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Klein's Conjecture for Contact Automorphisms of the Three-Dimensional Affine Space

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1. Introduction

The ground field k is of characteristic 0.

Let (x, y, p) be three affine coordinates. The Pfaffian form

$$\omega = dy - p\,dx \tag{1.1}$$

is said to be a *contact form* of the three-dimensional space. A birational transformation T of the three-dimensional (x, y, p)-space defined by

$$x' = f(x, y, p), \quad y' = g(x, y, p), \quad p' = h(x, y, p) \tag{1.2}$$

is said to be a *contact Cremona transformation* of the (x, y)-plane if the transform $T^*(\omega)$ of the contact form (1.1) is proportional to this form:

$$T^*(\omega) = \rho(x, y, p) \cdot \omega, \tag{1.3}$$

where $\rho(x, y, p)$ is a nonzero rational function. Our reference to a "plane" may seem mistaken, but classically it means an action of the transformation on plane contact elements $((x, y), p)$ consisting of a point (x, y) and a slope p at the point. We will say that $\rho(x, y, p)$ is the *multiplier* of T. The contact transformation T is said to be a *contact affine transformation* (or *contact polynomial automorphism*) if T and its inverse T^{-1} are polynomial. For a contact affine transformation of $\mathbb{A}^3$, the multiplier is a nonzero constant (see Lemma 2.1 for contact transformations of odd-dimensional spaces).

Example 1.1. Let

$$x' = f(x, y), \qquad y' = g(x, y) \tag{1.4}$$

be a Cremona transformation of the (x, y)-plane. It is possible to extend (1.4) to a contact transformation

$$x' = f(x, y), \quad y' = g(x, y), \quad p' = h(x, y, p), \tag{1.5}$$

where

$$h(x, y, p) = \frac{p\dfrac{\partial g}{\partial y} + \dfrac{\partial g}{\partial x}}{p\dfrac{\partial f}{\partial y} + \dfrac{\partial f}{\partial x}}.$$

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See [I, 4.5, p. 103] for a description of such an extension for a one-parameter group. According to tradition, we will say that (1.5) is a *contact extension of point transformation* (1.4) or, more briefly, that (1.5) is a *point transformation*.

EXAMPLE 1.2. It is not hard to verify that the transformation

$$L\colon \quad x' = p, \quad y' = xp - y, \quad p' = x \tag{1.6}$$

is involutive and contact. It is the *Legendre transformation* (see [I, 2.5, pp. 40–41]). The Legendre transformation belongs to the set of duality transformations. All duality transformations are conjugate by extended plane projective collineations to the Legendre transformation. See [K, Sec. 62] for a description of space duality transformations as contact transformations. According to [P, p. 125], the connection between the reciprocity defined by a quadric and the Legendre transformation was observed by Michel Chasles.

The following conjecture may be found in [K, Sec. 75.1, p. 300].

KLEIN'S CONJECTURE. *The group of contact Cremona transformations of the projective plane is generated by the subgroup of point contact transformations and by the Legendre transformation.*

REMARK 1.3. The conjecture's formulation here is more explicit than Klein's original description of his *principle*. In [K, p. 300] Klein comments on an example of decomposition of a contact transformation (it was the pedal transformation) and writes as follows:

> *Wir entnehmen aus unserem Beispiel daher das folgende allgemeine Prinzip: Um Beispiele ein eindeutiger Berührungstransformation herzustellen, braucht man nur eine beliebige dualistische Transformation mit einer beliebigen Cremona Transformation verbunden.*

Later authors stated the conjecture without a reference to Klein. For example, Keller [Ke, p. 651] writes that he does not know a birational contact transformation of the plane that cannot be presented as a composition of point Cremona transformations and duality transformations:

> *Korrelationen und Cremona Transformationen und alles daraus Zusammengestzen sind Berührungstransformationen. Eine birationale Berührungstransformation die nicht dieser Gruppe angehört, ist nur nicht bekannt.*

Some other authors (e.g. Hermann [He]) attribute the conjecture to Keller but not to Klein.

I note Klein's sorrowful remark at the end of [K, Sec. 75.1]. He says that so far we do not have a general theory of one-to-one algebraic contact transformations: "Eine allgemeine Theorie der eindeutigen und algebraischen Berührungstransformationen sheint noch nicht entwickelt zu sein." Moreover, I add that Klein stated his conjecture for the first time in his lectures on higher geometry, printed lithographically (the first publication of [K]) in 1893.

Our result is the following theorem.

THEOREM 1.4. *Any polynomial contact automorphism of the affine (x, y, p)-space is a composition of some extended point polynomial automorphisms of the (x, y)-plane and of some number of Legendre transformations.*

In Section 4 we present a description of the structure of a polynomial contact transformation of multidimensional affine space.

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2. General Definitions and Lemmas

We begin with general multidimensional formulations of the basic definitions. Here we consider the odd-dimensional affine space $\mathbb{A}^{2n+1}$ with point coordinates

$$(\mathbf{x}, y, \mathbf{p}) = (x_1, \dots, x_n, y, p_1, \dots, p_n),$$

its even-dimensional (sub/quotient)space $\mathbb{A}^{2n}$ with coordinates

$$(\mathbf{x}, \mathbf{p}) = (x_1, \dots, x_n, p_1, \dots, p_n),$$

the differential form on $\mathbb{A}^{2n+1}$,

$$\omega(\mathbf{x}, y, \mathbf{p}) = dy - p_1 dx_1 - \dots - p_n dx_n, \tag{2.1}$$

and the differential

$$\Omega = d\omega = dx_1 \wedge dp_1 + \dots + dx_n \wedge dp_n. \tag{2.2}$$

Certainly, (2.1) generalizes (1.1). A birational transformation T of the $(2n+1)$-dimensional affine space $\mathbb{A}^{2n+1}$ defined by

$$\begin{aligned} x_i' &= f_i(\mathbf{x}, y, \mathbf{p}), \\ y' &= g(\mathbf{x}, y, \mathbf{p}), \\ p_i' &= h_i(\mathbf{x}, y, \mathbf{p}), \end{aligned} \tag{2.3}$$

where $1 \leq i \leq n$, is said to be a *contact Cremona transformation* of the space if the transform $T^*(\omega)$ of the contact form (2.1) is proportional to the form

$$T^*(\omega) = \rho(\mathbf{x}, y, \mathbf{p}) \cdot \omega, \tag{2.4}$$

where $\rho(\mathbf{x}, y, \mathbf{p})$ is a nonzero rational function.

The function $\rho(\mathbf{x}, y, \mathbf{p})$ is the *multiplier* of T. The contact Cremona transformation T is said to be a *contact affine transformation* if T and its inverse T^{-1} are polynomial. By Lemma 2.1, the multiplier of any contact affine (biregular) transformation is a nonzero constant. A birational transformation S of the $2n$-dimensional affine space $\mathbb{A}^{2n}$ defined by

$$\begin{aligned} x'_i &= f_i(\mathbf{x}, \mathbf{p}), \\ p'_i &= h_i(\mathbf{x}, \mathbf{p}) \end{aligned} \tag{2.5}$$

is said to be a *conformally symplectic Cremona transformation* of the space if the image $S^*(\Omega)$ of the symplectic form (2.2) is proportional to the form

$$S^*(\Omega) = \sigma(\mathbf{x}, \mathbf{p}) \cdot \Omega, \tag{2.6}$$

where $\sigma(\mathbf{x}, \mathbf{p})$ is a nonzero rational function. The function σ is the *conformal multiplier* of S.

A few words about the proof of Lemma 2.1 to follow. According to [P, p. 138], an assertion as in our lemma but for the multidimensional case was proved by Lie (the proof is reproduced in [Ca, p. 109]). Carathéodory did not use exterior differential forms, and Polistchuk writes that the idea of applying differential forms in a proof is due to Frobenius and Cartan. We use this idea in the sequel.

LEMMA 2.1. *For a contact Cremona transformation T defined by* (2.3), *the determinant $J(T)$ of the Jacobian matrix*

$$M(T) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial p_1} & \cdots & \frac{\partial f_1}{\partial p_n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial y} & \frac{\partial f_n}{\partial p_1} & \cdots & \frac{\partial f_n}{\partial p_n} \\ \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial p_1} & \cdots & \frac{\partial g}{\partial p_n} \\ \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} & \frac{\partial h_1}{\partial y} & \frac{\partial h_1}{\partial p_1} & \cdots & \frac{\partial h_1}{\partial p_n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_n}{\partial x_1} & \cdots & \frac{\partial h_n}{\partial x_n} & \frac{\partial h_n}{\partial y} & \frac{\partial h_n}{\partial p_1} & \cdots & \frac{\partial h_n}{\partial p_n} \end{pmatrix} \tag{2.7}$$

is equal to $(n+1)$th *power of the multiplier*:

$$J(T) = \rho^{n+1}.$$

Proof. Indeed, if T is contact then

$$\begin{aligned} T^*(\Omega) &= T^*(d\omega) = dT^*(\omega) \\ &= d\rho \wedge \omega + \rho\Omega. \end{aligned} \tag{2.8}$$

Therefore,

$$\begin{aligned} &J(T)dx_1 \wedge \cdots \wedge dx_n \wedge dy \wedge dp_1 \wedge \cdots \wedge dp_n \\ &\quad = T^*(dx_1 \wedge \cdots \wedge dx_n \wedge dy \wedge dp_1 \wedge \cdots \wedge dp_n) \\ &\quad = \frac{1}{n!}T^*(\omega \wedge \Omega^{\wedge n}) = \frac{1}{n!}\rho^{n+1}\omega \wedge \Omega^{\wedge n} \\ &\quad = \rho^{n+1}dx_1 \wedge \cdots \wedge dx_n \wedge dy \wedge dp_1 \wedge \cdots \wedge dp_n. \end{aligned}$$ □

A parallel similar assertion with almost the same proof can be made for conformally symplectic transformations.

LEMMA 2.2. *For a conformally symplectic Cremona transformation S defined by* (2.5), *the determinant* $J(S)$ *of the Jacobian matrix*

$$M(S) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial p_1} & \cdots & \frac{\partial f_1}{\partial p_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial p_1} & \cdots & \frac{\partial f_n}{\partial p_n} \\ \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} & \frac{\partial h_1}{\partial p_1} & \cdots & \frac{\partial h_1}{\partial p_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_n}{\partial x_1} & \cdots & \frac{\partial h_n}{\partial x_n} & \frac{\partial h_n}{\partial p_1} & \cdots & \frac{\partial h_n}{\partial p_n} \end{pmatrix} \tag{2.9}$$

is equal to nth *power of the conformal multiplier*:

$$J(S) = \sigma^n.$$

We omit the proof.

We will say that, for a conformally symplectic Cremona transformation (2.5), there exists a *potential* $U = U(\mathbf{x}, \mathbf{p}) \in k(x_1, \dots, x_n, p_1, \dots, p_n)$ if the following identities are fulfilled:

$$\begin{aligned} \frac{\partial U}{\partial p_i} &= \sum_{k=1}^{n} h_k \frac{\partial f_k}{\partial p_i}, \\ \frac{\partial U}{\partial x_i} &= \sum_{k=1}^{n} h_k \frac{\partial f_k}{\partial x_i} - p_i\sigma; \end{aligned}$$

here $1 \le i \le n$ and σ is the conformal multiplier of the transformation. It is clear that, for any symplectic polynomial automorphism of $\mathbb{A}^{2n}$, the conformal multiplier σ is a constant and a potential exists.

If σ is a constant and if a potential for (2.5) exists, then

$$\begin{aligned} x_i' &= f_i(\mathbf{x},\mathbf{p}),\\ y' &= \sigma y + U(\mathbf{x},\mathbf{p}),\\ p_i' &= h_i(\mathbf{x},\mathbf{p}) \end{aligned} \tag{2.10}$$

is a contact transformation. We will say that the transformation (2.10) is a *contact lift* of (2.5). For such a lift (2.10), the multiplier ρ coincides with σ. Any potential is defined up to an additive constant. As a result, any lift is defined up to an element from group $\mathbf{T}_y(\mathbb{A}^{2n+1})$ of translations parallel to the y-axis:

$$x_i' = x_i, \quad y' = y + b, \quad p_i' = p_i, \tag{2.11}$$

where b is an element of the ground field k.

LEMMA 2.3. *Any contact polynomial automorphism of the affine* $(2n+1)$*-space is a contact lift of some conformally symplectic polynomial automorphism of the affine* $2n$*-space. That is, such a transformation is representable as* (2.10), *where* f_i, h_i, *and* U *are polynomials.*

Proof. For a polynomial contact automorphism T defined by (2.3), one can write

$$\det(M(T)) = \frac{\partial g}{\partial x_1}F_1 + \cdots + \frac{\partial g}{\partial x_n}F_n + \frac{\partial g}{\partial y}G + \frac{\partial g}{\partial p_1}H_1 + \cdots + \frac{\partial g}{\partial p_n}H_n,$$

where $F_1,\dots,F_n,G,H_1,\dots,H_n$ are the co-factors of elements in the $(n+1)$th row of matrix $M(T)$ in (2.7).

The plan of the proof is as follows. First, we must show that if the multiplier ρ is a constant then all the co-factors (with the exception of G) in the $(n+1)$th row of $M(T)$ vanish. But the co-factor G is equal to the nth power of the multiplier; therefore, according to Lemma 2.2, the $(n+1)$th element $\frac{\partial g}{\partial y}$ in the $(n+1)$th row of matrix is equal to the multiplier:

$$\frac{\partial g}{\partial y} = \rho.$$

Second, we must show that if the multiplier ρ of transformation (2.3) is a constant then the right-hand sides f_i and g_i of (2.3) do not depend on y—that is, the corresponding lines of formulas for (2.3) have the same form as the lines of (2.10).

The proof of Lemma 2.2 employed identity (2.8). Here the multiplier is a constant, so

$$T^*(\Omega) = \rho\Omega. \tag{2.12}$$

Therefore,

$$T^*(\Omega^{\wedge n}) = \rho^n\Omega^{\wedge n}$$

and

$$T^*(dx_1\wedge\cdots\wedge dx_n\wedge dp_1\wedge\cdots\wedge dp_n) = \rho^n\cdot dx_1\wedge\cdots\wedge dx_n\wedge dp_1\wedge\cdots\wedge dp_n.$$

This equality shows that the co-factor G is equal to ρ^n and that other co-factors vanish.

Second, identity (2.12) implies that

$$\sum_{i=1}^{n} \begin{vmatrix} \dfrac{\partial f_i}{\partial x_m} & \dfrac{\partial f_i}{\partial y} \\ \dfrac{\partial h_i}{\partial x_m} & \dfrac{\partial h_i}{\partial y} \end{vmatrix} = 0, \quad m = 1, \ldots, n,$$

or

$$\sum_{i=1}^{n} \frac{\partial h_i}{\partial y} \frac{\partial f_i}{\partial x_m} - \sum_{i=1}^{n} \frac{\partial f_i}{\partial y} \frac{\partial h_i}{\partial x_m} = 0, \quad m = 1, \ldots, n.$$

One may consider the latter identities as a system of homogeneous linear equations with unknown quantities $\partial h_i/\partial y$ and $\partial f_i/\partial y$. The determinant of the system is equal (up to a sign) to G (or to the determinant of a matrix of form (2.9)). Because of nonvanishing of the determinant, the solution to the linear system is trivial. Finally, it is necessary to observe that, owing to independence of f_i and g_i on y, the restriction of T to the $(\mathbf{x}, \mathbf{p})$-hyperplane is symplectic.

We add that a comparison of (2.12) and (2.6) implies the equality $\rho = \sigma$. □

REMARK 2.4. Regarding the second step of our proof: The vanishing of the partial derivatives by y admits an interpretation from the viewpoint of a general theory of contact varieties (see e.g. [H, Chap. 4]). On a general contact variety, the structure contact form ω defines a vector field V_ω by the condition

$$V_\omega(f) \cdot \omega \wedge (d\omega)^{\wedge n} = df \wedge (d\omega)^{\wedge n}.$$

For the case of our standard ω (see (2.1)), Cartan [C, Chap. XIII] used the notation $\{f\}$ instead of $V_\omega(f)$. Certainly, for the standard case the vector field is parallel to the y-axis, $\{f\} = \partial f/\partial y$. The vanishing of $V_\omega(f)$ means that f is a constant along the trajectories of the vector field V_ω, and f is a lift of a function defined on a symplectic quotient of the contact variety.

3. Proof of Theorem 1.4

We begin with some elementary examples.

EXAMPLE 3.1. Let $P(x)$ be a polynomial of x, and let

$$x' = x, \quad y' = y + P(x), \quad p' = p + \frac{dP}{dx}.$$

By Example 1.1, this transformation is a contact extension of a triangular point transformation

$$x' = x, \quad y' = y + P(x). \tag{3.1}$$

If we take

$$f(x, p) = x, \quad h(x, p) = p + \frac{dP}{dx}$$

to be an automorphism of the affine (x, p)-plane, then by a lift (in the sense of the comments after (2.10)) of the automorphism we obtain the transformation of Example 3.1.

EXAMPLE 3.2. The Legendre transformation (1.6) is a lift of the transposition of variables x and p.

EXAMPLE 3.3. More generally, the linear transformation

$$x' = Ap, \quad p' = Bx, \tag{3.2}$$

where $A \in k^*$ and $B \in k^*$, has the following lift:

$$x' = Ap, \quad y' = AB(px - y), \quad p' = Bx.$$

It is clear that the transformation (3.2) is a composition of an extended linear diagonal transformation and the Legendre transformation.

EXAMPLE 3.4. One of the lifts of the triangular transformation

$$x' = x + F(p), \quad p' = p, \tag{3.3}$$

where $F(p) \in k[p]$, is

$$x' = x + F(p), \quad y' = y + U(p), \quad p' = p,$$

where $U(p) \in k[p]$ and $dU/dp = pF(p)$. The latter transformation is a composition of two Legendre transformations L from (1.6) and of a point transformation. Indeed, if R is the point transformation

$$x' = x, \quad y' = y - U(x) + xF(x), \quad p' = p + F(x),$$

then the lift coincides with LRL.

REMARK 3.5. Lie preferred the following notation for the lift of (3.3):

$$x' = x + \frac{dW(p)}{dp}, \quad y' = y - W(p) + p\frac{dW(p)}{dp}, \quad p' = p.$$

Certainly, Lie considered $W(p)$ more general than a rational function with real coefficients. He proved in [L, Chap. 2, Thm. 11, p. 60] that such a commutative subgroup of contact transformations coincides with its own centralizer in the group of all contact transformations.

Proof of Theorem 1.4 (*continued*). It is enough to find a set of generators of the group of contact polynomial automorphisms of the affine 3-space such that any generator is decomposable into a composition of some extended point transformations and some number of the Legendre transformations.

According to a well-known theorem of Jung and Van der Kulk [J; V], the group of polynomial automorphisms of the (x, p)-plane is generated by transformations (3.2) and (3.3) from Examples 3.3 and 3.4, respectively. We saw that some contact lifts of the transformations exist. Any contact lift is defined up to a translation parallel to the y-axis as

$$x' = x, \quad y' = y + b, \quad p' = p. \tag{3.4}$$

By Lemma 2.3 and the theorem of Jung and Van der Kulk, the union
of the set of lifted transformations (3.3) and (3.2)
with the set of extended (3.1)
and with the set of translations (3.4)

generates the group of contact polynomial automorphisms of the affine 3-space. It is clear that any translation is a point transformation. Moreover, we saw that the lifts of (3.3) and (3.2) satisfy Klein's conjecture.

4. A Group Extension

We would like to say a few final words about the structure of the group of contact polynomial automorphisms of an odd-dimensional space.

Let $\mathrm{CSAut}(\mathbb{A}^{2n})$ denote the group of all conformally symplectic polynomial automorphisms of $\mathbb{A}^{2n}$ (see (2.5) and (2.6)), let $\mathrm{ContAut}(\mathbb{A}^{2n+1})$ denote the group of all contact polynomial automorphisms of $\mathbb{A}^{2n+1}$ (see (2.3) and (2.4)), and let $\mathbf{T}_y(\mathbb{A}^{2n+1})$ be the group of translations parallel to the y-axis (see (2.11)). The group of such translations is a subgroup $\mathrm{ContAut}(\mathbb{A}^{2n+1})$. Lemma 2.3 says that by omitting the middle line in (2.3) we obtain formulas of type (2.5). Thus we have a homomorphism of $\mathrm{ContAut}(\mathbb{A}^{2n+1})$ to $\mathrm{CSAut}(\mathbb{A}^{2n})$. The latter homomorphism is surjective. Hence we obtain the following result.

THEOREM 4.1. *The sequence of homomorphisms*

$$\{1\} \to \mathbf{T}_y(\mathbb{A}^{2n+1}) \to \mathrm{ContAut}(\mathbb{A}^{2n+1}) \to \mathrm{CSAut}(\mathbb{A}^{2n}) \to \{1\} \qquad (4.1)$$

is exact.

Proof. The theorem is a reformulation of Lemma 2.3. □

REMARK 4.2. In (4.1), the middle group is an extension of the abelian invariant subgroup $\mathbf{T}_y(\mathbb{A}^{2n+1})$ by $\mathrm{CSAut}(\mathbb{A}^{2n})$. One can describe such an extension with the help of an action of the quotient group on the kernel together with a system of factors (see [Ku, Sec. 48]). In our case, the action coincides with the multiplication by the Jacobian determinant; that is, if $\alpha \in \mathrm{CSAut}(\mathbb{A}^{2n})$ is the image of $g_\alpha \in \mathrm{ContAut}(\mathbb{A}^{2n+1})$ and if t is the translation (2.11), then $g_\alpha t g_\alpha^{-1}$ is defined by

$$x_i' = x_i, \quad y' = y + J(\alpha)b, \quad p_i' = p_i,$$

where $J(\alpha)$ is the Jacobian determinant of α. For a general extension, the system of factors is the function $m_{\alpha,\beta}$ of pairs of elements α, β of the quotient group with values in the kernel, and the function is defined (after a fixation of some representatives g_α) by the identity

$$g_\alpha g_\beta = m_{\alpha,\beta} g_{\alpha\beta}.$$

For our case, we can fix the representatives by the condition of vanishing of potentials at the origin $(\mathbf{0})$. After doing so, the constant $b_{\alpha,\beta}$ corresponding to the translation $m_{\alpha,\beta}$ is defined by

$$b_{\alpha,\beta} = U_\alpha(g_\beta(\mathbf{0})),$$

with $U_\alpha(\mathbf{x}, \mathbf{p})$ the potential of α vanishing at $(\mathbf{0})$. The produced family $m_{\alpha,\beta}$ is the system of factors defining extension (4.1).

References

[Ca] C. Carathéodory, *Variationsrechnung und partielle differential Gleichungen erster Ordnung,* Band I, Zweiter Auflage, Leipzig, 1956; English translation, *Calculus of variations and partial differential equations of the first order,* Chelsea, New York, 1982.

[C] E. Cartan, *Leçons sur les invariants intégraux,* Hermann, Paris, 1922.

[He] M. Hermann, *Über birationale Berührungstransformationen zweiter Ordnung,* Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe 8 (1958/1959), 375–386.

[H] N. E. Hurt, *Geometric quantization in action,* Math. Appl. (East European Ser.), 8, Reidel, Dordrecht, 1983.

[I] E. L. Ince, *Ordinary differential equations,* Dover, New York, 1956.

[J] H. W. Jung, *Über ganze birationale Transformationen der Ebene,* J. Reine Angew. Math. 184 (1942), 161–174.

[Ke] O. H. Keller, *Zur Theorie der ebenen Berührungstransformationen I,* Math. Ann. 120 (1947), 650–675.

[K] F. Klein, *Vorlesungen über höhere Geometrie,* 3rd ed., Springer, Berlin, 1926.

[Ku] A. G. Kurosh, *The theory of groups,* 3rd ed., Moscow, 1967; English translation (2 vols.), 2nd ed., Chelsea, New York, 1960.

[L] S. Lie, *Geometrie der Berührungstransformationen,* Leipzig, 1896; English translation, Chelsea, New York, 1977.

[P] E. M. Polischuk, *Sophus Lie,* Nauchno-Biograf. Lit., Nauka Leningrad. Otdel., Leningrad, 1983.

[V] W. Van der Kulk, *On polynomial rings in two variables,* Nieuw Arch. Wiskunde (3) 1 (1953), 33–41.

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