0} \quad \text{for } j = 1,\dots,n.$$

Restricting if necessary the neighborhood U of 0 on which we are working, the vector fields defined by $X^j = v^j - iJv^j$ $(1 \le j \le n)$ form a basis of the J-complex tangent space to $\{\rho = \rho(z)\}$ at any $z \in U$. Moreover, if $X^0 = v^0 - iJv^0$ then the family $X = (X^0, X^1, \dots, X^n)$ forms a basis of $(1,0)$ vector fields on U. Similarly, we construct a basis $X' = (X'^0, X'^1, \dots, X'^n)$ of $(1,0)$ vector fields on U' such that $(X'^1(w), \dots, X'^n(w))$ defines a basis of the J'-complex tangent space to $\{\rho' = \rho'(w)\}$ at any $w \in U'$. In the sequel, we will denote by $A(p_k)$ the matrix of the map dF_{p_k} with respect to $X(p_k)$ and $X'(F(p_k))$.

PROPOSITION 4 [5]. *The matrix $A(p_k)$ satisfies the following estimates*:

$$A(p_k) = \begin{pmatrix} O_{1,1}(1) & O_{1,n}(\mathrm{dist}(p_k, \partial D)^{1/2}) \\ O_{n,1}(\mathrm{dist}(p_k, \partial D)^{-1/2}) & O_{n,n}(1) \end{pmatrix}.$$

REMARK 3. The asymptotic behavior of $A(p_k)$ depends only on the distance from p_k to ∂D, not on the choice of the sequence $(p_k)_k$.

In the case of a biholomorphism, one immediately obtains a similar estimate for $(dF_{p_k})^{-1} = d(F^{-1})_{F(p_k)}$. For a proper map, the control of the inverse matrix comes from Proposition 4 and the control of the Jacobian.

PROPOSITION 5. *The matrix $A(p_k)$ is invertible, and its inverse verifies the following estimates*:

$$A(p_k)^{-1} = \begin{pmatrix} O_{1,1}(1) & O_{1,n}(\mathrm{dist}(p_k, \partial D)^{1/2}) \\ O_{n,1}(\mathrm{dist}(p_k, \partial D)^{-1/2}) & O_{n,n}(1) \end{pmatrix}.$$

Proof. The formula $A^{-1} = \frac{1}{\mathrm{Jac}\, F} \times {}^t\mathrm{com}\, A$, together with Lemma 9 and Theorem 1, shows that we need only derive estimates for the matrix $B = {}^t\mathrm{com}\, A$. The determinant extracted from A and appearing in the entry $B_{i,j}$ can be expressed by developing along the row number 0 and/or the column number 0 of A. This gives the statement according to Proposition 4. □

Lemma 11 provides, exactly as in [12, Prop. 4.5], more information on the entry $(0,0)$ in the matrix $A(p_k)$. Observe that X and X', and hence the matrix $A(p_k)$, were normalized by the condition $J(p_\infty) = \mathcal{J}_{st}$; hence we write $A(p_\infty, p_k)$ in place of $A(p_k)$.

PROPOSITION 6. *The entry* $(0,0)$ *of the matrix A verifies the following properties.*

- *Every cluster point of the function $z \mapsto A_{0,0}(p, z)$ is real when z tends to $p \in \partial D$.*
- *Given $z \in D$, let $p \in \partial D$ be such that* $\mathrm{dist}(z, \partial D) = \|p - z\|$; *then there exists a constant $A > 0$, independent of $z \in D$, such that $|A_{(0,0)}(p, z)| \geq A$.*

By means of Theorem 1 and Propositions 5 and 6, we may use the arguments of the proof of [12, Thm. 0.1]. As a consequence we obtain the following result.

THEOREM 3. *Let (D, J) and (D', J') be strictly pseudoconvex regions of the same dimension. Then every proper pseudoholomorphic map from D to D' has a $\mathcal{C}^1$-extension to $\bar{D}$.*

5.2. *Proof of Theorem 2 and Corollary 1*

We suppose that the conditions of Theorem 2 are satisfied. Since F is a local biholomorphism near the boundary, we can apply [3, Cor. 2] to the map $(F, {}^t(dF)^{-1})$ with $N = N^*D$ and $N' = N^*D'$. We obtain that the map $(F, {}^t(dF)^{-1})$ is locally of class $\mathcal{C}^{t_1 - 1}$, where $t_1 = \min(r-1, r')$. Likewise, the map $(F^{-1}, {}^t(dF))$ (which is well-defined near the boundary) is locally of class $\mathcal{C}^{t_2 - 1}$, where $t_2 = \min(r'-1, r)$.

Hence F is of class $\mathcal{C}^s$, where $s = \max(t_1, t_2)$. This gives $s = \max(r-1, r'-1)$ if $|r' - r| < 1$ and $s = \min(r, r')$ if $|r' - r| \geq 1$. Moreover,

$$\|F\|_{\mathcal{C}^{s-1}(\bar{D})} \leq c(s) \|(F, {}^t(dF)^{-1})\|_\infty \left(1 + \frac{c'}{\sqrt{\lambda_{N'}^{J'}}}\right).$$

This concludes the proof of Theorem 2.

Corollary 1 follows immediately, since the almost complex structure J' is defined near the boundary by $J'_q = dF_q \circ J \circ (dF_q)^{-1}$.

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