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Interpolation by Entire Functions with Growth Conditions

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0. Introduction

Let $p\colon \mathbb{C} \to [0,+\infty[$ be a weight (see Definition 1.1) and let $A_p(\mathbb{C})$ be the vector space of all entire functions satisfying $\sup_{z\in\mathbb{C}}|f(z)| \le \exp(-Bp(z)) < \infty$ for some constant $B > 0$. For instance, if $p(z) = |z|$, then $A_p(\mathbb{C})$ is the space of all entire functions of exponential type.

Following [3], the interpolation problem we are considering is as follows. Let $V = \{(z_j, m_j)\}_j$ be a multiplicity variety; that is, suppose $\{z_j\}_j$ is a sequence of complex numbers diverging to ∞, $|z_j| \le |z_{j+1}|$, and $\{m_j\}_j$ is a sequence of strictly positive integers. Let $\{w_{j,l}\}_{j,0\le l<m_j}$ be a doubly indexed sequence of complex numbers.

Under what conditions does there exist an entire function $f \in A_p(\mathbb{C})$ such that

$$\frac{f^{(l)}(z_j)}{l!} = w_{j,l} \quad \forall j,\ \forall 0 \le l < m_j?$$

In other words, if we denote by ρ the restriction operator defined on $A_p(\mathbb{C})$ by

$$\rho(f) = \left\{\frac{f^l(z_j)}{l!}\right\}_{j,0\le l<m_j},$$

what is the image of $A_p(\mathbb{C})$ by ρ?

We say that V is an *interpolating variety* when $\rho(A_p(\mathbb{C}))$ is the space of all doubly indexed sequences $W = \{w_{j,l}\}$ satisfying the growth condition

$$|w_{j,l}| \le A\exp(Bp(z_j)) \quad \forall j,\ \forall 0 \le l < m_j,$$

for certain constants $A, B > 0$.

We have the following important result.

Theorem 0.1 [2, Cor. 4.8]. *V is an interpolating variety for $A_p(\mathbb{C})$ if and only if, for some constants $A, B > 0$, the following conditions hold*:

(i) *for all $R > 0$, $N(0,R) \le AP(R) + B$*;
(ii) *for all $j \in \mathbb{N}$, $N(z_j, |z_j|) \le AP(z_j) + B$.*

Here, $N(z,r)$ denotes the integrated counting function of V in the disc of center z and radius r (see Definition 1.3).

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In [3], Berenstein and Taylor describe the space $\rho(A_p(\mathbb{C}))$ in the case where there exists a function $g \in A_p(\mathbb{C})$ such that $V = g^{-1}(0)$. They used groupings of the points of V with respect to the connected components of the set $\{|g(z)| \le \varepsilon \exp(-Bp(z))\}$ for some $\varepsilon, B > 0$ and the divided differences with respect to this grouping.

The main aim of this paper is to determine more explicitly the space $\rho(A_p(\mathbb{C}))$ in the more general case where condition (i) is satisfied. Clearly, (i) holds when V is not a uniqueness set for $A_p(\mathbb{C})$—that is, when there exists $f \in A_p(\mathbb{C})$ not identically equal to zero such that $V \subset f^{(-1)}(0)$. See [6] and [10] for similar results in the case where $p(z) = |z|^\alpha$.

As in [3] and [6], the divided differences will be important tools. Our condition will involve the divided differences with respect to the intersections of V with discs centered at the origin. To be more precise, the main theorem—stated in the case where all multiplicities equal 1 for simplicity—is as follows.

THEOREM 0.2. *Assume V verifies Theorem 0.1(i). Then* $W = \{w_j\}_j \in \rho(A_p(\mathbb{C}))$ *if and only if, for all* $R > 0$,

$$\left| \sum_{|z_k|<R} w_k \prod_{|z_m|<R,\, m\neq k} R \Big/ (z_k - z_m) \right| \le A \exp(Bp(R)),$$

where $A, B > 0$ *are positive constants depending only on V and W.*

We will denote by $\tilde{A}_p(V)$ the space of sequences $W = \{w_j\}_j$ satisfying the condition of Theorem 0.2. We will show that in general $\rho(A_p(\mathbb{C})) \subset \tilde{A}_p(V)$ and so we can consider $\rho\colon A_p(\mathbb{C}) \to \tilde{A}_p(V)$. In this context, the theorem states that condition (i) implies the surjectivity of ρ.

On the other hand, we will prove that condition (i) is actually equivalent to saying that V is not a uniqueness set or, in other words, that it is equivalent to the noninjectivity of ρ.

As a corollary of the main theorem, we will find the sufficiency in the geometric characterization of interpolating varieties given in Theorem 0.1.

The difficult part of the proof of the main theorem is the sufficiency. As in [4; 7; 11], we will follow a Bombieri–Hörmander approach based on L^2-estimates of the solution to the $\bar\partial$-equation. The scheme will be as follows. The condition on W gives a smooth interpolating function F with good growth, using a partition of unity and Newton polynomials (see Lemma 2.5). Then we are led to solve the $\bar\partial$ equation: $\bar\partial u = -\bar\partial F$ with L^2-estimates, using Hörmander's theorem [8]. To do so, we must construct a subharmonic function U with a convenient growth and with prescribed singularities on the points z_j (see Lemma 2.6). Following Bombieri [5], the fact that e^{-U} is not summable near the points $\{z_j\}$ forces u to vanish on the points z_j, and we are done by defining the interpolating entire function by $u + F$.

NOTATION. We use A, B, and C to denote positive constants whose actual value may change from one occurrence to the next. By $A(t) \lesssim B(t)$ we mean that there exists a constant $C > 0$, not depending on t, such that $A(t) \le CB(t)$. We use $A \simeq B$ to mean that $A \lesssim B \lesssim A$.

The notation $D(z, r)$ will be used for the Euclidean disk of center z and radius r. We denote $\partial f = \frac{\partial f}{\partial z}$ and $\bar\partial f = \frac{\partial f}{\partial \bar z}$. Then $\Delta f = 4\partial\bar\partial f$ denotes the Laplacian of f.

1. Preliminaries and Definitions

DEFINITION 1.1. A subharmonic function $p\colon \mathbb{C} \to \mathbb{R}_+$, is called a *weight* if, for some positive constants C,

(a) $\ln(1 + |z|^2) \le Cp(z)$,
(b) $p(z) = p(|z|)$,
(c) there exists a constant $C > 0$ such that $p(2z) \le Cp(z)$.

Property (c) is referred to as the *doubling property of the weight p*. It implies that $p(z) = O(|z|^\alpha)$ for some $\alpha > 0$.

Let $A(\mathbb{C})$ be the set of all entire functions. We consider the space

$$A_p(\mathbb{C}) = \{f \in A(\mathbb{C}) \text{ and, for all } z \in \mathbb{C},\quad |f(z)| \le A\exp(Bp(z)) \text{ for some } A > 0,\ B > 0\}.$$

REMARK 1.2. (i) Property (a) implies that $A_p(\mathbb{C})$ contains all polynomials.
(ii) Property (c) implies that $A_p(\mathbb{C})$ is stable under differentiation.

Here are some examples of weights.

- $p(z) = \ln(1 + |z|^2)$; then $A_p(\mathbb{C})$ is the space of all the polynomials.
- $p(z) = |z|$; then $A_p(\mathbb{C})$ is the space of entire functions of exponential type.
- $p(z) = |z|^\alpha$ $(\alpha > 0)$; then $A_p(\mathbb{C})$ is the space of all entire functions of order $\le \alpha$ and of finite type.

Let $V = \{(z_j, m_j)\}_{j\in\mathbb{N}}$ be a multiplicity variety. For a function $f \in A(\mathbb{C})$, we will write $V = f^{-1}(0)$ when f vanishes exactly on the points z_j with multiplicity m_j and $V \subset f^{-1}(0)$ when f vanishes on the points z_j (but possibly elsewhere) with multiplicity at least equal to m_j. We will say that V is a *uniqueness set* for $A_p(\mathbb{C})$ if there is no function $f \in A_p(\mathbb{C})$, except the zero function, such that $V \subset f^{-1}(0)$.

We now recall definitions of counting functions and integrated counting functions as follows.

DEFINITION 1.3. Let $V = \{(z_j, m_j)\}_j$ be a multiplicity variety. For $z \in \mathbb{C}$ and $r > 0$,

$$n(z, r) = \sum_{|z - z_j| \le r} m_j,$$

$$N(z, r) = \int_0^r \frac{n(z,t) - n(z,0)}{t}\,dt + n(z,0)\ln r = \sum_{0<|z-z_j|\le r} m_j \ln\frac{r}{|z - z_j|} + n(z,0)\ln r.$$

An application of Jensen's formula in the disc $D(0, R)$ shows that, if V is not a uniqueness set for $A_p(\mathbb{C})$, then the following condition holds:

$$\exists A > 0,\ \exists B > 0 : \forall R > 0,\ N(0, R) \le AP(R) + B. \tag{1}$$

We will later show that the converse property holds.

By analogy with the spaces $A(\mathbb{C})$ and $A_p(\mathbb{C})$, we define the spaces

$$A(V) = \{W = \{w_{j,l}\}_{j,\,0\le l<m_j} \subset \mathbb{C}\}$$

and

$$A_p(V) = \left\{ W = \{w_{j,l}\}_{j,\,0\le l<m_j} \subset \mathbb{C} : \right. \\ \left. \forall j,\ \sum_{l=0}^{m_j-1} |w_{j,l}| \le A\exp(Bp(z_j)) \text{ for some } A > 0,\ B > 0 \right\}.$$

The space $A_p(\mathbb{C})$ can be seen as the union of the Banach spaces

$$A_{p,B}(\mathbb{C}) = \left\{ f \in A(\mathbb{C}),\ \|f\|_B := \sup_{z\in\mathbb{C}} |f(z)| \exp(-Bp(z)) < \infty \right\}$$

and has a structure of an (LF)-space with the topology of the inductive limit. The analogue holds for $A_p(V)$.

REMARK 1.4 (cf. [1, Prop. 2.2.2]). Let f be a function in $A_p(\mathbb{C})$. Then, for some constants $A > 0$ and $B > 0$,

$$\forall z \in \mathbb{C}, \quad \sum_{k=0}^{\infty} \left| \frac{f^{(k)}(z)}{k!} \right| \le A\exp(Bp(z)).$$

As a consequence of this remark, we see that the restriction map

$$\rho\colon A_p(\mathbb{C}) \to A(V), \\ f \mapsto \left\{ \frac{f^l(z_j)}{l!} \right\}_{j,\,0\le l\le m_j-1}$$

maps $A_p(\mathbb{C})$ into $A_p(V)$; in general, however, the space $A_p(V)$ is larger than $\rho(A_p(\mathbb{C}))$. It is clear that ρ is injective if and only if V is a uniqueness set for $A_p(\mathbb{C})$.

When $\rho(A_p(\mathbb{C})) = A_p(V)$ we say that V is an interpolating variety for $A_p(\mathbb{C})$. As mentioned in the Introduction, Berenstein and Li have given the following geometric characterization of these varieties.

THEOREM 1.5 [2, Cor. 4.8]. *V is an interpolating variety for $A_p(\mathbb{C})$ if and only if conditions* (1) *and*

$$\exists A > 0,\ \exists B > 0 : \forall j \in \mathbb{N},\ N(z_j, |z_j|) \le A_p(z_j) + B \tag{2}$$

hold.

In this paper we are concerned with determining the subspace $\rho(A_p(\mathbb{C}))$ of $A(V)$ when condition (1) is verified.

To any $W = \{w_{j,l}\}_{j,\, 0\le l\le m_j-1} \in A(V)$ we associate the sequence of divided differences $\Phi(W) = \{\phi_{j,l}\}_{j,\, 0\le l\le m_j-1}$ defined by induction as follows. First, we establish the following notation:

$$\Pi_q(z) = \prod_{k=1}^{q} (z - z_k)^{m_k} \quad \text{for all } q \ge 1;$$

$$\phi_{1,l} = w_{1,l} \quad \text{for all } 0 \le l \le m_1 - 1,$$

$$\phi_{q,0} = \frac{w_{q,0} - P_{q-1}(z_q)}{\Pi_{q-1}(z_q)},$$

$$\phi_{q,l} = \frac{w_{q,l} - \frac{P_{q-1}^{(l)}(z_q)}{l!} - \sum_{j=0}^{l-1} \frac{1}{(l-j)!} \Pi_{q-1}^{(l-j)}(z_q)\phi_{q,j}}{\Pi_{q-1}(z_q)} \quad \text{for } 1 \le l \le m_q - 1,$$

where

$$P_{q-1}(z) = \sum_{j=1}^{q-1} \left(\sum_{l=0}^{m_j-1} \phi_{j,l}(z - z_j)^l \prod_{t=1}^{j-1} (z - z_t)^{m_t} \right).$$

REMARK 1.6. Actually, P_q is the polynomial interpolating the values $w_{j,l}$ at the points z_j with multiplicity m_j for $1 \le j \le q$. It is the unique polynomial of degree $m_1 + \cdots + m_q - 1$ such that

$$\frac{P_q^{(l)}(z_j)}{l!} = w_{j,l}$$

for all $1 \le j \le q$ and $0 \le l \le m_j - 1$.

EXAMPLES. (i) Let $W_0 = \{\delta_{1,j}\delta_{l,m_1-1}\}_{j,\, 0\le l<m_j}$. Using that $P_j(z)$ must coincide with $(z - z_1)^{m_1-1}\prod_{k=2}^{j-1}(z - z_j)^{m_j}$ and identifying the coefficient in front of $z^{m_1+\cdots+m_{j-1}+l-1}$, we find that

$$\phi_{1,1} = \phi_{1,2} = \cdots = \phi_{1,m_1-2} = 0, \qquad \phi_{1,m_1-1} = 1$$

and, for $0 \le l \le m_j - 1$ ($j \ge 2$),

$$\phi_{l,j} = (z_1 - z_j)^{-(l+1)} \prod_{k=2}^{j-1} (z_1 - z_k)^{-m_k}.$$

(ii) In the special case where $m_j = 1$ for all j and $W = \{w_j\}_j$, we have

$$\phi_j = \sum_{k=1}^{j} w_k \prod_{1\le l\le j,\, l\ne k} (z_k - z_l)^{-1} \quad \text{for all } j \ge 1.$$

To compute the coefficients, we may use that $P_j(z)$ must coincide with the Lagrange polynomial $\sum_{n=1}^{j} w_n \prod_{1\le k\le j,\, k\ne n} \frac{(z-z_k)}{(z_n-z_k)}$ and identify the coefficient in front of z^{j-1}.

Let us denote by $\tilde{A}_p(V)$ the subspace of $A(V)$ consisting of the elements $W \in A(V)$ such that the following condition holds:

$$\begin{aligned} &\text{for all } n \geq 0,\ |z_j| \leq 2^n \text{ and } 0 \leq l \leq m_j - 1, \\ &|\phi_{j,l}| 2^{n(l+m_1+\cdots+m_{j-1})} \leq A \exp(Bp(2^n)), \end{aligned} \tag{3}$$

where A and B are positive constants depending only on V and W.

We have chosen to use a covering of the complex plane by discs $D(0, 2^n)$, but we can replace 2^n by any R^n with $R > 1$.

LEMMA 1.7. *Assume that $z_1 = 0$. Then condition* (1) *holds if and only if*

$$W_0 = \{\delta_{1,j}\delta_{l,m_1-1}\}_{j,0\leq l<m_j} \in \tilde{A}_p(V).$$

Proof. Suppose that (1) is verified. Let $n \in \mathbb{N}$, $0 < |z_j| \leq 2^n$, and $0 \leq l \leq m_j - 1$. By definition, we have

$$N(0, 2^n) = \sum_{0<|z_k|\leq 2^n} m_k \ln \frac{2^n}{|z_k|} + m_1 \ln(2^n) \geq \ln\biggl(2^{n(m_1+\cdots+m_j)} \prod_{k=2}^{j} |z_k|^{-m_k}\biggr)$$

and

$$\begin{aligned} |\phi_{j,l}| = |z_j|^{m_j-l-1} \prod_{k=2}^{j} |z_k|^{-m_k} &\leq 2^{n(m_j-l-1)} \prod_{k=2}^{j} |z_k|^{-m_k} \\ &\leq \exp(N(0, 2^n)) 2^{-n(m_1+\cdots+m_{j-1}+l+1)}. \end{aligned}$$

We readily obtain the estimate (3), using that $N(0, 2^n) \leq A_p(2^n) + B$.

Conversely, let n be an integer. Using the estimate (3) when $j \geq 2$ is the number of distinct points $\{z_k\}$ in $D(0, 2^n)$ and $l = m_j - 1$, we obtain

$$\begin{aligned} N(0, 2^n) &= \ln\biggl(2^{n(m_1+\cdots+m_j)} \prod_{k=2}^{j} |z_k|^{-m_k}\biggr) \\ &= \ln(2^{n(m_1+\cdots+m_j)} |\phi_{j,m_j-1}|) \leq A_p(2^n) + B. \end{aligned}$$

Using this with $2^{n-1} \leq R < 2^n$ and the doubling property of p, we can deduce the estimate for $N(0, R)$. □

We define the norm

$$\|W\|_B = \sup_n \|W^{(n)}\|_n \exp(-Bp(2^n)),$$

where

$$\|W^{(n)}\|_n = \sup_{|z_j|\leq 2^n} \sup_{0\leq l\leq m_j-1} |\phi_{j,l}| 2^{-n(l+m_1+\cdots+m_{j-1})}.$$

The space $\tilde{A}_p(V)$ can also be seen as an (LF)-space that is an inductive limit of the Banach spaces

$$\tilde{A}_{p,B}(V) = \{W \in A(V),\ \|W\|_B < \infty\}.$$

We are now ready to state the main results.

PROPOSITION 1.8. *The restriction operator* ρ *maps* $A_p(\mathbb{C})$ *continuously into* $\tilde{A}_p(V)$.

PROPOSITION 1.9. *Under the assumption of condition* (1), $\tilde{A}_p(V)$ *is a subspace of* $A_p(V)$.

PROPOSITION 1.10. *If conditions* (1) *and* (2) *are verified, then* $\tilde{A}_p(V) = A_p(V)$.

THEOREM 1.11. *If condition* (1) *holds, then*

$$\tilde{A}_p(V) = \rho(A_p(\mathbb{C})).$$

In other words, condition (1) *implies that the map* $\rho\colon A_p(\mathbb{C}) \to \tilde{A}_p(V)$ *is surjective.*

The combination of Proposition 1.10 and Theorem 1.11 shows easily the sufficiency in Theorem 1.5.

Using the results given so far, we can deduce our next theorem.

THEOREM 1.12. *The following assertions are equivalent.*

(i) V *is not a uniqueness set for* $A_p(\mathbb{C})$.
(ii) *The map* ρ *is not injective.*
(iii) V *verifies condition* (1).
(iv) *The sequence* $W_0 = \{\delta_{1,j}\delta_{l,m_1-1}\}_{j,0\le l<m_j}$ *belongs to* $\rho(A_p(\mathbb{C}))$.

In particular, Theorem 1.2 shows that condition (1) is equivalent to the existence of a function $f \in A_p(\mathbb{C})$ such that $V \subset f^{-1}(0)$. Combined with Theorem 1.11, it shows both that (a) if ρ is not injective then it is surjective and (b) if the image contains W_0 then it contains the whole $\tilde{A}_p(V)$.

Proof of Theorem 1.12. As mentioned previously, it is clear that (i) is equivalent to (ii) and that (i) implies (iii).

(iv) implies (i): We have a function $f \in A_p(\mathbb{C})$ not identically equal to 0 such that $f^{(l)}(z_j) = 0$ for all $j \neq 1$ and for all $0 \le l < m_j$. The function g defined by $g(z) = (z - z_1)^{m_1} f(z)$ belongs to $A_p(\mathbb{C})$, thanks to property (i) of the weight p, and vanishes on every z_j with multiplicity at least m_j.

(iii) implies (iv): Up to a translation, we may suppose that $z_1 = 0$. By Lemma 1.7, we know that $W_0 \in \tilde{A}_p(\mathbb{C})$. By Theorem 1.11, $W_0 \in \rho(A_p(\mathbb{C}))$. □

2. Proofs of the Main Results

Proof of Proposition 1.8. We first recall some definitions about the divided differences and the Newton polynomials. We refer the reader to [1, Chap. 6.2] or [9, Chap. 6] for more details.

Let $f \in A(\mathbb{C})$ and let $x_1, \dots, x_q$ be distinct points of $\mathbb{C}$. The qth divided difference of the function f with respect to the points $x_1, \dots, x_q$ is defined by

$$\Delta^{q-1} f(x_1, \dots, x_q) = \sum_{j=1}^{q} f(z_j) \prod_{1 \le k \le q,\, k \ne j} (x_j - x_k)^{-1},$$

and the Newton polynomial of f of degree $q-1$ is

$$P(z) = \sum_{j=1}^{q} \Delta^{j-1} f(x_1, \dots, x_j) \prod_{k=0}^{j-1} (z - x_k);$$

it is the unique polynomial of degree $q-1$ such that $P_q(z) = f(x_j)$ for all $1 \le j \le q$. When x_j, $1 \le j \le q$, are each one repeated l_j times, the divided differences are defined by

$$\Delta^{l_1+\cdots+l_q-1} f(\underbrace{x_1, \dots, x_1}_{l_1}, \dots, \underbrace{x_{q-1}, \dots, x_{q-1}}_{l_{q-1}}, \underbrace{x_q, \dots, x_q}_{l_q}) = \frac{1}{l_1! \cdots l_q!} \frac{\partial^{l_1+\cdots+l_j}}{\partial x_1^{l_1} \cdots \partial x_q^{l_q}} \Delta^{q-1} f(x_1, \dots, x_q).$$

The corresponding Newton polynomial is the unique polynomial of degree $l_1 + \cdots + l_q - 1$ such that, for all $0 \le j \le q$ and $0 \le l \le l_j - 1$,

$$P^{(l)}(x_j) = f^{(l)}(x_j).$$

We have the following estimate.

LEMMA 2.1 [1, Lemma 6.2.9]. *Suppose $f \in A(\mathbb{C})$, Ω is an open set of $\mathbb{C}$, $\delta > 0$, and $x_1, \dots, x_k$ are in $\Omega_0 = \{z \in \Omega : d(z, \Omega^c) > \delta\}$. Then*

$$|\Delta^{k-1} f(x_1, \dots, x_k)| \le \frac{2^{k-1}}{\delta^{k-1}} \sup_{z \in \Omega} |f(z)|.$$

Let $B > 0$ be fixed and $f \in A_{p,B}(\mathbb{C})$.

Let n be a fixed integer. Let $|z_j| \le 2^n$ and $0 \le l \le m_j - 1$. We consider the divided differences of f with respect to the points $z_1, \dots, z_j$, where each z_k, $1 \le k \le j-1$, is repeated m_k times and each z_j is repeated l times. Let $M_{j,l} = m_1 + \cdots + m_{j-1} + l$; then the divided differences are

$$\phi_{j,l} = \Delta^{M_{j,l}} f(\underbrace{z_1, \dots, z_1}_{m_1 \text{ times}}, \dots, \underbrace{z_{j-1}, \dots, z_{j-1}}_{m_{j-1} \text{ times}}, \underbrace{z_j, \dots, z_j}_{l+1 \text{ times}}).$$

Using Lemma 2.1 with $\Omega = D(0, 2^{n+2})$, $\delta = 2^{n+1}$, and $k = M_{j,l} + 1$, we obtain

$$|\phi_{j,l}| \le 2^{-nM_{j,l}} \|f\|_B \exp(Bp(2^{n+2})) \le 2^{-nM_{j,l}} \|f\|_B \exp(B'p(2^n)).$$

Hence

$$\|\rho(f)\|_{B'} \le \|f\|_B,$$

and this concludes the proof of Proposition 1.8. $\square$

Before proceeding with the proofs of the main results, we shall need the following lemmas.

LEMMA 2.2. *Condition* (1) *implies that there exist constants* $A, B > 0$ *such that, for all* $R > 0$,

$$n(0,R) \le A_p(R) + B.$$

Proof. Using property (c) of the weight, we have

$$n(0,R) \le 2\int_R^{2R} \frac{n(0,t)}{t}\,dt \le 2N(0,2R) \le A_p(2R) + B \le A_p(R) + B. \quad \square$$

LEMMA 2.3. *Let* W *be an element of* $A(V)$ *and let* q *be in* $\mathbb{N}^*$. *We suppose that for all* $1 \le j \le q$, *for all* $n \in \mathbb{N}$ *such that* $|z_q| \le 2^n$, *and for all* $0 \le l \le m_j - 1$,

$$|\phi_{j,l}| 2^{n(l+m_1+\cdots+m_{j-1})} \le A\exp(Bp(2^n)),$$

where A *and* B *are positive constants depending only on* V *and* W.

Then there exist constants $A, B > 0$ *depending only on* V *and* W *and such that, for all* $n \in \mathbb{N}$ *and* $|z| \le 2^n$,

$$\sum_{l=0}^{+\infty} \frac{|P_q^{(l)}(z)|}{l!} \le A\exp(Bp(2^n)) \sum_{j=1}^{q} 2^{2(m_1+\cdots+m_j)},$$

$$\sum_{l=0}^{+\infty} \frac{|\Pi_q^{(l)}(z)|}{l!} \le 2^{(n+2)(m_1+\cdots+m_q)}.$$

Proof. If $|z| \le 2^{n+1}$, then for $j = 1, \ldots, q$ and $|z - z_j| \le 2^{n+2}$ we have

$$\begin{aligned} |P_q(z)| &\le \sum_{j=1}^{q} 2^{(n+2)(m_1+\cdots+m_{j-1})} \sum_{l=0}^{m_j-1} |\phi_{j,l}| 2^{(n+2)l} \\ &\le A\exp(Bp(2^n)) \sum_{j=1}^{q} 2^{2(m_1+\cdots+m_j)} \end{aligned}$$

and

$$|\Pi_q(z)| = \prod_{j=1}^{q} |z - z_j|^{m_j} \le 2^{(n+2)(m_1+\cdots+m_q)}.$$

Now for $|z| \le 2^n$, if $|z - w| \le 2$ then $|w| \le 2^{n+1}$. By the preceding inequalities and Cauchy inequalities, for all $l \ge 0$ it follows that

$$\frac{|P_q^{(l)}(z)|}{l!} \le \frac{1}{2^l} \max_{|z-w|\le 2} |P_q(w)| \le \frac{1}{2^l} A\exp(Bp(2^n)) \sum_{j=1}^{q} 2^{2(m_1+\cdots+m_j)},$$

from which we readily obtain the desired estimate for P_q. Using Cauchy estimates once again for the function Π_q yields the second inequality. $\square$

Proof of Proposition 1.9. We assume that condition (1) holds. Let

$$W = \{w_{j,l}\}_{j,\,0\le l\le m_j-1} \in \tilde{A}_p(V).$$

Let $q \ge 1$ and let n be the integer such that $2^{n-1} \le |z_q| < 2^n$. We know that $P_q^{(l)}(z_q)/l! = w_{q,l}$ for every $0 \le l \le m_{q-1}$. By Lemma 2.3,

$$\sum_{l=0}^{m_q-1} |w_{q,l}| \le \sum_{l=0}^{+\infty} \frac{|P_q^{(l)}(z_q)|}{l!} \le A\exp(Bp(2^n)) \sum_{j=1}^{q} 2^{2(m_1+\cdots+m_j)}.$$

By Lemma 2.2, $m_1+\cdots+m_j \le n(0,|z_j|) \le A_p(z_j)+B$. Using that $q \le n(0,|z_q|) \le A_p(z_q)+B$, we obtain

$$\sum_{l=0}^{m_q-1} |w_{q,l}| \le A\exp(Bp(2^n)) \le A\exp(Bp(z_q));$$

that is, $W \in A_p(V)$. □

Proof of Theorem 1.10. We assume that conditions (1) and (2) are fulfilled. We already have $\tilde{A}_p(V) \subset A_p(V)$ by Proposition 1.9. Before proving the reverse inclusion, we need the following useful consequences of (1) and (2).

LEMMA 2.4. *There exist constants $A, B > 0$ such that, for all $j \in \mathbb{N}^*$ and all $n \in \mathbb{N}$ such that $|z_j| \le 2^n$:*

(i) $2^{nm_j} \le A|z_j|^{m_j}\exp(Bp(2^n))$ *and* $2^{n(m_1+\cdots+m_j)} \le A|z_j|^{m_1+\cdots+m_j}\exp(Bp(2^n))$;
(ii) $|z_j|^{m_j} \le A\exp(Bp(z_j))$;
(iii) $\prod_{k=1}^{j-1}|z_j-z_k|^{-m_k} \le A\exp(Bp(2^n))2^{-n(m_1+\cdots+m_{j-1})}$.

Proof. (i) For $0 < |z_j| \le 2^n$,

$$N(0,2^n) \ge \sum_{0<|z_k|\le 2^n} m_k \ln\frac{2^n}{|z_k|} \ge m_j \ln\frac{2^n}{|z_j|}.$$

We readily obtain the result by condition (1).

The second inequality is obtained in the same way after noting that

$$N(0,2^n) \ge \sum_{k=1}^{j} m_k \ln\frac{2^n}{|z_k|} \ge \left(\ln\frac{2^n}{|z_j|}\right)\sum_{k=1}^{j} m_k.$$

(ii) This is a simple consequence of condition (2):

$$m_j \ln|z_j| \le N(z_j,|z_j|) \le A_p(z_j)+B.$$

(iii) This also is a consequence of condition (2):

$$\sum_{k=1}^{j-1} m_k \ln\frac{|z_j|}{|z_j-z_k|} \le \sum_{0<|z_k-z_j|\le|z_j|} m_k \ln\frac{|z_j|}{|z_j-z_k|} = N(z_j,|z_j|) \le A_p(z_j)+B.$$

Using (i), we deduce that

$$
\begin{aligned}
\prod_{k=1}^{j-1} |z_j - z_k|^{-m_k} &\leq A \exp(Bp(z_j)) |z_j|^{-(m_1+\cdots+m_{j-1})} \\
&\leq A 2^{-n(m_1+\cdots+m_{j-1})} \exp(Bp(2^n)). \qquad \square
\end{aligned}
$$

Let $W = \{w_{j,l}\}_{j,0\leq l\leq m_j-1}$ be in $A_p(V)$. In order to show that W verifies (3), we use Lemma 2.3 to show (by induction on $q \geq 1$) the following property: For all $n \in \mathbb{N}$ such that $|z_q| \leq 2^n$ and for all $0 \leq l \leq m_q - 1$,

$$
|\phi_{q,l}| 2^{n(l+m_1+\cdots+m_{q-1})} \leq A \exp(Bp(2^n)),
$$

where A and B are positive constants depending only on V and W.

Suppose $q = 1$. Then, for $|z_1| \leq 2^n$ and $0 \leq l \leq m_1 - 1$, we have

$$
\begin{aligned}
|\phi_{1,l}| = |w_{1,l}| &\leq A \exp(Bp(z_1)) \\
&\leq A \exp(Bp(z_1)) 2^{-nl} 2^{nm_1} \leq A \exp(Bp(2^n)) 2^{-nl},
\end{aligned}
$$

using parts (i) and (ii) of Lemma 2.4.

Now suppose the property holds for $1 \leq j \leq q-1$, and let $n \in \mathbb{N}$ be such that $|z_q| \leq 2^n$. Again we proceed by induction on l, $0 \leq l \leq m_q - 1$.

Let $l = 0$. By Lemmas 2.3 and 2.2, we have

$$
\begin{aligned}
|P_{q-1}(z_q)| &\leq A \exp(Bp(2^n)) \sum_{j=1}^{q-1} 2^{2(m_1+\cdots+m_j)} \\
&\leq (q-1) 2^{2(m_1+\cdots+m_{q-1})} \leq A \exp(Bp(2^n)).
\end{aligned}
$$

By Lemma 2.4(iii),

$$
|\Pi_{q-1}(z_q)|^{-1} = \prod_{k=1}^{q-1} |z_q - z_k|^{-m_k} \leq A \exp(Bp(2^n)) 2^{-n(m_1+\cdots+m_{q-1})}.
$$

We deduce that

$$
|\phi_{q,0}| \leq A \exp(Bp(2^n)) 2^{-n(m_1+\cdots+m_{q-1})}.
$$

Now suppose the estimate true for $0 \leq j \leq l-1$. Using the inequalities from Lemmas 2.3 and 2.2 yields

$$
\sum_{j=0}^{l-1} \left| \frac{\Pi_{q-1}^{(l-j)}(z_q)}{(l-j)!} \phi_{q,j} \right| \leq A \exp(Bp(2^n))
$$

and

$$
\left| \frac{P_{q-1}^{(l)}(z_q)}{l!} \right| \leq A \exp(Bp(2^n)).
$$

As for $l = 0$, we use Lemma 2.4(iii) to complete the proof of Theorem 1.10. $\square$

Proof of Theorem 1.11. We already showed the necessity in Theorem 1.8, so we now prove the sufficiency.

We assume condition (1). Let $W = \{w_{j,l}\}_{j,0\leq l\leq m_j-1}$ be an element of $\tilde{A}_p(V)$. Let $\mathcal{X}$ be a smooth cutoff function such that $\mathcal{X}(x) = 1$ if $|x| \leq 1$ and $\mathcal{X}(x) = 0$ if $|x| \geq 4$.

Set $\mathcal{X}_n(z) = \mathcal{X}(|z|^2/2^{2n})$ for $n \in \mathbb{N}$, $\rho_0 = \mathcal{X}_0$, and $\rho_{n+1} = \mathcal{X}_{n+1} - \mathcal{X}_n$. It is clear that the family $\{\rho_n\}_n$ forms a partition of unity, that the support of $\mathcal{X}_n$ is contained in the disk $|z| \leq 2^{n+1}$, and that the support of ρ_n is contained in the annulus $\{2^{n-1} \leq |z| \leq 2^{n+1}\}$ for $n \geq 1$.

We will denote by q_n the number of distinct points z_j in $D(0, 2^n)$; that is, $q_n = \sum_{|z_j| \leq 2^n} 1$.

LEMMA 2.5. *There exists a $\mathcal{C}^\infty$ function F on $\mathbb{C}$ such that, for certain constants $A, B > 0$:*

(i) $F^{(l)}(z_j)/l! = w_{j,l}$ *for all $j \in \mathbb{N}$ with $0 \leq l \leq m_j - 1$;*
(ii) $|F(z)| \leq A \exp(Bp(z))$ *for all $z \in \mathbb{C}$;*
(iii) $\bar{\partial} F = 0$ *on $D(0,1)$ and, for any $n \geq 2$ and $2^{n-2} \leq |z| < 2^{n-1}$,*

$$|\bar{\partial} F(z)| \leq A 2^{-n(m_1 + \cdots + m_{q_n})} \prod_{k=1}^{q_n} |z - z_k|^{m_k} \exp(Bp(2^n)).$$

Proof. We set

$$F(z) = \sum_{n \geq 2} \rho_{n-2}(z) P_{q_n}(z),$$

where

$$P_q(z) = \sum_{j=1}^{q} \left(\sum_{l=0}^{m_j - 1} \phi_{j,l} (z - z_j)^l \right) \prod_{k=1}^{j-1} (z - z_k)^{m_k}.$$

This is the Newton polynomial mentioned in Remark 1.6.

(i) For all $j \geq 1$ and $0 \leq l \leq m_j - 1$, if z_j is in the support of ρ_{n-2} then $P_{q_n}^{(l)}(z_j) = l!\, w_{j,l}$. Thus

$$\begin{aligned} F^{(l)}(z_j) &= \sum_{n \geq 2} \left(\sum_{k=0}^{l} C_l^k \rho_{n-2}^{(l-k)}(z_j) k!\, w_{j,k} \right) \\ &= \sum_{k=0}^{l} C_l^k k!\, w_{j,k} \left(\sum_n \rho_n \right)^{(l-k)} (z_k) = l!\, w_{j,l}. \end{aligned}$$

(ii) For $z \geq 1$, let $n \geq 2$ be the integer such that $2^{n-2} \leq |z| < 2^{n-1}$. Then we have

$$F(z) = \rho_{n-2}(z) P_{q_n}(z) + \rho_{n-1}(z) P_{q_{n+1}}(z).$$

For all $0 \leq j \leq q_n$ we have $|z_j| \leq 2^n$ and $|z - z_j| \leq 2^{n+1}$. Using Lemma 2.3, condition (1), and property (c) of the weight yields

$$|P_{q_n}(z)| \lesssim \exp(Bp(2^n)) \leq A \exp(Bp(2^n)) \leq A \exp(Bp(z)).$$

The same estimation holds for $P_{q_{n+1}}$, so

$$|F(z)| \lesssim \exp(Bp(z)).$$

(iii) Now, we want to estimate $\bar{\partial} F$. It is clear that $F(z) = P_{q_2}(z)$ on $D(0,1)$. Let $|z| \geq 1$ and let n be the integer such that $2^{n-2} \leq |z| < 2^{n-1}$. Then

$$\bar{\partial} F(z) = \bar{\partial} \rho_{n-2}(z) P_{q_n}(z) + \bar{\partial} \rho_{n-1}(z) P_{q_{n+1}}(z).$$

Since z is outside the supports of $\bar{\partial}\mathcal{X}_{n-3}$ and $\bar{\partial}\mathcal{X}_{n-1}$, it follows that

$$\bar{\partial}F(z) = -\bar{\partial}\mathcal{X}_{n-2}(z)(P_{q_{n+1}}(z) - P_{q_n}(z)) = \prod_{k=1}^{q_n}(z - z_k)^{m_k} G_n(z),$$

where

$$G_n(z) = -\bar{\partial}\mathcal{X}_{n-2}(z) \sum_{j=q_n+1}^{q_{n+1}} \prod_{k=q_n+1}^{j-1} (z - z_k)^{m_k} \left(\sum_{l=0}^{m_j-1} \phi_{j,l}(z - z_j)^l \right).$$

For $k \leq q_{n+1}$ we have $|z - z_k| \leq 2^{n+2}$. Hence, using the estimate given by (3) and then Lemma 2.2, we can show that

$$|G_n(z)| A \exp(Bp(2^n)) 2^{-n(m_1+\cdots+q_n)} \sum_{j=q_n+1}^{q_{n+1}} 2^{m_{q_n+1}+\cdots+m_j} \lesssim \exp(Bp(2^n)) 2^{-n(m_1+\cdots+m_{q_n})}.$$

We readily obtain the desired estimate. □

While looking for a holomorphic interpolating function of the form $f = F + u$, we are led to the $\bar{\partial}$-problem

$$\bar{\partial}u = -\bar{\partial}F,$$

which we solve using Hörmander's theorem [8, Thm. 4.2.1].

The interpolation problem is then reduced to the following lemma.

LEMMA 2.6. *There exists a subharmonic function U such that, for certain constants $A, B > 0$:*

(i) $U(z) \simeq m_j \log|z - z_j|^2$ *near* z_j;
(ii) $U(z) \leq A_p(z) + B$ *for all* $z \in \mathbb{C}$;
(iii) $|\bar{\partial}F(z)|^2 \exp(-U(z)) \leq A \exp(B(p(z))$ *for all* $z \in \mathbb{C}$.

Admitting this lemma for a moment, we proceed with the proof of the theorem.

By Hörmander's theorem [8, Thm. 4.4.2] we can find a $\mathcal{C}^\infty$ function u such that $\bar{\partial}u = -\bar{\partial}F$ and, denoting by $d\lambda$ the Lebesgue measure,

$$\int_{\mathbb{C}} \frac{|u(w)|^2 \exp(-U(w) - A_p(w))}{(1 + |w|^2)^2} \, d\lambda(w) \leq \int_{\mathbb{C}} |\bar{\partial}F|^2 \exp(-U(w) - Ap(w)) \, d\lambda(w).$$

By property (a) of the weight p, there exists a $C > 0$ such that

$$\int_{\mathbb{C}} \exp(-Cp(w)) \, d\lambda(w) < \infty.$$

Thus, using (ii) of the lemma and the estimate on $|\bar{\partial}F(z)|^2$, we see that the last integral is convergent if A is large enough. By condition (iii) we know that, near z_j, $\exp(-U(w))(w - z_j)^l$ is not summable for $0 \leq l \leq m_j - 1$. Hence necessarily $u^{(l)}(z_j) = 0$ for all j and $0 \leq l \leq m_j - 1$; as a result, $f^{(l)}(z_j)/l! = w_j^l$.

Now we must verify that f has the desired growth. By the mean value inequality,

$$|f(z)| \lesssim \int_{D(z,1)} |f(w)|\,d\lambda(w) \lesssim \int_{D(z,1)} |F(w)|\,d\lambda(w) + \int_{D(z,1)} |u(w)|\,d\lambda(w).$$

Let us estimate the latter two integrals, which we denote by I_1 and I_2.

For $w \in D(z,1)$,

$$|F(w)| \lesssim \exp(Bp(w)) \lesssim \exp(Cp(z)).$$

Then

$$I_1 \lesssim \exp(Cp(z)).$$

To estimate I_2, we use Cauchy–Schwarz inequality,

$$I_2^2 \le J_1 J_2,$$

where

$$J_1 = \int_{D(z,1)} |u(w)|^2 \exp(-U(w) - Bp(w))\,d\lambda(w),$$

$$J_2 = \int_{D(z,1)} \exp(U(w) + Bp(w))\,d\lambda(w).$$

Then

$$\begin{aligned} J_1 &\lesssim \int_{\mathbb{C}} |u(w)|^2 \exp(-U(w) - Bp(w))\,d\lambda(w) \\ &\lesssim \int_{\mathbb{C}} \frac{|u(w)|^2 \exp(-U(w))}{(1+|w|^2)^2}\,d\lambda(w) < +\infty, \end{aligned}$$

by property (a) of p as long as $B > 0$ is chosen large enough.

To estimate J_2, we use condition (i) of the lemma and property (b) of the weight p. For $w \in D(z,1)$,

$$\exp(U(w) + Bp(w)) \le \exp(Cp(w)) \lesssim \exp(A_p(z)).$$

We easily deduce that $J_2 \lesssim \exp(A_p(z))$ and hence, finally, that $f \in A_p(\mathbb{C})$; this completes the proof of Theorem 1.11. □

Proof of Lemma 2.6. For the sake of simplicity and up to a homotethy, we may assume that $|z_k| > 2$ for all $z_k \neq 0$. In the definition of the following functions V_n, we will assume $z_1 \neq 0$; otherwise, we may add the term $m_1 \ln|z|$ to each V_n. We set

$$V_n(z) = \sum_{0<|z_j|\le 2^n} m_j \log \frac{|z-z_j|^2}{|z_j|^2};$$

then

$$V(z) = \sum_{n \ge 2} \rho_{n-2}(z) V_n(z).$$

First we will show that V verifies (i), (ii), and (iii). Then we estimate ΔV from below and add a correcting term W. The subharmonic function U will be of the form $V + W$.

(i) Let $|z_k|$ be such that $2^{m-1} < |z_k| < 2^{m+1}$. For $2^{m-1} < |z| < 2^{m+1}$,

$$V(z) = \rho_{m-1}(z)V_{m+1}(z) + \rho_m(z)V_{m+2}(z) + \rho_{m+1}(z)V_{m+3}(z).$$

Because the ρ_n form a partition of unity, it is clear that $V(z) - m_k \ln|z - z_k|^2$ is continuous in a neighborhood of z_k. Observe that V is smooth on $\{|z| \le 2\}$ since we have assumed that all $|z_j| > 2$.

(ii) Let $n \ge 2$ and $2^{n-2} \le |z| < 2^{n-1}$. Then

$$V(z) = \rho_{n-2}(z)V_n(z) + \rho_{n-1}(z)V_{n+1}(z).$$

For all $|z_j| < 2^n$, we have $|z - z_j| < 2^{n+1}$. Thus,

$$V_n(z) \le \sum_{|z_j| \le 2^n} m_j \log \frac{2^{n+1}}{|z_j|} \le N(0, 2^{n+1}).$$

Finally, we obtain that

$$V(z) \le N(0, 2^{n+1}) + N(0, 2^{n+2}) \lesssim p(2^n) \lesssim p(z)$$

by condition (1) and property (c) of the weight.

(iii) We have

$$\frac{-V(z)}{2} = \sum_{|z_j| \le 2^n} m_j \ln \frac{|z_j|}{|z - z_j|} + \rho_{n+1}(z) \sum_{2^n < |z_j| \le 2^{n+1}} m_j \ln \frac{|z_j|}{|z - z_j|}.$$

Note that for all $2^n < |z_j| \le 2^{n+1}$ we have $|z - z_j| > 2^n - 2^{n-1} = 2^{n-1}$. We obtain

$$\begin{aligned} \frac{-V(z)}{2} &\le \sum_{|z_j| \le 2^n} m_j \ln \frac{2^n}{|z - z_j|} + \ln 4 \sum_{2^n < |z_j| \le 2^{n+1}} m_j \\ &\le \ln\left(2^{n(m_1 + \cdots + m_{q_n})} \prod_{j=1}^{q_n} |z - z_j|^{-m_j} \right) + \ln(A \exp(Bp(2^n))) \qquad (4) \end{aligned}$$

for certain constants $A, B > 0$, using Lemma 2.2. Finally, combining this inequality with Lemma 2.5(iii) yields

$$|\bar{\partial} F(z)| \exp(-V(z)/2) \lesssim \exp(Bp(2^n)) \lesssim \exp(Bp(z)).$$

Now, in order to obtain a lower bound of the Laplacian, we compute $\Delta V(z)$ as

$$\Delta V = \sum_{n \ge 2} \rho_{n-2} \Delta V_n + 2 \operatorname{Re}\left(\sum_n \bar{\partial} \rho_{n-2} \partial V_n \right) + \sum_{n \ge 2} \partial \bar{\partial} \rho_{n-2} V_n.$$

The first sum is positive because every V_k is subharmonic.

Let us estimate the second and third sums, which we denote respectively by $B(z)$ and $C(z)$. Let $n \ge 2$ and $2^{n-2} \le |z| < 2^{n-1}$; then, since z is outside the supports of $\bar{\partial}\mathcal{X}_{n-3}$ and $\bar{\partial}\mathcal{X}_{n-1}$, we have

$$B(z) = 2 \operatorname{Re}[\bar{\partial}\mathcal{X}_{n-2}(z) \partial(V_n(z) - V_{n+1}(z))],$$

$$C(z) = \partial\bar{\partial}\mathcal{X}_{n-2}(z)(V_n(z) - V_{n+1}(z)).$$

Moreover,

$$V_n(z) - V_{n+1}(z) = \sum_{2^n < |z_j| \le 2^{n+1}} m_j \log \frac{|z - z_j|^2}{|z_j|^2},$$

$$\partial(V_n(z) - V_{n+1}(z)) = \sum_{2^n < |z_j| \le 2^{n+1}} m_j \frac{1}{z - z_j},$$

and

$$|\bar{\partial}\mathcal{X}_{n-2}(z)| \lesssim \frac{1}{2^n}, \qquad |\partial\bar{\partial}\mathcal{X}_{n-2}(z)| \lesssim \frac{1}{2^{2n}}.$$

For z in the support of $\bar{\partial}\mathcal{X}_{n-2}$ we have $|z| \le 2^{n-1}$, and for $2^n \le |z_j| < 2^{n+1}$ it follows that $2^{n-1} \le |z - z_j| \le 2^{n+2}$. Therefore,

$$|\partial\bar{\partial}\mathcal{X}_{n-2}(z)(V_{n+1}(z) - V_n(z))| \lesssim \frac{n(0, 2^{n+1}) - n(0, 2^n)}{2^{2n}},$$

$$|\bar{\partial}\mathcal{X}_{n-2}(z)\partial(V_{n+1}(z) - V_n(z)| \lesssim \frac{n(0, 2^{n+1}) - n(0, 2^n)}{2^{2n}}.$$

Finally,

$$\Delta V(z) \gtrsim -\frac{n(0, 2^{n+1}) - n(0, 2^n)}{2^{2n}} \gtrsim -\frac{n(0, 2^3|z|) - n(0, 2|z|)}{|z|^2}.$$

To construct the correcting term W, we begin by putting $W(z) = g(2^3|z|)$ and

$$f(t) = \int_0^t n(0, s)\, ds, \qquad g(t) = \int_0^t \frac{f(s)}{s^2}\, ds.$$

The following inequalities are easy to see:

$$f(t) \le tn(0, t), \qquad g(t) \le \int_0^t \frac{n(0, s)}{s}\, ds = N(0, s).$$

Thus, by condition (1) and property (c),

$$W(z) \le N(0, 2^3|z|) \lesssim p(2^3 z) \lesssim p(z).$$

We must now estimate the Laplacian of W. Let $t = 2^3|z|$. Then

$$\Delta W(z) = \frac{1}{t} g'(t) + g''(t) = \frac{1}{t^2}\left(f'(t) - \frac{f(t)}{t} \right)$$

and

$$\begin{aligned} f(t) = \int_0^t n(0, s)\, ds &= \int_0^{t/4} n(0, s)\, ds + \int_{t/4}^t n(0, s)\, ds \\ &\le \frac{t}{4} n\left(0, \frac{t}{4}\right) + t\left(1 - \frac{1}{4}\right) n(0, t). \end{aligned}$$

Therefore,

$$f'(t) - \frac{f(t)}{t} = n(0, t) - \frac{f(t)}{t} \ge \frac{1}{4}\left(n(0, t) - n\left(0, \frac{t}{4}\right) \right)$$

and

$$\Delta W(z) \gtrsim \frac{n(0, 2^3|z|) - n(0, 2|z|)}{|z|^2}.$$

Now, the desired function will be of the form

$$U(z) = V(z) + \alpha W(z),$$

where α is a sufficiently large positive constant. □

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