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Holomorphic Maps on Projective Spaces and Continuations of Fatou Maps

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1. Introduction

Let f be a holomorphic map of the complex projective space $\mathbb{P}^n$ of dimension n onto itself. We assume that f is of degree $d \geq 2$. As in the one-dimensional case, the Fatou set Ω for f is defined by

$$\Omega := \{p \in \mathbb{P}^n \mid \{f^j\}_{j\geq 0} \text{ is a normal family on a neighborhood of } p\}.$$

It is sometimes useful to consider the normality of the sequence $\{f^j\}_{j\geq 0}$ when restricted to local analytic sets of lower dimension. We introduce a slightly more general concept as follows.

Definition. A holomorphic map φ from a complex analytic space R into $\mathbb{P}^n$ is said to be a *Fatou map* for f if $\{f^j \circ \varphi\}_{j\geq 0}$ is a normal family.

This definiton was given in [U3] and [R] independently. The following facts are immediate. An open set V in $\mathbb{P}^n$ is contained in the Fatou set Ω if and only if the inclusion map $V \hookrightarrow \mathbb{P}^n$ is a Fatou map. If $\varphi\colon R \to \mathbb{P}^n$ is a holomorphic map with $\varphi(R) \subset \Omega$, then φ is a Fatou map.

In this paper we prove the following two theorems.

Theorem 1. *Let S be a Riemann surface and let E be a closed polar set in S. If $\varphi\colon S-E \to \mathbb{P}^n$ is a Fatou map, then φ can be extended to a Fatou map $\check{\varphi}\colon S \to \mathbb{P}^n$.*

Theorem 2. *Let S be a Riemann surface and let E be a closed polar set in S. If $\varphi\colon S \to \mathbb{P}^n$ is a holomorphic map and if $\varphi(S-E)$ is contained in the Fatou set Ω, then $\varphi(S)$ is also contained in Ω.*

Here E is said to be *polar* if it is locally expressed as the set on which a subharmonic function takes the value $-\infty$.

Theorem 1 may be viewed as an analogue of theorems concerning continuation of a holomorphic map from $S-E$ into a compact Riemann surface of genus ≥ 2 [N; Su1] or, more generally, into a complex manifold whose universal cover is a bounded domain of certain type [Su2]. Our proof is an adaptation of Suzuki's idea.

As an application of these theorems, we make a remark concerning a result due to Fornæss and Sibony [FS3]. A connected component of the Fatou set Ω is called

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a *Fatou component,* and a Fatou component U is said to be *recurrent* if there is a point $p_0 \in U$ whose forward orbit $\{f^j(p_0)\}_{j\geq 0}$ has a subsequence convergent to a point in U. A recurrent Fatou component is periodic—that is, invariant under some iterate of f. Hence the classification of the recurrent Fatou components reduces to that of recurrent and invariant Fatou components. The following theorem was proved in [FS3].

THEOREM (Fornæss–Sibony). *Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a holomorphic map of degree ≥ 2. Then a recurrent and invariant Fatou component U of f is of one of the following three types.*

(1) *U is the immediate basin of some attracting* (*or superattracting*) *fixed point in U.*
(2) *There is a complex one-dimensional closed submanifold Σ of U that is biholomorphic to either a disk, a punctured disk, or an annulus, and $\{f^j|_U\}_{j\geq 0}$ contains a subsequence convergent to a holomorphic map $\psi : U \to \Sigma$ such that $\psi|_\Sigma$ is the identity.*
(3) *U is a rotation domain* (*Siegel domain*). *In other words, $\{f^j|_U\}_{j\geq 0}$ contains a subsequence that is uniformly convergent on compact sets to the identity map* id_U *of U.*

We can now prove the following statement.

THEOREM 3. *In case* (2) *of the preceding theorem, the manifold Σ is not biholomorphic to the punctured disk.*

Indeed, suppose that Σ is biholomorphic to the punctured disk $\Delta - \{0\}$. Then the inclusion map $\varphi : \Delta - \{0\} \to \mathbb{P}^2$ is a Fatou map and can be extended to a Fatou map $\check{\varphi} : \Delta \to \mathbb{P}^2$ by Theorem 1. Then the point $\check{\varphi}(0)$ is contained in the Fatou set Ω by Theorem 2. This contradicts the fact that Σ is a closed submanifold of U.

REMARK. Examples of case (2) with Σ biholomorphic to a disk (or an annulus) can be constructed as follows.

(a) Let $g : \mathbb{P}^1 \to \mathbb{P}^1$ be a holomorphic map of degree d with a Siegel disk (or a Herman ring) U_0. We express g as $[z_0, z_1] \mapsto [g_0(z_0, z_1), g(z_0, z_1)]$ in homogeneous coordinates. Define $f : \mathbb{P}^2 \to \mathbb{P}^2$ by

$$[z_0, z_1, z_2] \mapsto [g_0(z_0, z_1), g_1(z_0, z_1), z_2^d].$$

Then the Fatou component U of f that contains the set $\Sigma = \{[z_0, z_1, 0] \mid [z_0, z_1] \in U_0\}$ has the desired property.

(b) Let $g : \mathbb{P}^1 \to \mathbb{P}^1$ be a holomorphic map that has an attracting fixed point with its immediate basin U_1 and a Siegel disk (or a Herman ring) U_2. The 2-fold symmetric product of $\mathbb{P}^1$ can be identified with $\mathbb{P}^2$; this identification induces a map $\varpi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$, which is a 2-fold branched cover. The map $(g, g) : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ can be factored to a holomorphic map $f : \mathbb{P}^2 \to \mathbb{P}^2$, and $\varpi(U_1 \times U_2)$ is a Fatou component of f with the desired property (cf. [M; U1]).

2. Lifts of Fatou Maps

In order to state characterizations of Fatou maps that are analogous to those of Fatou sets, we first recall briefly some basic notions on holomorphic dynamics on $\mathbb{P}^n$. For more detail, see [FS1; FS2; HP; S; U2].

We denote $\mathcal{X} = \mathbb{C}^{n+1} - \{0\}$ and $\pi : \mathcal{X} \to \mathbb{P}^n$ the canonical projection. For a holomorphic map $f : \mathbb{P}^n \to \mathbb{P}^n$, there exists a homogeneous polynomial map $F : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ of degree d such that $F^{-1}(0) = \{0\}$ and $\pi \circ F|_{\mathcal{X}} = f \circ \pi$. Such an F is unique up to a constant factor.

The Green function h for F is defined by

$$h(z) := \lim_{j\to\infty} \frac{1}{d^j} \log\|F^j(z)\|, \quad z \in \mathbb{C}^{n+1}.$$

The function h is a plurisubharmonic function on $\mathbb{C}^{n+1}$ with logarithmic pole at 0 and is of the form

$$h(z) = \log\|z\| + \xi(\pi(z)).$$

Here $\|z\|$ denotes the Euclidean norm of $z \in \mathbb{C}^{n+1}$ and ξ is a continuous function on $\mathbb{P}^n$. Furthermore, h behaves under composition of F as

$$h(F(z)) = d \cdot h(z).$$

The Fatou set Ω is characterized in terms of h as follows. Let

$$\mathcal{H} = \{z \in \mathbb{C}^{n+1} \mid h \text{ is pluriharmonic in a neighborhood of } z\};$$

then we have $\pi^{-1}(\Omega) = \mathcal{H}$. Let

$$\mathcal{A} = \{z \in \mathbb{C}^{n+1} \mid F^j(z) \to 0 \ (j \to \infty)\}$$

be the basin of attraction to 0 of the map F. This set $\mathcal{A}$ coincides with the set $\{z \in \mathbb{C}^{n+1} \mid h(z) < 0\}$.

Now let R be an anayltic space and let $\varphi : R \to \mathbb{P}^n$ be a holomorphic map. By a *holomorphic lift* of φ we mean a holomorphic map $\Phi : R \to \mathcal{X}$ such that $\pi \circ \Phi = \varphi$.

THEOREM 4. *For a holomorphic map $\varphi : R \to \mathbb{P}^n$, the following properties are equivalent.*

(i) *φ is a Fatou map for f.*
(ii) *The sequence $\{f^j \circ \varphi\}_{j\geq 0}$ contains a subsequence that is uniformly convergent on compact sets.*
(iii) *If V is an open set in R and if $\Phi_V : V \to \mathcal{X}$ is a holomorphic lift of $\varphi|_V$, then $h \circ \Phi_V$ is a pluriharmonic function on V.*
(iv) *For any point $\zeta \in R$, there exist an open set V containing ζ and a holomorphic lift Φ_V of $\varphi|_V$ such that $h \circ \Phi_V = 0$ identically (in other words, $\Phi_V(V)$ is contained in the boundary $\partial\mathcal{A}$ of $\mathcal{A}$).*

We omit the proof, which proceeds exactly as in [U2].

For an analytic space R, we denote by $\pi_{\hat{R}}\colon \hat{R} \to R$ the universal covering of R. We identify its covering transformation group $\Gamma_{\hat{R}}$ with the fundamental group $\pi_1(R)$. Similarly, we denote by $\pi_{\tilde{R}}\colon \tilde{R} \to R$ the homology covering of R and identify its covering transformation group $\Gamma_{\tilde{R}}$ with the homology group $H_1(R, \mathbb{Z})$ (see e.g. [ASa]).

THEOREM 5. *Let $\varphi\colon R \to \mathbb{P}^n$ be a Fatou map for f. Then there exists a holomorphic map $\tilde{\Phi}$ from the homology covering $\tilde{R}$ of R into $\mathcal{X}$ such that the diagram*

$$\begin{tikzcd} \tilde{R} \arrow[r, "\tilde{\Phi}"] \arrow[d, "\pi_{\tilde{R}}"'] & \mathcal{X} \arrow[d, "\pi"] \\ R \arrow[r, "\varphi"'] & \mathbb{P}^n \end{tikzcd}$$

commutes and such that $h \circ \tilde{\Phi} = 0$ identically. Moreover, there is a group homomorphism $t\colon \Gamma_{\tilde{R}} \to U(1)$ such that

$$\tilde{\Phi} \circ \gamma = t(\gamma)\tilde{\Phi} \quad \textit{for } \gamma \in \Gamma_{\tilde{R}},$$

where $U(1)$ denotes the multiplicative group of complex numbers of modulus 1.

Proof. Let V be a connected open set in R and let $\Phi_{V,i}\colon V \to \mathcal{X}$ be holomorphic maps such that $\pi \circ \Phi_{V,i} = \varphi|_V$ and $h \circ \Phi_{V,i} = 0$ for $i = 1, 2$. Then there is a nonvanishing holomorphic function $\tau(p)$ on V such that $\Phi_{V,2}(\zeta) = \tau(\zeta)\Phi_{V,1}(\zeta)$. Since

$$h \circ \Phi_{V,2}(\zeta) = \log|\tau(\zeta)| + h \circ \Phi_{V,1}(\zeta),$$

it follows that $\log|\tau(\zeta)| = 0$ identically on V. Hence $\tau(\zeta)$ is a constant of modulus 1.

Thus, by analytic continuation, we have a holomorphic map $\hat{\Phi}$ from the universal cover $\hat{R}$ to $\mathcal{X}$ such that $h \circ \hat{\Phi} = 0$ and $\pi \circ \hat{\Phi} = \varphi \circ \pi_{\hat{R}}$. There is also a homomorphism $\Gamma_{\hat{R}} \to U(1)$ such that $\hat{\Phi} \circ \gamma = t(\gamma)\hat{\Phi}$ for $\gamma \in \Gamma_{\hat{R}}$.

Because $U(1)$ is commutative, the homomorphism t is factored to a homomorphism from $\Gamma_{\tilde{R}}$, and the $\hat{\Phi}$ descends to a map $\tilde{\Phi}\colon \tilde{R} \to \mathcal{X}$. □

3. Proof of Theorem 1

Let R be a Riemann surface and let $\varphi\colon R \to \mathbb{P}^n$ be a nonconstant Fatou map for f. By Theorem 5 we have a holomorphic map $\tilde{\Phi}$ from the homology cover $\tilde{R}$ into $\mathcal{X}$ with $h \circ \tilde{\Phi} = 0$; in other words, $\tilde{\Phi}(\tilde{R}) \subset \partial\mathcal{A}$. For later use we choose and fix positive numbers r_1, r_2 such that

$$\partial\mathcal{A} \subset \{z \in \mathbb{C}^{n+1} \mid r_1 < \|z\| < r_2\}.$$

Now we denote by

$$\omega = \frac{i}{2}\sum_{j=0}^{n} dz_j \wedge d\bar{z}_j = \frac{1}{4}dd^c\|z\|^2$$

the fundamental form corresponding to the Euclidean metric on $\mathbb{C}^{n+1}$, where $d^c = i(\bar{\partial} - \partial)$. Then $\tilde{\Phi}$ induces a form

$$\omega_{\tilde{R}} = \tilde{\Phi}^*\omega = \frac{1}{4}dd^c\|\tilde{\Phi}\|^2$$

on $\tilde{R}$. Since $\|\tilde{\Phi}\|^2$ is invariant under the covering transformation group $\Gamma_{\tilde{R}}$, we have a function K on R such that $K \circ \pi_{\tilde{R}} = \|\tilde{\Phi}\|^2$. Then the form

$$\omega_R = \frac{1}{4}dd^cK$$

on R satisfies $\pi_{\tilde{R}}^*\omega_R = \omega_{\tilde{R}}$. The metrics corresponding to $\omega, \omega_{\tilde{R}}, \omega_R$ will be denoted (respectively) by $ds, ds_{\tilde{R}}, ds_R$. For a relatively compact domain D in R, we denote by area(D) the area with respect to ω_R; for a smooth arc γ, we denote by $\mathrm{len}(\gamma)$ the length with respect to ds_R.

Let $\gamma = \gamma_1 + \cdots + \gamma_q$ be a sum of oriented smooth closed curves $\gamma_1, \dots, \gamma_q$ in R. We can regard γ as an element of $H_1(R, \mathbb{Z})$ and also as an element of Γ.

Lemma 6. *For γ as just described,*

$$|t(\gamma) - 1| \le \frac{1}{r_1}\,\mathrm{len}(\gamma).$$

Proof. First we consider the case where γ is a single closed curve. Let $\tilde{\gamma}$ be a lift of γ to $\tilde{R}$. If we denote by a the initial point of the arc $\tilde{\gamma}$ then its terminal point is $\gamma(a)$, the image of a under the covering transformation corresponding to γ. The path $\tilde{\Phi}(\tilde{\gamma})$ joins the points $\tilde{\Phi}(a)$ and $\tilde{\Phi}(\gamma(a))$ in $\mathbb{C}^{n+1}$. Hence

$$\|\tilde{\Phi}(\gamma(a)) - \tilde{\Phi}(a)\| \le \mathrm{len}(\tilde{\Phi}(\tilde{\gamma})) = \mathrm{len}(\tilde{\gamma}) = \mathrm{len}(\gamma).$$

Since $\tilde{\Phi}(\gamma(a)) = t(\gamma)\tilde{\Phi}(a)$, it follows that

$$|t(\gamma) - 1| = \left|\frac{\tilde{\Phi}(\gamma(a))}{\tilde{\Phi}(a)} - 1\right| \le \frac{1}{\|\tilde{\Phi}(a)\|}\,\mathrm{len}(\gamma) \le \frac{1}{r_1}\,\mathrm{len}(\gamma).$$

Now, if $\gamma = \gamma_1 + \cdots + \gamma_q$ then $t(\gamma) = t(\gamma_1)\cdots t(\gamma_q)$. Since the $t(\gamma_i)$ are of absolute value 1, we have

$$\begin{aligned}|t(\gamma) - 1| &= |t(\gamma_1)\cdots t(\gamma_q) - 1| \le \sum_{j=1}^{q}|t(\gamma_i) - 1| \\ &\le \frac{1}{r_1}\sum_{i=1}^{q}\mathrm{len}(\gamma_i) = \frac{1}{r_1}\,\mathrm{len}(\gamma).\end{aligned}$$

□

LEMMA 7. *If D is a relatively compact domain in R with smooth boundary ∂D, then*

$$\operatorname{area}(D) \le \frac{r_2}{2} \operatorname{len}(\partial D).$$

Proof. Because $d^c\|z\|^2 = i\sum(z_j d\bar{z}_j - \bar{z}_j dz_j)$, we have $|d^c\|z\|^2| \le 2r_2 ds$ in the ball $\{\|z\| < r_2\}$. Hence $|d^c K| \le 2r_2 ds_R$ on R. Therefore,

$$\begin{aligned}\operatorname{area}(D) &= \frac{1}{4}\int_D dd^c K = \frac{1}{4}\int_{\partial D} d^c K \le \frac{1}{4}\int_{\partial D} |d^c K| \\ &\le \frac{r_2}{2}\int_{\partial D} ds_R = \frac{r_2}{2}\operatorname{len}(\partial D).\end{aligned}$$

□

Let Δ be the unit disk in $\mathbb{C}$, and let E be a compact polar set in Δ. We choose s_0 $(0 < s_0 < 1)$ such that $E \subset \Delta(s_0)$. By the Evans–Selberg theorem, there is a harmonic function $u(\zeta)$ on $\Delta(s_0) - E$ such that $u(\zeta) \to 0$ as $\zeta \to \partial D(s_0)$ and $u(\zeta) \to +\infty$ as $\zeta \to E$ (see e.g. [T]). Let

$$\begin{aligned} D_\lambda &= \{\zeta \in \Delta(s_0) - E \mid u(\zeta) < \lambda\}, \\ C_\lambda &= \{\zeta \in \Delta(s_0) - E \mid u(\zeta) = \lambda\}. \end{aligned}$$

We suppose that $\Delta - E$ is equipped with a conformal metric and denote by $\operatorname{area}(D_\lambda)$ and $\operatorname{len}(C_\lambda)$ the area and length with respect to this metric.

LEMMA 8.

$$\liminf_{\lambda \to +\infty} \frac{\operatorname{len}(C_\lambda)}{\operatorname{area}(D_\lambda)} = 0.$$

Proof. This is a standard area-length argument (see [Su1]), but we include a proof here for completeness. We can assume that $\int_{C_\lambda} d^c u = 1$. Write $\omega = \rho^2 du \wedge d^c u$ with $\rho \ge 0$. Then, by the Schwarz inequality,

$$\operatorname{len}(C_\lambda)^2 = \left(\int_{C_\lambda} \rho\, d^c u\right)^2 \le \left(\int_{C_\lambda} d^c u\right)\left(\int_{C_\lambda} \rho^2\, d^c u\right) = \frac{d}{d\lambda}\operatorname{area}(D_\lambda).$$

For any $\varepsilon > 0$, the measure of the set $e(\varepsilon) = \{\lambda > 1 \mid \operatorname{len}(C_\lambda) \ge \varepsilon \operatorname{area}(D_\lambda)\}$ is bounded by

$$\int_{e(\varepsilon)} \frac{d\operatorname{area}(D_\lambda)}{\operatorname{len}(C_\lambda)^2} \le \frac{1}{\varepsilon^2}\int_{e(\varepsilon)} \frac{d\operatorname{area}(D_\lambda)}{\operatorname{area}(D_\lambda)^2} < +\infty.$$

Hence the lemma follows. □

Now we prove Theorem 1. Since the theorem concerns a local property, it suffices to consider the case where $S = \Delta$ and E is a compact subset of Δ. We also assume that the Fatou map $\varphi\colon \Delta - E \to \mathbb{C}^n$ is extended holomorphically to $\partial\Delta$. We set $R = \Delta - E$ and introduce the conformal metric ω_R on R.

By Lemma 7 we have

$$\operatorname{area}(D_\lambda) \le \frac{r_2}{2}(\operatorname{len}(C_\lambda) + \operatorname{len}(C_0)),$$

and, by Lemma 8,

$$\frac{2}{r_2} \leq \liminf_{\lambda \to \infty} \frac{\operatorname{len}(C_\lambda)}{\operatorname{area}(D_\lambda)} + \lim_{\lambda \to \infty} \frac{\operatorname{len}(C_0)}{\operatorname{area}(D_\lambda)},$$

since $\operatorname{area}(D_\lambda)$ is an increasing function. Hence $\operatorname{area}(D_\lambda)$ is bounded, and we may use Lemma 8 again to conclude that

$$\liminf_{\lambda \to +\infty} \operatorname{len}(C_\lambda) = 0.$$

Let γ be a closed Jordan curve in $R = \Delta - E$. We choose λ sufficiently large so that γ is contained in D_λ. Let γ' be the union connected components of C_λ that are inside of γ. Then γ is homologous to γ' with suitable orientation. Thus, by Lemma 6,

$$|t(\gamma) - 1| = |t(\gamma') - 1| \leq \frac{1}{r_1} \operatorname{len}(\gamma') \leq \frac{1}{r_1} \operatorname{len}(C_\lambda).$$

Since the inferior limit of the right-hand side is 0, we conclude that $t(\gamma) = 1$.

As a result, $\tilde{\Phi}$ is invariant under the covering transformations of $\tilde{R}$ and so we have a holomorphic lift $\Phi\colon R \to \mathcal{X}$ of φ. Because Φ is bounded, it can be extended to a holomorphic map $\check{\Phi}\colon \Delta \to \mathcal{X}$ with $\check{\Phi}(\Delta) \subset \partial\mathcal{A}$. Thus $\check{\varphi} = \pi \circ \check{\Phi}\colon \Delta \to \mathbb{P}^n$ is the desired Fatou map. This completes the proof of Theorem 1.

4. Proof of Theorem 2

We begin with a lemma.

LEMMA 9. *Let $\Delta(s_0)$ be a disk of radius s_0 in $\mathbb{C}$, and let B be a domain in $\mathbb{C}^m$. Suppose that H is a plurisubharmonic function on $\Delta(s_0) \times B$ that satisfies the following conditions*:

(1) *there exists a real number s_1 $(0 < s_1 < s_0)$ such that H is pluriharmonic on* $(\Delta(s_0) - \overline{\Delta(s_1)}) \times B$;

(2) *there exists a point w_0 such that $H(z, w_0)$ is a harmonic function of $z \in \Delta$.*

Then H is pluriharmonic on $\Delta \times B$.

Proof. We fix a number r with $s_1 < r < s_0$ and define

$$H^*(z, w) = \frac{1}{2\pi} \int_0^{2\pi} H(re^{i\theta}, w) \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}\, d\theta$$

by Poisson integral with respect to the first variable. Then H^* is real-analytic on $\Delta(r) \times B$. For each $z \in \Delta(r)$ fixed, H^* is pluriharmonic in B. For each $w \in B$ fixed, H^* is harmonic in $\Delta(r)$ and the boundary value on $\partial\Delta(r)$ coincides with H.

Hence, for each $w \in B$ fixed, $H - H^*$ is subharmonic in $\Delta(r)$ and takes the boundary value 0 on $\partial\Delta(r)$. It follows that $H - H^* \leq 0$ on $\Delta(r) \times B$. On the other hand, for each $z \in \Delta(r)$, the function $H - H^*$ is pluriharmonic in B and takes the value 0 at w_0. Thus we have $H = H^*$ on $\Delta(r) \times B$ by the maximum principle.

Because $H = H^*$ is real-analytic on $\Delta(r) \times B$ and pluriharmonic on a nonempty open set, it is pluriharmonic on all of $\Delta(r) \times B$. This proves the lemma. $\square$

Now, to prove Theorem 2, it suffices to consider the case where $R = \Delta$ and E is a compact subset of Δ. For a holomorphic map $\varphi\colon \Delta \to \mathbb{P}^n$ we can choose a holomorphic lift $\Phi\colon \Delta \to \mathcal{X}$. By assumption, we have $\varphi(\Delta - E) \subset \Omega$ and so $\Phi(\Delta - E) \subset \pi^{-1}(\Omega) = \mathcal{H}$.

Choose s_0 $(0 < s_0 < 1)$ such that $E \subset \Delta(s_0)$, and choose $\varepsilon > 0$ such that the ε-neighborhood of $\Phi(\{|z| = s_0\})$ is in $\mathcal{H}$. We define

$$H(z,w) = h(\Phi(z) + w), \quad (z,w) \in \Delta \times B,$$

where h is the Green function and $B = \{w \in \mathbb{C}^{n+1} \mid \|w\| < \varepsilon\}$. Then H is continuous and plurisubharmonic on $\Delta \times B$ and satisfies condition (1) in Lemma 9. Since $H(z,0) = h(\Phi(z))$ is continuous on Δ and harmonic on $\Delta - E$, it is harmonic on Δ and thus satisfies condition (2).

It follows from Lemma 9 that H is pluriharmonic on $\Delta \times B$. This implies that h is pluriharmonic in the neighborhood of $\Phi(z)$. Therefore, $\Phi(\Delta) \subset \mathcal{H}$ and hence $\varphi(\Delta) \subset \Omega$. This completes the proof of Theorem 2.

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