

On Abel Maps of Stable Curves

LUCIA CAPORASO & EDUARDO ESTEVES

1. Introduction

We construct Abel maps for a stable curve X . Namely, for each one-parameter deformation of X to a smooth curve having regular total space and for each $d \geq 1$, we construct by specialization a map $\alpha_X^d: \dot{X}^d \rightarrow \overline{P}_X^d$, where $\dot{X} \subseteq X$ is the smooth locus and where \overline{P}_X^d is the coarse moduli scheme for equivalence classes of degree- d “semibalanced” line bundles on semistable curves having \underline{X} as a stable model. For $d = 1$, we show that α_X^1 extends to a map $\overline{\alpha}_X^1: X \rightarrow \overline{P}_X^1$ and does not depend on the choice of the deformation. Finally, we give a precise description of when $\overline{\alpha}_X^1$ is injective.

The theory of Abel maps for smooth curves goes back to the nineteenth century. In the modern language, let C be a smooth projective curve and let $\text{Pic}^d C$ be its degree- d Picard variety parameterizing line bundles of degree d on C . For each $d > 0$ there exists a remarkable morphism, often called the d th Abel map:

$$\begin{aligned} C^d &\longrightarrow \text{Pic}^d C, \\ (p_1, \dots, p_d) &\longmapsto \mathcal{O}_C\left(\sum p_i\right). \end{aligned}$$

This map has been extensively studied and used in the literature. For $d = 1$ it gives, after the choice of a “base” point on C , the Abel–Jacobi embedding $C \hookrightarrow \text{Pic}^1 C \cong \text{Pic}^0 C$ (unless $C \cong \mathbb{P}^1$). For an interesting historic survey see [K1] or [K2].

What about Abel maps for singular curves? Abel maps were constructed for all integral curves in [AK] and were further studied in [EGK1; EGK2; EK]. In [AK] it is shown that the first Abel map of an integral singular curve is an embedding into its compactified Picard scheme. However, almost nothing is known for reducible curves—not even when they are stable. This lack of knowledge is all the more regrettable given the importance of stable curves in moduli theory.

In this paper we construct Abel maps for stable curves. As we see it, Abel maps should satisfy the following natural properties. First, they should have a geometric meaning. More explicitly, recall that for a smooth curve C the d th Abel map is the “moduli map” defined by a natural line bundle on $C^d \times C$; see Section 2.5. We want a similar property to hold for singular curves as well.

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Second, Abel maps should vary continuously in families. In particular, given a one-parameter family of smooth curves specializing to a singular curve, we expect the d th Abel maps of the smooth fibers to specialize to the d th Abel map of the singular fiber.

Both requirements turn out to be nontrivial. In order to address the second one, we view stable curves as limits of smooth ones. So let X be a stable curve and let $f: \mathcal{X} \rightarrow B$ be a family of curves over a local one-dimensional regular base B with regular total space \mathcal{X} , smooth generic fiber, and X as closed fiber. We observe that there exists a canonical way to partially extend the d th Abel map of the generic fiber of f by using the Néron model N_f^d of the degree- d Picard scheme of that fiber. The Néron mapping property yields a close relative of the d th Abel map of X defined on the nonsingular locus $\dot{X}^d \subseteq X^d$ and that we call the d th *Abel–Néron map* of X ; see Lemma 2.6. The target of this map is the closed fiber of N_f^d rather than the Picard scheme of X .

Néron models appeared first in [N]; good references in a more modern language are [R] and [BLR]. The great advantage of Abel–Néron maps is their naturality, which is obtained directly from the universal property of Néron models. However, they have two major drawbacks. First, they do not have any a priori modular interpretation. Second, they are not defined on the whole X^d .

To attack these problems, we consider the geometric compactified Picard scheme introduced in [C1] and further studied in [C2]. If X is suitably general—more precisely, “ d -general” (Definition 3.6)—then there exists a proper B -scheme \overline{P}_f^d that is a coarse moduli space for equivalence classes of degree- d “semibalanced” line bundles on semistable curves having the fibers of f as stable models; see Section 3.8. These are line bundles whose multidegree satisfies certain inequalities; see Definition 3.2. It is shown in [C2] that \overline{P}_f^d contains N_f^d as a dense open subscheme. Thus, \overline{P}_f^d gives not only a geometrically meaningful description of N_f^d but also a completion of it; alternatively, see (6) and (14).

So, assume for now that X is d -general. Let \overline{P}_X^d be the closed fiber of \overline{P}_f^d . As explained in Section 3.8 (see also [C2, Sec. 5]), \overline{P}_X^d does not depend on the choice of f . Because N_f^d sits inside \overline{P}_f^d , we obtain our d th Abel map $\alpha_X^d: \dot{X}^d \rightarrow \overline{P}_X^d$ (see Theorem 3.10).

The map α_X^d is modular, but an explicit description for it is hard to exhibit in full generality; we do this only for curves with two components (see Proposition 3.12). The case $d = 1$ turns out to be easier. By means of Theorem 4.6 we give an explicit description of the line bundle defining α_X^1 . Using this description, we show in Corollary 4.10 that α_X^1 does not depend on the choice of f , a remarkable property not to be expected in general for $d > 1$; see Remark 3.14. (More precisely, this property holds for d smaller than a certain invariant of the graph of X ; see [C3].)

Using our modular description of α_X^1 , we construct a completion for it as a regular map $\overline{\alpha}_X^1: X \rightarrow \overline{P}_X^1$ (Theorem 5.5). Finally, we prove that α_X^1 is as close as it can be to an injection (see Proposition 5.9 for the precise statement).

Finally, suppose that X is not 1-general; then g is even and $[X]$ lies in a proper closed subset of \overline{M}_g (see Proposition 3.15). In this case, \overline{P}_f^1 fails to contain N_f^1 ; nevertheless our existence results, suitably modified, do extend (see Section 5.10) whereas uniqueness and injectivity results (like Proposition 5.9) may fail. In this

case the setup is significantly more complicated for standard technical reasons (presence of non-GIT-stable points or of nonfine moduli spaces.) This is why we chose to work first under the assumption of 1-generality and later to indicate, in Section 5.10 and Remark 5.13, how to modify proofs and statements so that they include the special case.

Constructing Abel maps for reducible curves presents difficulties not found for integral curves, difficulties that are due to the lack of natural, separated target spaces. The use of Néron models as target spaces is not new in the literature: in [Ed], Abel–Jacobi maps for nodal curves were studied by means of the Néron mapping property in a manner similar to what we do here with our Abel–Néron maps. However, Néron models are seldom proper and so we cannot expect that Abel maps to Néron models will be complete. In this framework, our contribution is that of bringing compactified Picard schemes into the picture. This enables us to compactify Néron models and hence to obtain a target space into which complete Abel maps might be considered. In fact, we prove that α_X^1 extends over the whole of X . For $d \geq 2$ the completion problem is more subtle and still open. A preliminary study, as well as the case of irreducible curves, shows that completing α_X^d would require one to blow up the source—that is, to blow up X^d .

The paper is organized as follows. Section 2 is devoted to preliminaries of various types. In Section 3 we describe degree- d Abel–Néron maps to the compactified Picard scheme. In Section 4 we establish the modular description of the Abel–Néron map in degree 1 and show that it is independent of the choice of the deformation. Finally, in Section 5 we construct the completed degree-1 Abel map, give it a modular description, and study under what circumstances it is injective. Finally, from Section 5.10 to the end of the paper, we explain how to handle the special case of curves that are not 1-general.

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2. Néron Models of Picard Schemes

2.1. SETUP. We work over a fixed algebraically closed field k . Unless stated otherwise, all schemes are assumed to be locally of finite type over k .

For us, a *curve* is a reduced and connected projective scheme of dimension 1. Mostly we will deal with *nodal curves*—that is, curves whose only singularities are nodes. A *regular pencil (of curves)* is a flat projective morphism $f: \mathcal{X} \rightarrow B$ between connected regular schemes such that $\dim B = 1$, every geometric fiber of f is a curve, and f is smooth over a dense open subscheme of B .

We call a regular pencil $f: \mathcal{X} \rightarrow B$ *local* if $B = \operatorname{Spec} R$, where R is a discrete valuation ring over k having k as residue field. If X is the closed fiber, we will also say that f is a *regular smoothing of X* . For each regular pencil $f: \mathcal{X} \rightarrow B$ we let $K := k(B)$, the field of rational functions of B , and denote by \mathcal{X}_K the generic fiber of f . Notice that \mathcal{X}_K is a smooth curve over K .

Given any morphism $f: \mathcal{X} \rightarrow B$ and any integer $d \geq 1$, let $f_d: \mathcal{X}_B^d \rightarrow B$ denote the d th fibered power of \mathcal{X} over B . If f is a regular pencil, we use $\dot{\mathcal{X}}_B^d$ to denote the open subset of \mathcal{X}_B^d where f_d is smooth; thus

$$\dot{\mathcal{X}}_B^d := \mathcal{X}_B^d \setminus \operatorname{Sing}(f_d).$$

If f is a local regular pencil, let \dot{X}^d denote the closed fiber of $\dot{\mathcal{X}}_B^d \rightarrow B$; then

$$\dot{X}^d = \{(p_1, \dots, p_d) : p_i \in X \setminus X_{\text{sing}}\}.$$

Given any morphism $f: \mathcal{X} \rightarrow B$ and any B -scheme T , the base change of f to T is denoted by $f_T: \mathcal{X}_T \rightarrow T$.

2.2. THE RELATIVE PICARD SCHEME. Let $f: \mathcal{X} \rightarrow B$ be a regular pencil and d an integer. The closed fibers of f are geometric by our general assumption, and the general fiber is smooth. Hence the irreducible components of the fibers of f are geometrically irreducible. By a theorem of Mumford (see [BLR, Thm. 2, p. 210]), the relative Picard scheme Pic_f of f exists and is locally of finite type over B . Furthermore, Pic_f is formally smooth over B by [BLR, Prop. 2, p. 232] and thus smooth over B by [BLR, Prop. 6, p. 37].

Let Pic_f^d be the degree- d Picard scheme of f , the open subscheme of Pic_f parameterizing line bundles of relative degree d . Given any B -scheme S and any line bundle \mathcal{L} on \mathcal{X}_S of f_S -relative degree d , there is a *moduli map* associated to \mathcal{L} ,

$$\begin{aligned} \mu_{\mathcal{L}}: S &\longrightarrow \text{Pic}_f^d, \\ s &\longmapsto L_s, \end{aligned} \tag{1}$$

where $L_s \in \text{Pic}^d X_s$ is the restriction of \mathcal{L} to the fiber $X_s := f_S^{-1}(s)$. The map $\mu_{\mathcal{L}}$ determines \mathcal{L} up to tensoring with pullbacks of line bundles from S . Observe that to a map $S \rightarrow \text{Pic}_f^d$ there does not necessarily correspond a line bundle on \mathcal{X}_S , although the line bundle will exist, for example, if f admits a section; see [BLR, Prop. 4, p. 204].

2.3. NÉRON MODELS OF PICARD SCHEMES. Let $f: \mathcal{X} \rightarrow B$ be a regular pencil and d an integer. Recall that a basic characteristic (and a drawback for various applications) of the Picard scheme Pic_f^d is that it is not separated over B if f has reducible special fibers. One way to fix this is to introduce the Néron model:

$$N_f^d := \text{N}(\text{Pic}^d \mathcal{X}_K).$$

The Néron model is a smooth, separated (possibly not proper) scheme of finite type over B , with generic fiber equal to $\text{Pic}^d \mathcal{X}_K$, that satisfies a fundamental mapping property by which it is uniquely determined. Namely, for every smooth B -scheme Z , each map $Z_K \rightarrow \text{Pic}^d \mathcal{X}_K$ extends uniquely to a map $Z \rightarrow N_f^d$; see [BLR, Def. 1, p. 12].

The existence of N_f^d for any regular pencil f is likely well known. Since this result is fundamental for our work yet we could not find a precise statement for reference, we sketch a proof of it here using results in [BLR].

First, assume that f is local—that is, assume B is the spectrum of a discrete valuation ring R . Then there is a Néron model of $\text{Pic}^d \mathcal{X}_K$ over B that is equal to Pic_f^d if f is smooth. Indeed, since $\text{Pic}^d \mathcal{X}_K$ is a $(\text{Pic}^0 \mathcal{X}_K)$ -torsor, by descent theory we may assume that R is a strictly Henselian ring (see [BLR, Cor. 3, p. 158]). In this case, f admits a section through its smooth locus by [BLR, Prop. 5, p. 47]. This section can be used to produce a B -isomorphism $\text{Pic}_f^d \rightarrow \text{Pic}_f^0$. We may therefore

assume that $d = 0$. In this case, there is a Néron model of $\text{Pic}^0 \mathcal{X}_K$ over B because $\text{Pic}^0 \mathcal{X}_K$ is an Abelian variety over K ; see [BLR, Cor. 2, p. 16]. Furthermore, if f is smooth then Pic_f^0 is an Abelian B -scheme and hence is the Néron model of $\text{Pic}^0 \mathcal{X}_K$ over B by [BLR, Prop. 8, p. 15].

Now consider the general case. Let $U \subseteq B$ be the largest open subscheme over which f is smooth, and set $h := f|_{f^{-1}(U)}$. As we have seen, Pic_h^d restricts to the Néron model of $\text{Pic}^d \mathcal{X}_K$ locally around each point of U . Then Pic_h^d is the Néron model of $\text{Pic}^d \mathcal{X}_K$ over U by [BLR, Prop. 4, p. 13]. Finally, the local existence of Néron models of $\text{Pic}^d \mathcal{X}_K$ around each point of B and the existence of the Néron model over the dense open subscheme $U \subseteq B$ together imply the (global) existence of the Néron model over B by [BLR, Prop. 1, p. 18].

Since Pic_f^d is smooth over B , a first consequence of the mapping property of the Néron model N_f^d is the existence of a canonical B -morphism

$$q_f: \text{Pic}_f^d \rightarrow N_f^d, \quad (2)$$

which is the identity on the generic fiber.

Assume now that the geometric fibers of f are nodal. Let X be a closed fiber of f . In the description of the Néron model, and also in our paper, an important role is played by the following subgroup $\text{Tw}_f X \subseteq \text{Pic}^0 X$ of (isomorphism classes of) distinguished line bundles:

$$\text{Tw}_f X := \frac{\{\mathcal{O}_{\mathcal{X}}(D)|_X : D \in \text{Div } \mathcal{X} \text{ with } \text{Supp } D \subset X\}}{\cong} \subset \text{Pic}^0 X.$$

The divisors D appearing in this expression are simply sums with integer coefficients of the components of X , which are Cartier divisors of \mathcal{X} because \mathcal{X} is regular. Line bundles in $\text{Tw}_f X$ are called *twisters*. Here is a useful observation:

$$\forall T, T' \in \text{Tw}_f X, \quad T = T' \iff \deg T = \deg T', \quad (3)$$

where \deg denotes the multidegree of a line bundle on X . More precisely, let $X = \bigcup_{i=1}^{\gamma} C_i$ be the decomposition into irreducible components; then the multidegree $\deg L$ of $L \in \text{Pic } X$ is defined as $\deg L := (\deg_{C_1} L, \dots, \deg_{C_{\gamma}} L)$.

Because twisters are specializations of the trivial line bundle of the generic fiber, $\mathcal{O}_{\mathcal{X}_K}$, all of them must be identified in any separated quotient of Pic_f^0 . In particular, $q_f(T) = q_f(\mathcal{O}_X)$ for each $T \in \text{Tw}_f X$.

We shall now identify multidegrees that differ by multidegrees of twisters. Let γ be the number of irreducible components of X , and set

$$\Lambda_X := \{\deg T : T \in \text{Tw}_f X\} \subset \mathbb{Z}^{\gamma}.$$

Define now an equivalence relation “ \equiv ” on multidegrees by setting

$$\underline{d} \equiv \underline{d}' \iff \underline{d} - \underline{d}' \in \Lambda_X.$$

The set of multidegree classes $\underline{d} + \Lambda_X$ with fixed total degree $|\underline{d}| := \sum d_i$ equal to d is denoted by Δ_X^d . Thus,

$$\Delta_X^d := \frac{\{\underline{d} \in \mathbb{Z}^{\gamma} : |\underline{d}| = d\}}{\equiv}. \quad (4)$$

It is well known that Δ_X^0 is a finite group and a purely combinatorial invariant of X , called the degree class group of X in [C1] but known before as the group of connected components of N_f^0 (see [BLR, Thm. 1, p. 274] or [R, Thm. 8.1.2, p. 64]). In addition, for each d there is a (noncanonical) bijection between the set of connected components of N_f^0 and that of N_f^d as well as a (nonunique) bijection $\Delta_X^0 \rightarrow \Delta_X^d$ obtained by summing with any multidegree \underline{d} where $|\underline{d}| = d$.

For each $\delta \in \Delta_X^d$, let \underline{d} be any multidegree representing δ and set

$$\mathrm{Pic}_f^\delta := \mathrm{Pic}_f^{\underline{d}} \subset \mathrm{Pic}_f^d, \quad (5)$$

where $\mathrm{Pic}_f^{\underline{d}}$ parameterizes line bundles with fixed multidegree \underline{d} on X . The particular choice of representative \underline{d} is not important; see [C2, Sec. 3.9].

Assume now that f is a regular smoothing of X . At this point we are able to describe the Néron model of $\mathrm{Pic}^d \mathcal{X}_K$:

$$N_f^d \cong \frac{\coprod_{\delta \in \Delta_X^d} \mathrm{Pic}_f^\delta}{\sim_K}, \quad (6)$$

where \sim_K denotes the gluing along the generic fiber equal to $\mathrm{Pic}^d \mathcal{X}_K$; see [C2, Lemma 3.10].

Let N_X^d denote the closed fiber of N_f^d . Observe that N_X^d is a disjoint union of finitely many copies of the generalized Jacobian of X : picking a representative \underline{d}^δ for each class $\delta \in \Delta_X^d$, we have

$$N_X^d \cong \coprod_{\delta \in \Delta_X^d} \mathrm{Pic}^{\underline{d}^\delta} X.$$

Although this isomorphism is not canonical, we see that the scheme structure of N_X^d does not depend on f . The closed points of N_X^d are in one-to-one correspondence with the degree- d line bundles on X modulo twistors. In particular, for $d = 0$ we have $q_f^{-1}(q_f(\mathcal{O}_X)) = \mathrm{Tw}_f X$.

2.4. NÉRON MAPS. Let $f: \mathcal{X} \rightarrow B$ be a regular pencil. Let T be a B -scheme and let \mathcal{L} be a line bundle on \mathcal{X}_T of relative degree d over T . Let $\mu_{\mathcal{L}}: T \rightarrow \mathrm{Pic}_f^d$ be the moduli map of \mathcal{L} as defined in Section 2.2. Consider the composition

$$\overline{\mu}_{\mathcal{L}}: T \xrightarrow{\mu_{\mathcal{L}}} \mathrm{Pic}_f^d \xrightarrow{q_f} N_f^d.$$

We call $\overline{\mu}_{\mathcal{L}}$ the *Néron map* of \mathcal{L} . Notice that \mathcal{L} is certainly not determined by its Néron map, not even modulo pullbacks of line bundles on T . In fact, if $D \subset \mathcal{X}$ is a Cartier divisor entirely supported on a closed fiber of f , then $\mathcal{L} \otimes \mathcal{O}_{\mathcal{X}_T}(D_T)$ has the same Néron map as \mathcal{L} because $N_f^d \rightarrow B$ is separated.

2.5. ABEL–NÉRON MAPS. Let us recall the precise definition of the Abel map of a smooth curve, using the setup of [GIT, Sec. 6, pp. 118, 119].

Let $h: \mathcal{C} \rightarrow S$ be a smooth curve over a scheme S (i.e., a smooth morphism whose fibers are curves). For each integer $d \geq 1$, let \mathcal{C}_S^d be the d th fibered power of \mathcal{C} over S . Then there is a canonical S -morphism

$$\begin{aligned} C_S^d &\longrightarrow \operatorname{Pic}_h^d \\ C_s^d \ni (p_1, \dots, p_d) &\longmapsto \mathcal{O}_{C_s}(p_1 + \dots + p_d) \end{aligned} \quad (7)$$

defined over each $s \in S$ by taking a d -tuple of points of the fiber C_s to the line bundle associated to their sum, which we shall call the d th Abel map of h . Recall that the map in (7) is the moduli map of a natural line bundle on $C_S^d \times_S \mathcal{C}$ —namely, the one associated to the Cartier divisor $\sum_1^d S_i$, where each S_i is the image of the i th natural section σ_i of the first projection $C_S^d \times_S \mathcal{C} \rightarrow C_S^d$ given by

$$\sigma_i(p_1, \dots, p_d) = ((p_1, \dots, p_d), p_i). \quad (8)$$

We may apply this construction to a regular pencil $f: \mathcal{X} \rightarrow B$. First of all, since \mathcal{X}_K is smooth over K , we may consider the d th Abel map of \mathcal{X}_K :

$$\alpha_K^d: \mathcal{X}_K^d \rightarrow \operatorname{Pic}^d \mathcal{X}_K. \quad (9)$$

This map extends to a map $\dot{\mathcal{X}}_B^d \rightarrow \operatorname{Pic}_f^d$. Indeed, the extension is the moduli map of the line bundle associated to the Cartier divisor E^d of $\dot{\mathcal{X}}_B^d \times_B \mathcal{X}$, where E^d is the sum of the images of the d natural sections $\sigma_1, \dots, \sigma_d$ given by (8) of the first projection $\dot{\mathcal{X}}_B^d \times_B \mathcal{X} \rightarrow \dot{\mathcal{X}}_B^d$. Composing with the map q_f of (2), we obtain the Néron map of $\mathcal{O}_{\dot{\mathcal{X}}_B^d \times_B \mathcal{X}}(E^d)$ (see Section 2.4), which is also an extension of α_K^d .

The first simple but crucial observation is the following (well-known) fact.

LEMMA 2.6 (Definition). *Let $f: \mathcal{X} \rightarrow B$ be a regular pencil. For each integer $d \geq 1$ there exists a unique morphism, which we call the d th Abel–Néron map of f ,*

$$N(\alpha_K^d): \dot{\mathcal{X}}_B^d \rightarrow N(\operatorname{Pic}^d \mathcal{X}_K) = N_f^d,$$

whose restriction to the generic fiber is α_K^d . The map $N(\alpha_K^d)$ is the Néron map of $\mathcal{O}_{\dot{\mathcal{X}}_B^d \times_B \mathcal{X}}(E^d + D)$ for every Cartier divisor D of $\dot{\mathcal{X}}_B^d \times_B \mathcal{X}$ supported on any finite number of closed fibers of $\dot{\mathcal{X}}_B^d \times_B \mathcal{X} \rightarrow B$.

Proof. The existence and uniqueness of an extension of α_K^d is a straightforward consequence of the Néron mapping property [BLR, Def. 1, p. 12]: since $\dot{\mathcal{X}}_B^d$ is smooth over B and has generic fiber \mathcal{X}_K^d , the Abel map α_K^d admits a unique extension $N(\alpha_K^d): \dot{\mathcal{X}}_B^d \rightarrow N(\operatorname{Pic}^d \mathcal{X}_K)$.

Since also the Néron map of $\mathcal{O}_{\dot{\mathcal{X}}_B^d \times_B \mathcal{X}}(E^d + D)$, for any D as described in the lemma, extends α_K^d , the last statement follows from the fact that N_f^d is separated over B . \square

3. Abel Maps to Balanced Picard Schemes

We want to give a modular interpretation of the Abel–Néron maps and, at the same time, study the problem of completing them. To do this we shall use some results of [C2], where Néron models are glued together over the moduli space of stable curves and are endowed with a geometrically meaningful completion. Our moduli problem is centered around Definition 3.2 (to follow). First, we recall a few concepts.

3.1. Let X be a nodal curve of arithmetic genus $g \geq 2$. Denote by ω_X its canonical (or dualizing) bundle. For each proper subcurve $Z \subsetneq X$, which is always assumed to be complete, let $Z' := \bar{X} \setminus \bar{Z}$ and $k_Z := \#Z \cap Z'$. Also, let $w_Z := \deg_Z \omega_X$. If Z is connected, denote by g_Z its arithmetic genus and recall that

$$w_Z = 2g_Z - 2 + k_Z, \quad (10)$$

a well-known identity that can be proved using adjunction.

We call X *semistable* (resp. *stable*) if $k_Z \geq 2$ (resp. $k_Z \geq 3$) for each smooth rational component Z of X . Those Z for which $k_Z = 2$ are called *exceptional*. A semistable curve is called *quasistable* if two exceptional components never meet each other. If X is semistable then it has a *stable model*—that is, a stable curve \bar{X} and a map $X \rightarrow \bar{X}$ contracting all exceptional components. We may also say that X is *semistable over \bar{X}* .

If X is semistable, it follows from (10) that $w_Z \geq 0$ for each subcurve $Z \subseteq X$, with equality if and only if Z is a union of exceptional components.

A *family of semistable* (resp. *stable*, resp. *quasistable*) *curves* is a flat projective map $f: \mathcal{X} \rightarrow B$ whose geometric fibers are semistable (resp. stable, resp. quasistable) curves. A line bundle of degree d on such a family $f: \mathcal{X} \rightarrow B$ is a line bundle on \mathcal{X} whose restriction to each fiber has degree d .

DEFINITION 3.2. Let X be a semistable curve of arithmetic genus $g \geq 2$, and let $L \in \text{Pic}^d X$.

- (i) We say that L and its multidegree $\deg L$ are *semibalanced* if, for each connected proper subcurve $Z \subsetneq X$, the so-called Basic Inequality holds:

$$m_Z(d) \leq \deg_Z L \leq M_Z(d), \quad (11)$$

where

$$M_Z(d) := \frac{dw_Z}{2g-2} + \frac{k_Z}{2}$$

and

$$m_Z(d) := \begin{cases} M_Z(d) - k_Z & \text{if } Z \text{ is not an exceptional component,} \\ 0 & \text{if } Z \text{ is an exceptional component.} \end{cases}$$

- (ii) We call L and $\deg L$ *balanced* if they are semibalanced and if, for each exceptional component $E \subset X$,

$$\deg_E L = 1.$$

- (iii) We call L and $\deg L$ *stably balanced* if they are balanced and if, for each connected proper subcurve $Z \subsetneq X$ such that $\deg_Z L = m_Z(d)$, the complement Z' is a union of exceptional components.
- (iv) We denote by B_X^d the set of balanced multidegrees on X and by \tilde{B}_X^d the subset of stably balanced ones (in [C2], B_X^d denotes the semibalanced multidegrees).
- (v) A line bundle (of degree d) on a family of semistable curves is called *semibalanced* (balanced or stably balanced) if its restriction to each geometric fiber of the family is.

REMARK 3.3. Definition 3.2 is [C2, Def. 4.6], which originates from [C1, Def. 3.1] and [CCaCo, Section 5.1.1]. In [CCaCo] the Basic Inequality is

$$\left| \deg_Z L - \frac{dw_Z}{2g-2} \right| \leq \frac{k_Z}{2},$$

and the extra condition for when Z is an exceptional component is imposed separately. For convenience, the exceptional cases are included in the set of inequalities (11). Abusing the terminology, we still call (11) the Basic Inequality.

We mention some basic useful consequences of the definition as follows.

(A) If X is stable, then Definitions 3.2(i) and (ii) coincide; that is, a semibalanced line bundle is always balanced.

(B) A balanced line bundle can exist on a semistable curve X only if X is quasi-stable. Indeed, let $Z \subset X$ be a connected chain of exceptional components. If L is a semibalanced line bundle on X , then L has degree 0 on every component of Z except possibly one, where L may have degree 1. However, if L is balanced then L cannot have degree 0 on any component of Z . Hence $Z \cong \mathbb{P}^1$ and $\deg_Z L = 1$.

(C) To check whether L is semibalanced it is enough to check whether $\deg_Z L \geq m_Z(d)$ for every connected proper subcurve $Z \subsetneq X$ and whether $\deg_Z L \geq 0$ if Z is exceptional. Indeed, let Z be such a subcurve and let Y_1, \dots, Y_n denote the connected components of Z' . By hypothesis,

$$\deg_{Y_i} L \geq m_{Y_i}(d) \geq \frac{dw_{Y_i}}{2g-2} - \frac{k_{Y_i}}{2}$$

for each i . Since $k_{Y_1} + \dots + k_{Y_n} = k_{Z'} = k_Z$, summing up the preceding inequalities yields

$$\deg_{Z'} L \geq \frac{dw_{Z'}}{2g-2} - \frac{k_Z}{2}.$$

Now, since $\deg_Z L + \deg_{Z'} L = d$ and $w_Z + w_{Z'} = 2g - 2$, it follows that

$$\deg_Z L = d - \deg_{Z'} L \leq d - \frac{dw_{Z'}}{2g-2} + \frac{k_Z}{2} = \frac{dw_Z}{2g-2} + \frac{k_Z}{2}.$$

3.4. In [C2, Lemma 4.4] it is proved that each multidegree class has a semibalanced representative. More precisely, fix an integer d and let X be a stable curve. Recall the notation in (4) and Definition 3.2(iv). Then [C2, Lemma 4.4] implies that the following natural map is surjective:

$$\begin{aligned} [\dots]: B_X^d &\longrightarrow \Delta_X^d, \\ \underline{d} &\longmapsto [\underline{d}], \end{aligned} \tag{12}$$

where the brackets denote classes. We shall say that X is “ d -general” if the map (12) is bijective (see Definition 3.6).

3.5. The moduli problem for balanced line bundles was introduced and studied in [C1] in order to compactify the universal Picard scheme over M_g . That compactification was constructed as a GIT-quotient. We need not recall all the details of this construction here. However, we shall note that there exist morphisms

$$H_d \xrightarrow{\pi_d} H_d/G \xrightarrow{\phi_d} \bar{M}_g,$$

where H_d is an open subscheme of a suitable Hilbert scheme acted upon by an algebraic group G , the map π_d is a GIT-quotient map, and ϕ_d is a surjective, projective morphism. The quotient scheme H_d/G , denoted by $\bar{P}_{d,g}$, is integral and projective. The fiber of ϕ_d over a general smooth curve X is exactly $\text{Pic}^d X$; see [C1, Thm. 6.1].

Denote by $U \subseteq \bar{P}_{d,g}$ the nonempty open subscheme over which π_d restricts to a geometric GIT-quotient. That is, all fibers over points in U are G -orbits, and all stabilizers are finite and reduced.

DEFINITION 3.6. Let X be a stable curve of arithmetic genus $g \geq 2$. We say that X is d -general if any of the following equivalent conditions hold.

- (i) $\phi_d^{-1}(X) \subset U$.
- (ii) The class map (12) of Section 3.4 is bijective.
- (iii) Every balanced line bundle on X is stably balanced; that is, $\tilde{B}_X^d = B_X^d$.

The equivalence of these three conditions follows from [C1, Lemma 6.1].

It is known that *all stable curves of genus g are d -general if and only if the integers d and g satisfy $(d - g + 1, 2g - 2) = 1$; see [C1, Prop. 6.2].*

3.7. We need to recall when two semibalanced line bundles are defined to be equivalent. Let X be a stable curve, and let X_1 and X_2 be two semistable curves having X as a stable model. For each $i = 1, 2$, let L_i be a semibalanced line bundle on X_i . Let Y_i be the semistable curve obtained by contracting all exceptional components of X_i where L_i has degree 0. Then there is a unique line bundle M_i on Y_i whose pullback to X_i is L_i . Since L_i is semibalanced, it follows that M_i is balanced and Y_i is quasistable. Let $F_i \subseteq Y_i$ be the union of all the exceptional components of Y_i , and let $\tilde{Y}_i := F_i'$, the complementary subcurve of F_i in Y_i . Then L_1 and L_2 are equivalent if there is an isomorphism $Y_1 \rightarrow Y_2$ such that $M_1|_{\tilde{Y}_1} \cong M_2|_{\tilde{Y}_2}$ under the identification given by that isomorphism. Observe that M_i is equivalent to L_i for $i = 1, 2$.

Thus, every equivalence class always includes a balanced line bundle N on a quasistable curve Y . The quasistable curve Y is unique, but N is not. What is unique is the restriction of N to \tilde{Y} , the complementary subcurve of the union F of all the exceptional components of Y . The quasistable curve Y and $N|_{\tilde{Y}}$ determine the equivalence class. The restriction $N|_F$ is also unique, since a balanced line bundle must have degree 1 on every exceptional component. Hence, our equivalence relation disregards the gluing data of the bundles over the points in $\tilde{Y} \cap F$.

If $\mathcal{X} \rightarrow B$ is a family of semistable curves, then two semibalanced line bundles \mathcal{L}_1 and \mathcal{L}_2 on $\mathcal{X} \rightarrow B$ are called equivalent if and only if their restrictions to every geometric fiber of $\mathcal{X} \rightarrow B$ are equivalent in the sense just explained.

3.8. Let d and g be integers, with $g \geq 2$. Assume first that $d - g + 1$ and $2g - 2$ are coprime, so that every stable curve of arithmetic genus g is d -general. Then the construction summarized in Section 3.5 can be improved by considering stacks.

More precisely, there exist two (modular) Deligne–Mumford stacks, $\overline{\mathcal{P}}_{d,g}$ and $\mathcal{P}_{d,g}$, each equipped with a natural and strongly representable morphism to $\overline{\mathcal{M}}_g$. (To tie in with Section 3.5, we mention that $\overline{\mathcal{P}}_{d,g}$ is the quotient stack $[H_d/G]$.) The following properties hold (see [C2, Sec. 5] for details).

(A) For each (d -general) stable curve X , denote by P_X^d and \overline{P}_X^d the fibers of $\mathcal{P}_{d,g}$ and $\overline{\mathcal{P}}_{d,g}$ over X . Since $\mathcal{P}_{d,g}$ and $\overline{\mathcal{P}}_{d,g}$ are strongly representable over $\overline{\mathcal{M}}_g$, both P_X^d and \overline{P}_X^d are quasiprojective schemes. The first, P_X^d , is the fine moduli scheme of degree- d balanced line bundles on X ; the second, \overline{P}_X^d , is the coarse moduli scheme of equivalence classes of degree- d semibalanced line bundles on semistable curves having X as stable model (see Section 3.7). Actually, \overline{P}_X^d is not far from being a fine moduli scheme; see property (C). The fiber P_X^d lies naturally inside \overline{P}_X^d as an open and dense subscheme.

(B) Let $f: \mathcal{X} \rightarrow B$ be any family of (d -general) stable curves of genus g , and consider the schemes

$$P_f^d := B \times_{\overline{\mathcal{M}}_g} \mathcal{P}_{d,g} \quad \text{and} \quad \overline{P}_f^d := B \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{P}}_{d,g}.$$

(That these are indeed schemes follows, again, from the fact that the maps $\mathcal{P}_{d,g} \rightarrow \overline{\mathcal{M}}_g$ and $\overline{\mathcal{P}}_{d,g} \rightarrow \overline{\mathcal{M}}_g$ are strongly representable.) We have a natural inclusion $P_f^d \subset \overline{P}_f^d$. See (13) for an explicit description of P_f^d when f is a local regular pencil. As for \overline{P}_f^d , the following statement holds. Let T be a B -scheme, $\mathcal{Y} \rightarrow T$ a family of semistable curves having $f_T: \mathcal{X}_T \rightarrow T$ as a stable model, and \mathcal{L} a semibalanced line bundle of degree d on $\mathcal{Y} \rightarrow T$; then to each triple $(T, \mathcal{Y} \rightarrow T, \mathcal{L})$ there corresponds a moduli map,

$$\hat{\mu}_{\mathcal{L}}: T \rightarrow \overline{P}_f^d,$$

taking each geometric point t of T to the equivalence class of the restriction of \mathcal{L} to the (geometric) fiber of \mathcal{Y} over t . We call $\hat{\mu}_{\mathcal{L}}$ the *moduli map* of \mathcal{L} . Note that the image of $\hat{\mu}_{\mathcal{L}}$ is contained in P_f^d if and only if \mathcal{L} has degree 0 on every exceptional component of every geometric fiber of $\mathcal{Y} \rightarrow T$.

(C) The scheme P_f^d is a fine moduli scheme (see [C2, Cor. 5.14, Rem. 5.15]). Also, \overline{P}_f^d is not far from being a fine moduli scheme. In fact, it is endowed with a quasiuniversal pair $(\mathcal{Z} \rightarrow \overline{P}_f^d, \mathcal{N})$, where $\mathcal{Z} \rightarrow \overline{P}_f^d$ is a family of quasistable curves having $f_{\overline{P}_f^d}: \mathcal{X}_{\overline{P}_f^d} \rightarrow \overline{P}_f^d$ as stable model and where \mathcal{N} is a balanced line bundle of degree d on $\mathcal{Z} \rightarrow \overline{P}_f^d$ that has a role similar to that of a Poincaré bundle. Indeed, for each triple $(T, \mathcal{Y} \rightarrow T, \mathcal{L})$ as described in (B) there is a map $\mathcal{Y} \rightarrow \mathcal{Z}$ such that the diagram of maps

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow \\ T & \xrightarrow{\hat{\mu}_{\mathcal{L}}} & \overline{P}_f^d \end{array}$$

is commutative and such that \mathcal{L} is equivalent to the pullback of \mathcal{N} to \mathcal{Y} ; see Section 3.7. (The map $\mathcal{Y} \rightarrow \mathcal{Z}$ is certainly not uniquely determined, which is why

we call the pair $(\mathcal{Z} \rightarrow \overline{P}_f^d, \mathcal{N})$ quasiuniversal.) Furthermore, if $\mathcal{Y} \rightarrow T$ is a family of quasistable curves and if \mathcal{L} is balanced, then the map $\mathcal{Y} \rightarrow \mathcal{Z}$ can be chosen such that the preceding diagram is a fibered product diagram.

REMARK 3.9. If $(d - g + 1, 2g - 2) \neq 1$, then almost everything in Section 3.8 works over the open subset of \overline{M}_g parameterizing d -general stable curves. For a proof, it suffices to argue exactly as for [C2, Thm. 5.9] after replacing \overline{M}_g with the substack of d -general curves and replacing the stacks $\overline{\mathcal{P}}_{d,g}$ and $\overline{\mathcal{P}}_{d,g}$ with the corresponding substacks (over d -general curves).

The only assertion in Section 3.8 that does not hold is the existence in (C) of a Poincaré line bundle, which will not be used in this paper.

We may now return to our study of Abel maps.

THEOREM 3.10. *Let $f: \mathcal{X} \rightarrow B$ be a regular pencil of d -general stable curves. Then there exists a canonical map*

$$\alpha_f^d: \dot{\mathcal{X}}^d \rightarrow P_f^d$$

that restricts to the d th Abel map on the generic fiber.

Proof. We may glue local extensions of the d th Abel map of \mathcal{X}_K because they are unique. Hence we may assume that f is local; let X be the closed fiber of f . In this case, the explicit description of P_f^d is

$$P_f^d = \frac{\coprod_{\underline{d} \in B_X^d} \text{Pic}_f^{\underline{d}}}{\sim_K} \quad (13)$$

(see [C2, Cor. 5.14]), where, as in Section 2.3, \sim_K means gluing over the generic fiber.

As we know from (6) in Section 2.3, N_f^d is described in a similar way. Indeed, by [C2, Thm. 6.1] we have the canonical isomorphism

$$\varepsilon_f^d: P_f^d \xrightarrow{\cong} N_f^d. \quad (14)$$

To describe it precisely is straightforward: For each $\underline{d} \in B_X^d$, the restriction of ε_f^d to $\text{Pic}_f^{\underline{d}}$ is the natural isomorphism

$$\text{Pic}_f^{\underline{d}} \xrightarrow{\cong} \text{Pic}_f^{[\underline{d}]} \subset N_f^d$$

that restricts to the identity on the generic fibers. The isomorphism ε_f^d is completely described because, since X is d -general, the class map $B_X^d \rightarrow \Delta_X^d$ is bijective (by Definition 3.6). To conclude, use Lemma 2.6 and (14) to define $\alpha_f^d := (\varepsilon_f^d)^{-1} \circ N(\alpha_K^d)$. \square

We call α_f^d the d th Abel map of f . The natural problem now is to describe α_f^d as the moduli map of a balanced line bundle on $\pi: \dot{\mathcal{X}}^d \times_B \mathcal{X} \rightarrow \dot{\mathcal{X}}^d$. Since P_f^d is a fine moduli scheme, this should be possible. In fact, the proof of [C1, Prop. 4.1,

p. 621] can be used to produce an algorithm for determining the necessary twistors with which to tensor $\mathcal{O}_{\dot{X}^d \times_B \mathcal{X}}(E^d)$ in order to obtain the balanced line bundle.

However, it turns out that in general this line bundle is difficult to describe explicitly. In Section 4 we will do so for $d = 1$, and in Section 3.11 we will do so for every d in a special case.

3.11. TWO-COMPONENT CURVES. Let X be a stable curve with only two irreducible components, C_1 and C_2 . Let g be the arithmetic genus of X and let $\delta := \#C_1 \cap C_2$. Set

$$m := \lceil m_{C_1}(d) \rceil = \left\lceil \frac{dw_{C_1}}{2g-2} - \frac{\delta}{2} \right\rceil$$

(see Definition 3.2), where $\lceil x \rceil$ denotes the ceiling of a real number x (i.e., the smallest integer not smaller than x). The set B_X^d of balanced multidegrees of X satisfies

$$B_X^d \supseteq \{(m, d-m), (m+1, d-m-1), \dots, (m+\delta-1, d-m-\delta+1)\}, \quad (15)$$

with equality if and only if $m_{C_1}(d)$ is not integer and X is d -general.

For each integer a , define $r(a)$ to be the integer determined by the following two conditions:

$$0 \leq r(a) < \delta \quad \text{and} \quad a - m \equiv r(a) \pmod{\delta}.$$

PROPOSITION 3.12. *Let X be a stable curve with exactly two irreducible components, C_1 and C_2 . Let $\delta := \#C_1 \cap C_2$. For each regular smoothing $f: \mathcal{X} \rightarrow B$ of X , let*

$$\mathcal{L}^{(d)} := \mathcal{O}_{\dot{X}^d \times_B \mathcal{X}} \left(E^d + \sum_{a=0}^d \frac{a - m - r(a)}{\delta} C_1^a \times C_2^{d-a} \times C_1 \right),$$

where (abusing notation) we view

$$C_1^a \times C_2^{d-a} \times C_1 \subset X^d \times X \subset \mathcal{X}^d \times_B \mathcal{X}$$

as a Cartier divisor of $\dot{X}^d \times_B \mathcal{X}$ by restriction. Then $\mathcal{L}^{(d)}$ is balanced on $f_{\dot{X}^d}: \dot{X}^d \times_B \mathcal{X} \rightarrow \dot{X}^d$. Furthermore, if X is d -general then the d th Abel map $\alpha_f^d: \dot{X}^d \rightarrow P_f^d$ is the moduli map of $\mathcal{L}^{(d)}$.

Proof. Since each $C_1^a \times C_2^{d-a} \times C_1$ is supported over the closed point of B , the line bundle $\mathcal{L}^{(d)}$ coincides with $\mathcal{O}_{\dot{X}^d \times_B \mathcal{X}}(E^d)$ over the generic point of B . This implies that $\mathcal{L}^{(d)}$ and $\mathcal{O}_{\dot{X}^d \times_B \mathcal{X}}(E^d)$ have the same Néron map.

If X is d -general, then α_f^d is defined and coincides with the Néron map of $\mathcal{O}_{\dot{X}^d \times_B \mathcal{X}}(E^d)$ on \mathcal{X}_K^d . On the other hand, if the line bundle $\mathcal{L}^{(d)}$ is balanced on $\pi: \dot{X}^d \times_B \mathcal{X} \rightarrow \dot{X}^d$, then there is an associated moduli map $\hat{\mu}_{\mathcal{L}^{(d)}}: \dot{X}^d \rightarrow P_f^d$ by Section 3.8(B). Since $\hat{\mu}_{\mathcal{L}^{(d)}}$ coincides with the Néron map of $\mathcal{L}^{(d)}$ on \mathcal{X}_K^d , it follows that $\hat{\mu}_{\mathcal{L}^{(d)}} = \alpha_f^d$. Thus, it suffices to prove that $\mathcal{L}^{(d)}$ is balanced.

To verify this, we must compute the multidegree of the restriction of $\mathcal{L}^{(d)}$ to every singular fiber of $\pi: \dot{X}^d \times \mathcal{X} \rightarrow \dot{X}^d$. Of course, all singular fibers lie over \dot{X}^d . So let $p \in \dot{X}^d$, and denote by X_p the fiber $\pi^{-1}(p)$. The point p determines

a unique pair of nonnegative integers (a_0, b_0) such that $a_0 + b_0 = d$ and $p \in C_1^{a_0} \times C_2^{b_0}$. Then, identifying X_p with X in the natural way, we have

$$(E^d \cdot C_1, E^d \cdot C_2) = (a_0, b_0) = (a_0, d - a_0). \quad (16)$$

To compute the intersection degrees with C_1 and C_2 of the remaining summands defining $\mathcal{L}^{(d)}$, notice first that, since $p \in C_1^{a_0} \times C_2^{b_0}$, the only nonzero degrees come from the summand indexed by $a = a_0$. Now, using

$$(C_1^{a_0} \times C_2^{b_0} \times C_1) \cdot (C_1, C_2) = (-\delta, \delta)$$

and (16), we obtain

$$\deg_{X_p} \mathcal{L}^{(d)} = (\deg \mathcal{L}^{(d)}|_{C_1}, \deg \mathcal{L}^{(d)}|_{C_2}) = (m + r(a_0), d - m - r(a_0)),$$

which is balanced because $0 \leq r(a_0) < \delta$; see (15). \square

EXAMPLE 3.13. Let X be a “split” curve of arithmetic genus g ; that is, $X = C_1 \cup C_2$ with $C_i \cong \mathbb{P}^1$ and $\#C_1 \cap C_2 = g + 1$. Then, for each $d = 1, \dots, g$ and any regular smoothing $f: \mathcal{X} \rightarrow B$ of X , the map α_f^d is the moduli map of $\mathcal{O}_{\dot{\mathcal{X}}^d \times_B \mathcal{X}}(E^d)$. In particular, given $p_1, \dots, p_d \in \dot{X}$ we have, independently of f ,

$$\alpha_f^d(p_1, \dots, p_d) = \mathcal{O}_X(p_1 + \dots + p_d).$$

A split curve is d -general if and only if $d \equiv g \pmod{2}$. Actually, the conclusion of Example 3.13 is valid regardless of X being d -general, because in any case $\mathcal{O}_{\dot{\mathcal{X}}^d \times_B \mathcal{X}}(E^d)$ is stably balanced on $f_{\dot{\mathcal{X}}^d}$.

REMARK 3.14. The case of split curves is, in a sense, special. In general, we should expect the restriction $\alpha_f^d|_{\dot{\mathcal{X}}^d}$ of the d th Abel map of Proposition 3.12 to depend on the choice of smoothing f . For a simple concrete example of this dependence, consider the case $d = 2$ and $\delta = 2$. Then X is stable and 2-general if C_1 and C_2 have distinct positive arithmetic genera. Indeed, X is stable because the genera of C_1 and C_2 are positive. Moreover, since they are distinct, it follows that $w_{C_i}/(g - 1)$ is never an integer for any $i = 1, 2$. Hence the Basic Inequality is always strict and so X is 2-general.

If C_1 has smaller genus then $m = 0$. Thus $r(0) = 0$ and $r(1) = 1$ but $r(2) = 0$. Let $p, q \in C_1 \setminus C_1 \cap C_2$. Then

$$\alpha_f^2(p, q) = \mathcal{O}_X(p + q) \otimes \mathcal{O}_X(C_1)|_X.$$

Now, as f varies through all smoothings of X , the restriction $\mathcal{O}_X(C_1)|_X$ varies through all line bundles restricting to $\mathcal{O}_{C_1}(-r_1 - r_2)$ and $\mathcal{O}_{C_2}(r_1 + r_2)$, where r_1 and r_2 are the nodes of X . As a result, $\alpha_f^2(p, q)$ depends on the choice of f .

Denote by Σ_g^d the locus in \bar{M}_g of curves that are not d -general. Then Σ_g^d is a proper closed subset of \bar{M}_g ; in fact, in the notation of Section 3.5, Σ_g^d is the image via ϕ_d of the complement of U . As we mentioned in Section 3.5, Σ_g^d is empty if and only if $(d - g + 1, 2g - 2) = 1$. Since the rest of our paper is devoted to Abel maps for $d = 1$, we conclude this section by describing the locus of curves that are not 1-general.

PROPOSITION 3.15. *Let $g \geq 2$.*

- (i) *If g is odd then Σ_g^1 is empty.*
- (ii) *If g is even then Σ_g^1 is the closure in \bar{M}_g of the locus of curves X such that $X = C_1 \cup C_2$, where C_1 and C_2 are smooth and of the same genus and where $\#C_1 \cap C_2$ is odd.*

Proof. If g is odd then

$$(1 - g + 1, 2g - 2) = (g - 2, g - 1) = 1,$$

so part (i) follows; see Definition 3.6.

For part (ii), let X be a stable curve. Suppose first that X has the description given in (ii). Then

$$\left(\frac{1 - \delta}{2}, \frac{1 + \delta}{2} \right) \in B_X^1 \setminus \tilde{B}_X^1,$$

where $\delta := k_{C_1} = k_{C_2}$. So X is in Σ_g^1 . Since Σ_g^1 is closed in \bar{M}_g , it contains the closure of the locus defined in (ii).

Suppose now that X is in Σ_g^1 —that is, there is a line bundle L on X such that $\deg L \in B_X^1 \setminus \tilde{B}_X^1$. Then there is a connected, proper subcurve $Z \subsetneq X$ such that either $m_Z(1)$ or $M_Z(1)$ is equal to $\deg_Z L$. Then both $m_Z(1)$ and $M_Z(1)$ are integers. Thus

$$\frac{w_Z}{w} + \frac{k_Z}{2} \in \mathbb{Z}, \quad (17)$$

where $w := \deg \omega_X = 2g - 2$. Now, since X is stable, it follows that

$$0 < \frac{w_Z}{w} < 1. \quad (18)$$

In particular, w_Z/w is never an integer and thus (17) implies that k_Z is odd. Since k_Z is odd, (17) and (18) immediately yield $w_Z/w = 1/2$.

If Z' is connected then, since $w_{Z'}/w = 1/2$ as well, we have $g_Z = g_{Z'}$. Since both Z and Z' are limits of smooth curves, X lies in the closure of the locus described in (ii).

Thus it remains to show that Z' is connected. Let Z'_1, \dots, Z'_m be the connected components of Z' . Notice that

$$k_{Z'_1} + \dots + k_{Z'_m} = k_{Z'} = k_Z. \quad (19)$$

Suppose by contradiction that $m > 1$. Then

$$0 < \frac{w_{Z'_i}}{w} < \frac{1}{2} \quad (20)$$

for each i . Since $\deg L \in B_X^1$, we have

$$\frac{w_{Z'_i}}{w} - \frac{k_{Z'_i}}{2} \leq \deg_{Z'_i} L \leq \frac{w_{Z'_i}}{w} + \frac{k_{Z'_i}}{2}$$

for each i . Using (20), we obtain

$$\frac{1 - k_{Z'_i}}{2} \leq \deg_{Z'_i} L \leq \frac{k_{Z'_i}}{2}$$

for each i . Summing up and using (19) then yields

$$\frac{m - k_Z}{2} \leq \deg_{Z'} L \leq \frac{k_Z}{2}.$$

Now, since $\deg L = 1$, we must have

$$\frac{2 - k_Z}{2} \leq \deg_Z L \leq \frac{2 - m + k_Z}{2}. \quad (21)$$

Suppose first that $\deg_Z L = M_Z(1)$; that is, suppose $\deg_Z L = (1 + k_Z)/2$. Then (21) implies $1 + k_Z \leq 2 - m + k_Z$ and hence $m \leq 1$, a contradiction. Finally, suppose $\deg_Z L = m_Z(1)$, that is, $\deg_Z L = (1 - k_Z)/2$. Then (21) implies $2 - k_Z \leq 1 - k_Z$, which is also a contradiction. \square

4. Geometric Interpretation of the First Abel Map

The following diagram represents the families we shall deal with in this section, starting from a regular pencil of stable curves $f: \mathcal{X} \rightarrow B$:

$$\begin{array}{ccccc} \dot{\mathcal{X}} \times_B \mathcal{X} & \xrightarrow{\subset} & \mathcal{X}_B^2 & \longrightarrow & \mathcal{X} \\ \dot{\pi} \downarrow & & \pi \downarrow & & \downarrow f \\ \dot{\mathcal{X}} & \xrightarrow{\subset} & \mathcal{X} & \longrightarrow & \overline{P}_f^d, \end{array}$$

where π is the projection onto the first factor.

We denote by $\Delta \subset \mathcal{X}_B^2$ the diagonal; its restriction to $\dot{\mathcal{X}} \times_B \mathcal{X}$ is a Cartier divisor. Denote by $\mathcal{O}_{\dot{\mathcal{X}} \times_B \mathcal{X}}(\Delta)$ the associated line bundle, which we may view as a family of degree-1 line bundles on the fibers of $\dot{\pi}$. Recall that the first Abel map of the generic fiber of f is the moduli map of the restriction of $\mathcal{O}_{\dot{\mathcal{X}} \times_B \mathcal{X}}(\Delta)$; see Section 2.5. We want to interpret the first Abel map α_f^1 (defined in Theorem 3.10) as the moduli map of a balanced line bundle on $\dot{\pi}$, which will necessarily be a (possibly trivial) twist of $\mathcal{O}_{\dot{\mathcal{X}} \times_B \mathcal{X}}(\Delta)$.

In fact, we shall see that $\mathcal{O}_{\dot{\mathcal{X}} \times_B \mathcal{X}}(\Delta)$ may fail to be balanced over points of a singular fiber X of f if X has separating nodes. To fix this, we will tensor $\mathcal{O}_{\dot{\mathcal{X}} \times_B \mathcal{X}}(\Delta)$ by twistors supported on “tails” (see Definition 4.1).

We need a few preliminary results that hold for any curve X , including those that have singularities other than nodes. For the sake of future applications of the techniques developed in this paper, from now until Section 4.5—and in Lemma 4.8, Definition 4.11, and Lemma 4.13—we shall be in this more general situation; that is, X will be any (reduced, connected, and projective) curve over an algebraically closed field.

Let r be a node of X and let $X_r^\nu \rightarrow X$ be the normalization of X at r only. If X_r^ν is not connected, then r is called a *separating node* of X .

DEFINITION 4.1. Let X be a curve of arithmetic genus g . A proper subcurve $Q \subsetneq X$ will be called a *tail* of X if Q intersects the complementary subcurve Q' in a separating node r of X . We say that Q is *attached to* r or that r *generates* Q . A tail Q of X will be called *small* if $g_Q < g/2$ and *large* if $g_Q > g/2$. Let

$$\mathcal{Q}(X) := \{Q \subset X : Q \text{ is a small tail of } X\}.$$

If X has no separating node (e.g., if X is smooth) then $\mathcal{Q}(X) = \emptyset$.

If r is a separating node of X , then X_r^\vee has two connected components that are isomorphic to the two tails generated by r ; hence every tail is connected.

For every tail $Q \subset X$ we have that $g = g_Q + g_{Q'}$. Therefore, at least one of the two tails attached to a separating node has arithmetic genus at most $g/2$. If the curve X is stable and 1-general, then it follows from Proposition 3.15 that no tail of X can have genus equal to $g/2$ or, in other words, that every tail of X is either small or large.

REMARK 4.2. Let r be a separating node of X that generates the tails Q and Q' . If $Z \subset X$ is a connected subcurve not containing r , then Z is entirely contained in either Q or Q' .

LEMMA 4.3. Let X be a curve and let Q_1 and Q_2 be two tails of X . Then

$$Q_1 \cup Q_2 = X \quad \text{or} \quad Q_1 \cap Q_2 = \emptyset \quad \text{or} \quad Q_1 \subseteq Q_2 \quad \text{or} \quad Q_2 \subseteq Q_1.$$

Proof. For each $i = 1, 2$, let r_i be the separating node of X generating Q_i . If $r_1 = r_2$, then either $Q_1 = Q_2'$ (and hence $Q_1 \cup Q_2 = X$) or $Q_1 = Q_2$. So we may assume that $r_1 \neq r_2$.

Now $r_1 \notin Q_2$ or $r_1 \notin Q_2'$. Suppose first that $r_1 \notin Q_2$. Since Q_2 is connected, either $Q_2 \subset Q_1$ or $Q_2 \subset Q_1'$ by Remark 4.2. If $Q_2 \subset Q_1'$, then $Q_1 \cap Q_2 \subseteq Q_1 \cap Q_1' = \{r_1\}$ and hence $Q_1 \cap Q_2 = \emptyset$ because $r_1 \notin Q_2$. So either $Q_2 \subset Q_1$ or $Q_1 \cap Q_2 = \emptyset$.

The case where $r_1 \notin Q_2'$ is treated similarly: either $Q_2' \subset Q_1'$, in which case $Q_1 \subset Q_2$; or $Q_1' \cap Q_2' = \emptyset$, in which case $Q_1 \cup Q_2 = X$. \square

LEMMA 4.4. Let X be a curve and Q a tail of X . Then, for any two line bundles L_1 on Q and L_2 on Q' , there is (up to isomorphism) a unique line bundle L on X such that $L|_Q \cong L_1$ and $L|_{Q'} \cong L_2$.

Proof. Let r be the separating node of X to which Q is attached. For each isomorphism $\mu: L_1|_{\{r\}} \rightarrow L_2|_{\{r\}}$, let L be the kernel of the composition

$$\phi_\mu: L_1 \oplus L_2 \rightarrow L_1|_{\{r\}} \oplus L_2|_{\{r\}} \xrightarrow{\tilde{\mu}} L_2|_{\{r\}},$$

where $\tilde{\mu} := (-\mu, 1)$. Because ϕ_μ is surjective, $L \neq L_1 \oplus L_2$. Also, since μ is an isomorphism, $L \neq L_1(-r) \oplus L_2$ and $L \neq L_1 \oplus L_2(-r)$. Since r is a node of X , it follows that L is a line bundle, $L|_Q \cong L_1$, and $L|_{Q'} \cong L_2$.

Conversely, if N is a line bundle on X for which there exist isomorphisms $\lambda_1: N|_Q \rightarrow L_1$ and $\lambda_2: N|_{Q'} \rightarrow L_2$, then N is the kernel of ϕ_μ , where $\mu := \lambda_2|_{\{r\}} \circ \lambda_1^{-1}|_{\{r\}}$.

Finally, if $\mu': L_1|_{\{r\}} \rightarrow L_2|_{\{r\}}$ is another isomorphism, then the kernel of ϕ_μ is carried isomorphically to the kernel of $\phi_{\mu'}$ by the automorphism

$$(a, 1): L_1 \oplus L_2 \rightarrow L_1 \oplus L_2,$$

where a is the unique scalar such that $\mu = a\mu'$. \square

4.5. TWISTERS ON TAILS. Let X be a curve. By Lemma 4.4, for each tail Q of X there is a unique (up to isomorphism) line bundle on X whose restrictions to Q and Q' are $\mathcal{O}_Q(-r)$ and $\mathcal{O}_{Q'}(r)$, where r is the separating node of X generating Q . Denote this bundle by $\mathcal{O}_X(Q)$.

For each formal sum $\sum a_Q Q$ of tails Q with coefficients $a_Q \in \mathbb{Z}$, set

$$\mathcal{O}_X(\sum a_Q Q) := \bigotimes \mathcal{O}_X(Q)^{\otimes a_Q}.$$

If X is a nodal curve and a closed fiber of a regular pencil $f: \mathcal{X} \rightarrow B$, then

$$\mathcal{O}_X(\sum a_Q Q)|_X \cong \mathcal{O}_X(\sum a_Q Q). \quad (22)$$

So *twisters supported on tails do not depend on the chosen regular pencil*. To check (22) it is enough to observe that, for each tail Q of X , since

$$\mathcal{O}_X(Q)|_Q \cong \mathcal{O}_Q(-r) \quad \text{and} \quad \mathcal{O}_X(Q)|_{Q'} \cong \mathcal{O}_{Q'}(r)$$

it follows by Lemma 4.4 that $\mathcal{O}_X(Q)|_X \cong \mathcal{O}_X(Q)$.

To state the main result of this section we need some notation similar to that used in Proposition 3.12. Let $f: \mathcal{X} \rightarrow B$ be a regular pencil and let Z be a subcurve of X , where $X \subset \mathcal{X}$ is a singular fiber of f . Then Z is a divisor of \mathcal{X} . The restriction π_Z of the first projection $\pi: \mathcal{X}_B^2 \rightarrow \mathcal{X}$ over Z is the trivial family

$$\mathcal{X}_B^2 \supset Z \times X \xrightarrow{\pi_Z} Z.$$

Hence, for any other subcurve $Z_1 \subseteq X$, the product $Z \times Z_1$ can be viewed as a Weil divisor of \mathcal{X}_B^2 . Now, since the open subscheme $\dot{\mathcal{X}} \times_B \mathcal{X} \subset \mathcal{X}_B^2$ is regular, the restriction of $Z \times Z_1$ to this subscheme is a Cartier divisor. Let $\mathcal{O}_{\dot{\mathcal{X}} \times_B \mathcal{X}}(Z \times Z_1)$ denote the associated line bundle. Using this notation, we have an explicit description of the map $\alpha_f^1: \dot{\mathcal{X}} \rightarrow P_f^1$ defined in Theorem 3.10.

THEOREM 4.6. *Let $f: \mathcal{X} \rightarrow B$ be a regular pencil of stable curves. Then the line bundle*

$$\mathcal{L}^{(1)} := \mathcal{O}_{\dot{\mathcal{X}} \times_B \mathcal{X}} \left(\Delta + \sum_{\substack{b \in B \\ Q \in \mathcal{Q}(X_b)}} Q \times Q \right)$$

is balanced on $\dot{\pi}: \dot{\mathcal{X}} \times_B \mathcal{X} \rightarrow \dot{\mathcal{X}}$. Furthermore, assume that the fibers of f are 1-general. Then the following two statements hold.

- (i) *The morphism $\alpha_f^1: \dot{\mathcal{X}} \rightarrow P_f^1$ is the moduli map of $\mathcal{L}^{(1)}$.*
- (ii) *If \mathcal{M} is a balanced line bundle on $\dot{\pi}: \dot{\mathcal{X}} \times_B \mathcal{X} \rightarrow \dot{\mathcal{X}}$ having α_f^1 as moduli map, then $\mathcal{M} \cong \mathcal{L}^{(1)}$ up to pullbacks from $\dot{\mathcal{X}}$.*

REMARK 4.7. Except for part (ii), the theorem generalizes to curves that are not 1-general. See Section 5.10 and Remark 5.13.

Proof of Theorem 4.6. The divisor $\sum Q \times Q$ is entirely supported on a union of closed fibers of $\mathcal{X} \times_B \mathcal{X} \rightarrow B$. By Lemma 2.6, the Néron maps of $\mathcal{L}^{(1)}$ and $\mathcal{O}(\Delta)$ are equal. Now, if the fibers of f are 1-general, then α_f^1 agrees with the Néron map of $\mathcal{O}(\Delta)$ on \mathcal{X}_K . On the other hand, if $\mathcal{L}^{(1)}$ is balanced on $\dot{\pi}: \dot{\mathcal{X}} \times_B \mathcal{X} \rightarrow \dot{\mathcal{X}}$, then there is an associated moduli map $\hat{\mu}_{\mathcal{L}^{(1)}}: \dot{\mathcal{X}} \rightarrow P_f^1$ by Section 3.8(B). Since $\hat{\mu}_{\mathcal{L}^{(1)}}$ coincides with the Néron map of $\mathcal{L}^{(1)}$ on \mathcal{X}_K , it follows that $\hat{\mu}_{\mathcal{L}^{(1)}} = \alpha_f^1$. Thus, to prove part (i) it is enough to prove that $\mathcal{L}^{(1)}$ is balanced on $\dot{\pi}$. And to prove part (ii) it is also enough to show that $\mathcal{L}^{(1)}$ is balanced, since P_f^1 is a fine moduli scheme; see Section 3.8(C).

Let us now prove that $\mathcal{L}^{(1)}$ is indeed balanced. We need only check this on each singular fiber of f , whence we may assume f is local. Let X be the closed fiber. It suffices to consider the singular fibers of the first projection

$$\dot{\pi}: \dot{\mathcal{X}} \times_B \mathcal{X} \rightarrow \dot{\mathcal{X}},$$

which are all isomorphic to X . Let $p \in X$ be a nonsingular point, and set $L_p^{(1)} := \mathcal{L}^{(1)}|_{\dot{\pi}^{-1}(p)}$. Then

$$L_p^{(1)} \cong \mathcal{O}_X(p) \otimes \mathcal{O}_X \left(\sum_{\substack{Q \in \mathcal{Q}(X) \\ p \in Q}} Q \right). \quad (23)$$

We conclude by using Lemma 4.9(ii) and observing that, because X is stable, a semibalanced line bundle on X is necessarily balanced; see Remark 3.3(A). \square

The next two lemmas are needed to complete the proof of Theorem 4.6.

LEMMA 4.8. *Let X be a curve and let Z be a connected, proper subcurve. Let*

$$Q_1 \subset Q_2 \subset \cdots \subset Q_{n-1} \subset Q_n \quad (24)$$

be a chain of tails of X , and let r_i be the separating node of X generating Q_i for each $i = 1, \dots, n$. Then

$$-1 \leq \deg_Z \mathcal{O}_X \left(\sum Q_i \right) \leq 1.$$

Furthermore, the extremes are attained if and only if there is a unique j such that $r_j \in Z \cap Z'$. In this case, the lower bound is attained if $Z \subseteq Q_j$ and the upper bound is attained if $Z \subseteq Q_j'$.

Proof. If $r_\ell \notin Z$ for any $\ell = 1, \dots, n$, then $\deg_Z \mathcal{O}_X \left(\sum Q_i \right) = 0$. Suppose now that Z contains at least one r_ℓ . Let i and j be the smallest and greatest integers such that $r_i \in Z$ and $r_j \in Z$, respectively. Since Z is connected, Z also contains all the irreducible components of X containing r_{i+1}, \dots, r_{j-1} . In particular, $r_\ell \notin Z \cap Z'$ for any $\ell = i+1, \dots, j-1$.

If $Z \cap Z'$ contains both r_i and r_j or neither of them, then $\deg_Z \mathcal{O}_X \left(\sum Q_i \right) = 0$. If $Z \cap Z'$ contains r_i but not r_j , then $\deg_Z \mathcal{O}_X \left(\sum Q_i \right) = 1$ and $Z \subseteq Q_i'$. At last, if $Z \cap Z'$ contains r_j but not r_i , then $\deg_Z \mathcal{O}_X \left(\sum Q_i \right) = -1$ and $Z \subseteq Q_j$. \square

LEMMA 4.9. *Let X be a semistable curve and p a nonsingular point. Then the following two statements hold.*

- (i) The line bundle $\mathcal{O}_X(p)$ is semibalanced if and only if p does not belong to any small tail of X .
(ii) The line bundle

$$L_p^{(1)} := \mathcal{O}_X(p) \otimes \mathcal{O}_X \left(\sum_{\substack{Q \in \mathcal{Q}(X) \\ p \in Q}} Q \right)$$

is semibalanced.

Proof. The “if” part of (i) is a consequence of (ii), since the sum of tails in (ii) is zero when p does not belong to any small tail. As for the “only if” part, recall from Section 3.1 that

$$w_Z = 2g_Z - 2 + k_Z \quad (25)$$

for every connected proper subcurve $Z \subset X$. In particular,

$$w_Q < g - 1 \quad \text{for every small tail } Q \text{ of } X. \quad (26)$$

Therefore, if p is contained in a small tail Q then

$$M_Q(1) < 1 = \deg_Q \mathcal{O}_X(p).$$

Hence the Basic Inequality (11) is not satisfied for Q , so $\mathcal{O}_X(p)$ is not semibalanced.

We need only prove (ii) now. First, since X is semistable, it follows that $w_Z \geq 0$ for every subcurve $Z \subseteq X$; see Section 3.1. As a consequence,

$$w_{Z_1} \leq w_{Z_2} \quad \text{for all subcurves } Z_1 \text{ and } Z_2 \text{ of } X \text{ with } Z_1 \subseteq Z_2. \quad (27)$$

Let Q_1, \dots, Q_n be the small tails of X containing p , and let r_1, \dots, r_n be their generating nodes. Since $w_{Q_i} + w_{Q_j} < 2g - 2$ by (26), we have $Q_i \cup Q_j \neq X$ for each i and j . By Lemma 4.3, we may (up to reordering) assume that

$$Q_1 \subset Q_2 \subset \dots \subset Q_{n-1} \subset Q_n.$$

Let $N := \mathcal{O}_X(\sum_1^n Q_i)$; so $L_p^{(1)} = \mathcal{O}_X(p) \otimes N$. Let Z be any connected, proper subcurve of X . Then $\deg_Z N \geq -1$ by Lemma 4.8 and hence $\deg_Z L_p^{(1)} \geq -1$. As pointed out in Remark 3.3(C), we need only show that $\deg_Z L_p^{(1)} \geq m_Z(1)$ and that $\deg_Z L_p^{(1)} \geq 0$ if Z is exceptional.

First, suppose $\deg_Z N = -1$. By Lemma 4.8, there is j such that $r_j \in Z$ and $Z \subseteq Q_j$. By (26) and (27),

$$w_Z \leq w_{Q_j} < g - 1 \quad (28)$$

and hence $m_Z(1) \leq 0$. Thus, if $p \in Z$,

$$\deg_Z L_p^{(1)} = 0 \geq m_Z(1).$$

Suppose $p \notin Z$; then $Z \neq Q_j$. Since $r_j \in Z$, either $k_Z \geq 3$ or Z is a tail of Q_j . Now, if Z were a tail of Q_j , then $\overline{Q_j} \setminus \overline{Z}$ would be a tail of X contained in Q_j and thus be a small tail. Since $p \in \overline{Q_j} \setminus \overline{Z}$, we would have $\overline{Q_j} \setminus \overline{Z} = Q_i$ for some $i < j$, or $Z = \overline{Q_j} \setminus \overline{Q_i}$. But then $\deg_Z N = 0$, a contradiction. Therefore, $k_Z \geq 3$. In particular, Z is not an exceptional component of X . It follows now from (28) that $m_Z(1) < -1$ and hence

$$\deg_Z L_p^{(1)} \geq -1 > m_Z(1).$$

Second, suppose $\deg_Z N \geq 0$. Then $\deg_Z L_p^{(1)} \geq 0$. By (27) we have that $w_Z \leq w_X = 2g - 2$. Hence, if Z is not a large tail of X , then $m_Z(1) \leq 0$ and so

$$\deg_Z L_p^{(1)} \geq 0 \geq m_Z(1).$$

On the other hand, suppose that Z is a large tail. In any case, $m_Z(1) \leq 1/2$. Thus, if $p \in Z$,

$$\deg_Z L_p^{(1)} \geq 1 \geq 1/2 \geq m_Z(1). \quad (29)$$

Finally, if $p \notin Z$ then p lies on Z' , which is a small tail of X . Thus $Z' = Q_j$ for some j and hence $Z = Q'_j$. It follows that $\deg_Z N = 1$, so (29) holds as well.

Let X be a 1-general stable curve, and let $\dot{X} := X \setminus X_{\text{sing}}$. For any regular smoothing f of X , let

$$\alpha_X^1 := \alpha_f^1|_{\dot{X}}: \dot{X} \rightarrow P_X^1.$$

The notation is not ambiguous by the following consequence of Theorem 4.6.

COROLLARY 4.10. *Let X be a 1-general stable curve. Then α_X^1 does not depend on f . In fact, for each nonsingular point $p \in X$ we have*

$$\alpha_X^1(p) = \mathcal{O}_X(p) \otimes \mathcal{O}_X \left(\sum_{\substack{Q \in \mathcal{Q}(X) \\ p \in Q}} Q \right).$$

Proof. The expression for $\alpha_X^1(p)$ follows from (23) in the proof of Theorem 4.6. By Section 4.5, the map α_X^1 does not depend on f . \square

If X is free from separating nodes then α_X^1 is injective. This follows immediately from Lemma 4.13, which will also be used in the proof of Proposition 5.9, a more general and precise statement. For the lemma and the proposition, the following definition is used.

DEFINITION 4.11. Let X be a curve. A rational, smooth component C of X is called a *separating line* if C intersects $\overline{X \setminus C}$ in separating nodes of X . More generally, a connected subcurve $Z \subseteq X$ of arithmetic genus 0 is called a *separating tree of lines* if Z intersects $\overline{X \setminus Z}$ in separating nodes of X .

4.12. Let X be a curve, and let $Z \subsetneq X$ be a proper connected subcurve such that Z intersects $\overline{X \setminus Z}$ in separating nodes of X . Then the connected components of $\overline{X \setminus Z}$ are tails of X . In addition, if r is a separating node of Z then r is a separating node of X .

A curve of arithmetic genus 0 is a curve of compact type (i.e., a nodal curve with every node separating) whose irreducible components are smooth and rational. So, if Z is a separating tree of lines, then every node of Z is a separating node of Z and hence of X . It follows that every connected subcurve of Z is also a separating tree of lines. In particular, every irreducible component of Z is a separating line.

We shall later need the following lemma.

LEMMA 4.13. *Let X be a curve, and let p and q be distinct nonsingular points of X . Let $C \subseteq X$ be the irreducible component containing p . Then there is an isomorphism $\mathcal{O}_X(p) \cong \mathcal{O}_X(q)$ if and only if C contains q and is a separating line of X .*

Proof. Assume first that C contains q and is a separating line of X . Since C is smooth and rational, $\mathcal{O}_C(p) \cong \mathcal{O}_C(q)$. We may thus assume $C \neq X$. Since C meets $C' := \overline{X \setminus C}$ in separating nodes, we can apply Lemma 4.4 a few times to show that a line bundle on X is uniquely determined by its restrictions to C and to C' . Since $\mathcal{O}_X(p)$ and $\mathcal{O}_X(q)$ restrict to isomorphic line bundles on C and to the trivial line bundle on C' , it follows that $\mathcal{O}_X(p) \cong \mathcal{O}_X(q)$.

Conversely, suppose $\mathcal{O}_X(p) \cong \mathcal{O}_X(q)$. Since $\mathcal{O}_X(p)$ has degree 1 on C , so has $\mathcal{O}_X(q)$ and hence $q \in C$ as well. Now, since $\mathcal{O}_X(p) \cong \mathcal{O}_X(q)$, in particular we have $\mathcal{O}_C(p) \cong \mathcal{O}_C(q)$. Since $p \neq q$, it follows from [AK, Thm. 8.8, p. 108] that $C \cong \mathbb{P}^1$.

If $C = X$ then we are done. So suppose that $C \neq X$, and let $C' := \overline{X \setminus C}$. Also, suppose by contradiction that $C \cap C'$ is not made of separating nodes of X . Then there is a connected subcurve $Z \subseteq C'$ such that $C \cap Z$ is a scheme of length at least 2.

Since X is connected, the restriction $\tau: H^0(X, \mathcal{O}_X(p)) \rightarrow H^0(C, \mathcal{O}_C(p))$ is injective—but it is not surjective. Indeed, if a nonconstant $\sigma \in H^0(C, \mathcal{O}_C(p))$ could be extended to $\tilde{\sigma} \in H^0(X, \mathcal{O}_X(p))$, then $\tilde{\sigma}$ would have to be constant on Z and hence σ would be constant on $C \cap Z$. This is impossible because $C \cong \mathbb{P}^1$ and because σ is a nonconstant section of $\mathcal{O}_{\mathbb{P}^1}(1)$. So τ is injective but not surjective, and hence $h^0(X, \mathcal{O}_X(p)) = 1$. Since $\mathcal{O}_X(p) \cong \mathcal{O}_X(q)$, it follows that $p = q$, an absurdity. \square

5. Completing the First Abel Map

The main result of this section is Theorem 5.5. We shall prove it first in a simpler case in Proposition 5.2, where we have a neater statement concerning the modularity (see Remark 5.3).

As in Section 4, certain basic results of this section hold in more generality for curves having singularities other than nodes. Apart from the notation set in Section 5.1 below, these results are concentrated in Section 5.4.

5.1. Fix a nodal curve X . For each node $r \in X$, let $\nu_r: X_r^\nu \rightarrow X$ be the partial normalization of X at r , and denote by \hat{X}_r the connected nodal curve obtained by adding to X_r^ν a smooth rational curve E_r connecting the two points of $\nu_r^{-1}(r)$. Thus,

$$\hat{X}_r = X_r^\nu \cup E_r$$

and there is a natural surjection $\sigma_r: \hat{X}_r \rightarrow X$ such that $\sigma_r(E_r) = \{r\}$ and $(\sigma_r)|_{X_r^\nu} = \nu_r$ (so that σ_r is an isomorphism away from E_r). The nodal curve \hat{X}_r will be considered up to those automorphisms of E_r that fix the two attaching points $E_r \cap X_r^\nu$.

Assume now that X is a 1-general stable curve, and let r be a node of X . Let $\bar{r} \in E_r$ be any point distinct from the attaching points. If X has no separating

nodes, then the line bundle $\mathcal{O}_{\hat{X}_r}(\bar{r}) \in \text{Pic}^1 \hat{X}_r$ is balanced (by Lemma 4.9) and hence determines a point of $\overline{P}_X^1 \setminus P_X^1$; see Section 3.8(B). This point does not depend on the choice of \bar{r} because, in any case, the restriction of $\mathcal{O}_{\hat{X}_r}(\bar{r})$ to X_r^\vee is trivial; see Section 3.7. We shall therefore denote it by ℓ_r .

PROPOSITION 5.2. *Let $f: \mathcal{X} \rightarrow B$ be a regular pencil of 1-general stable curves that are free from separating nodes. Then $\alpha_f^1: \dot{\mathcal{X}} \rightarrow P_f^1$ extends to an injection*

$$\overline{\alpha_f^1}: \mathcal{X} \rightarrow \overline{P_f^1}$$

such that $\overline{\alpha_f^1}(r) = \ell_r \in \overline{P}_X^1$ for each node r of each closed fiber X of f .

REMARK 5.3. More precisely, the proof will show that $\overline{\alpha_f^1}$ is the moduli map of the line bundle $\mathcal{O}_{\mathcal{Y}}(\tilde{\Delta})$, where $\mathcal{Y} \rightarrow \mathcal{X}_B^2$ is a partial resolution of singularities and $\tilde{\Delta}$ is the proper transform in \mathcal{Y} of the diagonal Δ ; see Section 5.4.

Proof of Proposition 5.2. Denote by $\rho: \mathcal{Y} \rightarrow \mathcal{X}_B^2$ the partial resolution of singularities described in Section 5.4, from where we take some of the properties mentioned here. The map ρ is an isomorphism away from the points (r, r) for $r \in \mathcal{X} \setminus \dot{\mathcal{X}}$. On the other hand, if $r \in \mathcal{X} \setminus \dot{\mathcal{X}}$ then $\rho^{-1}(r, r)$ is a copy of \mathbb{P}^1 . In addition, composing ρ with the first projection π ,

$$\mathcal{Y} \xrightarrow{\rho} \mathcal{X}_B^2 \xrightarrow{\pi} \mathcal{X},$$

we obtain a family of quasistable curves $\mathcal{Y} \rightarrow \mathcal{X}$ having $\pi: \mathcal{X}_B^2 \rightarrow \mathcal{X}$ as a stable model.

For each closed fiber X of f and each $r \in X_{\text{sing}} \subset \mathcal{X}$, let Y_r be the fiber of $\pi \circ \rho$ over r . Then $Y_r = \hat{X}_r$, where \hat{X}_r is as defined in Section 5.1. On the other hand, each fiber of $\pi \circ \rho$ over $\dot{\mathcal{X}}$ is the same as the corresponding fiber of π .

Let $\tilde{\Delta} \subset \mathcal{Y}$ be the proper transform of Δ . By property 5.4(ii), the map ρ restricts to an isomorphism between $\tilde{\Delta}$ and Δ . Also, $\tilde{\Delta}$ meets each fiber $Y_r = \hat{X}_r$ of $\pi \circ \rho$ over any $r \in \mathcal{X} \setminus \dot{\mathcal{X}}$ transversally at a nonsingular point \bar{r} contained in the exceptional component E_r .

To prove that α_f^1 extends, we prove two claims: first, that $\mathcal{O}_{\mathcal{Y}}(\tilde{\Delta})$ is balanced on $\pi \circ \rho: \mathcal{Y} \rightarrow \mathcal{X}$ and thus induces a morphism $\overline{\alpha_f^1}: \mathcal{X} \rightarrow \overline{P_f^1}$, its moduli map; and second (to show that $\overline{\alpha_f^1}$ extends α_f^1) that the restriction of $\mathcal{O}_{\mathcal{Y}}(\tilde{\Delta})$ to each fiber of $\pi \circ \rho$ over $\dot{\mathcal{X}}$ is isomorphic to the corresponding restriction of $\mathcal{L}^{(1)}$, whose moduli map is α_f^1 by Theorem 4.6.

We may now assume that f is local. Let X be its closed fiber. Since $\tilde{\Delta}$ intersects Y_r transversally at \bar{r} , for each $r \in X_{\text{sing}}$ we have

$$\mathcal{O}_{\mathcal{Y}}(\tilde{\Delta})|_{Y_r} \cong \mathcal{O}_{\hat{X}_r}(\bar{r}), \quad (30)$$

which is balanced by Lemma 4.9. In addition, for each nonsingular point $p \in X$ we have

$$\mathcal{O}_{\mathcal{Y}}(\tilde{\Delta})|_{Y_p} \cong \mathcal{O}_X(p),$$

which (also by Lemma 4.9) is balanced and isomorphic to the corresponding restriction of $\mathcal{L}^{(1)}$. Therefore, $\mathcal{O}_{\mathcal{Y}}(\tilde{\Delta})$ induces a moduli map

$$\overline{\alpha_f^1}: \mathcal{X} \rightarrow \overline{P_f^1},$$

which extends α_f^1 . Observe that (30) also shows that $\overline{\alpha_f^1}(r) = \ell_r$ for each $r \in X_{\text{sing}}$.

To show that $\overline{\alpha_f^1}$ is injective, it suffices to consider singular points of X —by Lemma 4.13 and the fact that $\overline{\alpha_f^1}(r) \in \overline{P_X^1} \setminus P_X^1$ for every node $r \in X$. Now, if $r \in X_{\text{sing}}$ then $\overline{\alpha_f^1}(r)$ represents a balanced line bundle on \hat{X}_r . Hence, two different nodes r and r' of X are mapped to two points of $\overline{P_X^1}$ corresponding to balanced line bundles on different quasistable curves, \hat{X}_r and $\hat{X}_{r'}$. Therefore $\overline{\alpha_f^1}(r) \neq \overline{\alpha_f^1}(r')$; see Section 3.7. \square

5.4. RESOLUTION OF SINGULARITIES. In the proof of Proposition 5.2 we used a partial resolution of singularities of \mathcal{X}_B^2 , which we shall now describe in detail and in more generality.

Let $f: \mathcal{X} \rightarrow B$ be a regular pencil. The 3-fold \mathcal{X}_B^2 is singular at the points (r_1, r_2) , where r_1 and r_2 are (not necessarily distinct) singular points of the same closed fiber of f .

Let X be a closed fiber of f , and let r_1 and r_2 be nodes of X . Because f is regular, locally around r_i the surface \mathcal{X} is formally equivalent to the surface in \mathbb{A}^3 given by the equation $x_i y_i = t$, where t denotes a local parameter of B at the closed point covered by X . Pulling back these local equations to \mathcal{X}_B^2 under the two projection maps $\mathcal{X}_B^2 \rightarrow \mathcal{X}$ and then abusing the same notation, we obtain that \mathcal{X}_B^2 is formally equivalent, locally around (r_1, r_2) , to the 3-fold in \mathbb{A}^5 given the equations

$$x_1 y_1 = t,$$

$$x_2 y_2 = t.$$

If $r_1 = r_2$, then the diagonal $\Delta \subset \mathcal{X}_B^2$ contains (r_1, r_2) , and we may assume that it is given locally around (r_1, r_2) by

$$x_1 y_1 = t,$$

$$x_2 = x_1,$$

$$y_2 = y_1.$$

Locally around (r_1, r_2) , we may eliminate t and view \mathcal{X}_B^2 as the cone $C \subset \mathbb{A}^4$ over the smooth quadric in \mathbb{P}^3 given by $x_1 y_1 = x_2 y_2$. Also, if $r_1 = r_2$ then we may view Δ as the plane $D \subset \mathbb{A}^4$ given by $x_2 = x_1$ and $y_2 = y_1$. Notice that C is singular only at the origin. To resolve this singularity we need only blow up a plane in C containing the origin. Any plane will do, but let us blow up the plane given by $x_1 = x_2 = 0$. The blowup is the nonsingular 3-fold $\tilde{C} \subset \mathbb{P}^1 \times \mathbb{A}^4$ given by the equations

$$\xi_2 x_1 = \xi_1 x_2,$$

$$\xi_1 y_1 = \xi_2 y_2,$$

where ξ_1, ξ_2 are homogeneous coordinates of \mathbb{P}^1 . The blowup $\gamma: \tilde{C} \rightarrow C$ is isomorphic to C away from the origin. In addition, the fiber F over the origin is given by $x_1 = x_2 = y_1 = y_2 = 0$ and hence is isomorphic to \mathbb{P}^1 .

The exceptional divisor E of the blowup \tilde{C} is given by $x_2 = 0$ where $\xi_2 \neq 0$ and by $x_1 = 0$ where $\xi_1 \neq 0$. In particular, $F \subset E$. Now, since $\xi_2 x_1 = \xi_1 x_2$, summing the divisor given by $\xi_1 = 0$ to E yields the principal divisor given by $x_1 = 0$. Thus $E \cdot F = -1$.

Suppose $r_1 = r_2$. Then $\gamma^{-1}(D)$ is given by $x_1(\xi_1 - \xi_2) = y_2(\xi_1 - \xi_2) = 0$ where $\xi_1 \neq 0$ and by $x_2(\xi_1 - \xi_2) = y_1(\xi_1 - \xi_2) = 0$ where $\xi_2 \neq 0$. Hence $\gamma^{-1}(D)$ is the union of the Cartier divisor given by $\xi_1 = \xi_2$ and the fiber F . The strict transform \tilde{D} of D is thus a Cartier divisor intersecting F transversally at a point.

For $i = 1, 2$, let $\phi_i: \tilde{C} \rightarrow \mathbb{A}^2$ be the composition of γ with the projection onto the plane having coordinates x_i, y_i . The fiber of ϕ_1 over the origin is given by $x_1 = y_1 = \xi_1 x_2 = \xi_2 y_2 = 0$. It is the union of F and the affine lines N_1 , given by $x_1 = y_1 = \xi_1 = y_2 = 0$, and N_2 , given by $x_1 = y_1 = \xi_2 = x_2 = 0$. The lines N_1 and N_2 do not meet, and F intersects each N_i transversally at a single point. Also, ϕ_2 maps N_1 and N_2 isomorphically onto the lines $y_2 = 0$ and $x_2 = 0$, respectively.

The exceptional divisor E contains N_2 and intersects N_1 transversally. Because $\xi_1 \neq 0$ on N_2 , we have $E \cdot N_2 = 0$. If $r_1 = r_2$, then the strict transform \tilde{D} meets neither N_1 nor N_2 , and it intersects F transversally. (An analogous description holds if we reverse the roles of ϕ_1 and ϕ_2 .)

We now consider the global picture. Let \mathcal{I}_Δ denote the ideal sheaf of the diagonal $\Delta \subset \mathcal{X}_B^2$, and let $\check{\mathcal{I}}_\Delta$ denote the dual sheaf:

$$\check{\mathcal{I}}_\Delta := \text{Hom}(\mathcal{I}_\Delta, \mathcal{O}_{\mathcal{X}_B^2}).$$

Since \mathcal{I}_Δ is a sheaf of ideals, it follows that $\check{\mathcal{I}}_\Delta$ is a sheaf of fractional ideals of \mathcal{X}_B^2 . A piece of notation: For each open subscheme $U \subseteq \mathcal{X}_B^2$ and each sheaf of fractional ideals \mathcal{M} of U , consider its powers \mathcal{M}^n and form the associated sheaf of Rees algebras,

$$\mathcal{R}(\mathcal{M}) := \mathcal{O}_U \oplus \mathcal{M} \oplus \mathcal{M}^2 \oplus \cdots \oplus \mathcal{M}^n \oplus \cdots.$$

Set $\mathcal{Y} := \text{Proj}_{\mathcal{X}_B^2}(\mathcal{R}(\check{\mathcal{I}}_\Delta))$, and let $\rho: \mathcal{Y} \rightarrow \mathcal{X}_B^2$ be the structure map.

We may view ρ as a blowup. In fact, for any open subscheme $U \subseteq \mathcal{X}_B^2$ over which there is an embedding $\iota: \check{\mathcal{I}}_\Delta|_U \rightarrow \mathcal{L}$ into an invertible sheaf \mathcal{L} , we may view $\rho: \rho^{-1}(U) \rightarrow U$ as the blowup of U along the closed subscheme $V \subseteq U$ whose sheaf of ideals $\mathcal{I}_V|_U$ satisfies $\iota(\check{\mathcal{I}}_\Delta|_U) = \mathcal{I}_V|_U \mathcal{L}$. In other words, ι induces an isomorphism over U :

$$\text{Proj}_U(\mathcal{R}(\mathcal{I}_V|_U)) \longrightarrow \text{Proj}_U(\mathcal{R}(\check{\mathcal{I}}_\Delta)|_U).$$

In the same vein, for each invertible sheaf of ideals $\mathcal{J} \subseteq \mathcal{O}_U$ we have that $\text{Hom}(\mathcal{I}_\Delta|_U, \mathcal{J}) = \check{\mathcal{I}}_\Delta|_U \mathcal{J}$, and thus we obtain a canonical isomorphism over U :

$$\text{Proj}_U(\mathcal{R}(\check{\mathcal{I}}_\Delta)|_U) \longrightarrow \text{Proj}_U(\mathcal{R}(\text{Hom}(\mathcal{I}_\Delta|_U, \mathcal{J}))).$$

Since \mathcal{I}_Δ is invertible away from the points (r, r) for $r \in \mathcal{X} \setminus \check{\mathcal{X}}$, it follows from the preceding description that ρ is an isomorphism away from these same points.

Around the points (r, r) , where r is a node of a closed fiber of f , the map ρ is also formally equivalent to the blowup just described because

$$\mathrm{Hom}_A((x_1 - x_2, y_1 - y_2), (x_1 - x_2)) = (x_1, x_2),$$

where $A := k[[x_1, x_2, y_1, y_2]]/(x_1 y_1 - x_2 y_2)$. Then all the foregoing properties, verified locally, yield global properties of ρ . Indeed, assume that the fibers of f are nodal. (It would actually be enough to assume that the fibers are Gorenstein.) Then, recalling that $\pi: \mathcal{X}_B^2 \rightarrow \mathcal{X}$ denotes the first projection, we have the following properties.

(i) The composition

$$\pi \circ \rho: \mathcal{Y} \rightarrow \mathcal{X}$$

is a family of curves whose fiber Y_r over a point r of a closed fiber X of f is X if r is nonsingular and is \hat{X}_r (as described in Section 5.1) if r is a node.

(ii) Let $\tilde{\Delta} \subset \mathcal{Y}$ denote the proper transform of Δ . Then, for each node r of each closed fiber X of f , the transform $\tilde{\Delta}$ intersects the fiber Y_r transversally at a point lying in the exceptional component $E_r = \rho^{-1}(r, r)$.

(iii) Let Q be a tail of a closed fiber X of f , and let r be the node of X generating Q . Let

$$\tilde{Q}^2 := \rho^{-1}(Q \times Q).$$

Then \tilde{Q}^2 is a Cartier divisor of \mathcal{Y} containing E_r . Furthermore,

$$\tilde{Q}^2 \cdot E_r = -1, \quad \tilde{Q}^2 \cdot \hat{Q} = -1, \quad \text{and} \quad \tilde{Q}^2 \cdot \hat{Q}' = 1.$$

Here (using the notation of Section 5.1) $\hat{Q} := \sigma_r^{-1}(Q)$ and $\hat{Q}' := \overline{\hat{X}_r \setminus \hat{Q}}$; that is, \hat{Q} is the tail of \hat{X}_r mapping to Q and containing E_r , and \hat{Q}' is the complementary tail.

We may now generalize Proposition 5.2.

THEOREM 5.5. *Let $f: \mathcal{X} \rightarrow B$ be a regular pencil of 1-general stable curves. Then there exists a morphism*

$$\overline{\alpha}_f^1: \mathcal{X} \rightarrow \overline{P}_f^1$$

extending $\alpha_f^1: \dot{\mathcal{X}} \rightarrow P_f^1$. If r is a node of a closed fiber X of f , then $\overline{\alpha}_f^1(r) \in P_X^1$ if and only if r is a separating node of X .

REMARK 5.6. The result extends to curves that are not 1-general. See Section 5.10 and Remark 5.13.

Proof of Theorem 5.5. As in the proof of Theorem 3.10, we may work locally around each singular fiber. So assume that f is local and let X denote its closed fiber.

The new difficulty with respect to Proposition 5.2 is that, if X has separating nodes, then α_f^1 is the moduli map of a nontrivial twist of the diagonal (by Theorem 4.6) and hence the same must hold for its completion. Fortunately, however, the divisors we need for the twist are already present in the partial resolution of singularities $\rho: \mathcal{Y} \rightarrow \mathcal{X}_B^2$ described in Section 5.4.

Namely, let Q_1, \dots, Q_m be all the small tails of X . Let $\tilde{\Delta} \subset \mathcal{Y}$ be the strict transform of Δ , and set $\tilde{Q}_i^2 := \rho^{-1}(Q_i \times Q_i)$ for $i = 1, \dots, m$. As seen in Section 5.4, all the \tilde{Q}_i^2 and $\tilde{\Delta}$ are Cartier divisors. Define the line bundle

$$\mathcal{M} := \mathcal{O}_{\mathcal{Y}}(\tilde{\Delta} + \tilde{Q}_1^2 + \dots + \tilde{Q}_m^2). \quad (31)$$

We claim that \mathcal{M} is semibalanced on the composition $\pi \circ \rho: \mathcal{Y} \rightarrow \mathcal{X}$ of ρ with the first projection $\pi: \mathcal{X}_B^2 \rightarrow \mathcal{X}$. Once the claim is proved, we may let $\alpha_f^1: \mathcal{X} \rightarrow \overline{P}_f^1$ be the moduli map of \mathcal{M} ; see Section 3.8(B).

To prove the claim, first observe that ρ is an isomorphism over $\dot{\mathcal{X}} \times_B \mathcal{X}$; whence

$$\mathcal{M}|_{\dot{\mathcal{X}} \times_B \mathcal{X}} \cong \mathcal{O}_{\dot{\mathcal{X}} \times_B \mathcal{X}}(\Delta + Q_1 \times Q_1 + \dots + Q_m \times Q_m),$$

which is balanced (by Theorem 4.6) and defines α_f^1 . As a result, once \mathcal{M} is shown to be semibalanced we have $\alpha_f^1|_{\dot{\mathcal{X}}} = \alpha_f^1$.

Now let $r \in X_{\text{sing}}$. The fiber $Y_r := (\pi \circ \rho)^{-1}(r)$ is equal to \hat{X}_r by property 5.4(i). Also, $\tilde{\Delta}$ intersects Y_r transversally at a point \bar{r} of the exceptional component $E_r = \rho^{-1}(r, r)$ by property 5.4(ii).

For each $i = 1, \dots, m$, let r_i be the separating node of X generating Q_i . Let

$$\hat{Q}_i := \sigma_r^{-1}(Q_i) \subset \hat{X}_r,$$

and let \hat{Q}_i' be its complement in \hat{X}_r . Then \hat{Q}_i is a small tail of \hat{X}_r dominating Q_i , and \hat{Q}_i contains E_r if and only if $r \in Q_i$. If $r = r_i$ then also $\hat{Q}_i \setminus E_r$ is a small tail of \hat{X}_r . These are all the small tails of \hat{X}_r : the subcurves $\hat{Q}_1, \dots, \hat{Q}_m$ together with $\hat{Q}_i \setminus E_r$ in case $r = r_i$.

For each $i = 1, \dots, m$, the subscheme $Q_i \times Q_i \subset \mathcal{X}_B^2$ is a Cartier divisor away from (r_i, r_i) . Identifying Y_r with \hat{X}_r , we claim that

$$\mathcal{O}_{\mathcal{Y}}(\tilde{Q}_i^2)|_{Y_r} \cong \begin{cases} \mathcal{O}_{\hat{X}_r} & \text{if } r \notin Q_i, \\ \mathcal{O}_{\hat{X}_r}(\hat{Q}_i) & \text{if } r \in Q_i. \end{cases} \quad (32)$$

In fact, if $r \notin Q_i$ then \tilde{Q}_i^2 does not meet Y_r and hence (32) holds. Suppose now that $r \in Q_i$, and recall that \hat{Q}_i is a tail of \hat{X}_r . Let s_i denote the generating node of \hat{Q}_i . If $r \neq r_i$ then, since $Q_i \times Q_i$ is a Cartier divisor of \mathcal{X}_B^2 at (r, r_i) , we have

$$\mathcal{O}_{\mathcal{Y}}(\tilde{Q}_i^2)|_{\hat{Q}_i} \cong \mathcal{O}_{\hat{Q}_i}(-s_i) \quad \text{and} \quad \mathcal{O}_{\mathcal{Y}}(\tilde{Q}_i^2)|_{\hat{Q}_i'} \cong \mathcal{O}_{\hat{Q}_i'}(s_i). \quad (33)$$

The same restrictions are achieved with $\mathcal{O}_{\hat{X}_r}(\hat{Q}_i)$. Thus (32) follows from Lemma 4.4. Finally, if $r = r_i$ then (33) still holds by property 5.4(iii) and hence (32) follows in the same way. The proof of (32) is complete.

Observe that Q_i contains r if and only if \hat{Q}_i contains \bar{r} . In addition, if $r = r_i$ then $\bar{r} \notin \hat{Q}_i \setminus E_r$. Since $\hat{Q}_1, \dots, \hat{Q}_m$ (together with $\hat{Q}_i \setminus E_r$ if $r = r_i$) are all the small tails of \hat{X}_r , it follows from (32) that

$$\mathcal{M}|_{Y_r} \cong \mathcal{O}_{\hat{X}_r}(\bar{r}) \otimes \mathcal{O}_{\hat{X}_r} \left(\sum_{\substack{Q \in \mathcal{Q}(\hat{X}_r) \\ \bar{r} \in Q}} Q \right),$$

which is semibalanced by Lemma 4.9. Our claim is proved, which finishes our proof of the existence of α_f^1 .

To prove the second statement of the theorem, it suffices to show that, for any node $r \in X$,

$$\deg_{E_r} \mathcal{M} = \begin{cases} 1 & \text{if } r \text{ is not separating,} \\ 0 & \text{otherwise.} \end{cases}$$

To prove this, notice that if $r \neq r_i$ then $\tilde{Q}_i^2 \cdot E_r = 0$, whereas if $r = r_i$ then $\tilde{Q}_i^2 \cdot E_r = -1$ by property 5.4(iii). Since at any rate $\tilde{\Delta} \cdot E_r = 1$, the degree of $\mathcal{M}|_{E_r}$ is 1 unless $r = r_i$ for some i , in which case the degree is 0. Since X is 1-general, each separating node of X generates a small tail and so is equal to r_i for some i . Therefore, $\alpha_f^1(r) \in P_X^1$ if and only if r is a separating node. \square

EXAMPLE 5.7. Let X be a curve of compact type with two components, C_1 and C_2 . Then C_1 and C_2 are smooth and $C_1 \cap C_2 = \{r\}$, where r is the unique node of X . Assume $g_{C_1} < g_{C_2}$. Then $\hat{X}_r = C_1 \cup E \cup C_2$ and $\mathcal{Q}(\hat{X}_r) = \{C_1, C_1 \cup E\}$, where $E = \mathbb{P}^1$. The line bundle \mathcal{M} in the proof of Theorem 5.5, whose moduli map is α_f^1 , satisfies

$$\mathcal{M} = \mathcal{O}_{\mathcal{Y}}(\tilde{\Delta} + \tilde{C}_1^2).$$

In this case, it is easy to describe the completed Abel map. First notice that, essentially by Lemma 4.4, there is a canonical isomorphism $P_X^1 \cong \text{Pic}^0 C_1 \times \text{Pic}^1 C_2$. Hence, a point $\ell \in \overline{P}_X^1$ is represented by a pair (L_1, L_2) with $L_i \in \text{Pic } C_i$. For $i = 1, 2$, let $q_i \in C_i$ lying above r . Then

$$\overline{\alpha}_f^1(p) = \begin{cases} (\mathcal{O}_{C_1}(p - q_1), \mathcal{O}_{C_2}(q_2)) & \text{if } p \in C_1, \\ (\mathcal{O}_{C_1}, \mathcal{O}_{C_2}(p)) & \text{if } p \in C_2. \end{cases}$$

In particular, $\overline{\alpha}_f^1(r) = (\mathcal{O}_{C_1}, \mathcal{O}_{C_2}(q_2))$. If we now compose $\overline{\alpha}_f^1|_{C_1}$ with the projection $\text{Pic}^0 C_1 \times \text{Pic}^1 C_2 \rightarrow \text{Pic}^0 C_1$, we obtain the classical Abel–Jacobi map of C_1 with base point q_1 :

$$\begin{aligned} C_1 &\longrightarrow \text{Pic}^0 C_1, \\ p &\longmapsto \mathcal{O}_{C_1}(p - q_1). \end{aligned}$$

The analogous composition for C_2 gives the first Abel map $C_2 \rightarrow \text{Pic}^1 C_2$.

Let X be a 1-general stable curve and f a regular smoothing of X . The restriction $\overline{\alpha}_f^1|_X$ coincides with $\alpha_f^1|_X$ and so (by Corollary 4.10) does not depend on f . Hence $\overline{\alpha}_f^1|_X$ does not depend on f either, and we may set $\overline{\alpha}_X^1 := \overline{\alpha}_f^1|_X$.

5.8. EXPLICIT DESCRIPTION OF THE COMPLETE ABEL MAP. Let X be a 1-general stable curve and let $p \in X$. We shall now explicitly describe $\overline{\alpha}_X^1(p)$ in formulas (34) and (35), following the proof of Theorem 5.5.

First some notation. Let P_1, \dots, P_m be all the small tails of X containing p . (The unusual naming of the tails using “ P ” rather than “ Q ” is to match the notation of the proof of Proposition 5.9.) By Lemma 4.3 we can write

$$P_m \subset P_{m-1} \subset \dots \subset P_2 \subset P_1.$$

Set $Z_i := \overline{P_i - P_{i+1}}$ for each $i = 1, \dots, m-1$ and set $Z_m := P_m$, so that $P_1 = \bigcup_1^m Z_i$. Also, put $Q := P'_m$, a large tail of X . Then

$$X = P_1 \cup Q = \bigcup_1^m Z_i \cup Q.$$

Let r_1, \dots, r_m be the separating nodes of X generating P_1, \dots, P_m . Observe that $Z_i \cap Z'_i = \{r_i, r_{i+1}\}$ if $i = 1, \dots, m-1$ and that $Z_m \cap Z'_m = \{r_m\}$. Hence each of the Z_i and Q meets the complementary curve in separating nodes of X . Therefore, by iterated use of Lemma 4.4, to give a line bundle on X it suffices to give its restrictions to all the Z_i and to Q .

We are now ready to describe $\overline{\alpha}_X^1(p)$ if p is a nonsingular point or a separating node of X (in which case, of course, $p = r_m$). Recall that, by Theorem 5.5, $\overline{\alpha}_X^1(p)$ corresponds to a line bundle on X . We have

$$\overline{\alpha}_X^1(p) = \{\mathcal{O}_Q(r_1), \mathcal{O}_{Z_1}(r_2 - r_1), \dots, \mathcal{O}_{Z_{m-1}}(r_m - r_{m-1}), \mathcal{O}_{Z_m}(p - r_m)\}. \quad (34)$$

Suppose that p is a nonseparating node of X . Then we know that $\overline{\alpha}_X^1(p)$ corresponds to a line bundle on \hat{X}_p . Let $E \subset \hat{X}_p$ be the exceptional component of \hat{X}_p , and let \widetilde{Z}_m denote the normalization of Z_m at p only. Retaining our previous notation, we have

$$\hat{X}_p = Q \cup Z_1 \cup \dots \cup Z_{m-1} \cup \widetilde{Z}_m \cup E.$$

Recall from Section 3.7 that $\overline{\alpha}_X^1(p)$ is uniquely determined by a line bundle L , of degree 0, on the complementary curve of E —that is (arguing as before), by the string of the restrictions of L to $Q, Z_1, \dots, Z_{m-1}, \widetilde{Z}_m$. We have

$$\overline{\alpha}_X^1(p) = \{\mathcal{O}_Q(r_1), \mathcal{O}_{Z_1}(r_2 - r_1), \dots, \mathcal{O}_{Z_{m-1}}(r_m - r_{m-1}), \mathcal{O}_{\widetilde{Z}_m}(-r_m)\}. \quad (35)$$

PROPOSITION 5.9. *Let X be a 1-general stable curve, and let p and q be distinct points of X . Then $\overline{\alpha}_X^1(p) = \overline{\alpha}_X^1(q)$ if and only if p and q belong to the same separating tree of lines of X .*

A similar result for the Abel–Jacobi map to the (degree-0) Jacobian is proved by Edixhoven in [Ed, Prop. 9.5]. His statement (necessarily) excludes the case where p or q is a nonseparating node, since there the target space of the map is a non-compactified Néron model.

Proof of Proposition 5.9. Suppose first that p and q belong to a separating tree of lines $Z \subset X$. Since Z is connected, in order to prove that $\overline{\alpha}_X^1(p) = \overline{\alpha}_X^1(q)$ it is enough to consider the case where p and q are nonsingular points of X in the same irreducible component C of Z . We know (see Section 4.12) that C is a separating line of X . Thus $\mathcal{O}_X(p) \cong \mathcal{O}_X(q)$ by Lemma 4.13. Since p and q lie on the same component, it follows that $\alpha_X^1(p) = \alpha_X^1(q)$ and hence $\overline{\alpha}_X^1(p) = \overline{\alpha}_X^1(q)$.

Conversely, suppose $\overline{\alpha}_X^1(p) = \overline{\alpha}_X^1(q)$. We claim that p and q are nonsingular points or separating nodes of X . Indeed, suppose by contradiction (and without loss of generality) that p is a nonseparating node of X . Then $\overline{\alpha}_X^1(p) \in \overline{P}_X^1 \setminus P_X^1$ by Theorem 5.5. Since $\overline{\alpha}_X^1(p) = \overline{\alpha}_X^1(q)$, it then follows from Theorem 5.5 that q is

also a nonseparating node of X . However, $\overline{\alpha_X^1}(p)$ and $\overline{\alpha_X^1}(q)$ correspond to balanced line bundles on different quasistable curves, \hat{X}_p and \hat{X}_q (see Section 5.8), so $\alpha_X^1(p) \neq \alpha_X^1(q)$ (see Section 3.7). The contradiction proves the claim.

Because p and q are nonsingular or separating nodes of X , both $\alpha_X^1(p)$ and $\alpha_X^1(q)$ are line bundles on X . Let P_1, \dots, P_m be the small tails containing p and Q_1, \dots, Q_n the small tails containing q . (We may have $m = 0$ or $n = 0$.) It follows from Lemma 4.3 (as in the proof of Theorem 4.6) that, up to reordering the tails,

$$P_m \subset P_{m-1} \subset \dots \subset P_2 \subset P_1 \quad \text{and} \quad Q_n \subset Q_{n-1} \subset \dots \subset Q_2 \subset Q_1.$$

Set $P_0 := Q_0 := X$. Let r_1, \dots, r_m be the separating nodes of X generating P_1, \dots, P_m and let s_1, \dots, s_n be those generating Q_1, \dots, Q_n . In addition, set $P_{m+1} := Q_{n+1} := \emptyset$, and put $r_{m+1} := p$ and $s_{n+1} := q$.

We may assume $m \leq n$ without loss of generality. Let i be the largest nonnegative integer such that $i \leq m$ and $P_j = Q_j$ for $j = 0, 1, \dots, i$. Then also $r_j = s_j$ for $j = 1, \dots, i$. We claim that $P_{i+1} \cap Q_{i+1} = \emptyset$. Indeed, if $i = m$ then P_{i+1} is already empty. Suppose $i < m$. If $P_{i+1} \subseteq Q_{i+1}$ then Q_{i+1} is a small tail containing p , and since

$$P_{i+1} \subseteq Q_{i+1} \subset Q_i = P_i$$

we have $P_{i+1} = Q_{i+1}$, contradicting the maximality of i . In a similar way, $Q_{i+1} \not\subseteq P_{i+1}$. Since $P_{i+1} \cup Q_{i+1} \neq X$, because P_{i+1} and Q_{i+1} are small tails, it follows from Lemma 4.3 that $P_{i+1} \cap Q_{i+1} = \emptyset$, proving our claim. In particular, $r_{i+1} \neq s_{i+1}$.

As $P_i = Q_i$, we may consider $Y := \overline{P_i} \setminus (P_{i+1} \cup Q_{i+1})$. Because P_{i+1} and Q_{i+1} do not meet, their union cannot be P_i , a connected subcurve of X . Thus Y is a subcurve of X . And Y is connected, since it is either equal to or a tail of $\overline{P_i} \setminus P_{i+1}$, which in turn is either equal to or a tail of $\overline{P_i}$, a tail of X .

Since $Y \subseteq \overline{P_i} \setminus P_{i+1}$, the restriction of $\overline{\alpha_X^1}(p)$ to Y is $\mathcal{O}_Y(r_{i+1} - r_i)$; see Section 5.8. Analogously, $\overline{\alpha_X^1}(q)$ restricts to $\mathcal{O}_Y(s_{i+1} - s_i)$. Since $\overline{\alpha_X^1}(p) = \overline{\alpha_X^1}(q)$ and $r_i = s_i$, it follows that $\mathcal{O}_Y(r_{i+1}) \cong \mathcal{O}_Y(s_{i+1})$. Since $r_{i+1} \neq s_{i+1}$, we see (by Lemma 4.13 applied to the curve Y) that r_{i+1} and s_{i+1} are contained in a separating line C of Y . Since $Y \cap Y'$ is made of separating nodes of X , so is $C \cap C'$; see Section 4.12. In other words, C is a separating line of X .

For $\ell = 1, \dots, m - i$, let $Y_\ell := \overline{P_{i+\ell}} \setminus P_{i+\ell+1}$. Then $\overline{\alpha_X^1}(p)$ restricts to $\mathcal{O}_{Y_\ell}(r_{i+\ell+1} - r_{i+\ell})$. On the other hand, since $Y_\ell \subset \overline{Q_i} \setminus Q_{i+1}$ for each ℓ yet neither $s_i \in Y_\ell$ nor $s_{i+1} \in Y_\ell$, it follows that $\overline{\alpha_X^1}(q)$ restricts to the trivial bundle \mathcal{O}_{Y_ℓ} . Applying Lemma 4.13 to the curve Y_ℓ , we obtain that $r_{i+\ell}$ and $r_{i+\ell+1}$ are contained in a separating line C_ℓ^p of Y_ℓ . As before, C_ℓ^p is also a separating line of X .

Similarly, for each $\ell = 1, \dots, n - i$ the points $s_{i+\ell}$ and $s_{i+\ell+1}$ are contained in a separating line C_ℓ^q of X . The union of all the separating lines,

$$C_{m-i}^p, \dots, C_2^p, C_1^p, C, C_1^q, C_2^q, \dots, C_{n-i}^q,$$

is a separating tree of lines containing p and q . □

To give a finer description of the map $\overline{\alpha_X^1}$ —and, in particular, to characterize when it is a closed embedding—requires additional techniques, such as the theory of theta functions from [E]. We have omitted such considerations in order to keep the paper at a reasonable length.

5.10. CURVES THAT ARE NOT 1-GENERAL. We conclude by discussing the case of stable curves that are not 1-general. Recall that such curves form a proper closed subset of \bar{M}_g , nonempty if and only if g is even; their combinatorial structure is described in Proposition 3.15. What kind of complications occur for curves that are not 1-general or not d -general?

The stack $\bar{\mathcal{P}}_{d,g}$ introduced in Section 3.8 for the case $(d - g + 1, 2g - 2) = 1$ is constructed as a quotient stack; that is, $\bar{\mathcal{P}}_{d,g} := [H_d/G]$, with notation as in Section 3.5. The same definition can be given for each d , and we thus obtain a quotient stack $[H_d/G]$. However, this stack presents some pathologies when non- d -general curves appear.

More precisely, recall from Section 3.5 that the scheme-theoretic quotient H_d/G is endowed with a natural surjective morphism $\phi_d: H_d/G \rightarrow \bar{M}_g$. The open subset of \bar{M}_g over which the quotient map $\pi_d: H_d \rightarrow H_d/G$ is a geometric quotient is exactly the locus of d -general curves. The problem is that, as soon as $\pi_d: H_d \rightarrow H_d/G$ fails to be a geometric quotient, the following pathologies occur:

- (i) $[H_d/G]$ fails to be a Deligne–Mumford stack;
- (ii) the natural map of stacks $[H_d/G] \rightarrow \bar{\mathcal{M}}_g$ fails to be representable; and
- (iii) Néron models are not parameterized by $[H_d/G]$.

However, when studying Abel maps we can still obtain some results. Since the stack $\bar{\mathcal{P}}_{d,g}$ behaves badly, let us consider the scheme $\bar{P}_{d,g} := H_d/G$ introduced in Section 3.5. As already mentioned, there is always a surjective morphism $\phi_d: \bar{P}_{d,g} \rightarrow \bar{M}_g$. By [C1, Thm. 6.1, p. 641], $\bar{P}_{d,g}$ is an integral projective scheme. It is also normal, being a GIT-quotient of H_d , which is nonsingular by [C1, Lemma 2.2, p. 609].

Although $\bar{P}_{d,g}$ is not a coarse moduli space—not even away from curves with nontrivial automorphisms— $\bar{P}_{d,g}$ does satisfy useful functorial properties. So let $f: \mathcal{X} \rightarrow B$ be a regular pencil of stable curves. Let $B \rightarrow \bar{M}_g$ be the associated map, and define

$$\bar{P}_f^d := B \times_{\bar{M}_g} \bar{P}_{d,g}.$$

If X is a closed fiber of f , denote by \bar{P}_X^d the corresponding fiber of \bar{P}_f^d over B . As mentioned before, \bar{P}_f^d may fail to contain the Néron model N_f^d . However, a functorial property holds: the moduli property given in Section 3.8(B) holds exactly as stated. More precisely, to any semibalanced line bundle \mathcal{L} on a family of semistable curves $\mathcal{Y} \rightarrow T$ having $\mathcal{X}_T \rightarrow T$ as a stable model, where T is any B -scheme, we can associate a canonical moduli map $\hat{\mu}_{\mathcal{L}}: T \rightarrow \bar{P}_f^d$ (see [C1, Prop. 8.1, p. 653]).

The main weakness when nongeneral curves are present is that different balanced line bundles on the same quasistable (or even stable) curve may be mapped to the same point in \bar{P}_f^d . We next give an example of this behavior with regard to Abel maps.

EXAMPLE 5.11. Let $X = C_1 \cup C_2$ be a curve of compact type as in Example 5.7. However, assume now that C_1 and C_2 have the same genus. Then X is not 1-general. As before, let r be the unique node of X , and let q_i be the point of C_i lying over r for $i = 1, 2$.

We shall now exhibit three nonequivalent balanced line bundles that correspond to the same point of \overline{P}_X^1 . Notice that, by Lemma 4.4, giving a line bundle on a curve of compact type is equivalent to giving a line bundle on each irreducible component of the curve.

Let $p \in C_1 \setminus \{q_1\}$. Our first line bundle is $L_1 \in \text{Pic } X$, corresponding to the pair

$$(\mathcal{O}_{C_1}(p), \mathcal{O}_{C_2}).$$

Let $Y := \hat{X}_r$; so $Y = C_1 \cup E \cup C_2$, where E is the exceptional component. Our second line bundle is $L_2 \in \text{Pic } Y$, corresponding to the triple

$$(\mathcal{O}_{C_1}(p - q_1), \mathcal{O}_E(q_E), \mathcal{O}_{C_2}),$$

where q_E is any point of E . Finally, our third line bundle is $L_3 \in \text{Pic } X$, corresponding to the pair

$$(\mathcal{O}_{C_1}(p - q_1), \mathcal{O}_{C_2}(q_2)).$$

We leave out the proof that L_1 , L_2 , and L_3 correspond to the same point of \overline{P}_X^1 and refer the reader to [C1, Sec. 7.2, Ex. 2, p. 645] for more details.

The preceding example shows that, if $f: \mathcal{X} \rightarrow B$ is a regular pencil with a fiber X that is not 1-general, then \overline{P}_f^1 and \overline{P}_X^1 are not coarse moduli schemes for balanced line bundles. However, we can still use our modular interpretation of the Abel map to obtain a map $\overline{\alpha}_f^1: \mathcal{X} \rightarrow \overline{P}_f^1$ restricting to the classical Abel map of \mathcal{X}_K . In fact, essentially the same line bundle \mathcal{M} given in (31) can be used to produce a moduli map $\overline{\alpha}_f^1: \mathcal{X} \rightarrow \overline{P}_f^1$. Most of the results in Sections 4 and 5 still hold—provided we change one definition, as explained in Section 5.12.

5.12. SMALL TAILS, AGAIN. Let X be a stable curve of arithmetic genus g . Taking into account the case where X is not 1-general, we need to adjust the definition of the set $\mathcal{Q}(X)$ in Definition 4.1. Suppose that X has a separating node that generates two tails Q and Q' of equal genus. It is easy to see that, if such a node exists, then it is unique. (By Proposition 3.15, a 1-general curve will never admit such a node.) We must add to the set $\mathcal{Q}(X)$ of small tails of X either Q or Q' , thus making an arbitrary choice between Q and Q' that nonetheless turns out to be completely irrelevant.

So, $\mathcal{Q}(X)$ is defined as the set of all small tails of X together with one tail of genus $g/2$, if any such tail exists.

REMARK 5.13. The following results of this paper hold with essentially the same proof as long as we use the modified definition of $\mathcal{Q}(X)$ for stable curves X of Section 5.12:

- (i) Theorem 4.6, excluding part (ii);
- (ii) Corollary 4.10;
- (iii) Theorem 5.5.

What is certainly lost is the possibility of interpreting the Abel map in a unique way. In other words: If $f: \mathcal{X} \rightarrow B$ is a regular pencil, then an extension $\overline{\alpha}_f^1: \mathcal{X} \rightarrow \overline{P}_f^1$ of the Abel map of \mathcal{X}_K is obtained as the moduli map of a semibalanced line bundle; however, the line bundle is not uniquely determined.

References

- [AK] A. Altman and S. Kleiman, *Compactifying the Picard scheme*, Adv. Math. 35 (1980), 50–112.
- [BLR] S. Bosch, W. Lüktebohmert, and M. Raynaud, *Néron models*, Ergeb. Math. Grenzgeb. (3), 21, Springer-Verlag, Berlin, 1990.
- [C1] L. Caporaso, *A compactification of the universal Picard variety over the moduli space of stable curves*, J. Amer. Math. Soc. 7 (1994), 589–660.
- [C2] ———, *Néron models and compactified Picard schemes over the moduli stack of stable curves*, Amer. J. Math. (to appear); math.AG/0502171.
- [C3] ———, *Naturality of Abel maps*, Manuscripta Math. 123 (2007), 53–71.
- [CCaCo] L. Caporaso, C. Casagrande, and M. Cornalba, *Moduli of roots of line bundles on curves*, Trans. Amer. Math. Soc. 359 (2007), 3733–3768.
- [Ed] B. Edixhoven, *On Néron models, divisors and modular curves*, J. Ramanujan Math. Soc. 13 (1998), 157–194.
- [E] E. Esteves, *Compactifying the relative Jacobian over families of reduced curves*, Trans. Amer. Math. Soc. 353 (2001), 3045–3095.
- [EGK1] E. Esteves, M. Gagné, and S. Kleiman, *Abel maps and presentation schemes*, Comm. Algebra 28 (2000), 5961–5992.
- [EGK2] ———, *Autoduality of the compactified Jacobian*, J. London Math. Soc. (2) 65 (2002), 591–610.
- [EK] E. Esteves and S. Kleiman, *The compactified Picard scheme of the compactified Jacobian*, Adv. Math. 198 (2005), 484–503.
- [K1] S. Kleiman, *What is Abel’s theorem anyway? The legacy of Niels Henrik Abel* (S. Kleiman, O. A. Laudal, eds.), pp. 395–440, Springer-Verlag, Berlin, 2004.
- [K2] ———, *The Picard scheme*, Fundamental algebraic geometry, Math. Surveys Monogr., 123, pp. 235–321, Amer. Math. Soc., Providence, RI, 2005.
- [GIT] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, 3rd ed., Ergeb. Math. Grenzgeb. (2), 34, Springer-Verlag, Berlin, 1994.
- [N] A. Néron, *Modèles minimaux des variétés abéliennes sur les corps locaux et globaux*, Inst. Hautes Études Sci. Publ. Math. 21 (1964), 5–128.
- [R] M. Raynaud, *Spécialisation du foncteur de Picard*, Inst. Hautes Études Sci. Publ. Math. 38 (1970), 27–76.

L. Caporaso
 Dipartimento di Matematica
 Università Roma Tre
 Largo S. L. Murialdo 1
 00146 Roma
 Italy
 caporaso@mat.uniroma3.it

E. Esteves
 Instituto Nacional de Matemática
 Pura e Aplicada
 Est. D. Castorina 110
 22460-320 Rio de Janeiro
 Brazil
 esteves@impa.br