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On Sections of Elliptic Fibrations

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1. Introduction

It is well known that two generic cubics P and Q in $\mathbb{C}P^2$ intersect each other in nine points $z_1, \dots, z_9$. By constructing the corresponding *pencil* of curves

$$\{sP + tQ \mid [s : t] \in \mathbb{C}P^1\}$$

one can define a map $f : \mathbb{C}P^2 - \{z_1, \dots, z_9\} \to \mathbb{C}P^1$. After blowing up $\mathbb{C}P^2$ at $\{z_1, \dots, z_9\}$ one can extend f to a Lefschetz fibration $\pi : E(1) = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ with nine distinguished sections and whose generic fiber is an elliptic curve. Our aim in this paper is to describe an analogous construction in the smooth category, but unfortunately we do not know whether our construction arises from an *algebraic* pencil of curves. Nevertheless, many 4-manifold topologists were curious about such a differential topological construction (e.g., this was posed explicitly as a question in [4]).

Let $\Gamma^s_{g,k}$ denote the mapping class group of a compact connected orientable genus-g surface with k boundary components and s marked points, so that diffeomorphisms and isotopies of the surface are assumed to fix the marked points and the points on the boundary. (We will drop k if the surface is closed and drop s if there are no fixed points.) A product $\prod_{i=1}^m t_i$ of right-handed Dehn twists in Γ_g provides a genus-g Lefschetz fibration $X \to D^2$ over the disk with closed fibers. If $\prod_{i=1}^m t_i = 1$ in Γ_g then the fibration closes up to a fibration over the sphere S^2. A lift of the relation $\prod_{i=1}^m t_i = 1$ to Γ^k_g shows the existence of k disjoint sections of the induced Lefschetz fibration. The self-intersection of the jth section is $-n_j$ if $\prod_{i=1}^m t_i = t_{\delta_1}^{n_1} \cdots t_{\delta_k}^{n_k}$ in $\Gamma_{g,k}$ for some positive integers $n_1, \dots, n_k$, where the t_{δ_i} are right-handed Dehn twists along circles parallel to the boundary components of the surface at hand (cf. [3]).

On the other hand, an expression of the form $\prod_{i=1}^m t_i = t_{\delta_1} \cdots t_{\delta_k}$ in $\Gamma_{g,k}$ naturally describes a Lefschetz pencil: the relation determines a Lefschetz fibration with k disjoint sections, where each section has self-intersection -1, and after blowing these sections down we get a Lefschetz pencil (cf. [4]). Conversely, blowing up the base locus of a Lefschetz pencil yields a Lefschetz fibration that can be captured

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(together with the exceptional divisors of the blow-ups, which are all sections now) by a relation of this type.

In this paper we find relations of the form $\prod_{i=1}^{12} t_i = t_{\delta_1} \cdots t_{\delta_k}$ in $\Gamma_{1,k}$ for $4 \leq k \leq 9$, generalizing the well-known cases $k = 1, 2, 3$. A relation of this type naturally induces a Lefschetz pencil, and by blowing up we obtain an elliptic Lefschetz fibration with k disjoint sections. Moreover, by taking the nth power of our relation (for $n \geq 2$) we have

$$\left(\prod_{i=1}^{12} t_i\right)^n = t_{\delta_1}^n \cdots t_{\delta_k}^n \in \Gamma_{1,k}$$

for $4 \leq k \leq 9$. Once again this relation induces an elliptic Lefschetz fibration $E(n) \to S^2$ with $12n$ singular fibers and k disjoint sections, where the self-intersection of each section is equal to $-n$.

The reader is advised to consult [2; 6; 8] for background material on Lefschetz fibrations and pencils. To simplify notation in the rest of this paper, we will denote a right-handed Dehn twist along a curve α also by α. A left-handed Dehn twist along α will be denoted by $\bar{\alpha}$. We will multiply the Dehn twists from right to left; that is, $\beta\alpha$ means that we first apply α then β.

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2. Lantern Relation for the Four-holed Sphere

Consider the four-holed sphere in Figure 1. Then we have the relation

$$\delta_1\delta_2\delta_3\delta_4 = \gamma\sigma\alpha$$

in $\Gamma_{0,4}$, which was discovered by Dehn [1]. It was rediscovered by Johnson [7] and named the lantern relation. We will freely use this relation on any subsurface (of another surface) that is homeomorphic to a sphere with four holes. The depiction of the lantern relation on the four-holed sphere in Figure 1 will be convenient in the subsequent discussions. The lantern relation is classically proven by comparing the actions of both sides on a suitable system of curves whose complement is a disc.

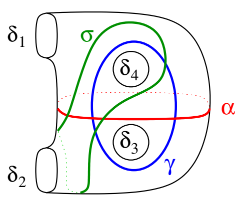

Figure 1 Four-holed sphere with boundary $\{\delta_1, \delta_2, \delta_3, \delta_4\}$

3. Relations on a Torus with Holes

In this section we will generalize the well-known one-holed torus relation to a relation on the k-holed torus for $2 \le k \le 9$. We will give all the details in each case since the relation for $k+1$ holes is derived by using the relation for k holes with $1 \le k \le 8$. The relations in the cases $k = 2, 3$ are also known, but we compute them anyway for the sake of completeness and to show our method.

We note that if two circles are disjoint then the corresponding Dehn twists commute. Also, if two circles α and β intersect transversely at one point, then the corresponding Dehn twists satisfy the braid relation: $\alpha\beta\alpha = \beta\alpha\beta$.

3.1. ONE-HOLED TORUS. If α and β are two circles on a torus with one boundary δ_1 that intersect each other transversely at one point, then

$$(\alpha\beta)^6 = \delta_1.$$

We call this the one-holed torus relation (it turns out that this relation was known to Dehn [1] in a slightly different form). The one-holed torus relation, just like the lantern relation, is proven by comparing the actions of both sides on a suitable system of curves whose complement is a disc.

3.2. TWO-HOLED TORUS. Consider the two-holed torus in Figure 2. By the lantern relation, we have

$$\alpha_2^2\delta_1\delta_2 = \gamma_1\sigma_1\alpha_1.$$

The one-holed torus relation is

$$\gamma_1 = (\alpha_2\beta)^6.$$

Observe that $\sigma_1 = \bar\beta\bar\alpha_2\bar\alpha_2\alpha_1\beta\bar\alpha_1\alpha_2\alpha_2\beta$. Then we have

$$\begin{aligned}
\delta_1\delta_2 &= \bar\alpha_2\bar\alpha_2\gamma_1\sigma_1\alpha_1\\
&= \bar\alpha_2\bar\alpha_2(\alpha_2\beta\alpha_2\beta\alpha_2\beta\alpha_2\beta\alpha_2\beta\alpha_2\beta)(\bar\beta\bar\alpha_2\bar\alpha_2\alpha_1\beta\bar\alpha_1\alpha_2\alpha_2\beta)\alpha_1\\
&= \bar\alpha_2\bar\alpha_2\alpha_2\alpha_2\beta\alpha_2\alpha_2\beta\alpha_2\beta\alpha_2\beta\bar\alpha_2\alpha_1\beta\bar\alpha_1\alpha_2\alpha_2\beta\alpha_1\\
&= \beta\alpha_2\alpha_2\beta\alpha_2\alpha_2\beta\alpha_1\beta\bar\alpha_1\alpha_2\alpha_2\beta\alpha_1\\
&= \beta\alpha_2\alpha_2\beta\alpha_2\alpha_2\alpha_1\beta\alpha_2\alpha_2\beta\alpha_1\\
&= \beta\alpha_2\beta\alpha_2\beta\alpha_2\alpha_1\beta\alpha_2\alpha_2\beta\alpha_1\\
&= \alpha_2\beta\alpha_2\alpha_2\beta\alpha_2\alpha_1\beta\alpha_2\alpha_2\beta\alpha_1\\
&= \alpha_2\alpha_2\beta\alpha_2\alpha_1\beta\alpha_2\alpha_2\beta(\alpha_1\alpha_2\beta)\\
&= \alpha_2\beta\alpha_2\beta\alpha_1\beta\alpha_2\alpha_2\beta(\alpha_1\alpha_2\beta)\\
&= \alpha_2\beta\alpha_2\alpha_1\beta\alpha_1\alpha_2\alpha_2\beta(\alpha_1\alpha_2\beta)\\
&= \alpha_2\beta\alpha_1\alpha_2\beta\alpha_2(\alpha_1\alpha_2\beta)(\alpha_1\alpha_2\beta)\\
&= \alpha_2\beta\alpha_1\beta\alpha_2\beta(\alpha_1\alpha_2\beta)^2\\
&= \alpha_2\alpha_1\beta(\alpha_1\alpha_2\beta)(\alpha_1\alpha_2\beta)^2\\
&= (\alpha_1\alpha_2\beta)^4.
\end{aligned}$$

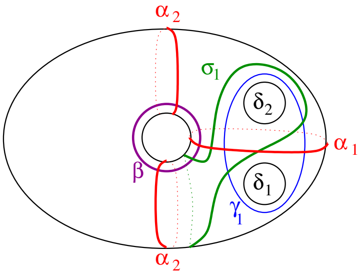

Figure 2 Two-holed torus with boundary $\{\delta_1, \delta_2\}$

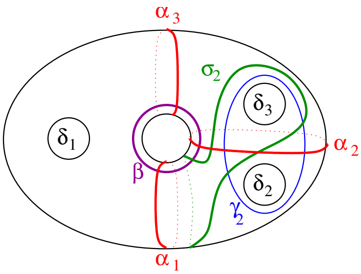

Figure 3 Three-holed torus with boundary $\{\delta_1, \delta_2, \delta_3\}$

3.3. THREE-HOLED TORUS. Consider the three-holed torus in Figure 3. By the lantern relation,

$$\alpha_3\alpha_1\delta_2\delta_3 = \gamma_2\sigma_2\alpha_2,$$

and by the two-holed torus relation,

$$\delta_1\gamma_2 = (\alpha_1\alpha_3\beta)^4.$$

Note that $\sigma_2 = \bar{\beta}\bar{\alpha}_1\bar{\alpha}_3\alpha_2\beta\bar{\alpha}_2\alpha_3\alpha_1\beta$. Then

$$\begin{aligned}\delta_1\delta_2\delta_3 &= \bar{\alpha}_1\bar{\alpha}_3\delta_1\gamma_2\sigma_2\alpha_2\\ &= \bar{\alpha}_1\bar{\alpha}_3(\alpha_1\alpha_3\beta\alpha_1\alpha_3\beta\alpha_1\alpha_3\beta\alpha_1\alpha_3\beta)(\bar{\beta}\bar{\alpha}_1\bar{\alpha}_3\alpha_2\beta\bar{\alpha}_2\alpha_3\alpha_1\beta)\alpha_2\\ &= \beta\alpha_1\alpha_3\beta\alpha_1\alpha_3\beta\alpha_2\beta\bar{\alpha}_2\alpha_3\alpha_1\beta\alpha_2\\ &= \beta\alpha_1\alpha_3\beta\alpha_1\alpha_3\alpha_2\beta\alpha_3\alpha_1\beta\alpha_2.\end{aligned}$$

Using the appropriate braid and commutation relations (as we did when deriving the two-holed torus relation) it follows that

$$\delta_1\delta_2\delta_3 = (\alpha_1\alpha_2\alpha_3\beta)^3.$$

This relation was called the star relation in [5].

3.4. FOUR-HOLED TORUS. The lantern relation for the sphere with boundary $\{\alpha_4, \alpha_2, \delta_3, \delta_4\}$ in Figure 4 is

$$\alpha_4\alpha_2\delta_3\delta_4 = \gamma_3\sigma_3\alpha_3.$$

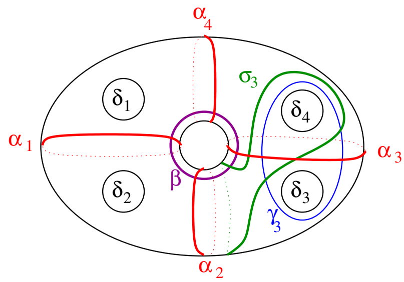

Figure 4 Four-holed torus with boundary $\{\delta_1, \delta_2, \delta_3, \delta_4\}$

The relation on the three-holed torus with boundary $\{\delta_1, \delta_2, \gamma_3\}$ given in Section 3.3 is

$$\delta_1\delta_2\gamma_3 = (\alpha_1\alpha_2\alpha_4\beta)^3.$$

Here we identify the curves $(\alpha_1, \alpha_2, \alpha_3)$ in Figure 3 with the curves $(\alpha_1, \alpha_2, \alpha_4)$ in Figure 4. Combining, we obtain

$$\begin{aligned}\delta_1\delta_2\delta_3\delta_4 &= \delta_1\delta_2\bar{\alpha}_2\bar{\alpha}_4\gamma_3\sigma_3\alpha_3\\ &= \bar{\alpha}_2\bar{\alpha}_4\delta_1\delta_2\gamma_3\sigma_3\alpha_3\\ &= \bar{\alpha}_2\bar{\alpha}_4(\alpha_1\alpha_2\alpha_4\beta)^3\sigma_3\alpha_3\\ &= \alpha_1\beta(\alpha_1\alpha_2\alpha_4\beta)^2\sigma_3\alpha_3\\ &= (\alpha_1\alpha_2\alpha_4\beta)^2\sigma_3\alpha_3\alpha_1\beta.\end{aligned}$$

REMARK. Although we will not need it in the rest of the paper, by plugging in

$$\sigma_3 = \bar{\beta}\bar{\alpha}_4\bar{\alpha}_2\alpha_3\beta\bar{\alpha}_3\alpha_2\alpha_4\beta$$

it is easy to see that this relation may also be written in a more symmetric form as

$$\delta_1\delta_2\delta_3\delta_4 = (\alpha_1\alpha_3\beta\alpha_2\alpha_4\beta)^2.$$

3.5. FIVE-HOLED TORUS. In Figure 5, the lantern relation for the sphere with boundary $\{\alpha_5, \alpha_3, \delta_4, \delta_5\}$ is

$$\alpha_5\alpha_3\delta_4\delta_5 = \gamma_4\sigma_4\alpha_4.$$

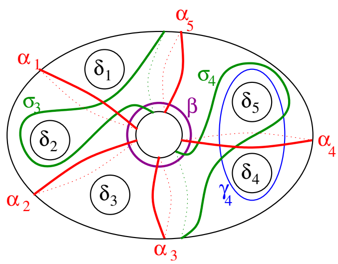

Figure 5 Five-holed torus with boundary $\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$

The relation on the four-holed torus with boundary $\{\delta_1, \delta_2, \delta_3, \gamma_4\}$ given in Section 3.4 is

$$\delta_1\delta_2\delta_3\gamma_4 = (\alpha_3\alpha_5\alpha_2\beta)^2\sigma_3\alpha_1\alpha_3\beta.$$

Here we identify the curves $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ in Figure 4 with the curves $(\alpha_3, \alpha_5, \alpha_1, \alpha_2)$ in Figure 5. Combining then yields

$$\begin{aligned}\delta_1\delta_2\delta_3\delta_4\delta_5 &= \bar{\alpha}_3\bar{\alpha}_5\delta_1\delta_2\delta_3\gamma_4\sigma_4\alpha_4\\ &= \bar{\alpha}_3\bar{\alpha}_5(\alpha_3\alpha_5\alpha_2\beta)^2\sigma_3\alpha_1\alpha_3\beta\sigma_4\alpha_4\\ &= \alpha_2\beta\alpha_3\alpha_5\alpha_2\beta\sigma_3\alpha_1\alpha_3\beta\sigma_4\alpha_4\\ &= \alpha_2\alpha_3\alpha_5\beta\sigma_3\alpha_1\alpha_3\beta\sigma_4\alpha_4\alpha_2\beta.\end{aligned}$$

3.6. SIX-HOLED TORUS. The lantern relation for the sphere with boundary $\{\alpha_6, \alpha_4, \delta_5, \delta_6\}$ in Figure 6 is

$$\alpha_6\alpha_4\delta_5\delta_6 = \gamma_5\sigma_5\alpha_5.$$

The relation for the five-holed torus with boundary $\{\delta_1, \delta_2, \delta_3, \delta_4, \gamma_5\}$ given in Section 3.5 is

$$\delta_1\delta_2\delta_3\delta_4\gamma_5 = \alpha_4\alpha_6\alpha_2\beta\sigma_3\alpha_3\alpha_6\beta\sigma_4\alpha_1\alpha_4\beta.$$

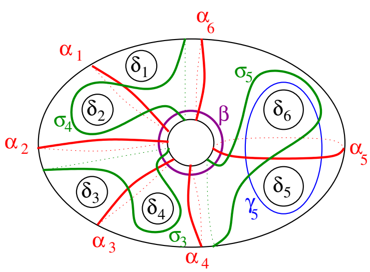

Figure 6 Six-holed torus with boundary $\{\delta_1, \delta_2, \dots, \delta_6\}$

We identify the curves $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ in Figure 5 with the curves $(\alpha_3, \alpha_4, \alpha_6, \alpha_1, \alpha_2)$ in Figure 6. Combining, we have

$$\begin{aligned}\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6 &= \bar{\alpha}_4\bar{\alpha}_6\delta_1\delta_2\delta_3\delta_4\gamma_5\sigma_5\alpha_5\\ &= \bar{\alpha}_4\bar{\alpha}_6\alpha_4\alpha_6\alpha_2\beta\sigma_3\alpha_3\alpha_6\beta\sigma_4\alpha_1\alpha_4\beta\sigma_5\alpha_5\\ &= \alpha_2\beta\sigma_3\alpha_3\alpha_6\beta\sigma_4\alpha_1\alpha_4\beta\sigma_5\alpha_5\\ &= \beta_2\alpha_2\sigma_3\alpha_3\alpha_6\beta\sigma_4\alpha_1\alpha_4\beta\sigma_5\alpha_5\\ &= \alpha_2\alpha_3\alpha_6\beta\sigma_4\alpha_1\alpha_4\beta\sigma_5\alpha_5\beta_2\sigma_3,\end{aligned}$$

where $\beta_2 = \alpha_2\beta\bar{\alpha}_2$.

3.7. SEVEN-HOLED TORUS. The lantern relation for the sphere with boundary $\{\alpha_7, \alpha_5, \delta_6, \delta_7\}$ in Figure 7 is

$$\alpha_7\alpha_5\delta_6\delta_7 = \gamma_6\sigma_6\alpha_6.$$

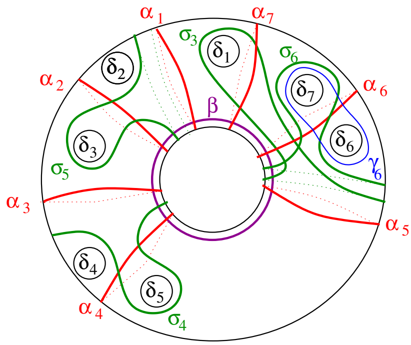

Figure 7 Seven-holed torus with boundary $\{\delta_1, \delta_2, \ldots, \delta_7\}$

The relation on the six-holed torus with boundary $\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \gamma_6\}$ given in Section 3.6 is

$$\delta_1\delta_2\delta_3\delta_4\delta_5\gamma_6 = \alpha_5\alpha_7\alpha_3\beta\sigma_4\alpha_4\alpha_1\beta\sigma_5\alpha_2\beta_5\sigma_3,$$

where we use the identification $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \to (\alpha_4, \alpha_5, \alpha_7, \alpha_1, \alpha_2, \alpha_3)$ to go from Figure 6 to Figure 7. We may then combine to obtain

$$\begin{aligned}
\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7 &= \bar{\alpha}_5\bar{\alpha}_7\delta_1\delta_2\delta_3\delta_4\delta_5\gamma_6\sigma_6\alpha_6 \\
&= \bar{\alpha}_5\bar{\alpha}_7\alpha_5\alpha_7\alpha_3\beta\sigma_4\alpha_4\alpha_1\beta\sigma_5\alpha_2\beta_5\sigma_3\sigma_6\alpha_6 \\
&= \alpha_3\beta\sigma_4\alpha_4\alpha_1\beta\sigma_5\alpha_2\beta_5\sigma_3\sigma_6\alpha_6 \\
&= \beta_3\alpha_3\sigma_4\alpha_4\alpha_1\beta\sigma_5\alpha_2\beta_5\sigma_3\sigma_6\alpha_6 \\
&= \beta_3\sigma_4\alpha_3\alpha_4\alpha_1\beta\sigma_5\alpha_2\beta_5\sigma_3\sigma_6\alpha_6 \\
&= \alpha_3\alpha_4\alpha_1\beta\sigma_5\alpha_2\beta_5\sigma_3\sigma_6\alpha_6\beta_3\sigma_4,
\end{aligned}$$

where $\beta_3 = \alpha_3\beta\bar{\alpha}_3$ and $\beta_5 = \alpha_5\beta\bar{\alpha}_5$.

3.8. Eight-holed Torus. The lantern relation for the sphere with boundary $\{\alpha_8, \alpha_6, \delta_7, \delta_8\}$ in Figure 8 is

$$\alpha_8\alpha_6\delta_7\delta_8 = \gamma_7\sigma_7\alpha_7.$$

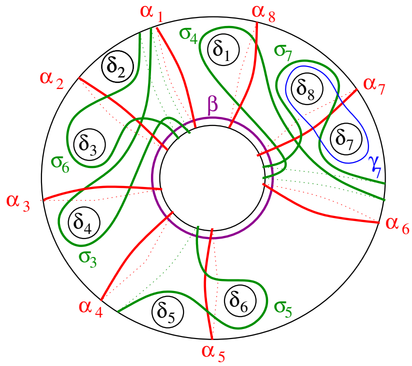

Figure 8 Eight-holed torus with boundary $\{\delta_1, \delta_2, \ldots, \delta_8\}$

The relation on the seven-holed torus with boundary $\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \gamma_7\}$ given in Section 3.7 is

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\gamma_7 = \alpha_6\alpha_8\alpha_4\beta\sigma_5\alpha_5\beta_1\sigma_3\sigma_6\alpha_2\beta_6\sigma_4,$$

where we use the identification $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) \rightarrow (\alpha_4, \alpha_5, \alpha_6, \alpha_8, \alpha_1, \alpha_2, \alpha_3)$ to go from Figure 7 to Figure 8. By combining, we get

$$\begin{aligned}\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8 &= \bar{\alpha}_6\bar{\alpha}_8\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\gamma_7\sigma_7\alpha_7\\ &= \bar{\alpha}_6\bar{\alpha}_8\alpha_6\alpha_8\alpha_4\beta\sigma_5\alpha_5\beta_1\sigma_3\sigma_6\alpha_2\beta_6\sigma_4\sigma_7\alpha_7\\ &= \alpha_4\beta\sigma_5\alpha_5\beta_1\sigma_3\sigma_6\alpha_2\beta_6\sigma_4\sigma_7\alpha_7\\ &= \beta_4\alpha_4\sigma_5\alpha_5\beta_1\sigma_3\sigma_6\alpha_2\beta_6\sigma_4\sigma_7\alpha_7\\ &= \beta_4\sigma_5\alpha_4\alpha_5\beta_1\sigma_3\sigma_6\alpha_2\beta_6\sigma_4\sigma_7\alpha_7\\ &= \alpha_4\alpha_5\beta_1\sigma_3\sigma_6\alpha_2\beta_6\sigma_4\sigma_7\alpha_7\beta_4\sigma_5,\end{aligned}$$

where $\beta_1 = \alpha_1\beta\bar{\alpha}_1$, $\beta_4 = \alpha_4\beta\bar{\alpha}_4$, and $\beta_6 = \alpha_6\beta\bar{\alpha}_6$.

3.9. NINE-HOLED TORUS. The lantern relation for the sphere with boundary $\{\alpha_9, \alpha_7, \delta_8, \delta_9\}$ in Figure 9 is

$$\alpha_9\alpha_7\delta_8\delta_9 = \gamma_8\sigma_8\alpha_8.$$

The relation on the eight-holed torus with boundary $\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \gamma_8\}$ given in Section 3.8 is

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\gamma_8 = \alpha_7\alpha_9\beta_4\sigma_3\sigma_6\alpha_5\beta_1\sigma_4\sigma_7\alpha_2\beta_7\sigma_5,$$

where we identify $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8)$ with $(\alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_9, \alpha_1, \alpha_2, \alpha_3)$ to go from Figure 8 to Figure 9. Combining then yields

$$\begin{aligned}\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8\delta_9 &= \bar{\alpha}_7\bar{\alpha}_9\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\gamma_8\sigma_8\alpha_8\\ &= \bar{\alpha}_7\bar{\alpha}_9\alpha_7\alpha_9\beta_4\sigma_3\sigma_6\alpha_5\beta_1\sigma_4\sigma_7\alpha_2\beta_7\sigma_5\sigma_8\alpha_8\\ &= \beta_4\sigma_3\sigma_6\alpha_5\beta_1\sigma_4\sigma_7\alpha_2\beta_7\sigma_5\sigma_8\alpha_8,\end{aligned}$$

where $\beta_1 = \alpha_1\beta\bar{\alpha}_1$, $\beta_4 = \alpha_4\beta\bar{\alpha}_4$, and $\beta_7 = \alpha_7\beta\bar{\alpha}_7$.

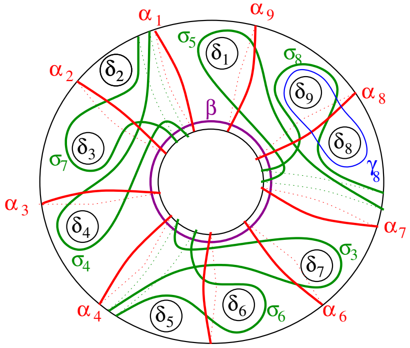

Figure 9 Nine-holed torus with boundary $\{\delta_1, \delta_2, \ldots, \delta_9\}$

REMARK. The curious reader might wonder why we stopped at $k = 9$. First of all, our process will not allow us to go any further because we will not have the canceling right-handed Dehn twists to kill off the left-handed Dehn twists that appear in the appropriate lantern relations. In fact, there is a good reason for this: An elliptic fibration $E(1) \to S^2$ admits at most nine disjoint sections (all with negative self-intersections). So we conclude that there is no such relation for a k-holed torus with $k \geq 10$.

4. Sections of the Elliptic Fibrations

First we consider the case $k = 4$. The relation

$$\delta_1\delta_2\delta_3\delta_4 = (\alpha_1\alpha_2\alpha_4\beta)^2\sigma_3\alpha_3\alpha_1\beta$$

in $\Gamma_{1,4}$ that we derived in Section 3.4 induces the word $(\alpha^3\beta)^3 = 1$ in Γ_1, which gives us an elliptic Lefschetz fibration on the elliptic surface $E(1) = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$. To draw a Kirby diagram (cf. [6]) of this elliptic fibration we start with a 0-handle, attach two 1-handles (see Figure 10), and then attach a 2-handle, which yields $D^2 \times T^2$. A torus fiber of the trivial fibration $D^2 \times T^2 \to D^2$ can be viewed in Figure 10 as follows. Take the obvious disk on the page, attach two 2-dimensional 1-handles (going through two 4-dimensional 1-handles) and cap off by a 2-dimensional disk. Then we draw the curves that appear in the monodromy of the elliptic fibration on parallel copies of this fiber. Observe that these curves are the attaching curves of some 2-handles. By attaching all twelve of these 2-handles with framing one less than the page framing we get an elliptic Lefschetz fibration over D^2 with twelve singular fibers, which then can be closed off to an elliptic Lefschetz fibration over S^2. We illustrate the four disjoint sections s_1, s_2, s_3, s_4 of the induced fibration in Figure 10. (Imagine replacing the s_i in Figure 10 with holes where they intersect the page and embedding the curves in Figure 4 into distinct fibers.) For each $i = 1, 2, 3, 4$, the curve s_i bounds two disks—one in the neighborhood of a regular fiber and one outside of that neighborhood—from which we

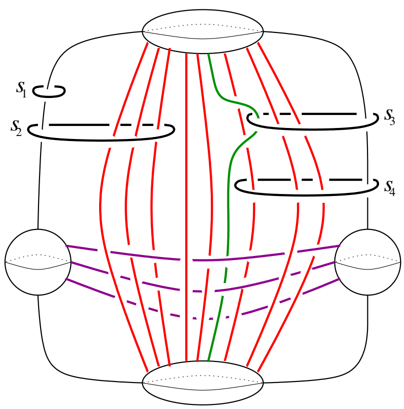

Figure 10 Elliptic Lefschetz fibration $E(1) \to S^2$ with four disjoint sections

obtain a section of the elliptic Lefschetz fibration by gluing these two disks along their common boundary s_i.

Similarly, we can draw the Kirby diagrams corresponding to the relations we derived for $k = 5, 6, \ldots, 9$ and explicitly indicate the locations of the k disjoint sections of $E(1) \to S^2$ in these diagrams. We skip the cases $k = 5, \ldots, 8$ and jump to the case $k = 9$ (see Figure 11). The relation

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8\delta_9 = \sigma_4\sigma_7\alpha_2\beta_7\sigma_5\sigma_8\alpha_8\beta_4\sigma_3\sigma_6\alpha_5\beta_1$$

on the nine-holed torus induces the word $(\alpha^3\beta_\alpha)^3 = 1$ in the mapping class group Γ_1; this gives us an elliptic Lefschetz fibration on the elliptic surface $E(1) = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$, where $\beta_\alpha = \alpha\beta\bar{\alpha}$ (which is indeed a right-handed Dehn twist). Note that we cyclicly permuted the curves in the equation derived in Section 3.9 to obtain the relation just displayed.

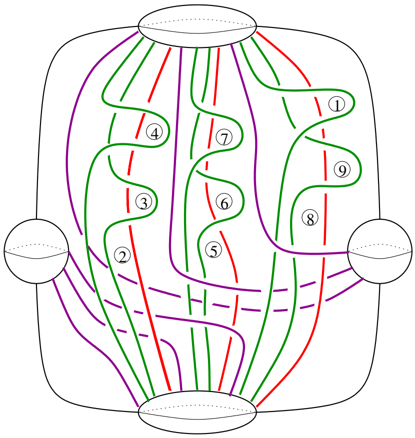

Figure 11 Elliptic Lefschetz fibration $E(1) \to S^2$ with nine disjoint sections (intersection points of sections with the regular fiber are indicated by circled numbers corresponding to the boundary components $\delta_1, \ldots, \delta_9$ of the nine-holed torus)

Finally, for $4 \leq k \leq 9$, by taking the nth power of our relation for the k-holed torus we can find k disjoint sections of the corresponding elliptic fibration on the elliptic surface $E(n)$ for any $n \geq 1$.

5. Final Comments

Suppose that the product $\delta_1\delta_2 \cdots \delta_k$, where δ_i denotes a right-handed Dehn twist along a curve parallel to the ith boundary component of a surface with k boundary components, can be expressed as a product of right-handed Dehn twists along interior (i.e., nonboundary parallel) curves on the surface. We will call such a relation in the corresponding mapping class group a *boundary–interior relation*. The technique we applied to derive a boundary–interior relation in $\Gamma_{1,k}$ $(2 \leq k \leq 9)$ can be easily generalized to derive a boundary–interior relation in $\Gamma_{g,k}$ for $g \geq 2$. As elaborated in this paper, one can start with a certain boundary–interior relation in $\Gamma_{g,1}$ and then derive a boundary–interior relation in $\Gamma_{g,2}$, and then a boundary-interior relation in $\Gamma_{g,3}$, and so forth. In fact, by applying our trick, it might be possible to derive a boundary–interior relation in $\Gamma_{g,k+1}$ if we are given a boundary–interior relation in $\Gamma_{g,k}$. Hence our method can be applied to construct additional sections of a given Lefschetz fibration in certain situations.

It is intriguing to note that, once we fix a boundary–interior relation in $\Gamma_{g,1}$ for some $g \geq 2$, then (for simple homological reasons) there is a maximum k for which we obtain a boundary–interior relation in $\Gamma_{g,k}$ by applying our method. It appears that this number depends not only on g but also on the initial boundary–interior relation in $\Gamma_{g,1}$.

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