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Singularities Appearing on Generic Fibers of Morphisms between Smooth Schemes

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Introduction

The goal of this paper is to explore the structure of singularities that occur on generic fibers in positive characteristics. As an application of our general results, we shall determine which rational double points do and which do not occur on generic fibers.

In some sense, our starting point is Sard's lemma from differential topology. It states that the critical values of a differential map between differential manifolds form a set of measure 0. As a consequence, any general fiber of a differential map is itself a differential manifold. The analogy in algebraic geometry is as follows. Let k be an algebraically closed ground field of characteristic $p \geq 0$, and suppose $f: S \to B$ is a morphism between smooth integral schemes. Then the generic fiber S_η is a regular scheme of finite type over the function field $E = \kappa(\eta)$.

In characteristic 0, this implies that S_η is smooth over E. Moreover, the absolute Galois group $G = \mathrm{Gal}(\bar{E}/E)$ acts on the geometric generic fiber $S_{\bar\eta}$ with quotient isomorphic to S_η. In other words, to understand the generic fiber it suffices to understand the geometric generic fiber, which is again smooth over an algebraically closed field, together with its Galois action.

The situation is more complicated in characteristic $p > 0$. The reason is that over nonperfect fields the notion of regularity, which depends only on the scheme and not on the structure morphism, is weaker than the notion of geometric regularity, which coincides with formal smoothness. Here it easily happens that the geometric generic fiber $S_{\bar\eta}$ acquires singularities. This special effect already plays a crucial role in the extension of Enriques' classification of surface to positive characteristics: as Bombieri and Mumford [3] showed, there are *quasi-elliptic fibrations* for $p = 2$ and $p = 3$, which are analogous to elliptic fibrations but have a cusp on the geometric generic fiber.

We call a proper morphism $f: S \to B$ of smooth algebraic schemes a *quasi-fibration* if $\mathcal{O}_B = f_*(\mathcal{O}_S)$ and if the generic fiber S_η is not smooth. The existence of quasi-fibrations should by no means be viewed as pathological. Rather, they involve some fascinating geometry and apparently offer new freedom to achieve geometrical constructions that are impossible in characteristic 0. The theory of quasi-fibrations, however, is still in its infancy. In [15, Rem. 1.2], Kollár asks

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whether or not there are Fano contractions on 3-folds whose geometric generic fibers are nonnormal del Pezzo surfaces. Some results in this direction appear in [22]. Mori and Saito [19] studied Fano contractions whose geometric generic fibers are nonreduced quadrics. Examples of quasi-fibrations involving minimally elliptic singularities appear in [23] in connection with Beauville's generalized Kummer varieties.

Of course, nonsmoothness of the generic fiber S_η leads to unusual complications. However, singularities appearing on the geometric generic fiber $S_{\bar\eta}$ are not arbitrary. First and foremost, they are locally of complete intersection; hence many powerful methods from commutative algebra apply. Yet they also satisfy far more restrictive conditions, and the goal of this paper is to analyze these conditions. Hirokado [14] started such an analysis, and he characterized those rational double points in odd characteristics that appear on geometric generic fibers. His approach was to study the closed fibers S_b ($b \in B$) and their deformation theory. Our approach is somewhat different: we look at the generic fiber S_η and deliberately work over the function field $\kappa(\eta)$.

In fact, we will mainly work in the following abstract setting. Given a field F in characteristic $p > 0$ and a subfield E such that the field extension $E \subset F$ is purely inseparable, we consider F-schemes X of finite type that descend to regular E-schemes Y; that is, $X \simeq Y \otimes_E F$. Our first results on such schemes X are as follows. In codimension 2, the local fundamental groups are trivial and the torsion of the local class groups are p-groups. Moreover, the Tjurina numbers are divisible by p, the stalks of the Jacobian ideal have finite projective dimension, and the tangent sheaf Θ_X is locally free in codimension 2. These conditions are comparatively straightforward but already give strong conditions on the singularities. The following restriction on the cotangent sheaf was a bit of a surprise to me; it is the first main result of this paper.

THEOREM. *If an F-scheme X descends to a regular scheme then, for each point $x \in X$ of codimension 2, the stalk $\Omega^1_{X/F,x}$ contains an invertible direct summand. In other words, $\Omega^1_{X/F,x} \simeq \mathcal{O}_{X,x} \oplus M$ for some $\mathcal{O}_{X,x}$-module M.*

Note that any torsion-free module of finite type over an integral local Noetherian ring is an extension of some ideal by a free module. Such extensions, however, do not necessarily split, so the preceding result puts a nontrivial condition on the cotangent sheaf.

As an application of all these results, we shall determine which rational double points appear on surfaces descending to regular schemes and which do not. This was already settled by Hirokado [14] in the case of odd characteristics. Recall that Artin [2] classified rational double points in positive characteristics. Here the isomorphism class is determined not only by a Dynkin diagram but sometimes also by some additional integral parameter r (this is the case for $p = 2, 3, 5$). It turns out that the situation is most challenging in characteristic 2: besides the A_n-singularities, which behave as in characteristic 0, there are the following isomorphism classes:

$$D_n^r \text{ with } 0 \leq r \leq \lfloor n/2 \rfloor - 1 \quad \text{and} \quad E_6^0, E_6^1, E_7^0, \dots, E_7^3, E_8^0, \dots, E_8^4.$$

For simplicity, I state the second main result of this paper only in characteristic 2, which answers a question of Hirokado [14].

THEOREM. *In characteristic* 2, *the rational double points that appear on surfaces and descend to regular surfaces are*: A_{2^e-1} *with* $e \geq 1$; D_n^0 *with* $n \geq 4$; *and* E_n^0 *with* $n = 6, 7, 8$.

Observe that this includes (but does not coincide with) all rational double points that are purely inseparable double coverings of a smooth scheme. Shepherd-Barron calls them *special* and shows [24] that they play an important role in the geometry of surfaces in characteristic 2. The D_{2m+1}^0-singularities are not special but nevertheless descend to regular surfaces. I also wish to point out that all members of our list have a tangent sheaf that is locally free, although there are other rational double points with locally free tangent sheaf.

The paper proceeds as follows. Section 1 sets up notation and gives some elementary examples and results; in the next four sections, we analyze F-schemes X that descend to regular E-schemes Y. In Section 2 we treat the local fundamental groups. In Section 3, we shall see that integer-valued invariants attached to the singularities of X like Tyurina numbers are multiples of p. Section 4 deals with the finite projective dimension of sheaves obtained from the cotangent sheaf $\Omega^1_{X/F}$. Here we will also see that the tangent sheaf Θ_X is locally free in codimension 2. Our first main result appears in Section 5, where we show that each stalk of the cotangent sheaf $\Omega_{X/F}$ in codimension 2 contains an invertible summand. This gives a powerful and easy-to-use criterion for complete intersections defined by a single equation. We will exploit this in Section 6, where we determine which rational double points are possible on schemes descending to regular schemes.

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1. Purely Inseparable Descent

Let F be a field of characteristic $p \geq 0$ and let $E \subset F$ be a subfield. For each E-scheme Y, base-change gives an F-scheme $X = Y \otimes_E F$. Conversely, given an F-scheme X, one may ask whether there exists an E-scheme Y with the property $X \simeq Y \otimes_E F$. If this is the case, we shall say that X *descends* along $E \subset F$. Then the fiber product $R = X \times_Y X$ defines an equivalence relation on X, and we may view $Y = X/R$ as the corresponding quotient. The topic of this paper is schemes that are not themselves regular but do descend to regular schemes.

DEFINITION 1.1. We say that a locally Noetherian F-scheme X *descends to a regular scheme* if there exist a subfield $E \subset F$ and a regular E-scheme Y with $X \simeq Y \otimes_E F$.

This notion is of little interest in characteristic 0, for then any locally Noetherian scheme descending to a regular scheme is itself regular (this follows from [8, Prop. 6.7.4]). Hence assume from now on that we are in characteristic $p > 0$. Throughout, X usually denotes a locally Noetherian F-scheme.

LEMMA 1.2. *Suppose that the F-scheme X descends to a regular scheme. Then there is a subfield E inside F and a regular E-scheme Y with $X \simeq Y \otimes_E F$ such that the field extension $E \subset F$ is purely inseparable.*

Proof. We have $X \simeq Y_0 \otimes_{E_0} F$ with a regular E_0-scheme Y_0 for some subfield $E_0 \subset F$. Choose a transcendence basis $f_\alpha \in F$ $(\alpha \in I)$ for this field extension, and let E be the separable algebraic closure of $E_0(f_\alpha)_{\alpha \in I}$ inside F. Then the field extension $E \subset F$ is purely inseparable, and the scheme $Y = Y_0 \otimes_{E_0} E$ remains regular by [8, Prop. 6.7.4]. □

We now discuss an example to see how this might happen. Suppose $v_0, v_1, \dots, v_n$ is a collection of indeterminates and let $f \in E[v_0, v_1, \dots, v_n]$ be a polynomial of the form $f = v_0^q + g$, where $q = p^e$ is a power of the characteristic and where $g \in (v_1, \dots, v_2)^2$ does not involve the indeterminate v_0 and has neither constant nor linear terms. Suppose there is an element $\lambda \in E$ that is not a pth power in E but does become a qth power in F. Consider the two affine schemes

$$Y = \operatorname{Spec} E[v_0, v_1, \dots, v_n]/(f - \lambda), \qquad X = \operatorname{Spec} F[v_0, v_1, \dots, v_n]/(f).$$

Let $y \in Y$ be the closed point defined by $v_1 = \cdots = v_n = 0$. Note that this is not a rational point; rather, its residue field is $\kappa(y) = E(\lambda^{1/q})$. Let $x \in X$ be the rational point given by $v_0 = v_1 = \cdots = v_n = 0$.

PROPOSITION 1.3. *With the preceding assumptions, the local ring $\mathcal{O}_{Y,y}$ is regular, the local ring $\mathcal{O}_{X,x}$ is not regular, and $\mathcal{O}_{X,x} \simeq \mathcal{O}_{Y,y} \otimes_E F$ holds. Hence some open neighborhood of the singularity $x \in X$ descends to a regular scheme.*

Proof. To check that $\mathcal{O}_{Y,y}$ is regular, set $\mathbb{A}^n_E := \operatorname{Spec} E[v_1, \dots, v_n]$ and consider the canonical morphism $\varphi\colon Y \to \mathbb{A}^n_E$. This is flat of degree q, and the fiber over the origin $0 \in \mathbb{A}^n_E$ is isomorphic to the spectrum of $E[v_0]/(v_0^q - \lambda)$, which is a field. In particular, the base $\mathbb{A}^n_E$ and the fiber $\varphi^{-1}(0)$ are both regular schemes. According to [7, Prop. 17.3.3], this implies that the local ring $\mathcal{O}_{Y,y}$ is regular.

By our assumptions on f, we have

$$x \in \operatorname{Spec} k[v_0, v_1, \dots, v_n]/(v_0, v_1, \dots, v_n)^2 \subset X.$$

Therefore, $\operatorname{edim}(\mathcal{O}_{X,x}) \geq n + 1 > n = \dim(\mathcal{O}_{X,x})$. It follows that the local ring $\mathcal{O}_{X,x}$ is not regular. Finally, the automorphism of $F[v_0, v_1, \dots, v_n]$ defined by the substitution $v_0 \mapsto v_0 + \lambda^{1/q}$ maps $f - \lambda$ to f, whence $X \simeq Y \otimes_E F$, and this isomorphism gives $\mathcal{O}_{X,x} \simeq \mathcal{O}_{Y,y} \otimes_E F$. □

REMARK 1.4. There are singularities of different structure that descend to regular schemes—for example, rational double points of type D^0_{2m+1} in characteristic 2. See Section 6.

Schemes descending to regular schemes are not arbitrary. Rather, they satisfy several strong conditions. Recall that a locally Noetherian scheme S is called *locally of complete intersection* if, for each point $s \in S$, the formal completion is of the form $\mathcal{O}^{\wedge}_{S,s} \simeq R/(f_1, \dots, f_r)$, where R is a regular complete local Noetherian ring and $f_1, \dots, f_r \in R$ is a regular sequence.

PROPOSITION 1.5. *Suppose that a locally Noetherian F-scheme X descends to a regular scheme. Then X is locally of complete intersection.*

Proof. Clearly, regular schemes are locally of complete intersection. According to [10, Cor. 19.3.4], the property of being locally of complete intersection is preserved under base field extensions. □

Suppose k is a ground field of characteristic $p > 0$, and let S be a smooth k-scheme. Suppose $f : S \to B$ is a morphism onto an integral k-scheme of finite type, and let $\eta \in B$ be the generic point. Then the generic fiber S_η is regular because the morphism $S_\eta \to S$ is flat with regular fibers. It follows that the geometric generic fiber $S_{\bar{\eta}}$, which is of finite type over $F = \overline{\kappa(\eta)}$, descends to a regular scheme. We have the following converse.

PROPOSITION 1.6. *Let X be a connected F-scheme of finite type that descends to a regular scheme. Then there exist smooth connected $\mathbb{F}_p$-schemes of finite type S and B, a dominant morphism $f : S \to B$, and an inclusion of fields $\kappa(\eta) \subset F$ with $X \simeq S_\eta \otimes_{\kappa(\eta)} F$. Here $\eta \in B$ denotes the generic point.*

Proof. Write $X = Y \otimes_E F$ for some regular E-scheme Y. The E-scheme Y is of finite type because the F-scheme X is [12, Exp. VIII, Prop. 3.3]. We may thus assume that the field E is finitely generated over its prime field $\mathbb{F}_p$ according to [9, Thm. 8.8.2]. Then $E = \kappa(B)$ is the function field of some integral $\mathbb{F}_p$-scheme B of finite type. Moreover, Y is isomorphic to the generic fiber of some dominant morphism $f : S \to B$. Shrinking S and B, we may assume that S and B are regular. Then they are also smooth, because the finite field $\mathbb{F}_p$ is perfect. □

Now suppose that S is smooth and proper. We call a morphism $f : S \to B$ onto another proper k-scheme B a *quasi-fibration* if $\mathcal{O}_B = f_*(\mathcal{O}_S)$ and if the generic fiber S_η is not smooth. Hence the geometric generic fiber $S_{\bar{\eta}}$ is singular but descends to a regular scheme. The most prominent examples of quasi-fibrations are quasi-elliptic surfaces.

2. Local Fundamental Groups

Let F be a field of characteristic $p > 0$, and let X be a locally Noetherian F-scheme. To avoid endless repetition of the same hypothesis, *we suppose throughout this section that X descends to a regular scheme*. In other words, there is a subfield E such that this field extension $E \subset F$ is purely inseparable, and a regular E-scheme Y with $X = Y \otimes_E F$. The goal of this section is to find out what this implies for fundamental groups attached to X. The main idea is to use the Zariski–Nagata purity theorem.

To start with, we recall the definition of fundamental groups in algebraic geometry. Suppose that S is a connected scheme. We shall denote by $\mathrm{Et}(S)$ the category of finite étale morphisms $S' \to S$. It is a *Galois category* in the sense of [12, Exp. V, Sec. 5.1] and hence equivalent to the category $\mathcal{C}(\pi)$ of finite sets endowed with a continuous action of a profinite group π. This is deduced as follows. Suppose $a\colon \mathrm{Spec}(\Omega) \to S$ is a base point for some separably closed field Ω. Then we have a fiber functor $\mathrm{Et}(S) \to (\mathrm{Set})$, $S' \mapsto S'_a$. The *fundamental group* $\pi_1(S,a)$ is defined as the automorphism group of this fiber functor. As explained in [12, Exp. V, Sec. 4], this makes the fundamental group profinite, and the fiber functor yields an equivalence $\mathrm{Et}(S) \to \mathcal{C}(\pi_1(S,a))$. The choice of different base points yields isomorphic fundamental groups, but the isomorphism is unique only up to inner automorphisms. By abuse of notation, we sometimes write $\pi_1(S)$ when speaking about group-theoretical properties that depend only on isomorphism classes of groups.

Return now to our situation $X = Y \otimes_E F$. Let a be a geometric point on X and b the induced geometric point on Y. We have the following basic fact.

LEMMA 2.1. *The canonical homomorphism $\pi_1(X,a) \to \pi_1(Y,b)$ is an isomorphism of topological groups.*

Proof. By assumption, the field extension $E \subset F$ is purely inseparable. Consider first the special case that $E \subset F$ is finite as well. Then [12, Exp. IX, Thm. 4.10] tells us that the pull-back functor $\mathrm{Et}(Y) \to \mathrm{Et}(X)$, $Y' \mapsto Y' \times_Y X$ is an equivalence of categories. In the general case write $F = \bigcup F_\lambda$ as a union of subfields that are finite over E. Using [9, Thm. 8.8.2] allows us to infer that the pull-back functor $Y' \mapsto Y' \times_Y X$ is still an equivalence of categories. The assertion follows. $\square$

This lemma leads to the following result.

THEOREM 2.2. *Suppose our F-scheme X is local and henselian with closed point $x \in X$. Let $A \subset X$ be a closed subset of codimension ≥ 2 and let $U \subset X$ be its complement. Then U is connected and its fundamental group $\pi_1(U)$ is isomorphic to the absolute Galois group of the residue field $\kappa(x)$.*

Proof. If follows from Proposition 1.5 that the scheme X is Cohen–Macaulay and hence $\mathrm{depth}(\mathcal{O}_{X,a}) \geq 2$ for all points $a \in A$. By Hartshorne's connectedness theorem [11, Exp. III, Thm. 3.6], the complement U is connected. Let $V \subset Y$ be the open subset corresponding to $U \subset X$, and consider the following chain of restriction and pull-back functors:

$$\mathrm{Et}(U) \xleftarrow{r_1} \mathrm{Et}(V) \xleftarrow{r_2} \mathrm{Et}(Y) \xleftarrow{r_3} \mathrm{Et}(X) \xleftarrow{r_4} \mathrm{Et}(x).$$

According to Lemma 2.1, the pull-back functors r_1, r_3 are equivalences. By assumption, Y is regular, so we may apply the Zariski–Nagata theorem (see [11, Exp. X, Thm. 3.4] for a scheme-theoretic proof) and deduce that the restriction functor r_2 is an equivalence as well. Since X is henselian, the restriction functor r_4 is an equivalence (as explained in [10, Prop. 18.5.15]). The assertion follows. $\square$

Recall that a local scheme is called *strictly henselian* if it is henselian and if the residue field of the closed point is separably closed.

COROLLARY 2.3. *Let the assumptions of Theorem 2.2 be given. If X is strictly henselian, then U is connected and simply connected.*

Recall that, for a local scheme S, the *local fundamental group* $\pi_1^{\text{loc}}(S, a)$ is defined as the fundmental group of the complement of the closed point $s \in S$, where a denotes a base point on the scheme $S \setminus \{s\}$. We may apply the preceding corollary to the strict henselization $\operatorname{Spec}(\mathcal{O}_{X,x}^{\text{sh}})$ of a point $x \in X$, where X is an arbitrary locally Noetherian scheme.

COROLLARY 2.4. *Let $x \in X$ be a point of codimension ≥ 2. Then $\pi_1^{\text{loc}}(\mathcal{O}_{X,x}^{\text{sh}}) = 0$.*

Proof. In order to apply Corollary 2.3, we merely have to verify that the henselization $S := \operatorname{Spec}(\mathcal{O}_{X,x}^{\text{sh}})$ descends to a regular scheme. Indeed, let $y \in Y$ be the image of $x \in X$. The canonical map $\mathcal{O}_{Y,y}^{\text{sh}} \to \mathcal{O}_{X,x}^{\text{sh}}$ coming from the universal property of strict henselizations on Y induces a map $\varphi\colon \mathcal{O}_{Y,y}^{\text{sh}} \otimes_E F \to \mathcal{O}_{X,x}^{\text{sh}}$. The scheme $\mathcal{O}_{Y,y}^{\text{sh}} \otimes_E F$ is clearly local with separably closed residue field, and it follows from [10, Thm. 18.5.11] that it is henselian as well. Therefore, the universal property of strict henselization on X yields the desired inverse map $\psi\colon \mathcal{O}_{X,x}^{\text{sh}} \to \mathcal{O}_{Y,y}^{\text{sh}} \otimes_E F$. $\square$

For an arbitrary locally Noetherian scheme S we define the *class group* as $\operatorname{Cl}(S) = \varinjlim \operatorname{Pic}(U)$, where the filtered direct limit runs over all open subsets $U \subset S$ such that $\operatorname{depth}(\mathcal{O}_{S,s}) \geq 2$ for all $s \in S \setminus U$. When S is normal, this is indeed the group of Weil divisors modulo principal divisors [11, Exp. XI, Prop. 3.7.1]. For nonnormal or even nonreduced schemes, however, our definition here seems to be more suitable.

COROLLARY 2.5. *Let $x \in X$ be a point of codimension ≥ 2. Then the torsion part of the class group $\operatorname{Cl}(\mathcal{O}_{X,x}^{\text{sh}})$ is a p-group.*

Proof. Set $S = \operatorname{Spec}(\mathcal{O}_{X,x}^{\text{sh}})$. It follows from Proposition 1.5 that the scheme S is Cohen–Macaulay. Hence the points $s \in S$ with $\operatorname{depth}(\mathcal{O}_{S,s}) \geq 2$ are precisely the points of codimension ≥ 2. Seeking a contradiction, we suppose that there is an open subset $U \subset S$ whose complement has codimension ≥ 2 and also an invertible $\mathcal{O}_U$-module $\mathcal{L}$ whose order $l > 1$ in $\operatorname{Pic}(U)$ is finite but not a p-power. Passing to a suitable multiple, we may assume that l is a prime number different from p. Choosing a trivialization of $\mathcal{L}^{\otimes l}$, we obtain an $\mathcal{O}_U$-algebra structure on the coherent $\mathcal{O}_U$-module $\mathcal{A} = \mathcal{O}_U \oplus \mathcal{L} \oplus \cdots \oplus \mathcal{L}^{\otimes(l-1)}$. Its relative spectrum $U' = \operatorname{Spec}(\mathcal{A})$ is a finite étale Galois covering $U' \to U$ of degree l. This covering is nontrivial, from which it follows that U is not simply connected. On the other hand, Corollary 2.3 tells us that U is simply connected—a contradiction. $\square$

We now specialize the preceding result to the case of normal surface singularities. Let $x \in X$ be a point of codimension 2, and assume that the local Noetherian ring $\mathcal{O}_{X,x}$ is normal. Let $r\colon \tilde{X} \to X$ be a resolution of singularities (whose existence we tacitly assume) and $D = r^{-1}(x)_{\text{red}}$ the reduced exceptional divisor. Consider the connected component $\text{Pic}^0_{D/\kappa(x)}$ of the Picard scheme, which parameterizes invertible sheaves on D with 0 degree on each integral component. We call the singularity $x \in X$ *unipotent* if the Picard scheme $\text{Pic}^0_{D/\kappa(x)}$ is unipotent. It is easy to verify that this does not depend on the choice of resolution of singularities. Note also that, to check this, we may use any divisor $D \subset \tilde{X}$ whose support is the full fiber. Rational singularities are examples of unipotent singularities.

COROLLARY 2.6. *Let $x \in X$ be a point of codimension* 2. *Then the singularity $x \in X$ is unipotent.*

Proof. Seeking a contradiction, we assume that the singularity $x \in X$ is not unipotent so that the group scheme $\text{Pic}^0_{D/k}$ is not unipotent, where $k = \kappa(x)$. Choose a separable closure $k \subset k^s$ and set $D^s = D \otimes_k k^s$. Then there must be an invertible sheaf $\mathcal{L}_0 \in \text{Pic}(D^s)$ of finite order l prime to p. It extends to an invertible sheaf $\mathcal{L}$ on the strict henselization $\mathcal{O}^{\text{sh}}_{X,x}$ of order l, as explained in [21, Lemma 2.2]. Restricting to the complement $U \subset \text{Spec}(\mathcal{O}^{\text{sh}}_{X,x})$ of the closed point, we obtain an invertible sheaf $\mathcal{L}_U \in \text{Pic}(U)$ of order l in contradiction to Corollary 2.5. □

COROLLARY 2.7. *Given the assumption of Corollary 2.6, let $D \subset \tilde{X}$ be the reduced fiber over the singularity $x \in X$ and let $D_i \subset D$ be its integral components. Then the schemes D_i are geometrically unibranch, and their intersection graph is a tree. If D_i is smooth, then it is isomorphic to a quadric in $\mathbb{P}^2$ and $h^1(\mathcal{O}_{D_i}) = 0$.*

Proof. If any of the conditions did not hold then the Picard scheme $\text{Pic}_{D/\kappa(x)}$ would not be unipotent, as explained in [4, Chap. 9]. □

REMARK 2.8. Recall that a normal point $x \in X$ of codimension 2 is called a *simple elliptic singularity* if the reduced fiber $D \subset \tilde{X}$ is an elliptic curve. Corollary 2.7 tells us that such singularities do not descend to E-schemes that are regular. Hirokado [14] showed this by using the equations for simple elliptic singularities.

3. Residue Fields and p-Divisibility

In this section, X denotes an F-scheme of finite type. We continue to assume throughout that X descends to a regular scheme. In other words, there is a subfield E such that this field extension $E \subset F$ is purely inseparable, and a regular E-scheme Y with $X = Y \otimes_E F$.

Let $x \in X$ be a point and $y \in Y$ its image. By our assumption, the local ring $\mathcal{O}_{Y,y}$ is regular. What can be said about the local ring $\mathcal{O}_{X,x}$ and the inclusion $\mathcal{O}_{Y,y} \subset \mathcal{O}_{X,x}$? We start with an observation on the residue fields. Consider the commutative diagram

$$\begin{array}{ccc} \kappa(y) & \longrightarrow & \kappa(x) \\ \uparrow & & \uparrow \\ E & \longrightarrow & F \end{array}$$

of fields, and choose an algebraic closure $\kappa(x) \subset \Omega$.

PROPOSITION 3.1. *Suppose $x \in X$ is not regular. Then the subfields $\kappa(y), F \subset \Omega$ are not linearly disjoint over E.*

Proof. Seeking a contradiction, we assume that the subfields $\kappa(y), F \subset \Omega$ are linearly disjoint. By definition, this means that the canonical map $\kappa(y) \otimes_E F \to \Omega$ is injective and so the ring $R = \kappa(y) \otimes_E F$ is integral. Using that the ring extension $\kappa(y) \subset R$ is integral, we infer that the ring R is actually a field. It follows that the fiber $X_y = \mathrm{Spec}(R)$ of the projection $X \to Y$ is a regular scheme. By our assumptions, $y \in Y$ is regular. By [8, Cor. 6.5.2], this implies that $x \in X$ is regular—a contradiction. □

This result has the following numerical consequence.

PROPOSITION 3.2. *Suppose $x \in X$ is not regular. Then the field extension $E \subset \kappa(y)$ is not separable. If, in addition, $x \in X$ is closed, then the characteristic p divides the degree $[\kappa(y) : E]$ of the image point $y \in Y$.*

Proof. Suppose that $E \subset \kappa(y)$ is separable. Then the ring $R = \kappa(y) \otimes_E F$ is reduced. It is also irreducible, because $E \subset F$ is purely inseparable. It follows that R is integral; hence the canonical map $R \to \Omega$ must be injective and so $\kappa(y), F \subset \Omega$ are linearly disjoint—contradiction.

If the point $x \in X$ is closed, then the field extension $\kappa(y) \subset E$ is finite and its degree is of the form $[\kappa(y) : E] = p^e[\kappa(y) : E]_s$, where the second factor is the separable degree and $e \geq 0$ is an integer. Because $E \subset \kappa(y)$ is not separable, we must have $e > 0$. □

We shall apply this result frequently in the following form.

LEMMA 3.3. *Let $\mathcal{F}_Y$ be a coherent zero-dimensional sheaf on Y supported on the nonsmooth locus, and let $\mathcal{F}_X = \mathcal{F}_Y \otimes_E F$ be the corresponding coherent sheaf on X. Then the number $h^0(X, \mathcal{F}_X)$ is a multiple of p.*

Proof. Recall that $h^0(X, \mathcal{F}_X)$ is the dimension of $H^0(X, \mathcal{F}_F)$ viewed as an F-vector space. By flat base-change, we have $h^0(Y, \mathcal{F}_Y) = h^0(X, \mathcal{F})$. The coherent zero-dimensional sheaf $\mathcal{F}_Y$ has a Jordan–Hölder series with factors isomorphic to residue fields $\kappa(y)$, where $y \in Y$ are nonsmooth closed points. By Proposition 3.2, the dimension of the E-vector space $\kappa(y)$ is a multiple of p, and the assertion follows. □

There are many such coherent sheaves $\mathcal{F}_X = \mathcal{F}_Y \otimes_E F$ coming from the sheaf of Kähler differentials, provided that the nonsmooth locus is zero-dimensional. We mention a few as follows.

PROPOSITION 3.4. *Suppose the nonsmooth locus of X is zero-dimensional. Then, for all $m > \dim(X)$, the numbers $h^0(X, \Omega^m_{X/k})$ are multiples of p.*

Proof. The coherent sheaf $\Omega^1_{Y/k}$ is locally free of rank $d = \dim(Y)$ on the smooth locus. For $m > \dim(Y)$, the coherent sheaf $\mathcal{F}_Y = \Omega^m_{Y/k}$ is therefore supported on the nonsmooth locus. Hence Lemma 3.3 applies. □

Next, consider the coherent sheafs $\mathcal{T}_m, \mathcal{T}'_m$ defined by the exact sequence

$$0 \longrightarrow \mathcal{T}_m \longrightarrow \Omega^m_{X/k} \longrightarrow (\Omega^m_{X/k})^{\vee\vee} \longrightarrow \mathcal{T}'_m \longrightarrow 0,$$

where the map in the middle is the canonical evaluation map into the bidual.

PROPOSITION 3.5. *Suppose the nonsmooth locus of X is zero-dimensional. Then, for all $m \geq 0$, the numbers $h^0(X, \mathcal{T}_m)$ and $h^0(X, \mathcal{T}'_m)$ are multiples of p.*

Proof. This follows as before from Lemma 3.3. □

Recall that the *Jacobian ideal* is defined as the dth Fitting ideal of $\Omega^1_{X/F}$, where $d = \dim(X)$. Here we tacitly assume that X is equidimensional. We call the closed subscheme $X' \subset X$ defined by the Jacobian ideal the *Jacobian subscheme*. It comprises the points $x \in X$ that are not smooth and puts a scheme structure on this locus, which is usually nonreduced.

PROPOSITION 3.6. *Suppose that the nonsmooth locus of X is zero-dimensional; that is, suppose the Jacobian subscheme $X' \subset X$ is zero-dimensional. Then the number $h^0(X, \mathcal{O}_{X'})$ is a multiple of p.*

Proof. Again this follows from Lemma 3.3. □

If $x \in X$ is an isolated nonsmooth point, then the local length $l(\mathcal{O}_{X',x})$ is called the *Tjurina number* of the singularity $x \in X$. We can see that the p-divisibility of Tjurina numbers is a necessary condition for an F-scheme to descend to regular schemes.

4. Finite Projective Dimension

We keep the assumptions from the preceding section with X an F-scheme of finite type that descends to a regular E-scheme Y, where $E \subset F$ is a purely inseparable field extension. The goal of this section is to exploit finer properties of coherent sheaves $\mathcal{F}_X = \mathcal{F}_Y \otimes_E F$ related to commutative algebra rather than field theory.

Suppose $\mathcal{M}$ is a coherent $\mathcal{O}_X$-module. Given a point $x \in X$, the *projective dimension* $\mathrm{pd}(\mathcal{M}_x)$ is defined as the infimum over all numbers n such that there exists a finite free resolution

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathcal{M}_x \longrightarrow 0,$$

where the F_n are free $\mathcal{O}_{X,x}$-modules of finite rank. If no such resolution exists then we write $\mathrm{pd}(\mathcal{M}_x) = \infty$.

LEMMA 4.1. *Let $\mathcal{F}_Y$ be a coherent sheaf on Y, and let $\mathcal{F}_X = \mathcal{F}_Y \otimes_E F$ be the induced sheaf on X. Then $\mathcal{F}_{X,x}$ has finite projective dimension for all $x \in X$.*

Proof. Let $y \in Y$ be the image of $x \in X$. Since $\mathcal{O}_{Y,y}$ is regular by assumption, it follows by Serre's criterion that the stalk $\mathcal{F}_{Y,y}$ has finite projective dimension. Since $X \to Y$ is flat, this implies that $\mathcal{F}_{X,x}$ has finite projective dimension as well. □

In particular, the stalks of $\Omega^1_{X/F}$ have finite projective dimension. From this we can immediately deduce the following facts.

PROPOSITION 4.2. *Any coherent $\mathcal{O}_X$-module obtained from $\Omega^1_{X/F}$ by taking tensor powers, symmetric powers, alternating powers, Fitting ideals, duals, biduals, kernels and cokernels of biduality maps, et cetera has stalks with finite projective dimension.*

In particular, the stalks of the Jacobian ideal $\mathcal{J} \subset \mathcal{O}_X$ have finite projective dimension. When the Jacobian subscheme $X' \subset X$ is zero-dimensional, we have as invariants the Tjurina numbers—that is, the lengths $l(\mathcal{O}_{X,x}/\mathcal{J}_x) \geq 0$ of the Jacobian subscheme at $x \in X$.

Recall that, for any coherent ideal $\mathcal{I} \subset \mathcal{O}_X$, we define the *bracket ideal* $\mathcal{I}^{[p]} \subset \mathcal{O}_X$ as the ideal whose stalks are generated by f^p with $f \in \mathcal{I}_x$.

THEOREM 4.3. *Suppose that the Jacobian subscheme $X' \subset X$ is zero-dimensional, and let $d = \dim(X)$. Then, for all closed points $x \in X$, the length formula $l(\mathcal{O}_{X,x}/\mathcal{J}_x^{[p]}) = p^d l(\mathcal{O}_{X,x}/\mathcal{J}_x)$ holds.*

Proof. The scheme X is locally of complete intersection by Proposition 1.5. If $x \in X$ is smooth then both lengths in question are 0, so it suffices to treat only the case $x \in X'$. Because X' is discrete, the ideal $\mathcal{J}_x \subset \mathcal{O}_{X,x}$ is $\mathfrak{m}_x$-primary. According to Miller's result [18, Cor. 5.2.3], the length formula $l(\mathcal{O}_{X,x}/\mathcal{I}_x^{[p]}) = p^d l(\mathcal{O}_{X,x}/\mathcal{I}_x)$ holds for an ideal $\mathcal{I}_x \subset \mathcal{O}_{X,x}$ defining a zero-dimensional subscheme if and only if $\mathcal{I}_x$ has finite projective dimension. Then Proposition 4.2 implies the assertion. □

EXAMPLE 4.4. Assume that X is a normal surface over F in characteristic $p = 2$. Suppose $x \in X$ is a rational point such that the local ring $\mathcal{O}_{X,x}$ is a rational double point of type E_8^3. When $k \subset \bar{k}$ is an algebraic closure we have $\mathcal{O}^{\wedge}_{X,\bar{x}} \simeq k[[x, y, z]]/(f)$, where $f = z^2 + x^3 + y^5 + y^3 z$ (as explained in Artin's paper [2]). Then

$$\mathcal{J}_{\bar{x}} = (x^2, y^4 + y^2 z, y^3) = (x^2, y^2 z, y^3).$$

From this one easily computes the lengths $l(\mathcal{O}_{X,x}/\mathcal{J}_x) = 10$ and $l(\mathcal{O}_{X,x}/\mathcal{J}_x^{[2]}) = 44$, which implies that the F-scheme X does not descend to an E-scheme that is regular. Note that the local fundamental group of the strict henselization of $x \in X$ is trivial, so Proposition 2.3 does not allow this conclusion.

Let us finally have a closer look the tangent sheaf $\Theta_X = \mathcal{H}om(\Omega^1_{X/k}, \mathcal{O}_X)$. Clearly, $\Theta_X = \Theta_Y \otimes_E F$ holds and so, by Proposition 4.2, the stalks of Θ_X have finite projective dimension. In small codimensions we also have the following statement.

PROPOSITION 4.5. *Let $x \in X$ be a point of codimension 2. Then the stalk $\Theta_{X,x}$ is a free $\mathcal{O}_{X,x}$-module.*

Proof. The scheme X is Cohen–Macaulay, whence $\operatorname{depth}(\mathcal{O}_{X,x}) = 2$. As a dual, the sheaf Θ_X satisfies Serre's condition (S_2) according to [13, Thm. 1.9]. In particular, $\operatorname{depth}(\Theta_{X,x}) \geq 2$. The stalks of Θ_X have finite projective dimension and so the Auslander–Buchsbaum formula,

$$\operatorname{pd}(\Theta_{X,x}) + \operatorname{depth}(\Theta_{X,x}) = \operatorname{depth}(\mathcal{O}_{X,x}),$$

holds. We infer that $\operatorname{pd}(\Theta_{X,x}) = 0$, which means that the stalk $\Theta_{X,x}$ is free. $\square$

REMARK 4.6. Lipman [17, Prop. 5.2] proved that, if Θ_X is locally free, than all irreducible components of the Jacobian subscheme $X' \subset X$ have codimension ≤ 2.

REMARK 4.7. We notice that there is a close connection between the length formula of Theorem 4.3 and the local freeness of the tangent sheaf in Proposition 4.5, which is independent of our assumption that X descends to a regular scheme. Suppose that S is the spectrum of $F[x, y, z]/(f)$ for some polynomial f, and assume that S is normal. As explained in [1, Thm. 6.2], dualizing the short exact sequence $0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega^1_{\mathbb{A}^3_E/E}|_S \to \Omega^1_S \to 0$ yields an exact sequence

$$0 \longrightarrow \Theta_S \longrightarrow \Theta_{\mathbb{A}^3_F}|_S \longrightarrow (\mathcal{I}/\mathcal{I}^2)^\vee \longrightarrow \mathcal{T}^1_S \longrightarrow 0,$$

where $\mathcal{I} = \mathcal{O}_{\mathbb{A}^3_F}(-S)$ and the cokernel $\mathcal{T}^1_S$ is isomorphic to the sheaf $\mathcal{E}xt^1(\Omega^1_{S/F}, \mathcal{O}_S)$ of first-order extension. This exact sequence shows that the annulator ideal of $\mathcal{T}^1_S$ coincides with the Jacobian ideal $\mathcal{J}$ and that $\mathcal{T}^1_S$ is an invertible module over $\mathcal{O}_S/\mathcal{J}$. It follows that $\operatorname{pd}(\Theta_S) = \operatorname{pd}(\mathcal{J}) - 2$; it follows also that Θ_S is locally free if and only if the length formula $l(\mathcal{O}_{S,s}/\mathcal{J}_s^{[p]}) = p^2 l(\mathcal{O}_{S,s}/\mathcal{J}_s)$ holds. Although this formula is fairly easy to check, finding an explicit basis of the stalks of the tangent sheaf seems to involve some nontrivial guesswork.

5. Invertible Summands of the Cotangent Sheaf

Let X be an F-scheme of finite type, and let $E \subset F$ be a subfield such that the field extension $E \subset F$ is purely inseparable. In previous sections we saw necessary conditions for X to descend to a regular E-scheme that involved local fundamental groups, degrees of residue fields, Jacobian ideals, and the tangent sheaf. However, these conditions are not always strong enough to rule out the possibility that X descends to a regular scheme. In this section we give another condition in terms of the sheaf of Kähler differentials $\Omega^1_{X/F}$. Together with the other criteria, this will be enough to settle the case of rational double points, which we shall do in Section 6.

We begin with the following simple observation from commutative algebra. Suppose R is a local ring. Let M be an R-module, let $M^\vee = \operatorname{Hom}(M, R)$ be the dual module, and let $\Phi\colon M \times M^\vee \to R$ be the canonical bilinear map $(a, f) \mapsto f(a)$.

LEMMA 5.1. *The following conditions are equivalent.*

(i) *The induced linear map $\Phi\colon M \otimes M^\vee \to R$ is surjective.*
(ii) *There are elements $a \in M$ and $f \in M^\vee$ with $f(a) \notin \mathfrak{m}_R$.*
(iii) *There is an R-module M' and a decomposition $M \simeq R \oplus M'$.*

Proof. The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are easy, so our task is to verify (ii) $\Rightarrow$ (iii). Suppose $f(a) \notin \mathfrak{m}_R$ for some $a \in M$ and $f \in M^\vee$. Then $f(a) \in R$ must be a unit because R is local. It follows that the map $M \to R$, $x \mapsto f(x)$ is surjective. Hence there is an exact sequence

$$0 \longrightarrow M' \longrightarrow M \xrightarrow{f} R \longrightarrow 0$$

with $M' = \mathrm{kern}(f)$. Such a sequence splits because R is free. $\square$

We say that an R-module M *has an invertible summand* if the equivalent conditions of the lemma hold. In the Noetherian situation, we may check this by passing back and forth to formal completions. More generally, we have the following result.

PROPOSITION 5.2. *Suppose M is of finite presentation, and let $R \to R'$ be a faithfully flat ring homomorphism. Then the R-module M has an invertible summand if and only if this holds for the induced R'-module $M' = M \otimes_R R'$.*

Proof. The condition is clearly necessary. Conversely, suppose that the evaluation map $M' \otimes (M')^\vee \to R'$ is surjective. Using that the canonical homomorphism $M^\vee \otimes_R R' \to (M \otimes_R R')^\vee$ is bijective, we infer that $M \otimes M^\vee \to R$ is surjective as well. $\square$

In our next theorem we apply this concept to the stalks $M = \Omega^1_{X,x}$ of the cotangent sheaf.

THEOREM 5.3. *Suppose our F-scheme of finite type X is equidimensional of dimension* $\dim(X) \geq 1$ *and descends to a regular scheme. Then, for all points $x \in X$ of codimension ≤ 2, the stalk $\Omega^1_{X,x}$ has an invertible summand.*

Proof. If $x \in X$ is smooth, then $\Omega^1_{X/k}$ is locally free at $x \in X$ of rank $\dim(X) \geq 1$; hence there is nothing to show. So assume that $x \in X$ is not smooth. Let $y \in Y$ be its image. It suffices to check that $\Omega^1_{Y/E,y}$ has an invertible summand. Our task is therefore to find a local vector field $\delta \in \Theta_{Y,y}$ together with a local section $s \in \mathcal{O}_{Y,y}$ with $\delta(s) = 1$.

First of all, we may assume that Y is affine. Let $Y' = \overline{\{y\}}$ be the reduced closure of the point $y \in Y$, and let $L = \kappa(y)$ be its residue field. Then the finitely generated field extension $E \subset L$ has transcendence degree $\mathrm{trdeg}_E(L) = \dim(Y')$. Let $E \subset K \subset L$ be the separable closure of $E(x_1, \ldots, x_m)$, where $x_1, \ldots, x_m \in L$ is a transcendence basis. Then the finite field extension $K \subset L$ is purely inseparable. According to Proposition 3.1, we have $K \neq L$. It follows from [5, Sec. 6, No. 3, Thm. 3] that $\Omega^1_{L/K} \neq 0$. In light of the exact sequence $\Omega^1_{L/E} \to \Omega^1_{L/K} \to 0$, there must be some $\lambda \in L$ with nonzero differential $d\lambda \in \Omega^1_{L/E}$. After replacing Y by a suitable affine open subscheme, we may extend the value $\lambda \in L = \kappa(y)$ to some section $s' \in H^0(Y', \mathcal{O}_{Y'})$. By the same token, we may further assume that the coherent $\mathcal{O}_{Y'}$-module $\Omega^1_{Y'/E}$ is locally free. Moreover, we may lift s' to a section $s \in H^0(Y, \mathcal{O}_Y)$. Then $ds(y) \neq 0$ as an element of the $\kappa(y)$-vector space $\Omega^1_{Y/E}(y)$.

Consider the biduality map $\Omega^1_{Y/E} \to (\Omega^1_{Y/E})^{\vee\vee}$, which sends differentials $\sum s_i df_i$ to the evaluation map $\delta \mapsto \delta\big(\sum s_i df_i\big)$. Let us denote this evaluation map by the symbol $\big(\sum s_i df_i\big)^{\vee\vee}$, which is thus a local section of $\Theta_Y^\vee$. Our aim now is to verify that the value at y of the evaluation map $(ds)^{\vee\vee}$, which is an element of $\Theta^\vee_{Y/E}(y)$, does not vanish. Here the problem is that the biduality map for Kähler differentials on Y is not necessarily bijective. However, we know that the biduality map for Kähler differentials on Y' is bijective. Before exploiting this, we make a brief digression to discuss biduality maps.

Suppose $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves on Y and Y', respectively. Set $\mathcal{F}' = \mathcal{F} \otimes \mathcal{O}_{Y'}$, and assume we have a homomorphism $f : \mathcal{F}' \to \mathcal{G}$. Consider the canonical restriction map $\mathcal{H}om(\mathcal{F}, \mathcal{O}_Y) \otimes \mathcal{O}_{Y'} \to \mathcal{H}om(\mathcal{F}', \mathcal{O}_{Y'})$. Applying it twice, we obtain a canonical map

$$\mathcal{H}om(\mathcal{H}om(\mathcal{F}, \mathcal{O}_Y), \mathcal{O}_Y) \otimes \mathcal{O}_{Y'} \to \mathcal{H}om(\mathcal{H}om(\mathcal{F}', \mathcal{O}_{Y'}), \mathcal{O}_{Y'}).$$

We may compose this with the map induced from $f : \mathcal{F}' \to \mathcal{G}$ to obtain a map

$$\varphi : \mathcal{H}om(\mathcal{H}om(\mathcal{F}, \mathcal{O}_Y), \mathcal{O}_Y) \longrightarrow \mathcal{H}om(\mathcal{H}om(\mathcal{G}, \mathcal{O}_{Y'}), \mathcal{O}_{Y'}).$$

This yields the commutative diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d, "f"'] & \mathcal{H}om(\mathcal{H}om(\mathcal{F}, \mathcal{O}_Y), \mathcal{O}_Y) \arrow[d, "\varphi"] \\
\mathcal{G} \arrow[r] & \mathcal{H}om(\mathcal{H}om(\mathcal{G}, \mathcal{O}_{Y'}), \mathcal{O}_{Y'}),
\end{tikzcd}$$

where the horizontal maps are the biduality maps on Y and Y', respectively.

We may apply this to the sheaves $\mathcal{F} = \Omega^1_{Y/E}$ and $\mathcal{G} = \Omega^1_{Y'/E}$ and the canonical surjection $\Omega^1_{Y/E} \otimes \mathcal{O}_{Y'} \to \Omega^1_{Y'/E}$. The result is the following commutative diagram:

$$\begin{tikzcd}
\Omega^1_{Y/E} \arrow[r] \arrow[d] & \mathcal{H}om(\mathcal{H}om(\Omega^1_{Y/E}, \mathcal{O}_Y), \mathcal{O}_Y) \arrow[d] \\
\Omega^1_{Y'/E} \arrow[r] & \mathcal{H}om(\mathcal{H}om(\Omega^1_{Y'/E}, \mathcal{O}_{Y'}), \mathcal{O}_{Y'}).
\end{tikzcd}$$

The lower horizontal map is bijective because $\Omega^1_{Y'/E}$ is locally free on Y'. Therefore, $(ds')^{\vee\vee}$ does not vanish at $y \in Y$. It follows that neither does $(ds)^{\vee\vee}$ vanish at y.

Now recall that, by Proposition 4.5, the stalk $\Theta_{Y,y}$ is free; hence $(\Omega^1_{Y/E})^{\vee\vee}_y$ is free as well. Set $e_1 = (ds)^{\vee\vee}$ and extend it to a basis $e_1, \dots, e_n \in (\Omega^1_{Y/E})^{\vee\vee}_y$. Let $\delta_1, \dots, \delta_n \in \Theta_{Y,y}$ be the corresponding dual basis. Then $\delta = \delta_1$ is the desired local vector field: we have $\delta(s) = \delta(ds) = \delta(ds)^{\vee\vee} = 1$ by construction. □

It follows from Lipman's results [17, Prop. 8.1] that $\Omega^1_{X/F}$ is torsion free if and only if our scheme X is geometrically normal. According to [6, Thm. 2.14], any torsion-free module of finite type over a local Noetherian ring is an extension of an ideal by a free module. But such extensions do not necessarily split, so the preceding result puts a nontrivial condition on the stalk $\Omega^1_{X/F,x}$.

It turns out that the result is well suited to ruling out that certain F-schemes descend to regular schemes. Suppose that X is two-dimensional, normal, and locally of complete intersection. Let $x \in X$ be a closed point with embedding dimension $\mathrm{edim}(\mathcal{O}_{X,x}) = 3$. Write $\mathcal{O}^\wedge_{X,x} = \operatorname{Spec} F[[u,v,w]]/(f)$ for some nonzero polynomial f. Using notation from physics, we write $f_u = \frac{\partial f}{\partial u}$ and so on for partial derivatives.

COROLLARY 5.4. *With assumptions as before, suppose that X descends to a regular scheme. Then, after a suitable permutation of the indeterminates u, v, w, the ideal $\mathfrak{a} = (f_u, f_v, f)$ inside the formal power series ring $F[[u,v,w]]$ is a parameter ideal and, furthermore, $f_w \in \mathfrak{a}$ holds.*

Proof. We may replace X by $\operatorname{Spec}(F[u,v,w]/(f)) \subset \mathbb{A}^3_F$ so that our point $x \in X$ corresponds to the origin $0 \in \mathbb{A}^3_F$. Now we have an exact sequence

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega^1_{\mathbb{A}^3_F/F}|_X \longrightarrow \Omega^1_{X/F} \longrightarrow 0,$$

where $\mathcal{I} = \mathcal{O}_{\mathbb{A}^3_F}(-X)$. Dualizing then yields an exact sequence

$$(\mathcal{I}/\mathcal{I}^2)^\vee \longleftarrow \Theta_{\mathbb{A}^3_F}|_X \longleftarrow \Theta_X \longleftarrow 0.$$

From this it follows that any local vector field $\delta \in \Theta_{X,x}$ is the restriction of some local vector field of the form

$$\tilde{\delta} = rD_u + sD_v + tD_w$$

subject to the condition $\tilde{\delta}(f) \in (f)_\mathfrak{m}$, where $\mathfrak{m} = (u,v,w)$. Here D_u, D_v, D_w denote the derivation given by taking partial derivatives. The coefficients are local sections $r, s, t \in \mathcal{O}_{\mathbb{A}^3,x} = F[u,v,w]_\mathfrak{m} \subset F[[u,v,w]]$.

By Theorem 5.3, the stalk $\Omega^1_{X,x}$ contains an invertible direct summand. Hence there is a derivation $\tilde{\delta} = rD_u + sD_v + tD_w$ such that (i) $\tilde{\delta}(f) = rf_u + sf_v + tf_w \in (f)_\mathfrak{m}$ and (ii) at least one of the local sections $\tilde{\delta}(u) = r$ and $\tilde{\delta}(v) = s$ and $\tilde{\delta}(w) = t$ is invertible inside $F[u,v,w]_\mathfrak{m}$. After a permutation of indeterminates, we may assume that $\tilde{\delta}(w) = t$ is invertible. From this we infer that $f_w \in (f_u, f_v, f)$ holds, where the ideals are considered inside the local ring $k[u,v,w]_\mathfrak{m}$ or, equivalently, in $F[[u,v,w]]$.

It remains to check that (f_u, f_v, f) is a parameter ideal. Indeed, the ideal (f_u, f_v, f_w, f) defines the nonsmooth locus of X. This locus is discrete, because X is a normal surface by assumption. It follows that $(f_v, f_w, f) = (f_u, f_v, f_w, f)$ is a parameter ideal in $F[[u,v,w]]$. □

6. Rational Double Points

The goal of this section is to determine which rational double points descend to regular schemes. Suppose that F is an algebraically closed ground field of characteristic $p > 0$, and let X be a normal surface over F. A singular point $x \in X$ is called a *rational double point* if the singularity is rational and Gorenstein (equivalently, rational and of multiplicity 2). Throughout this section, we tacitly assume that the field F is not algebraic over its prime field.

Rational double points are classified according to the intersection graph of the exceptional divisor on the minimal resolution of singularities; these intersection graphs correspond to the simply laced Dynkin diagrams A_n $(n \geq 1)$ and D_n $(n \geq 4)$ and E_6, E_7, E_8. In characteristic 0, this coincides with the isomorphism classification: Two rational double points are formally isomorphic if and only if they have the same Dynkin type. According to Artin [2], this is no longer true in characteristic $p = 2, 3, 5$. However, there are still only finitely many formal isomorphism classes, and Artin gave a list of formal equations for these isomorphism classes. Before going into details, we re-prove the following result of Hirokado [14].

THEOREM 6.1. (i) *A rational double point of type A_n $(n \geq 1)$ descends to a regular scheme if and only if $n + 1 = p^e$ for some exponent $e \geq 1$.*

(ii) *In characteristic $p \geq 3$, rational double points of type D_n do not descend to regular schemes.*

(iii) *For $p \geq 7$, rational double points of type E_n do not descend to regular schemes.*

Proof. Rational double points of type A_n are defined by $f = z^{n+1} - xy$. If $n+1 = p^e$, then this rational double point descends to regular schemes by Proposition 1.3. Conversely, suppose that $n+1$ is not a p-power. Then the local Picard group $\mathrm{Pic}^{\mathrm{loc}}$, which is cyclic of order $n + 1$, is not a p-group. By Corollary 2.5, such a rational double point does not appear on generic geometric fibers. The local Picard group of a rational double point of type D_4 is of order 4. Again using Corollary 2.5 we conclude that, for $p \geq 3$, rational double points of type D_n do not appear on geometric generic fibers. This settles claims (i) and (ii).

It remains to treat the E_n case. Here we must use local fundamental groups instead of local Picard groups. According to [2, Prop. 2.7], the local fundamental group π_1^{loc} of rational double points in characteristic $p \geq 7$ are *tame,* which simply means that they have order prime to the characteristic. It follows that π_1^{loc} is the maximal prime-to-p quotient of the corresponding local fundamental group in characteristic 0. Their orders are shown in Table 1 (cf. e.g. [16]). We infer that $\pi_1^{\mathrm{loc}} \neq 0$ in characteristic $p \geq 7$. Proposition 2.3 tells us that the rational double points of type E_n for $p \geq 7$ do not appear on geometric generic fibers. □

Let us now turn to rational double points of type D_n $(n \geq 4)$ in characteristic 2. According to [2], they are subdivided into $\lfloor n/2 \rfloor$ isomorphism classes D_n^r depending on an additional integral parameter $0 \leq r \leq \lfloor n/2 \rfloor - 1$. The formal equations $f(x, y, z) = 0$ are as follows.

$$
\begin{aligned}
D_{2m}^0 &: \quad f = z^2 + x^2y + xy^m. \\
D_{2m}^r &: \quad f = z^2 + x^2y + xy^m + xy^{m-r}z. \\
D_{2m+1}^0 &: \quad f = z^2 + x^2y + y^m z. \\
D_{2m+1}^r &: \quad f = z^2 + x^2y + y^m z + xy^{m-r}z.
\end{aligned}
\tag{1}
$$

REMARK 6.2. Note that the polynomial $f = z^2 + x^2y + xy^m$ defining D_{2m}^0 is contact equivalent to $g = z^2 + x^2y + xy^m + xy^m z$, which would be the case $r = 0$ in D_{2m}^r. This follows easily from the methods in [20, Sec. 2]. Recall that *contact*

Table 1 Local Fundamental and Picard Groups of Rational Double Points

Rational double point	A_n	D_n	E_6	E_7	E_8
π_1^{loc} for $p = 0$	cyclic	dihedral	$\tilde{A}_4$	$\tilde{S}_4$	$\tilde{A}_5$
order of π_1^{loc} for $p = 0$	$n+1$	$2(n-2)$	24	48	120
order of Pic^{loc}	$n+1$	4	3	2	1

equivalence means that the two equations define formally isomorphic singularities. The situation for D_n with n odd is similar. We thus see that it is not really necessary to distinguish the cases $r = 0$ and $r > 0$ when it comes to giving the formal equations.

THEOREM 6.3. *In characteristic $p = 2$, a rational double point of type D_n^r descends to a regular scheme if and only if $r = 0$.*

Proof. First, the condition is sufficient. Rational double points of type D_{2m}^0, which are formally given by $f = z^2 + x^2y + xy^m$, descend to regular schemes according to Proposition 1.3.

The case D_{2m+1}^0 is slightly more complicated because of the linear z-monomial that appears in the defining polynomial $f = z^2 + x^2y + y^m z$. Choose a nonperfect subfield $E \subset F$ and an element $\lambda \in E$ that is not a square. Consider the spectrum of $A = E[x, y, z]/(f + \lambda)$. The ring A is a flat $E[x, y]$-algebra of degree 2. The fiber ring $A/(x, y)A \simeq E(\lambda^{1/2})$ is regular, which implies that the ring A is regular. It remains to check that f and $f + \lambda$ define isomorphic singularities over the algebraically closed field F. In order to do so, we construct an automorphism of the ring $F[[x, y]][z]$ sending $f + \lambda$ to f. Clearly, the substitution $z \mapsto z + \lambda^{1/2}$ sends $f + \lambda$ to $f + \lambda^{1/2}y^m$. If $m = 2l + 1$ is odd, then the additional substitution $x \mapsto x + \lambda^{1/4}y^l$ achieves our goal. If the number $m = 2l$ is even, we make the substitution $z \mapsto z + \lambda^{1/4}y^l$ instead, which maps $f + \lambda^{1/2}y^m$ to $f + \lambda^{1/4}y^{m+m/2}$. Repeating substitutions of the latter kind puts us, after finitely many steps, in position to apply a substitution of the former kind.

Second, the condition is necessary. Set $S = \text{Spec}\, F[x, y, z]/(f)$, where f is the equation of a rational double point of type D_n^r with $1 \leq r \leq \lfloor n/2 \rfloor - 1$, as given in (1). Let $s \in S$ be the singular point. We shall verify that $\Omega^1_{S,s}$ contains no invertible summand. Toward this end we apply Corollary 5.4, which reduces our problem to a calculation involving the partial derivatives f_x, f_y, f_z. Suppose that $n = 2m$ is even. Then we compute

$$\begin{aligned}
f &= z^2 + x^2y + xy^m + xy^{m-r}z, \\
f_x &= y^m + y^{m-r}z, \\
f_y &= x^2 + mxy^{m-1} + (m-r)xy^{m-r-1}z, \\
f_z &= xy^{m-r}.
\end{aligned}$$

Consider the ideal $I_x = (f_y, f_z, f)$ inside the polynomial ring $F[x, y, z]$. Reducing modulo x yields $I_x \equiv (z^2)$, and we see that the induced ideal $I_x\mathcal{O}_{S,s}$ has height 1. An analogous argument applies to the ideal $I_y = (f_x, f_z, f)$, where we reduce modulo y. Finally, consider the ideal $I_z = (f_x, f_y, f)$. Computing modulo z, we have

$$I_z = (f_x, f_y, f) \equiv (y^m, x^2 + mxy^{m-1}, x^2 y) \equiv (y^m, x^2 + mxy^{m-1}).$$

It follows that the residue classes of $x^i y^j$ with $0 \leq i \leq 1$ and $0 \leq j \leq m-1$ form an F-vector space basis of the residue class ring $R = k[x, y]/(y^m, x^2 + mxy^{m-1})$. Obviously, the residue class of $f_z = xy^{m-r}$ is nonzero and the Artin ring R is local. This implies $f_z \notin I_z\mathcal{O}_{S,s}$. Invoking Corollary 5.4, we deduce that the rational double points of type D_{2m}^r $(r > 0)$ do not appear on geometric generic fibers.

The case of $n = 2m+1$ odd can be treated with similar arguments. Here we have

$$\begin{aligned}
f &= z^2 + x^2 y + y^m z + xy^{m-r} z,\\
f_x &= y^{m-r} z,\\
f_y &= x^2 + my^{m-1} z + (m-r)xy^{m-r-1} z,\\
f_z &= y^m + xy^{m-r}.
\end{aligned}$$

We proceed as before. Computing modulo x yields

$$I_x = (f_y, f_z, f) \equiv (my^{m-1} z, y^m, z^2),$$

and we infer that the residue class of the partial derivative $f_x = y^{m-r} z$ inside the local ring $R = k[y, z]/(my^{m-1} z, y^m, z^2)$ is nonzero provided $r \geq 2$ or m is even. This implies $f_x \notin I_x\mathcal{O}_{S,s}$. It remains to check the case $r = 1$ and m odd. Then we compute modulo y^m, xy to obtain $I_x \equiv (x^2 + y^{m-1} z, z^2)$, and we easily check that $f_x = y^{m-1} z$ is nonzero inside $R = k[x, y, z]/(x^2 + y^{m-1}, z^2, y^m, xy)$. Again $f_x \notin I_x\mathcal{O}_{S,s}$.

Computing modulo y, we have $I_y = (f_x, f_z, f) \equiv (z^2)$ and so $I_y\mathcal{O}_{S,s}$ has height 1. Finally, $I_z \equiv 0$ modulo x, z so that $I_z\mathcal{O}_{S,s}$ has height 1. According to Corollary 5.4, the rational double points of type D_{2m+1}^r with $r \geq 1$ do not appear on geometric generic fibers. □

We finally come to the rational double points of type E_6, E_7, E_8, which are the most challenging cases.

THEOREM 6.4. (i) *Suppose $p = 5$. Among the four rational double points of type E_n, only E_8^0 descends to a regular scheme.*

(ii) *Suppose $p = 3$. Among the seven rational double points of type E_n, only E_6^0 and E_8^0 descend to regular schemes.*

(iii) *Suppose $p = 2$. Among the eleven rational double points of type E_n, only E_7^0 and E_8^0 descend to regular schemes.*

Proof. The formal equations $f(x, y, z) = 0$ for rational double points were determined by Artin [2] and appear in Tables 2–4. Proposition 1.3 immediately tells

Table 2 Rational Double Points of Type E_n in Characteristic $p = 2$

	Formal relation	π_1^{loc}	$l(\mathcal{O}/\mathcal{J}), l(\mathcal{O}/\mathcal{J}^{[2]})$	Θ free
E_6^0	$z^2 + x^3 + y^2 z$	C_3	8, 32	yes
E_6^1	$z^2 + x^3 + y^2 z + xyz$	C_6	6, 28	no
E_7^0	$z^2 + x^3 + xy^3$	0	14, 56	yes
E_7^1	$z^2 + x^3 + xy^3 + x^2 yz$	0	12, 48	yes
E_7^2	$z^2 + x^3 + xy^3 + y^3 z$	0	10, 40	yes
E_7^3	$z^2 + x^3 + xy^3 + xyz$	C_4	8, 35	no
E_8^0	$z^2 + x^3 + y^5$	0	16, 64	yes
E_8^1	$z^2 + x^3 + y^5 + xy^3 z$	0	14, 56	yes
E_8^2	$z^2 + x^3 + y^5 + xy^2 z$	C_2	12, 48	yes
E_8^3	$z^2 + x^3 + y^5 + y^3 z$	0	10, 44	no
E_8^4	$z^2 + x^3 + y^5 + xyz$	$C_3 \rtimes C_4$	8, 37	no

Table 3 Rational Double Points of Type E_n in Characteristic $p = 3$

	Formal relation	π_1^{loc}	$l(\mathcal{O}/\mathcal{J}), l(\mathcal{O}/\mathcal{J}^{[3]})$	Θ free
E_6^0	$z^2 + x^3 + y^4$	0	9, 81	yes
E_6^1	$z^2 + x^3 + y^4 + x^2 y^2$	C_3	7, 71	no
E_7^0	$z^2 + x^3 + xy^3$	C_2	9, 81	yes
E_7^1	$z^2 + x^3 + xy^3 + x^2 y^2$	C_6	7, 75	no
E_8^0	$z^2 + x^3 + y^5$	0	12, 108	yes
E_8^1	$z^2 + x^3 + y^5 + x^2 y^3$	0	10, 99	no
E_8^2	$z^2 + x^3 + y^5 + x^2 y^2$	$\mathrm{SL}(2, \mathbb{F}_3)$	8, 85	no

Table 4 Rational Double Points of Type E_n in Characteristic $p = 5$

	Formal relation	π_1^{loc}	$l(\mathcal{O}/\mathcal{J}), l(\mathcal{O}/\mathcal{J}^{[5]})$	Θ free
E_6	$z^2 + x^3 + y^4$	A_4	6, 173	no
E_7	$z^2 + x^3 + xy^3$	S_4	7, 198	no
E_8^0	$z^2 + x^3 + y^5$	0	10, 250	yes
E_8^1	$z^2 + x^3 + y^5 + xy^4$	C_5	8, 239	no

us that the rational double points of type E_n^0, as stated in the assertion, descend to regular schemes. Our task is to argue that the remaining rational double points do not.

We begin by collecting further information from the tables. The third columns give the local fundamental groups, which were determined by Artin [2]. The fourth columns contain information on the Jacobian ideal $J \subset F[x,y,z]/(f)$ and the corresponding bracket ideal $J^{[p]}$. Recall that J and $J^{[p]}$ are induced from the ideals (f_x, f_y, f_z, f) and (f_x^p, f_y^p, f_z^p, f), respectively, inside the polynomial ring $F[x,y,z]$. Column three contains the length of the quotient by J and $J^{[p]}$. According to Theorem 4.3, the length formula

$$l(F[x,y,z]/(f_x, f_y, f_z, f)) = p^2 l(F[x,y,z]/(f_x^p, f_y^p, f_z^p, f))$$

holds if the singularity descends to a regular surface. By Remark 4.7, the length formula holds if and only if the tangent sheaf is locally free. The latter information appears in the last column of the tables. Using the information on the local Picard group for Proposition 2.4 and the information about the length for Theorem 4.3, we rule out all rational double points except the following three in characteristic $p = 2$.

$$\begin{aligned} E_7^1:&\quad f = z^2 + x^3 + xy^3 + x^2yz.\\ E_7^2:&\quad f = z^2 + x^3 + xy^3 + y^3z.\\ E_8^1:&\quad f = z^2 + x^3 + y^5 + xy^3z.\end{aligned} \tag{2}$$

For these points, the local fundamental group is trivial and the tangent sheaf is locally free. We shall discard them by showing that the cotangent sheaf does not contain an invertible direct summand at the singularity. As in the case of D_n-singularities, we will apply Corollary 5.4; here we treat the case of E_7^1. The partial derivatives are

$$\begin{aligned} f &= z^2 + x^3 + xy^3 + x^2yz,\\ f_x &= x^2 + y^3,\\ f_y &= xy^2 + x^2z,\\ f_z &= x^2y.\end{aligned}$$

Consider the ideal $I_x = (f_y, f_z, f)$ inside the power series ring $F[[x,y,z]]$. We have $I_x \subset (x,z)$, so I_x is not a parameter ideal. Next, consider the ideal

$$I_y = (f_x, f_z, f) = (x^2 + y^3, x^2y, z^2) = (x^2 + y^3, y^4, z^2).$$

The residue classes of the monomials $x^iy^jz^k$ with $0 \leq i,k \leq 1$ and $0 \leq j \leq 3$ constitute an F-vector space basis for the quotient ring $F[[x,y,z]]/I_y$. It follows that the residue class of $f_y = xy^2 + x^2z$ is nonzero and thus $f_y \notin I_y$. Finally, consider the ideal $I_z = (f_x, f_y, f) = (x^2+y^3, xy^2+y^3z, z^2+y^4z)$. Computing modulo z yields $I_z \equiv (x^2 + y^3, xy^2) = (x^2 + y^3, xy^2, y^5)$, and it is straightforward to see that the residue classes of $1, y, \ldots, y^4, x, xy$ form a F-vector space basis modulo

$I_z + (z)$. Therefore, $f_z = x^2y \equiv y^4$ is not contained in I_z. Using Corollary 5.4, we conclude that rational double points of type E_7^1 do not appear on geometric generic fibers. The remaining two cases E_7^2 and E_8^1 are similar and are left to the reader. □

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