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A Singular-Hyperbolic Closing Lemma

C. A. Morales

1. Introduction

The fundamental problem in the qualitative theory of dynamical systems is the study of the asymptotic behavior of the orbits in a given system. This problem gave rise to the concept of an *attracting set,* which (roughly speaking) is a place where a large set of positive orbits of the system go. On the other hand, the dynamical classification of the attracting sets is an extremely difficult problem that often requires extra structures. One such structure is *Smale's hyperbolicity,* which consists of a tangent bundle decomposition formed by a contracting and an expanding subbundle together with the flow's direction. There is now a rich theory of *hyperbolic attractors* (i.e., hyperbolic transitive attracting sets) dealing with the dynamical, geometric, ergodic, and topological properties of these objects. Nevertheless, this theory does not include such important examples as the *Henón attractor* [BeC], where there is a positive Lyapunov exponent at dense orbits; the *geometric Lorenz attractor* [AfByS; GW], where there is a critical point accumulated by periodic orbits; or even the *singular horseshoe* [LPa], which is not attracting but has properties resembling the Lorenz attractor. For the Lorenz attractor, R. Bowen, Y. Pesin, and Y. Sinai observed early on that it displays interesting properties (such as as robust transitivity and denseness of periodic orbits) that are usually associated with hyperbolic attractors. The explanation of these properties goes back to end of the nineties, when [ST] introduced what they called a *wild attractor*—that is, an example of an attractor in dimension 4 that simultaneously exhibits spiraling singularities and persistent homoclinic tangencies. The wild attractor has nothing to do with the hyperbolic properties of the Lorenz attractor, but it was noticed that the former displays a partially hyperbolic splitting *with volume-expanding central subbundle*. This subtle property was then used in [MPaP1] to define *singular-hyperbolic set* as a partially hyperbolic set with hyperbolic singularities and volume-expanding central subbundle. With this definition in hand, [MPaP1] proved that a C^1 robust transitive set with singularities is a singular-hyperbolic attractor for either the flow or the reverse flow (see also [MPaP3]). This result motivates the study of the dynamical properties of singular-hyperbolic sets taking the hyperbolic dynamical systems as a model.

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For example, a singular-hyperbolic attractor is not necessarily robustly transitive [MP], but the number of attractors arising from perturbing it is bounded by the number of equilibria it contains [M1]. On the other hand, [Co1] proved the uniform hyperbolicity of all C^2 singular-hyperbolic attractors with dense periodic orbits in dimension 3. Existence of Sinai–Ruelle–Bowen (SRB) measures for singular-hyperbolic attractors of class C^2 was proved independently in [APaPV] and [Co2]. It was also proved that a singular-hyperbolic attractor is *expanding* in the sense that its topological dimension is the dimension of its volume-expanding subbundle [M3]. Finally, [CaM] proved that there is a generic set of points in the basin of attraction of a singular-hyperbolic attractor whose omega-limit set contains a singularity. Therefore, a singular-hyperbolic attractor Λ with a single equilibrium of a C^r vector field X in a compact 3-manifold, although possibly nonrobustly transitive, *persists as a chain transitive Lyapunov stable set*. (Namely: For every neighborhood U of Λ and every vector field Y that is C^r-close to X, there is a chain transitive Lyapunov stable set A_Y of Y in U whose region of attraction is residual in U.) For further results see [BDV, Chap. 9] or the more recent book [APa] and references therein.

In this paper we further investigate the dynamics of the singular-hyperbolic sets. Indeed, we try to find a singular-hyperbolic counterpart to the following standard consequence of the classical *Anosov closing lemma for flows* [HaK]:

> *A recurrent point contained in a hyperbolic set either is a singularity or is approximated by periodic points.*

It is natural to ask if an analogous result holds for singular-hyperbolic sets and not only hyperbolic sets. There is an example of a transitive isolated (but not attracting) singular-hyperbolic set without periodic orbits [M2]. However, we have observed that this set and the geometric Lorenz attractor [AfByS; GW] are singular-hyperbolic sets that satisfy the following alternative statement:

> *A recurrent point contained in a singular-hyperbolic set is approximated by either periodic points or by points for which the omega-limit set is a singularity.*

We shall prove this alternative statement for every *attracting* singular-hyperbolic set. (We emphasize that transitivity of the attracting set is not necessary for obtaining this result.) Moreover, we prove that such a result is sharp by exhibiting an example of an attracting singular-hyperbolic set with a regular recurrent point that cannot be approximated by periodic points. Let us present these results in a precise way.

Let X be a C^1 vector field defined in a compact connected boundaryless three-dimensional manifold M. Let X_t ($t \in \mathbb{R}$) be the flow generated by X in M. The *omega-limit set* of p is the set $\omega(p)$ defined by

$$\omega(p) = \left\{ x \in M : x = \lim_{n\to\infty} X_{t_n}(p) \text{ for some sequence } t_n \to \infty \right\}.$$

We say that p is *recurrent* if $p \in \omega(p)$. By a *periodic point* of X we mean a point lying in a periodic orbit of X. Every periodic point is recurrent, but the converse does not hold.

A compact invariant set Λ is an *attracting set* if there is an open neighborhood U of Λ such that

$$\Lambda = \bigcap_{t \geq 0} X_t(U).$$

An *attractor* is a transitive attracting set.

DEFINITION 1.1. A compact invariant set H of X is *hyperbolic* if there are positive constants K, λ and a continuous invariant tangent bundle decomposition $T_H M = E^s_H \oplus E^X_H \oplus E^u_X$ such that the following statements hold.

- E^s_H is *contracting*; that is,

$$\|DX_t(x)/E^s_x\| \leq Ke^{-\lambda t} \quad \forall t > 0,\ \forall x \in H.$$

- E^u_H is *expanding*; that is,

$$\|DX_{-t}(x)/E^u_x\| \leq Ke^{-\lambda t} \quad \forall t > 0,\ \forall x \in H.$$

- E^X_H is tangent to the vector field X associated to X_t.

A hyperbolic set H is *saddle-type* if its contracting and expanding subbundles E^s_H and E^u_H never vanish; that is, $E^s_x \neq 0$ and $E^u_x \neq 0$ for every $x \in H$.

A closed orbit of X is hyperbolic if it is hyperbolic as a compact invariant set of X. Denote by

$$m(A) = \inf_{v \neq 0} \frac{\|Av\|}{\|v\|}$$

the minimum norm of a linear operator A.

DEFINITION 1.2. Let Λ be a compact invariant set of X. A continuous invariant splitting $T_\Lambda M = E_\Lambda \oplus F_\Lambda$ over Λ is *dominated* if there are positive constants K, λ such that

$$\frac{\|DX_t(x)/E_x\|}{m(DX_t(x)/F_x)} \leq Ke^{-\lambda t} \quad \forall t > 0,\ \forall x \in \Lambda.$$

We shall assume hereafter that E and F never vanish. A compact invariant set Λ is *partially hyperbolic* if it exhibits a dominated splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ such that E^s_Λ *is contracting*; that is,

$$\|DX_t(x)/E^s_x\| \leq Ke^{-\lambda t} \quad \forall t > 0,\ \forall x \in \Lambda.$$

Now we state the definition of singular-hyperbolic set [BDV; MPaP3].

DEFINITION 1.3. A *singular-hyperbolic set* Λ of X is a partially hyperbolic set with hyperbolic singularities and *volume-expanding central subbundle* E^c_Λ; that is,

$$|\det(DX_t(x)/E^c_x)| \geq K^{-1}e^{\lambda t} \quad \forall t > 0,\ \forall x \in \Lambda.$$

A singular-hyperbolic set is a hyperbolic set of saddle type if and only if it has no singularities. The most representative example of a singular-hyperbolic set with singularities is the geometric Lorenz attractor. Our main result is the following theorem.

THEOREM A. *Every recurrent point contained in an attracting singular-hyperbolic set of X is approximated either by periodic points or by points for which the omega-limit set is a singularity.*

An application of this result is as follows.

COROLLARY 1.4. *Every point of a singular-hyperbolic attractor can be approximated by points for which the omega-limit set is a singularity.*

Proof. Let Λ be a singular-hyperbolic attractor of X. Since Λ is transitive it follows that the unstable manifold of every periodic orbit in Λ intersects the stable manifold of a singularity [BaM3; MPa1]. Hence the conclusion of the corollary holds for every periodic point contained in Λ. Now choose $q \in \Lambda$ such that $\Lambda = \omega(q)$ and fix $x \in \Lambda$. It follows that q is recurrent and so, by Theorem A, q is approximated either by periodic points or by points for which the omega-limit set is a singularity. If q is approximated by points for which the omega-limit set is a singularity, then $x \in \Lambda = \omega(q)$ does also and we are done. If q is approximated by periodic points, then the point x is also approximated by points for which the omega-limit set is a singularity. Then $x \in \Lambda = \omega(q)$ is approximated by points for which the omega-limit set is a singularity, and we are done. □

Our second result is the following. Note that a point is called *regular* if it is not a singularity.

THEOREM B. *On every compact* 3*-manifold there exists a C^∞ vector field that exhibits an attracting singular-hyperbolic set with a regular recurrent point that cannot be approximated by periodic points.*

This theorem—together with the following examples—shows that the conclusion of Theorem A is sharp.

EXAMPLE 1.5. Attracting singular-hyperbolic sets exist for which none of its points can be accumulated by points for which the omega-limit set is a singularity. (Take, for instance, an attracting hyperbolic set of saddle-type.)

EXAMPLE 1.6. Every attracting singular-hyperbolic set contains a recurrent point that is approximated by periodic points. This follows because periodic orbits exist on every attracting singular-hyperbolic set [BaM1].

EXAMPLE 1.7. Every point of the geometric Lorenz attractor can be approximated both by periodic points and by points for which the omega-limit set is a singularity.

Theorem A is a direct consequence of Theorem 2.1 (see Section 2). Theorem 2.1, in turn, states that if Λ is an attracting singular-hyperbolic set and if $q \in \Lambda$ is a nonwandering point such that $\omega(q)$ has a singularity, then there is a point in its strong stable manifold $W^{ss}(q)$ that can be approximated either by periodic points or by points for which the omega-limit set is a singularity.

We now sketch the proof of Theorem 2.1. The proof is by contradiction; namely, we assume that there is a nonwandering point $q \in \Lambda$ such that $\omega(q)$ has a singularity but no point in $W^{ss}(q)$ can be approximated by periodic points or by points for which the omega-limit set is a singularity. Since $\omega(q)$ contains a singularity, we can assume that q belongs to a cross-section Σ that is close to a singularity. Such a section, which is analogous to the cross-section of the geometric Lorenz attractor, is equipped with a continuous foliation $\mathcal{F}^s$ that is the trace of the stable foliation of Λ on Σ. Now, because no point of $W^{ss}(q)$ can be approximated by periodic points, it follows from Theorem 3.9 that the leaf $\mathcal{F}^s_q$ of $\mathcal{F}^s$ containing q will be surrounded by some band $Q \subset \Sigma$ nearby $\mathcal{F}^s_q$; we call this the *adapted vertical band*. Since Q is nearby $\mathcal{F}^s_q$ we obtain that Q *does not intersect any periodic orbit*. However, given Theorem 3.15 and that no point of $W^{ss}(q)$ can be approximated by points for which the omega-limit set is a singularity, we can prove that any adapted vertical band nearby $\mathcal{F}^s_q$ (like Q) *does intersect a periodic orbit*. This contradiction will prove Theorem 2.1.

Our proof uses the attracting property of the singular-hyperbolic set simply to guarantee that the positive orbit of a point close to the set stays close to it (see Remarks 3.5 and 3.13). Because of this, it seems that both Theorem A and Theorem 2.1 are valid if we replace "attracting set" by "Lyapunov stable set" in the corresponding statements. We leave the details to the reader.

Our work is related to [ArP2], which proves a conjecture (stated in [M2, p. 618]) asserting that every singular-hyperbolic attractor is a homoclinic class. See also [ArP1] (the early version of [ArP2]) or [M4]. It would follow from such a result that the periodic orbits are dense on every *transitive* attracting singular-hyperbolic set. Theorem A does not assume transitivity but does yield denseness of periodic points or points for which the omega-limit set is a singularity instead of periodic points only. Regardless, our conclusion is sharp in the general case.

The paper is organized as follows. In Section 2 we reduce Theorem A to Theorem 2.1, and in Section 3 we prove Theorem 2.1. In Section 4 we prove Theorem B.

2. Proof of Theorem A Using Theorem 2.1

Let X be a C^1 vector field on a compact connected boundaryless 3-manifold M. Denote by $\mathrm{Per}(X)$ and $\mathrm{Sing}(X)$ the set of periodic points and singularities, respectively, of X.

A *nonwandering point* of X is a point $q \in M$ such that, for every $T > 0$ and every neighborhood W of q there is a $t > T$ satisfying $X_t(W) \cap W \neq \emptyset$. Observe that a recurrent point of X is a nonwandering point of X. Denote by $\Omega(X)$ the set of nonwandering points of X.

Let Λ be a singular-hyperbolic set of X. It follows from the invariant manifold theory [HPuSh] that each point $q \in \Lambda$ is contained in an immersed submanifold $W^{ss}(q) \subset M$ with tangent space E^s_q at q. It is known that the positive orbit of $z \in W^{ss}(q)$ is asymptotic to that of q:

$$\lim_{t\to\infty} d(X_t(q), X_t(z)) = 0.$$

Consequently, $\omega(z) = \omega(q)$ for all $z \in W^{ss}(q)$. Denote by $\mathrm{Cl}(A)$ the closure of A.

Observe that if Λ has no singularities then it is a hyperbolic set, and so, by the shadowing lemma for flows [HaK], every nonwandering point in Λ can be approximated by closed orbits (i.e., singularities or periodic orbits). As a result, every singular-hyperbolic set without singularities Λ satisfies the inequality

$$W^{ss}(q) \cap \mathrm{Cl}(\mathrm{Per}(X) \cup \mathrm{Sing}(X)) \neq \emptyset \quad \forall q \in \Lambda \cap \Omega(X).$$

One might expect this inequality to hold for all singular-hyperbolic sets (including the singular ones), but it does not [M2]. However, the counterexample in [M2] satisfies the alternative inequality

$$W^{ss}(q) \cap \mathrm{Cl}(W^s(\mathrm{Sing}(X))) \neq \emptyset \quad \forall q \in \Lambda \cap \Omega(X),$$

where

$$W^s(\sigma) = \{x \in M : \omega(x) = \sigma\}$$

and

$$W^s(\mathrm{Sing}(X)) = \bigcup_{\sigma \in \mathrm{Sing}(X)} W^s(\sigma).$$

It follows that both the hyperbolic sets and [M2] satisfy the following alternative inequality:

$$W^{ss}(q) \cap \mathrm{Cl}(\mathrm{Per}(X) \cup W^s(\mathrm{Sing}(X))) \neq \emptyset \quad \forall q \in \Lambda \cap \Omega(X).$$

Our next result proves this alternative inequality for all attracting singular-hyperbolic sets Λ and all $q \in \Lambda \cap \Omega(X)$ *whose omega-limit set contains a singularity.*

THEOREM 2.1. *Let Λ be an attracting singular-hyperbolic set of X. If $q \in \Lambda \cap \Omega(X)$ and $\omega(q) \cap \mathrm{Sing}(X) \neq \emptyset$, then*

$$W^{ss}(q) \cap \mathrm{Cl}(\mathrm{Per}(X) \cup W^s(\mathrm{Sing}(X))) \neq \emptyset.$$

Proof of Theorem A using Theorem 2.1. Let Λ be an attracting singular-hyperbolic set of a C^1 vector field X on a compact 3-manifold. To prove the theorem we must prove

$$q \in \mathrm{Cl}(\mathrm{Per}(X) \cup W^s(\mathrm{Sing}(X))) \tag{1}$$

for all recurrent points $q \in \Lambda$. For this we consider two cases according as $\omega(q)$ does or does not contain a singularity. If $\omega(q)$ has no singularities then $\omega(q)$ is a hyperbolic set [MPaP2]. In this case (1) holds by the shadowing lemma. Now assume that $\omega(q)$ contains a singularity. Note that $q \in \Omega(X)$ because q is recurrent. Hence $q \in \Lambda \cap \Omega(X)$ and so q satisfies the hypotheses of Theorem 2.1. Then, there exists a $z \in W^{ss}(q) \cap \mathrm{Cl}(\mathrm{Per}(X) \cup W^s(\mathrm{Sing}(X)))$ by Theorem 2.1. Since $z \in W^{ss}(q)$ we have $\omega(z) = \omega(q)$. But

$$\omega(z) \subset \mathrm{Cl}(\mathrm{Per}(X) \cup W^s(\mathrm{Sing}(X)))$$

since $z \in \mathrm{Cl}(\mathrm{Per}(X) \cup W^s(\mathrm{Sing}(X)))$, which is compact invariant. Therefore,

$$\omega(q) \subset \mathrm{Cl}(\mathrm{Per}(X) \cup W^s(\mathrm{Sing}(X)))$$

and then (1) holds because $q \in \omega(q)$. This proves the result. □

3. Proof of Theorem 2.1

Hereafter we fix a C^1 vector field X defined in a compact connected boundaryless 3-manifold M. Let Λ be an attracting singular-hyperbolic set of X and let $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ be the corresponding partially hyperbolic splitting. Since E^s_Λ is contracting, we can extend it to a continuous positively invariant contracting subbundle $E^s_{U(\Lambda)}$ defined in a full neighborhood $U(\Lambda)$ of Λ. By the invariant manifold theory [HPuSh] it follows that $E^s_{U(\Lambda)}$ is integrable—that is, tangent to a continuous contracting foliation in $U(\Lambda)$ whose leaf at $q \in \Lambda$ is the aforementioned submanifold $W^{ss}(q)$. For this reason we still denote this foliation by W^{ss} and use $W^{ss}(q)$ to denote the leaf of W^{ss} through $q \in U(\Lambda)$.

Hereafter we fix such a neighborhood $U(\Lambda)$. The proof of Theorem 2.1 will be presented in five segments as follows.

3.1. Adapted Cross-Sections

Given a submanifold W of M, we denote by ∂W and $\mathrm{Int}(W)$ the boundary and the interior, respectively, of W. A *transverse section* of X is a codimension-1 submanifold W for which there is a codimension-1 submanifold $\hat{W}$ of M such that $W \subset \mathrm{Int}(\hat{W})$ and $X(x) \notin T_x\hat{W}$ for every $x \in \mathrm{Int}(\hat{W})$. A *cross-section* of X is $D = \mathrm{Int}(W) \cup l$, where W is a transverse section of X and l is a finite disjoint union of intervals in ∂W. In this case we define $\mathrm{Int}(D) = \mathrm{Int}(W)$ and $\partial D = \partial W$.

If D is a cross-section of X, we define the return map

$$\Pi_D \colon \mathrm{Dom}(\Pi_D) \subset D \to D$$

associated to D by setting

$$\mathrm{Dom}(\Pi_D) = \{x \in D : \text{there is a } t > 0 \text{ such that } X_t(x) \in D\}$$

(this is the domain of Π_D) and defining

$$\Pi_D(x) = X_{t_D(x)}(x),$$

where $t_D \colon \mathrm{Dom}(\Pi_D) \to \mathbb{R}^+$ is the return time defined by

$$t_D(x) = \inf\{t > 0 : X_t(x) \in D\}.$$

Usually Π_D is not continuous owing to the possible intersection between ∂D and a positive trajectory with initial point in D. This motivates the following definition.

Definition 3.1. A cross-section D of X is *adapted* if, for every $x \in \mathrm{Dom}(\Pi_D)$, the positive orbit of X in between x and $\Pi_D(x)$ does not intersect ∂D. In other words,

$$X_t(x) \notin \partial D \quad \forall (x,t) \in \mathrm{Dom}(\Pi_D) \times (0, t_D(x)].$$

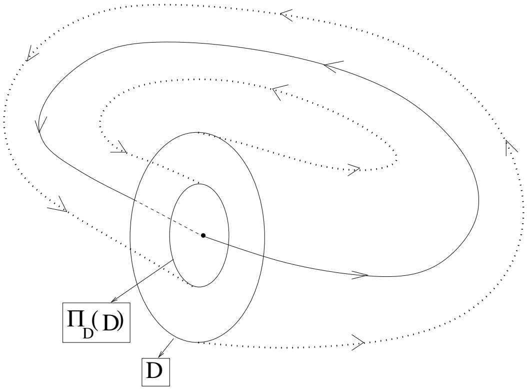

Figure 1 Adapted cross-section

Every boundaryless cross-section is adapted. A second example is the standard rectangular section of the geometric Lorenz attractor. Figure 1 shows a third example with $\mathrm{Dom}(\Pi_D) = D$.

The elementary lemma to follow gives the main properties of the adapted cross-sections.

LEMMA 3.2. *If D is an adapted cross-section of a C^r vector field X, $r \geq 1$, then* $\mathrm{Dom}(\Pi_D)$ *is open in D and Π_D is C^r.*

Proof. This is a direct consequence of the definition of adapted cross-section and the classical tubular flow box theorem [PalMe]. □

3.2. Singular Cross-Sections and Adapted Vertical Bands

A singularity σ of X is called *Lorenz-like* if its eigenvalues are real and satisfy

$$\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$$

up to some order $\lambda_1, \lambda_2, \lambda_3$. In particular, σ is hyperbolic, so the stable and unstable manifolds $W^s_X(\sigma)$ and $W^u_X(\sigma)$ of σ exist and are tangent at σ to the eigenspace associated to the set of eigenvalues $\{\lambda_2, \lambda_3\}$ and $\{\lambda_1\}$, respectively [HPuSh]. There is also an invariant manifold $W^{ss}_X(\sigma)$ that is tangent at σ to the eigenspace associated to $\{\lambda_2\}$. Note that $\dim(W^s_X(\sigma)) = 2$, $\dim(W^u_X(\sigma)) = 1$, and $\dim(W^{ss}_X(\sigma)) = 1$.

Take a linearizing coordinate system (x_1, x_2, x_3) in a neighborhood of σ as depicted in Figure 2. Note that $W^{ss}_X(\sigma)$ separates $W^s_X(\sigma)$ into two connected components, the top and the bottom. In the top component we consider a cross-section S^t_σ of X together with a curve l^t_σ as in Figure 2. Similarly, we consider a cross-section S^b_σ and a curve l^b_σ in the bottom component. We take the section S^*_σ to be diffeomorphic to $[-1, 1] \times [-1, 1]$ and the curve l^*_σ to be *contained* in $W^s_X(\sigma) \setminus W^{ss}_X(\sigma)$ for $* = t, b$. The positive flow lines of X at $S^t_\sigma \cup S^b_\sigma \setminus (l^t_\sigma \cup l^b_\sigma)$ exit a small neighborhood of σ passing through the cusp region in Figure 2. The positive orbits at $l^t_\sigma \cup l^b_\sigma$ go directly to σ. We note that the boundary of S^*_σ is formed by four curves,

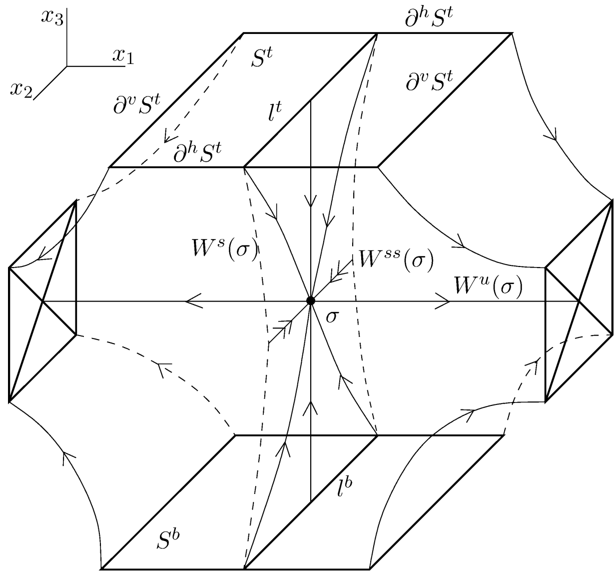

Figure 2 Singular cross-section

two of them transverse to l_σ^* and two of them parallel to l_σ^*. The unions of these curves are denoted by $\partial^v S_\sigma^*$ and $\partial^h S_\sigma^*$, respectively.

DEFINITION 3.3. A *singular cross-section associated to* σ is a cross-section Σ that is equal to S^t (or S^b) as just described. The curve l_σ^t (or l_σ^b, resp.) is called the *singular curve* of Σ.

Hereafter we fix a singular cross-section Σ of σ together with its singular curve l. Moreover, we assume that $\sigma \in \Lambda$ and that $\Sigma \subset U(\Lambda)$.

Since $\Sigma \subset U(\Lambda)$, it follows that, for all $x \in \Sigma$, the leaf $W^{ss}(x)$ of W^{ss} containing x is well-defined. Projecting $W^{ss}(x)$ into Σ through the flow of X yields a one-dimensional C^0 foliation $\mathcal{F}^s$ in Σ formed by compact intervals. Denote by $\mathcal{F}_x^s$ the leaf of $\mathcal{F}^s$ containing $x \in \Sigma$. It turns out that $\partial\Sigma$ can be chosen to be formed by four curves: two leaves of $\mathcal{F}^s$ and two curves transverse to l. We denote by $\partial^v\Sigma$ and $\partial^h\Sigma$, respectively, the union of these curves. The leaf space $\Sigma/\mathcal{F}^s$ of $\mathcal{F}^s$ is a compact interval. Therefore, we can give a natural order "$<$" on it. A leaf L will denote also the corresponding element of the leaf space.

A *vertical band* in Σ is a subset $V \subset \Sigma$ that is $\mathcal{F}^s$*-invariant*; namely, it is the union of leaves of $\mathcal{F}^s$. It follows that ∂V is formed by two leaves of $\mathcal{F}^s$, V^- and V^+, and two curves transverse to $\mathcal{F}^s$ contained in $\partial^h\Sigma$. The union of these curves will be denoted by $\partial^v V$ and $\partial^h V$, respectively. Note that $V^- < V^+$ in the natural order. If $L < L'$ are leaves of $\mathcal{F}^s$, we denote by $[L, L']$ (resp. (L, L')) the unique closed (resp. open) vertical band with $\partial^v[L, L'] = L \cup L'$ (resp. $\partial^v(L, L') = L \cup L'$).

A vertical band V will be an *open* or *closed* vertical band depending on whether $V^- \cup V^+ \subset V$ or $(V^- \cup V^+) \cap V = \emptyset$. In particular, a vertical band V satisfies

$V = [V^-, V^+]$ or (V^-, V^+) depending on whether V is closed or open. For any vertical band V we define the open vertical band

$$\mathrm{Int}^v(V) = (V^-, V^+).$$

By a *vertical band around a leaf* L we mean a vertical band V such that $L \subset \mathrm{Int}^v(V)$.

Observe that every vertical band is a cross-section of X. We can thus associate a return map Π_V with the domain $\mathrm{Dom}(\Pi_V)$ defined in Section 3.1. The following lemma gives some basic properties of these return maps.

LEMMA 3.4. *If Σ is close to its singular curve then, for every vertical band $V \subset \Sigma$ and every leaf L of $\mathcal{F}^s$ intersecting* $\mathrm{Dom}(\Pi_V)$, $L \subset \mathrm{Dom}(\Pi_V)$ *and, for all $x \in L$, the positive trajectory in between $\Pi_V(x)$ does not intersect $\partial^h V$. In particular, Π_V preserves $\mathcal{F}^s$. Moreover, the modulus of the derivative of Π_V in the direction transverse to $\mathcal{F}^s$ is greater than* 4.

Proof. Note that the return time $t_V(x)$ for $x \in \mathrm{Dom}(\Pi_V)$ is uniformly large for all vertical bands $V \subset \Sigma$ provided that Σ is close to its singular curve. From this we obtain the invariance of $\mathcal{F}^s$ because $E^s_{U(\Lambda)}$ is invariant contracting and $\Lambda \cap W^{ss}(\sigma) = \{\sigma\}$ (see [BaM1] for details). The last part of the lemma follows at once from the singular-hyperbolicity of Λ (see [MPa2]). □

REMARK 3.5. The proof of Lemma 3.4 uses that Λ is an attracting set. For instance, if Λ is not an attracting set then $\mathcal{F}^s$ may not be invariant, because $E^s_{U(\Lambda)}$ is only semi-invariant in this case.

DEFINITION 3.6. An *adapted vertical band* is an open vertical band in Σ that is also an adapted cross-section of X in the sense of Definition 3.1.

REMARK 3.7. Hereafter we fix a singular cross-section Σ close to its singular curve l. In particular, Σ satisfies the conclusion of Lemma 3.4.

3.3. Existence of Adapted Vertical Bands

First we give a sufficient condition for an open vertical band in Σ to be adapted.

LEMMA 3.8. *If $Q \subset \Sigma$ is an open vertical band whose closure $\bar{Q} = [Q^-, Q^+]$ satisfies*

$$\partial^v \bar{Q} \cap \Pi_{\bar{Q}}^{-1}(\mathrm{Int}(\bar{Q})) = \emptyset, \tag{2}$$

then Q is an adapted vertical band.

Proof. Assume by way of contradiction that Q is not adapted. Then there is $x \in \mathrm{Dom}(\Pi_Q)$ such that $X_t(x) \in \partial Q$ for some $t \in (0, t_Q(x)]$. By Lemma 3.4, $X_t(x) \in \partial^v Q$ and so the number

$$t_m = \max\{t \in (0, t_Q(x)] : X_t(x) \in \partial^v Q\}$$

is well-defined. Observe that

$$t_m < t_Q(x),$$

since Q is an open vertical band and $X_{t_m}(x) \in \partial^v Q$.

Define

$$z = X_{t_m}(x).$$

Since $\partial^v Q = \partial^v \bar{Q}$, we have

$$z \in \partial^v \bar{Q}.$$

Since $Q \subset \bar{Q}$ and $t_m < t_Q(x)$,

$$z \in \mathrm{Dom}(\Pi_{\bar{Q}}).$$

But the definition of $t_Q(x)$ and t_m implies that

$$X_s(x) \notin \bar{Q} \quad \forall s \in (t_m, t_Q(x)).$$

Therefore,

$$t_{\bar{Q}}(z) = t_Q(x) - t_m.$$

Then

$$\Pi_{\bar{Q}}(z) = \Pi_Q(x) \in Q = \mathrm{Int}(\bar{Q})$$

and so

$$z \in \Pi_{\bar{Q}}^{-1}(\mathrm{Int}(\bar{Q})).$$

It follows that

$$z \in \partial^v \bar{Q} \cap \Pi_{\bar{Q}}^{-1}(\mathrm{Int}(\bar{Q})),$$

whence

$$\partial^v \bar{Q} \cap \Pi_{\bar{Q}}^{-1}(\mathrm{Int}(\bar{Q})) \neq \emptyset;$$

this contradicts (2). The proof follows. □

Now we give a sufficient condition for the existence of adapted vertical bands in Σ.

THEOREM 3.9. *If L_0 is a leaf of $\mathcal{F}^s$ that does not intersect the closure of the periodic orbits of X, then there is an adapted vertical band $Q \subset \Sigma$ around L_0 that is arbitrarily close to L_0.*

Proof. By Lemma 3.8, it suffices to find an open vertical band Q around and arbitrarily close to L_0 whose closure $\bar{Q}$ satisfies (2). For this we proceed as follows. Since L_0 does not intersect the closure of the periodic orbits of X, we can choose a closed vertical band V, around and arbitrarily close to L_0, that does not intersect any periodic orbit of X. We can choose V such that

the leaves V^- and V^+ are equidistant to L_0 in the leaf space. $(*)$

Now we state a short claim whose proof follows from the techniques in [BaM1].

CLAIM 3.10. *There is a leaf $L^- \neq L_0$ in V such that*

$$L^- \cap \mathrm{Dom}(\Pi_V) = \emptyset. \tag{3}$$

Proof. If L^- does not exist then every leaf $L \neq L_0$ belongs to $\mathrm{Dom}(\Pi_V)$. Then Lemma 3.4 would imply that the map

$$F = \Pi_V / V \setminus L_0 \colon V \setminus L_0 \to V$$

is a λ*-hyperbolic triangular map* with $\lambda > 4$ and domain $V \setminus L_0$ [BaM1, Sec. 2.3, p. 271]. Moreover, by the proof of [BaM1, Prop. 2, p. 287], such a map satisfies hypotheses (H1) and (H2) (see [BaM1, Def. 9, p. 270]). Then, by [BaM1, Thm. 2, p. 273], such a map has a periodic point. It would follow that V intersects a periodic orbit, which is absurd. This contradiction proves the claim. $\square$

Let us continue with the proof of Theorem 3.9. Fix the leaf L^- found in Claim 3.10. We can assume without loss of generality that $L^- < L_0$ and then

$$L^- < L_0 < V^+. \tag{4}$$

To define the desired vertical band Q, we first define the leaves $Q^- < Q^+ \subset V$ and then

$$Q = (Q^-, Q^+).$$

To define Q^-, we simply set

$$Q^- = L^-.$$

To define Q^+, we first define the closed vertical band

$$W = [Q^-, V^+].$$

Next we proceed according to the following cases.

Case 1: $V^+ \cap \mathrm{Dom}(\Pi_W) = \emptyset$. In this case we define

$$Q^+ = V^+,$$

so $\bar{Q} = W$. Therefore, $\partial^v \bar{Q} \cap \mathrm{Dom}(\Pi_{\bar{Q}}) = \emptyset$ and then Q satisfies (2). It follows that Q is adapted. By (4) we have that Q is around L_0 and then we are done.

Case 2: $V^+ \cap \mathrm{Dom}(\Pi_W) \neq \emptyset$. Then $V^+ \subset \mathrm{Dom}(\Pi_W)$ by the invariance in Lemma 3.4. Obviously $\Pi_W(V^+) \subset L^- \cup V^+ \cup \mathrm{Int}^v(W)$ and so we have three possibilities:

$$\Pi_W(V^+) \subset L^- \quad \text{or} \quad \Pi_W(V^+) \subset V^+ \quad \text{or} \quad \Pi_W(V^+) \subset \mathrm{Int}^v(W).$$

In the first possibility we define

$$Q^+ = V^+.$$

Let us prove that Q so defined satisfies (2). Indeed, since $\Pi_W(V^+) \subset L^-$, it follows that

$$(Q^- \cup V^+) \cap \Pi_W^{-1}(\mathrm{Int}(W)) = \emptyset.$$

But

$$\partial^v \bar{Q} = \partial^v W = Q^- \cup V^+,$$

$\bar{Q} = W$, and $\mathrm{Int}(\bar{Q}) = \mathrm{Int}(W)$. Replacement then yields

$$\partial^v \bar{Q} \cap \Pi_{\bar{Q}}^{-1}(\mathrm{Int}(\bar{Q})) = \emptyset,$$

which is precisely (2). It follows that Q is adapted. By (4) we have that Q is around L_0 and then we are done.

In the second possibility, V^+ is an invariant leaf and so there should exist a periodic orbit passing through $V^+ \subset V$. This contradicts the fact that V does not intersect such orbits. Hence, this possibility cannot occur.

We thus arrive at the third possibility,

$$\Pi_W(V^+) \subset \mathrm{Int}^v(W).$$

It follows that there is an intermediary leaf

$$L^- \leq \tilde{V}^+ < V^+$$

such that the vertical band $\tilde{W}$ defined by

$$\tilde{W} = (\tilde{V}^+, V^+]$$

satisfies

$$\tilde{W} \subset \mathrm{Dom}(\Pi_W) \quad \text{and} \quad \Pi_W(\tilde{W}) \subset \mathrm{Int}^v(W).$$

In particular, $\Pi_W/\tilde{W}$ is continuous (actually C^1). Take the intermediary leaf $\tilde{V}^+$ so that $\tilde{W}$ is maximal with these properties.

We know from Lemma 3.4 that the derivative of Π_W along the direction transverse to $\mathcal{F}^s$ is greater than 4. Hence, the diameter of $\Pi_W(\tilde{W})$ in the direction transverse to $\mathcal{F}^s$ is at least twice that of $\tilde{W}$. Then $(*)$ implies

$$L_0 < \tilde{V}^+. \tag{5}$$

If $\tilde{V}^+ \cap \mathrm{Dom}(\Pi_W) = \emptyset$ then we define

$$Q^+ = \tilde{V}^+.$$

We can prove as before that the resulting band Q satisfies (2). It follows that Q is adapted. By (4) and (5) we have that Q is around L_0 and then we are done.

If $\tilde{V}^+ \cap \mathrm{Dom}(\Pi_W) \neq \emptyset$ then $\tilde{V}^+ \subset \mathrm{Dom}(\Pi_W)$ by invariance. Again we have

$$\Pi_W(\tilde{V}^+) \subset L^- \cup V^+ \cup \mathrm{Int}^v(W)$$

and so we have three situations:

$$\Pi_W(\tilde{V}^+) \subset L^- \quad \text{or} \quad \Pi_W(\tilde{V}^+) \subset V^+ \quad \text{or} \quad \Pi_W(\tilde{V}^+) \subset \mathrm{Int}^v(W).$$

In the first situation we define

$$Q^+ = \tilde{V}^+,$$

and the resulting band Q satisfies (2); hence it is adapted. By (4) and (5) we have that Q is around L_0 and then we are done.

To finish we prove that the remaining situations cannot occur. For the third possibility we simply observe that, if it did occur, we could then contradict the maximality of $\tilde{W}$ using the tubular flow box theorem. For the second possibility,

$$\Pi_W(\tilde{V}^+) \subset V^+, \tag{6}$$

we proceed as follows. By Lemma 3.4 there exists a one-dimensional map

$$f\colon \mathrm{Dom}(f) \subset W \to W$$

induced by Π_W in the leaf space (considering W as a subinterval in the leaf space). The inclusion (6) would imply that $f/(\tilde{V}^+, V^+]$ is orientation reversing with $f(\tilde{V}^+) = V^+$. Hence $f/(\tilde{V}^+, V^+]$ has a fixed point, which represents an invariant leaf of Π_W. Consequently, $(\tilde{V}^+, V^+]$ intersects a periodic orbit. Since $(\tilde{V}^+, V^+] \subset V$, we arrive at a contradiction because V does not intersect such orbits. This contradiction proves that (6) cannot occur, and Theorem 3.9 follows. □

3.4. Property $(\mathrm{P})_V$, *Stable Manifolds of the Singularities, and Periodic Orbits*

Recall that Σ is a singular cross-section close to its singular curve and therefore satisfies the conclusion of Lemma 3.4. The following is a minor modification of the Property (P) defined in [MPa1].

DEFINITION 3.11. Let $a \in \Sigma$ and let V be a closed vertical band in Σ. We say that $a \in \Sigma$ *satisfies Property* $(\mathrm{P})_V$ if there is an open interval $I \subset \Sigma$ transversal to $\mathcal{F}^s$ such that the following properties hold:

(i) $a \in \partial I$;
(ii) $\mathrm{Cl}(O_X^+(a)) \cap V = \emptyset$;
(iii) $O_X^+(x) \cap V \neq \emptyset$ for every $x \in I$.

Property $(\mathrm{P})_V$ gives a sufficient condition for a point a to be in the stable manifold of a singularity. More precisely, we have the following theorem (which corresponds to [BaM3, Thm. C]). Here we give only an outline of the proof, since the details will appear in [BaM3]. A strong version of this theorem can be found in [BaM2].

THEOREM 3.12. *If $a \in \Sigma$ satisfies Property $(\mathrm{P})_V$ for some closed vertical band $V \subset \Sigma$, then a is contained in the stable manifold of a singularity.*

Proof. We divide the proof into two steps.

Step 1: $\omega(a)$ contains a singularity. Suppose by contradiction that $\omega(a)$ has no singularities. Then, $\omega(a)$ is a hyperbolic set. Using $(\mathrm{P})_V$, one can prove that $\omega(a)$ has topological dimension 1, too. Then the classical hyperbolic theory applies to prove that $\omega(a)$ has a Markov partition $\mathcal{R}$ of arbitrarily small size. Given this partition, one can follow the proof of [MPa1, Thm. 5.2] in order to prove that $\omega(a)$ is a periodic orbit.

Indeed, consider the return map Π associated to this partition. Since $\mathcal{R}$ is a Markov partition of $\omega(a)$, we can assume without loss of generality that $a \in \mathcal{R}$ and that the interval I in Definition 3.11 belongs to $\mathcal{R}$. Define $a_n = \Pi^n(a)$ and $I_n = \Pi^n(I)$ for all $n \in \mathbb{N}$ large. Because I is transverse to the stable subbundle E^s_Λ of Λ that dominates the central subbundle E^c_Λ, the positive orbit of I is nearly tangent to E^c_Λ. We can therefore assume that I is tangent to E^c_Λ. It then follows from the volume expansion of E^c_Λ that the length of the intervals I_n is bounded away from 0. Now assume by contradiction that $\omega(a)$ is not a periodic orbit; then the sequence a_n has an accumulation point $b \in \mathcal{R}$. Take a subsequence $a_{n_k} \to b$. By considering the relative position of the strong stable manifolds of a_{n_k} close to b and using the aforementioned bound, we can prove that there are positive integers n_0, m_0 such that I_{n_0} intersects the strong stable manifold of a_{m_0}. This would yield a $z \in I$ whose positive orbit stays close to $\omega(a)$. In particular, such an orbit does not intersect V—a contradiction. This contradiction proves that $\omega(a)$ is a periodic orbit. On the other hand, $\omega(a)$ cannot be a periodic orbit because of the argument involving the inclination lemma [PalMe] in the third paragraph of [MPa1, p. 367].

Step 2: $\omega(a)$ is a singularity. The proof here consists of improving the aforementioned argument in [MPa1] to include the case when $\omega(a)$ contains a singularity. Indeed, we first notice that every singularity in $\omega(a)$ is Lorenz-like by [BaM1]. Since a satisfies $(\mathrm{P})_V$ for some V closed and since $\omega(a)$ contains a singularity, it follows that $\omega(a)$ has a *singular partition*—in other words, a finite disjoint collection $\mathcal{R}$ of cross-sections with $\omega(a) \cap \partial R = \emptyset$ (for all $R \in \mathcal{R}$) such that $\mathcal{R}$ intersects every regular orbit in $\omega(a)$. Such a partition (which plays the role of the Markov partition in [MPa1]) is obtained by choosing suitable singular cross-sections at the singularities of $\omega(a)$ and then taking a classical Markov partition outside the singularities. If now $\omega(a)$ were not a singularity, then the argument in [MPa1] would imply that $\omega(a)$ is a singularity, which is absurd. □

REMARK 3.13. The proof of Theorem 3.12 uses that Λ is an attracting set. For example, if Λ were not attracting, then $\omega(a)$ would be far from Λ and so we could not guarantee that the length of the intervals I_n in Step 1 is bounded away from 0.

LEMMA 3.14. *Let V be an* adapted *vertical band in Σ that does not intersect the stable manifold of the singularities. Then, for every closed vertical band $\bar{V} \subset \mathrm{Int}^v(V)$ and every $x \in \mathrm{Dom}(\Pi_V)$ with $\Pi_V(x) \subset \mathrm{Int}^v(\bar{V})$, there is a closed vertical band*

$$B = B_x \subset V$$

(see Figure 3) *around $\mathcal{F}^s_x$ such that*:

(i) $B \subset \mathrm{Dom}(\Pi_V)$ *and* Π_V/B *is continuous*;
(ii) $\Pi_V(B) \subset \bar{V}$;
(iii) $\Pi_V(\partial^v B) \subset \partial^v \bar{V}$.

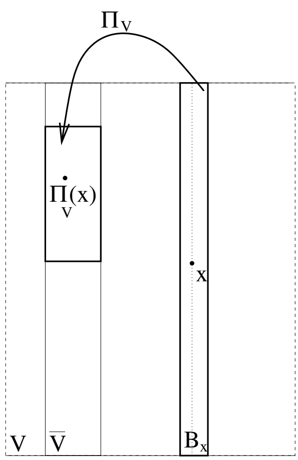

Figure 3 The band B_x

Proof. Denote by $\mathcal{B}$ the set of open vertical bands $B \subset V$ around $\mathcal{F}_x^s$ that satisfy (i) and (ii) of the lemma.

Because V is adapted, Π_V is continuous and $\mathrm{Dom}(\Pi_V)$ is open (Lemma 3.2). Therefore, $\mathcal{B} \neq \emptyset$ by the tubular flow box theorem [PalMe] since $\Pi_V(x) \in \bar{V} \subset \mathrm{Int}^v(V)$.

Endow $\mathcal{B}$ with the inclusion order. Then, by Zorn's lemma, there exists a maximal element (B_x^-, B_x^+) of $\mathcal{B}$. Define

$$B_x = [B_x^-, B_x^+].$$

We shall prove that B_x satisfies the conclusions of the lemma.

It follows from the definition of $\mathcal{B}$ that (B_x^-, B_x^+) is around $\mathcal{F}_x^s$. Hence B_x is around $\mathcal{F}_x^s$, too.

Now we claim that

$$B_x \subset V.$$

To prove this we need only check that

$$\partial^v B_x \subset V,$$

since

$$\mathrm{Int}^v(B_x) = (B_x^-, B_x^+) \subset V$$

by the definition of $\mathcal{B}$. Suppose by contradiction that

$$\partial^v B_x \not\subset V.$$

Because $(B_x^-, B_x^+) \subset V$, we have either

$$B_x^- \not\subset V \quad \text{or} \quad B_x^+ \not\subset V.$$

First we assume $B_x^- \not\subset V$; then $B_x^- = V^-$. Choose $a \in V^-$ and let $I \subset (B_x^-, B_x^+)$ be an open interval transverse to $\mathcal{F}^s$ such that $a \in \partial I$.

Observe that

$$\mathrm{Cl}(O_X^+(a)) \cap \bar{V} = \emptyset,$$

for otherwise (since $\bar{V} \subset V$) we would have $a \in \mathrm{Dom}(\Pi_V)$ and then could enlarge B_x, contradicting its maximality. On the other hand, $I \subset (B_x^-, B_x^+) \in \mathcal{B}$ and so

$$O_X^+(x) \cap \bar{V} \neq \emptyset \quad \forall x \in I.$$

It follows that a satisfies $(\mathrm{P})_{\bar{V}}$. Then, by Theorem 3.12, a is contained in the stable manifold of a singularity. This contradicts the hypothesis (since $a \in [V^-, V^+]$), a contradiction that proves $B_x^- \subset V$. We prove $B_x^+ \subset V$ analogously. Therefore, $\partial^v B_x \subset V$ and the claim follows.

Replacing V^* by B_x^* (for $* = +, -$) in the preceding argument, we obtain the inclusion

$$\partial^v B_x \subset \mathrm{Dom}(\Pi_V).$$

But $(B_x^-, B_x^+) \subset \mathrm{Dom}(\Pi_V)$ by the definition of $\mathcal{B}$, since $(B_x^-, B_x^+) \in \mathcal{B}$. Hence

$$B_x \subset \mathrm{Dom}(\Pi_V).$$

Because V is adapted, we have that Π_V/B_x is continuous since Π_V is (by Lemma 3.2). Then B_x satisfies (i).

Now we check that B_x satisfies (ii) and (iii). Note that (ii) is true in $\mathrm{Int}^v(B_x)$ by the definition of $\mathcal{B}$, since $\mathrm{Int}^v(B_x) = (B_x^-, B_x^+) \in \mathcal{B}$. Then (ii) follows because Π_V/B_x is continuous and $\bar{V}$ is a closed band. Since (B_x^-, B_x^+) is maximal, we obtain (iii). Lemma 3.14 follows. □

Now we state the main result of Section 3.4.

THEOREM 3.15. *Let V be an adapted vertical band in Σ such that $[V^+, V^-]$ does not intersect the stable manifold of the singularities. If V contains a nonwandering point, then V contains a periodic point.*

Proof. Assume that V contains a nonwandering point p. Then $\mathrm{Dom}(\Pi_V) \neq \emptyset$, so there is a $q \in V$ such that $\Pi_V(q)$ is defined.

Choose a sequence of closed vertical bands $\bar{V}_n \subset V$ satisfying the following properties for all n large:

(a) $q, \Pi_V(q) \in \mathrm{Int}^v(\bar{V}_n)$;
(b) $\bar{V}_n \subset \mathrm{Int}^v(\bar{V}_{n+1})$;
(c) the vertical boundary of $\bar{V}_n$ is $(1/n)$-close to the vertical boundary of V.

By hypothesis, $[V^-, V^+]$ does not intersect the stable manifolds of the singularities. We can thus apply Lemma 3.14 to the bands $V, \bar{V}_n$ and the point $x = q$ in order to obtain a sequence $B_{q,n} \subset V$ of closed vertical bands satisfying (i)–(iii) in that lemma. From (b) it follows that $B_{q,n} \subset \mathrm{Int}^v(B_{q,n+1})$ for all n, so the union

$$B_\infty = \bigcup_n B_{q,n}$$

is an open vertical band in V contained in $\mathrm{Dom}(\Pi_V)$. In addition, Π_V/B_∞ is continuous by Lemma 3.14(i).

On the other hand, Π_V preserves $\mathcal{F}^s$ and so, considering V as a subinterval of the leaf space I, we have the induced one-dimensional map

$$f\colon \mathrm{Dom}(f) \subset V \to V.$$

It turns out that f is continuous at the continuity points of Π_V because $\mathcal{F}^s$ is a continuous foliation.

The property (c), Lemma 3.14(iii), and the fact that Π_V preserves $\mathcal{F}^s$ together imply that the lateral limits

$$\lim_{L\to(B_\infty^-)+} f(L) \quad \text{and} \quad \lim_{L\to(B_\infty^+)-} f(L)$$

exist and belong to different elements of $\{V^+, V^-\}$. If $[B_\infty^-, B_\infty^+] \subset V$ then, by these limits and since Π_V/B_∞ is continuous, we have that f has a fixed point in V. This fixed point corresponds to a leaf whose image under Π_V falls into itself. Consequently, Π_V has a fixed point that corresponds to a periodic orbit intersecting V, so the result then follows in this case.

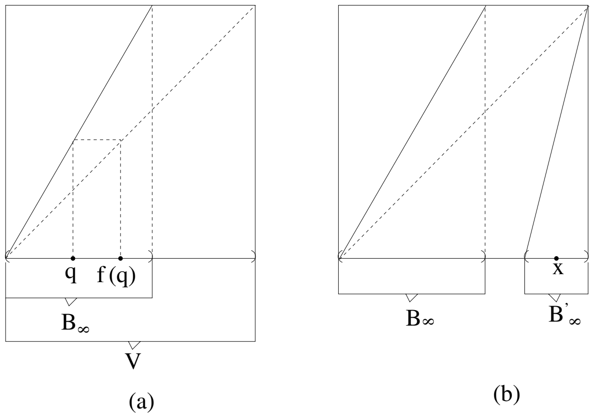

Figure 4 Graphs of f/B_∞ and $f/B_\infty \cup B'_\infty$

We can therefore now assume that

$$[B^-_\infty, B^+_\infty] \not\subset V.$$

Since $(B^-_\infty, B^+_\infty) \subset V$, we conclude that $B^-_\infty = V^-$ or $B^+_\infty = V^+$. We shall assume that $B^-_\infty = V^-$; the proof in the other case is similar.

Given $B^-_\infty = V^-$, we have $B^+_\infty \subset V$ because otherwise $B_\infty = V$ and then f has a fixed point, since it has a nonwandering point (say, p). In this case we are done. We can further assume that f/B_∞ is orientation preserving, for otherwise f/B_∞ would have a fixed point and again we are done. It follows that the graph of f/B_∞ in V is like that in Figure 4(a). But p is a nonwandering point and so $\mathrm{Int}^v(V \setminus B_\infty) \cap \mathrm{Dom}(f_V) \neq \emptyset$; hence we can choose $x \in \mathrm{Int}^v(V \setminus B_\infty) \cap \mathrm{Dom}(f_V)$.

Replacing q by x in the previous argument, we obtain an open vertical band B'_∞ containing x such that the graph of $f/B_\infty \cup B'_\infty$ is like that in Figure 4(b) (observe that q does not need to be nonwandering). The result then follows because a map like that in Figure 4(b) has infinitely many periodic points. This completes the proof of Theorem 3.15. $\square$

3.5. *Proof of Theorem 2.1*

Let Λ be an attracting singular-hyperbolic set of a C^1 vector field X on a closed 3-manifold. Let $q \in \Lambda \cap \Omega(X)$ be such that $\omega(q)$ contains a singularity σ. We must prove that $W^{ss}(q) \cap \mathrm{Cl}(\mathrm{Per}(X) \cup W^s(\mathrm{Sing}(X))) \neq \emptyset$. We can assume that q is not contained in the stable manifolds of the singularities, for otherwise $q \in W^s(\mathrm{Sing}(X))$ and then the result is obvious. In particular, q is a regular point and $q \notin W^s(\sigma)$. Because $\sigma \in \omega(q)$, we conclude that σ is Lorenz-like [BaM1]. From this we can assume that $q \in \Sigma$ for some singular cross-section Σ associated to σ that is arbitraritly close to its singular curve. It follows that Σ satisfies the conclusion of Lemma 3.4 and so the results of Sections 3.1–3.4 can be applied.

Now, to prove $W^{ss}(q) \cap \mathrm{Cl}(\mathrm{Per}(X) \cup W^s(\mathrm{Sing}(X))) \neq \emptyset$ we shall assume by contradiction that

$$W^{ss}(q) \cap \mathrm{Cl}(\mathrm{Per}(X) \cup W^s(\mathrm{Sing}(X))) = \emptyset.$$

Then

$$\mathcal{F}_q^s \cap \mathrm{Cl}(\mathrm{Per}(X) \cup W^s(\mathrm{Sing}(X))) = \emptyset. \tag{7}$$

In particular, $\mathcal{F}_q^s$ does not intersect the closure of the periodic orbits of X. By Theorem 3.9 applied to $L_0 = \mathcal{F}_q^s$, we can thus choose an adapted vertical band Q around and arbitrarily close to $\mathcal{F}_q^s$. Observe that Q does *not* intersect the periodic orbits of X.

On the other hand, since Q is close to $\mathcal{F}_q^s$, we can use (7) to choose Q such that $[Q^-, Q^+]$ does not intersect the stable manifold of the singularities of X in Λ. But Q contains q, which is nonwandering. Hence, by Theorem 3.15 applied to $V = Q$, we would have that Q *does* intersect a periodic orbit. This contradiction proves that $\mathcal{F}_q^s \cap \mathrm{Cl}(\mathrm{Per}(X) \cup W^s(\mathrm{Sing}(X))) \neq \emptyset$ and so $W^{ss}(q) \cap \mathrm{Cl}(\mathrm{Per}(X) \cup W^s(\mathrm{Sing}(X))) \neq \emptyset$. Theorem 2.1 follows. □

4. Regular Recurrent Points Far from Periodic Orbits on Attracting Singular-Hyperbolic Sets

The objective of this section is to prove Theorem B—that is, to construct on any compact 3-manifold an example of *attracting* singular-hyperbolic sets exhibiting a regular recurrent point that cannot be approximated by periodic orbits. This result is related to the example in [M2] of a *transitive isolated* singular-hyperbolic set without periodic orbits. It suffices to construct the example in $\mathbb{R}^3$, which we do in four steps.

4.1. Cherry Flows

Here we follow [Bo; PalMe]. Let N^0 be the two-dimensional torus. Let X^0 be a C^∞ vector field in N^0 that satisfies the following properties.

(A) X^0 has two singularities, a hyperbolic saddle S and a hyperbolic sink P.
(B) X^0 is transverse to a meridian circle Σ in N^0.
(C) One of the two orbits in $W^u(S) \setminus \{S\}$ belongs to $W^s(P)$ and so does not intersect Σ; the remaining orbit, denoted by O, intersects Σ in a first point c.
(D) The eigenvalues of the saddle S are such that the following statements hold. There is an open interval $(a, b) \subset \Sigma$ such that the positive orbit of $y \in (a, b)$ goes directly to P without re-intersecting Σ. The positive orbits of a or b also do not re-intersect Σ but go to S instead; in particular, $a, b \in W^s(S)$. Finally, the positive orbit of $y \in \Sigma \setminus [a, b]$ re-intersects Σ in a first point $f(y)$. This yields a Poincaré map $f\colon \Sigma \setminus [a, b] \to \Sigma$, which we require to be expanding (i.e., there is a $\lambda > 1$ such that $f'(y) > \lambda$). Moreover, $f'(y) \to \infty$ as $y \to a^-$ or $y \to b^+$.

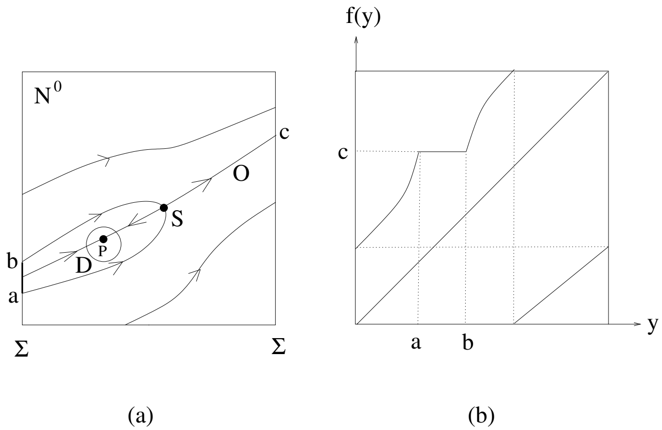

Figure 5 Cherry flow

The map f in (D) can be extended to the whole Σ by setting $f(y) = c$ for every $y \in [a, b]$. The resulting map $f : \Sigma \to \Sigma$ is then a continuous endomorphism of degree 1 in Σ. Therefore, f has a well-defined rotation number. The vector field X^0 is called *Cherry flow* if its associated f has an irrational rotation number. Figures 5(a) and 5(b) describe X^0 and f, respectively.

The following lemma summarizes the main properties of the Cherry flow to be used here; its proof can be found in [PalMe, p. 187]. We call a point $x \in \Sigma$ *regular for* X^0 if $X^0(x) \neq 0$.

LEMMA 4.1. *If X^0 is a Cherry flow, then*

(i) *X^0 has no periodic orbits, and*
(ii) *$\Lambda^0 = N^0 \setminus W^s(P)$ is a transitive set of X^0.*

Consequently, X^0 has a regular recurrent point $p \in \Lambda^0$.

4.2. Connected Sum

Consider the vector field Y in the closed disk T described in Figure 6(b). Note that Y has a hyperbolic repelling equilibrium at P'. Choose another closed disk $D' \subset \mathrm{Int}(T)$ with interior $\mathrm{Int}(D')$ containing P' such that Y is transverse to $l' = \partial D'$ pointing outward. Choose one more closed disk $D \subset N^0$ containing P in its interior such that X^0 is transverse to the boundary $l = \partial D$ of D pointing outward. These disks are shown in Figures 6(b) and 5(a), respectively.

Remove $\mathrm{Int}(D)$ from N^0 to obtain the manifold with boundary N^1 that is diffeomorphic to the punctured torus in Figure 6(a). Remove $\mathrm{Int}(D')$ from T and then glue the resulting manifold to N^1 by identifying l' and l. In this way we obtain the manifold in Figure 7(a), which is diffeomorphic to a punctured torus. The vector fields X^1 and Y (which are transverse to l and l', respectively) induce a

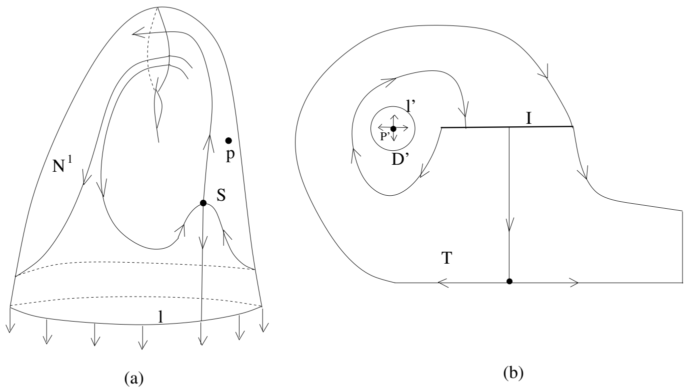

Figure 6 Deformed Cherry flow

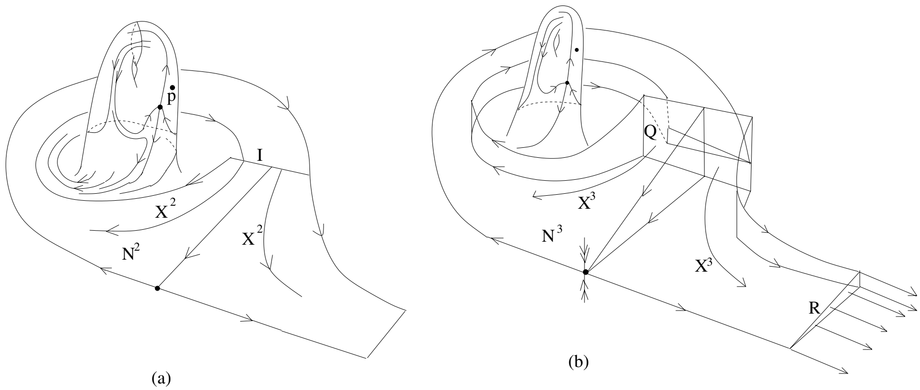

Figure 7 Connected sum

vector field X^2 in N^2 whose flow is depicted in Figure 7(a). The point p in Figure 7(a) represents the regular recurrent point p in Lemma 4.1. We fix a compact interval I as in Figure 7(b).

4.3. The Attracting Set

Now we define the attracting set Λ in Theorem B. Consider X^2 as in Section 4.2. Choose $\lambda_2 < 0$ and consider the vector field $F(x) = \lambda_2 \cdot x$ in $[-1, 1]$. Define X^3 as the vector field in $N^3 = N^2 \times [-1, 1]$ whose flow is given by

$$X_t^3(q, x) = (X_t^2(q), F_t(x)) \quad \forall (q, x) \in N^3.$$

The portrait face of X^3 is depicted in Figure 7(b). Observe that X^3 is transverse both to the square $Q = I \times [-1, 1]$ and to the cusp region R in the right-hand branch of N^3 shown in Figure 7(b).

Next we define a manifold U by flowing R back to N^3 as indicated in Figure 8. Notice that the resulting U is equipped with a vector field X induced by X^3, which is now transverse to the square $Q = I \times [-1,1]$ in Figure 8. Moreover, U has a fibration of the form $\{*\} \times [-1,1]$.

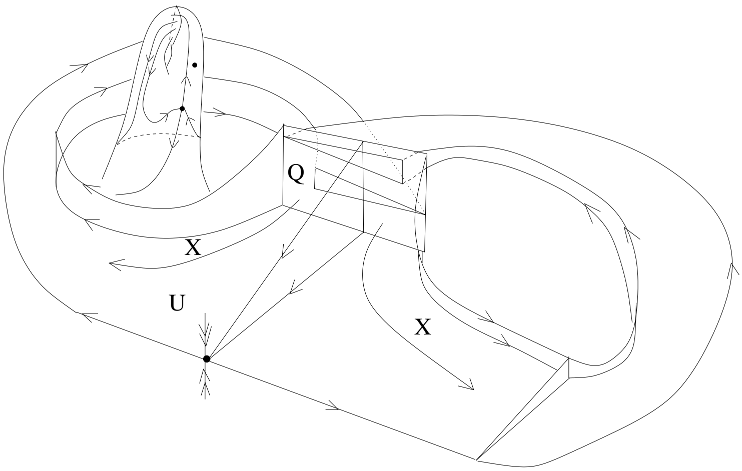

Figure 8 The attracting set

The construction is done in such a way that the positive orbits through Q go to a geometric Lorenz attractor L contained in U. Note that L is a singular-hyperbolic set with singular splitting

$$T_L U = F_L^s \oplus F_L^c,$$

where the subbundle F_L^s is tangent to the fibers $\{*\} \times [-1,1]$ in U.

Finally, we define

$$\Lambda = \bigcap_{t \geq 0} X_t(U). \tag{8}$$

Since $X_t(U) \subset U$ for all $t \geq 0$, it follows that Λ is an attracting set of X. We shall prove next that Λ is the attracting set required in Theorem B.

4.4. Proof of Theorem B

Let Λ be the attracting set defined in (8). Let p be the recurrent point in Lemma 4.1, and let $p \times 0 \in \Lambda$ be the corresponding regular recurrent point in Λ. Because p is not accumulated by periodic orbits, the same holds for $p \times 0 \in \Lambda$. Therefore, in order to prove the result, it suffices to prove that Λ is a singular-hyperbolic set. For this we proceed as follows.

Observe that $N^1 \times 0$ embeds into U and that

$$X_t(N^1 \times 0) \subset N^1 \times 0, \quad t \leq 0. \tag{9}$$

Therefore, the set

$$S = \bigcup_{t \geq 0} X_t(N^1 \times 0)$$

is a submanifold of U. Let TS be the tangent space of S. Define the splitting

$$T_S U = G^s_S \oplus G^c_S$$

over S by setting G^s_S as the line bundle tangent to the fibers $\{*\} \times [-1,1]$ and $G^c_S = TS$.

Now one can easily prove that

$$\Lambda = S \cup L, \tag{10}$$

where L is the geometric Lorenz attractor in Section 4.3. Recall that L has a singular-hyperbolic splitting $T_L U = F^s_L \oplus F^c_L$, with F^s_L tangent to the fibers $\{*\} \times [-1,1]$ in U. Then (10) allows us to define a splitting

$$T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$$

over Λ by setting

$$E^i_z = \begin{cases} F^i_z & \text{if } z \in L, \\ G^i_z & \text{if } z \in S \end{cases}$$

for $i = s$ or c. This splitting is clearly invariant. Moreover, E^s_Λ is contracting (and dominates E^c_Λ) if we choose λ_2 in Section 4.3 with modulus large enough.

We claim that E^c_Λ is volume expanding. Indeed, this volume expansiveness is clear in L, so we need only prove it in S. Now $E^c_S = TS$ by the definition of G^c_S. Let Λ^0 be the transitive set in Lemma 4.1. Then $\Lambda^0 \times 0 \subset S$ and so we have the decomposition

$$S = (\Lambda^0 \times 0) \cup (S \setminus (\Lambda^0 \times 0)).$$

It follows from the expansivity of f in (D) of Section 4.1 that $T_{\Lambda^0 \times 0} S$ is volume expanding. On the other hand, the points in $S \setminus (\Lambda^0 \times 0)$ are precisely the points in S whose positive orbits eventually fall into $T \times 0$.

Since the circle $l \times 0 \approx l' \times 0 \subset N^1 \times 0$ is transverse to the contracting subbundle of L (i.e., to the fibers $\{*\} \times [-1,1]$), we have that $T_{S \setminus (\Lambda^0 \times 0)} S$ is volume expanding, too. From this it follows that E^c_Λ is volume expanding, which concludes the proof. □

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Instituto de Matematica
Universidade Federal do Rio de Janeiro
P. O. Box 68530
21945-970 Rio de Janeiro
Brazil

morales@impa.br