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Invariant Metrics and Distances on Generalized Neil Parabolas

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1. Introduction and Results

In the survey paper [3], the authors asked for an effective formula for the Carathéodory distance $c_{A_{2,3}}$ on the Neil parabola $A_{2,3}$ (in the bidisc). Such a formula was presented in a more recent paper by Knese [4]. To repeat the main result of [4], we recall that the Neil parabola is given by $A_{2,3} := \{(z,w) \in \mathbb{D}^2 : z^2 = w^3\}$, where $\mathbb{D}$ denotes the open unit disc in the complex plane. Then there is the natural parameterization $p_{2,3} \colon \mathbb{D} \to A_{2,3}$, $p_{2,3}(\lambda) := (\lambda^3, \lambda^2)$. Moreover, let ρ denote the Poincaré distance of the unit disc. Recall that

$$\rho(\lambda,\mu) := \frac{1}{2}\log\frac{1+m_{\mathbb{D}}(\lambda,\mu)}{1-m_{\mathbb{D}}(\lambda,\mu)},$$

where

$$m_{\mathbb{D}}(\lambda,\mu) := \left|\frac{\lambda-\mu}{1-\lambda\bar{\mu}}\right|, \quad \lambda,\mu \in \mathbb{D}.$$

Let $\lambda,\mu \in \mathbb{D}$. Then Knese's result is

$$c_{A_{2,3}}(p_{2,3}(\lambda), p_{2,3}(\mu)) = \begin{cases} \rho(\lambda^2,\mu^2) & \text{if } |\alpha_0| \ge 1, \\ \rho\big(\lambda^2\frac{\alpha_0-\lambda}{1-\bar{\alpha}_0\lambda}, \mu^2\frac{\alpha_0-\mu}{1-\bar{\alpha}_0\mu}\big) & \text{if } |\alpha_0| < 1, \end{cases}$$

where $\alpha_0 := \alpha_0(\lambda,\mu) := \frac{1}{2}(\lambda + 1/\bar{\lambda} + \mu + 1/\bar{\mu})$. If $\lambda\mu = 0$ then the formula should be read as if $|\alpha_0| \ge 1$.

Observe that if λ and μ have a nonobtuse angle—that is, if $\operatorname{Re}(\lambda\bar{\mu}) \ge 0$—then $|\alpha_0(\lambda,\mu)| > 1$ (cf. Corollary 2).

Moreover, in [4] the formula for the Carathéodory–Reiffen pseudometric $\gamma_{A_{2,3}}$ is given as

$$\gamma_{A_{2,3}}((a,b);X) = \begin{cases} |X_2| & \text{if } a=b=0 \text{ and } |X_2| \ge 2|X_1|, \\ \frac{4|X_1|^2+|X_2|^2}{4|X_1|} & \text{if } a=b=0 \text{ and } |X_2| < 2|X_1|, \\ \frac{2|\lambda b|}{1-|b|^2} & \text{if } (a,b) \ne (0,0) \text{ and } X = \lambda(3a,2b),\ \lambda \in \mathbb{C}, \end{cases}$$

where $(a,b) \in A_{2,3}$ and $X \in T_{(a,b)}A_{2,3} :=$ the tangent space in (a,b) at $A_{2,3}$.

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We point out that these are the first effective formulas for the Carathéodory distance and the Carathéodory–Reiffen pseudodistance of a nontrivial complex space.

In this paper we will discuss more general Neil parabolas—namely, the spaces

$$A_{m,n} := \{(z,w) \in \mathbb{D}^2 : z^m = w^n\},$$
$$m,n \in \mathbb{N}, \quad m \le n, \quad m,n \text{ relatively prime.}$$

For short, we will call $A_{m,n}$ the *(m,n)-parabola*. As in the case of the classical Neil parabola, we have the following globally bijective holomorphic parameterization of $A_{m,n}$:

$$p_{m,n} \colon \mathbb{D} \to A_{m,n}, \qquad p_{m,n}(\lambda) := (\lambda^n, \lambda^m), \quad \lambda \in \mathbb{D}.$$

Observe that

$$q_{m,n} := p_{m,n}^{-1} \colon A_{m,n} \to \mathbb{D}$$

is given outside of the origin by $q_{m,n}(z,w) = z^k w^l$, where $k,l \in \mathbb{Z}$ are such that $kn + lm = 1$; furthermore, $q_{m,n}(0,0) = 0$. It is clear that $q_{m,n}$ is continuous on $A_{m,n}$ and holomorphic outside of the origin.

We will study the Carathéodory and the Kobayashi distances and also the Carathéodory–Reiffen and the Kobayashi–Royden pseudometrics of $A_{m,n}$. Let us now recall the objects to be dealt with in this paper:

$$m_{A_{m,n}}(\zeta,\eta) := \sup\{m_{\mathbb{D}}(f(\zeta), f(\eta)) : f \in \mathcal{O}(A_{m,n}, \mathbb{D})\}, \quad \zeta,\eta \in A_{m,n};$$

here $\mathcal{O}(A_{m,n}, \mathbb{D})$ denotes the family of holomorphic functions on $A_{m,n}$, that is, the family of those functions on $A_{m,n}$ that are locally restrictions of holomorphic functions on an open set in $\mathbb{C}^2$.

Observe that the Carathéodory distance $c_{A_{m,n}}$ is given by $c_{A_{m,n}}(\zeta,\eta) = \tanh^{-1} m_{A_{m,n}}(\zeta,\eta)$; moreover, $c_{\mathbb{D}} = \rho$. We must therefore study holomorphic functions on the (m,n)-parabola. We have the following bijection of $\mathcal{O}(A_{m,n},\mathbb{D})$ and a part $\mathcal{O}_{m,n}(\mathbb{D})$ of $\mathcal{O}(\mathbb{D},\mathbb{D})$, where

$$\mathcal{O}_{m,n}(\mathbb{D}) := \{h \in \mathcal{O}(\mathbb{D},\mathbb{D}) : h^{(s)}(0) = 0,\ s \in S_{m,n}\}$$

and $S_{m,n} := \{s \in \mathbb{N} : s + m + n \notin \mathbb{N}m + \mathbb{N}n\}$ (recall that $S_{1,n} = \emptyset$ and if $m \ge 2$ then $\max_{s \in S_{m,n}} s = nm - m - n$). To be precise, if $f \in \mathcal{O}(A_{m,n},\mathbb{D})$ then $f \circ p_{m,n} \in \mathcal{O}_{m,n}(\mathbb{D})$; conversely, if $h \in \mathcal{O}_{m,n}(\mathbb{D})$ then $h \circ q_{m,n} \in \mathcal{O}(A_{m,n},\mathbb{D})$.

These considerations yield the following description of the Caratheódory distance on $A_{m,n}$:

$$\begin{aligned}
&m_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) \\
&\quad = \max\{m_{\mathbb{D}}(h(\lambda), h(\mu)) : h \in \mathcal{O}_{m,n}(\mathbb{D})\} \\
&\quad = \max\{m_{\mathbb{D}}(h(\lambda), h(\mu)) : h \in \mathcal{O}_{m,n}(\mathbb{D}),\ h(0) = 0\} \\
&\quad = \max\{m_{\mathbb{D}}(\lambda^m h(\lambda), \mu^m h(\mu)) : h \in \mathcal{O}(\mathbb{D}, \bar{\mathbb{D}}),\ h^{(j)}(0) = 0,\ j + m \in S_{m,n}\}, \\
&\hspace{28em} \lambda,\mu \in \mathbb{D}.
\end{aligned}$$

We should like to point out that calculating the Carathéodory distance of a generalized Neil parabola may be viewed as the following interpolation problem for holomorphic functions on the unit disc. Let λ, μ be as before and let $\zeta, \eta \in \mathbb{D}$. Then there exists an $h \in \mathcal{O}_{m,n}(\mathbb{D})$ with $h(\lambda) = \zeta, h(\mu) = \eta$ if and only if $m_{\mathbb{D}}(\zeta, \eta) \le m_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))$. Note that $m_{A_{1,n}}(p_{1,n}(\lambda), p_{1,n}(\mu)) = m_{\mathbb{D}}(\lambda, \mu)$.

From the case of domains in $\mathbb{C}^n$ it is well known that the Carathéodory distance need not be an inner distance (see [2]). In the case of a generalized Neil parabola it turns out that the Carathéodory distance is an inner distance if and only if $m = 1$.

Recall that the associated inner distance is given by

$$c^i_{A_{m,n}}(\zeta, \eta) := \inf\{L_{c_{A_{n,m}}}(\alpha) : \alpha \text{ is a } \|\cdot\|\text{-rectifiable curve in } A_{m,n} \\ \text{connecting } \zeta, \eta\}, \quad \zeta, \eta \in A_{m,n},$$

where $L_{c_{A_{m,n}}}$ denotes the $c_{A_{m,n}}$-length. Obviously, $c_{A_{m,n}} \le c^i_{A_{m,n}}$. Then we have the following result for the inner distance.

THEOREM 1. *Let $\lambda, \mu \in \mathbb{D}$. Then*

$$c^i_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) = \begin{cases} c_{\mathbb{D}}(\lambda^m, \mu^m) & \textit{if } \operatorname{Re}(\lambda\bar{\mu}) \ge \cos(\pi/m)|\lambda\mu|, \\ c_{\mathbb{D}}(\lambda^m, 0) + c_{\mathbb{D}}(0, \mu^m) & \textit{otherwise}. \end{cases}$$

There is also the following comparison result between the Carathéodory distance and its associated inner one.

COROLLARY 2. *Let $\lambda, \mu \in \mathbb{D}$.*

(a) *If* $\operatorname{Re}(\lambda\bar{\mu}) \ge \cos(\pi/m)|\lambda\mu|$, *then*

$$c^i_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) = c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)).$$

(b) *If* $\operatorname{Re}(\lambda\bar{\mu}) < \cos(\pi/m)|\lambda\mu|$, *then*

$$c^i_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) = c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) \iff (\lambda\bar{\mu})^m < 0.$$

(c) *Hence, the following conditions are equivalent*:
- $c^i_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) = c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))$;
- $c^i_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) = c_{\mathbb{D}}(\lambda^m, \mu^m)$;
- $\operatorname{Re}(\lambda\bar{\mu}) \ge \cos(\pi/m)|\lambda\mu|$ *or* $(\lambda\bar{\mu})^m < 0$.

In particular, $c_{A_{m,n}}$ is not inner if $m > 1$.

Note that these results partially cover the result obtained by Knese. Moreover, observe that the condition $\operatorname{Re}(\lambda\bar{\mu}) \ge \cos(\pi/m)|\lambda\mu|$ in these results means geometrically that μ lies inside an angular sector around λ of opening angle equal to π/m (cf. Knese's result in [4]). And unlike the $A_{2,3}$ case, the new area $(\lambda\bar{\mu})^m < 0$ (i.e., the "rays" on which the angle between λ and μ is equal to $(2j-1)\pi/m$, $j = 2, \dots, m-1$) appears for $A_{m,n}$ with $m > 2$.

In order to prove Theorem 1, we must to calculate the Carathéodory–Reiffen metric $\gamma_{A_{m,n}}$ outside of the origin. First, recall its definition:

$$\gamma_{A_{m,n}}((z, w); X) := \max\{|f'(z, w)X| : f \in \mathcal{O}(A_{m,n}, \mathbb{D})\},$$

where $(z,w)\in A_{m,n}$ and X is a tangent vector in (z,w) at $A_{m,n}$. Note that if $(z,w)=\zeta=p_{m,n}(\lambda)$ and $\lambda\in\mathbb{D}\setminus\{0\}$, then the tangent space $T_\zeta(A_{m,n})$ at ζ is spanned by the vector $p'_{m,n}(\lambda)$. The same holds if $m=1$ and $\lambda=0$, whereas $T_0(A_{m,n})=\mathbb{C}^2$ if $m\geq 2$.

With the foregoing description of $\mathcal{O}(A_{m,n},\mathbb{D})$, we may recast this definition in a form that is appropriate for our uses here:

$$\gamma_{A_{m,n}}(p_{m,n}(\lambda);\,p'_{m,n}(\lambda))=\sup\left\{\frac{|h'(\lambda)|}{1-|h(\lambda)|^2}:h\in\mathcal{O}_{m,n}(\mathbb{D})\right\}.$$

Then we have the following result.

THEOREM 3. *Let $\lambda\in\mathbb{D}$. Then*

$$\gamma_{A_{m,n}}(p_{m,n}(\lambda);\,p'_{m,n}(\lambda))=\frac{m|\lambda|^{m-1}}{1-|\lambda|^{2m}}.$$

It follows from the preceding results (as in the case of domains in $\mathbb{C}^n$) that $\gamma_{A_{m,n}}$ is the infinitesimal form of $c_{A_{m,n}}$ outside the origin. More precisely, if $\lambda\in\mathbb{D}\setminus\{0\}$ then

$$\begin{aligned}\lim_{\mu\to\lambda}\frac{c_{A_{m,n}}(p_{m,n}(\lambda),p_{m,n}(\mu))}{|\lambda-\mu|}&=\lim_{\mu\to\lambda}\frac{c_{\mathbb{D}}(\lambda^m,\mu^m)}{|\lambda-\mu|}\\&=\frac{m|\lambda|^{m-1}}{1-|\lambda|^{2m}}=\gamma_{A_{m,n}}(p_{m,n}(\lambda);\,p'_{m,n}(\lambda)).\end{aligned}$$

Observe that the same holds if $m=1$ and $\lambda=0$.

On the other hand, note that

$$\gamma_{A_{m,n}}(0;X)=\max\{|f'(0)X|:f\in\mathcal{O}(A_{m,n},\mathbb{D}),\,f(0)=0\}.$$

For such f we have $f\circ p_{m,n}(\zeta)=\zeta^m h(\zeta)$ when $\zeta\in\mathbb{D}$, where $h\in\mathcal{O}(\mathbb{D},\bar{\mathbb{D}})$. Observe that

$$\frac{\partial f}{\partial w}(0)=h(0),\qquad\frac{\partial f}{\partial z}(0)=\frac{h^{(n-m)}(0)}{(n-m)!}\quad\text{for }m\geq 2.$$

Thus, if $X=(X_1,X_2)\in\mathbb{C}^2$, then

$$\begin{aligned}&\gamma_{A_{m,n}}(0;X)\\&=\max\left\{\left|X_1\frac{h^{(n)}(0)}{n!}+X_2\frac{h^{(m)}(0)}{m!}\right|:h\in\mathcal{O}_{m,n}(\mathbb{D}),\,h(0)=0\right\}\\&=\max\left\{\left|X_1\frac{h^{(n-m)}(0)}{(n-m)!}+X_2h(0)\right|:h\in\mathcal{O}(\mathbb{D},\bar{\mathbb{D}}),\,h^{(j)}(0)=0,\,j+m\in S_{m,n}\right\};\end{aligned}$$

in particular, $\gamma_{A_{m,n}}(0;X)=\|X\|$ if $X_1X_2=0$. Using the first equality above, we shall prove the following infinitesimal result at the origin.

PROPOSITION 4. *Let $X_{\lambda,\mu}:=(\lambda^n-\mu^n,\lambda^m-\mu^m)$. Then*

$$\lim_{\lambda,\mu\to 0,\lambda\neq\mu}\frac{c_{A_{m,n}}(p_{m,n}(\lambda),p_{m,n}(\mu))}{\gamma_{A_{m,n}}(0;X_{\lambda,\mu})}=1.$$

COROLLARY 5. *Let $m > 1$. Then there are points $\lambda, \mu \in \mathbb{D}$ such that*

$$c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) > \max\{\rho(\lambda^m, \mu^m), \rho(\lambda^{m+1}, \mu^{m+1})\}.$$

It turns out that the general calculation of the Carathéodory–Reiffen metric at the origin becomes much more difficult. The next proposition may give some flavor of the nature of these formulas.

PROPOSITION 6. *Let $X = (X_1, X_2) \in \mathbb{C}^2$. Then*

$$\gamma_{A_{3,4}}(0; X) = \begin{cases} |X_1| & \text{if } |X_1| \geq 2|X_2|, \\ |X_2| & \text{if } |X_2| \geq \sqrt{2}|X_1|, \\ |X_1|\frac{c^3 - 18c + (c^2+24)^{3/2}}{108} & \text{if } 1 < c := 2\frac{|X_2|}{|X_1|} < 2\sqrt{2}. \end{cases}$$

It seems rather difficult to calculate an effective formula of the Carathéodory distance of $A_{m,n}$. We do have its value at pairs of "opposite" points; more precisely, the following is true.

PROPOSITION 7. *Let $\lambda \in \mathbb{D}$, $\lambda \neq 0$. Then*

$$m_{A_{2,2k+1}}(p_{2,2k+1}(\lambda), p_{2,2k+1}(-\lambda)) = \frac{2|\lambda|^{2k+1}}{1 + |\lambda|^{4k+2}}.$$

Observe that now, unlike the previous cases, the number $n = 2k + 1$ appears in the formula.

Finally, the discussion of the Kobayashi distance and the Kobayashi–Royden metric on $A_{m,n}$ becomes comparably much simpler. Let us first recall the definitions of the Lempert function $\tilde{k}_{A_{m,n}}$, the Kobayashi distance $k_{A_{m,n}}$, and the Kobayashi–Royden metric $\kappa_{A_{m,n}}$:

- $\tilde{k}_{A_{m,n}}(\zeta, \eta) := \inf\{\rho(\lambda, \mu) : \lambda, \mu \in \mathbb{D}, \exists_{\varphi \in \mathcal{O}(\mathbb{D}, A_{m,n})} : \varphi(\lambda) = \zeta, \varphi(\mu) = \eta\}$, $\zeta, \eta \in A_{m,n}$;
- $k_{A_{m,n}} :=$ the largest distance on $A_{m,n}$ that is less than or equal to $\tilde{k}_{A_{m,n}}$;
- $\kappa_{A_{m,n}}(\zeta; X) := \inf\{\alpha \in \mathbb{R}_+ : \exists_{\varphi \in \mathcal{O}(\mathbb{D}, A_{m,n})} : \varphi(0) = \zeta, \alpha\varphi'(0) = X\}$, $\zeta \in A_{m,n}$, $X \in T_\zeta(A_{m,n})$.

We set $\tilde{k}_{A_{m,n}}(\zeta, \eta) := \infty$ or $\kappa_{A_{m,n}}(\zeta; X) := \infty$ if there are no respective discs φ.

Since $\mathcal{O}(\mathbb{D}, A_{m,n}) = \{p_{m,n} \circ \psi : \psi \in \mathcal{O}(\mathbb{D}, \mathbb{D})\}$, we have the formulas in our next proposition (see also [3; 4]).

PROPOSITION 8. *Let $\lambda, \mu \in \mathbb{D}$. Then*

$$k_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) = \tilde{k}_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) = \rho(\lambda, \mu).$$

If $\lambda \neq 0$, then $\kappa_{A_{m,n}}(p_{m,n}(\lambda); p'_{m,n}(\lambda)) = 1/(1 - |\lambda|^2)$.

Let $X = (X_1, X_2) \in T_0 A_{m,n} \setminus \{0\}$. Then

$$\kappa_{A_{m,n}}(0; X) = \begin{cases} |X_2| & \text{if } m = 1, \\ \infty & \text{otherwise.} \end{cases}$$

At the end of the paper we discuss a simple reducible variety.

2. Proofs and Additional Remarks

We start with the proof of Theorem 3, which will serve as the basic information for Theorem 1.

Proof of Theorem 3. Recall that

$$\gamma_{A_{m,n}}(p_{m,n}(\lambda);\, p'_{m,n}(\lambda)) = \max\left\{\frac{|h'(\lambda)|}{1-|h(\lambda)|^2} : h \in \mathcal{O}_{m,n}(\mathbb{D})\right\}.$$

Observe that if $\alpha \in \mathbb{D}$ and $\Phi_\alpha(\zeta) = \frac{\alpha-\zeta}{1-\bar{\alpha}\zeta}$, then $h_\alpha = \Phi_\alpha \circ h \in \mathcal{O}_{m,n}(\mathbb{D})$ (use, e.g., the Faà di Bruno formula) and

$$\frac{|h'_\alpha(\lambda)|}{1-|h_\alpha(\lambda)|^2} = \frac{|h'(\lambda)|}{1-|h(\lambda)|^2}.$$

Then

$$\begin{aligned}
&\gamma_{A_{m,n}}(p_{m,n}(\lambda);\, p'_{m,n}(\lambda))\\
&= \max\left\{\frac{|h'(\lambda)|}{1-|h(\lambda)|^2} : h \in \mathcal{O}_{m,n}(\mathbb{D}),\ h(0)=0\right\}\\
&= \max\left\{\frac{|(\lambda^m\tilde{h}(\lambda))'|}{1-|\lambda^m\tilde{h}(\lambda)|^2} : \tilde{h} \in \mathcal{O}(\mathbb{D},\bar{\mathbb{D}}),\ \tilde{h}^{(j)}(0)=0,\ j+m \in S_{m,n}\right\}\\
&= |\lambda|^{m-1}\max\left\{\frac{|mh(\lambda)+\lambda h'(\lambda)|}{1-|\lambda^m h(\lambda)|^2} : h \in \mathcal{O}(\mathbb{D},\bar{\mathbb{D}}),\ h^{(j)}(0)=0,\ j+m \in S_{m,n}\right\}\\
&= \frac{m|\lambda|^{m-1}}{1-|\lambda|^{2m}}.
\end{aligned}$$

The last equality follows because the unimodular constants are the only extremal functions for

$$\max\left\{\frac{|mh(\lambda)+\lambda h'(\lambda)|}{1-|\lambda^m h(\lambda)|^2} : h \in \mathcal{O}(\mathbb{D},\bar{\mathbb{D}})\right\}.$$

To prove this, observe that $(h(\lambda), h'(\lambda))$ varies on all pairs (a,b) satisfying $|b| \le (1-|a|^2)/(1-|\lambda|^2)$. Hence we must show that if $0 \le c, s < 1$ and $0 \le t \le t_s := (1-s^2)/(1-c^2)$ then $F(s,t) < F(1,0)$, where $F(s,t) = (ms+ct)/(1-c^{2m}s^2)$. Since $F(s,t) \le F(s,t_s)$, the problem may be reduced to the inequality

$$\frac{m(1-c^2)s + c(1-s^2)}{1-c^{2m}s^2} < \frac{m(1-c^2)}{1-c^{2m}} \iff \frac{c(1-c^{2m})}{m(1-c^2)} < \frac{1+c^{2m}s}{1+s}.$$

Given the inequality $\frac{1+c^{2m}}{2} < \frac{1+c^{2m}s}{1+s}$, it is clear that

$$\frac{c(1-c^{2m})}{m(1-c^2)} < \frac{1+c^{2m}}{2} \iff 2c\sum_{j=0}^{m-1} c^{2j} < m(1+c^{2m}).$$

Finally, after summing up the inequalities $1-c^{2j+1} > c^{2m-2j-1}(1-c^{2j+1})$ for $j = 0,\ldots,m-1$, the last inequality follows. $\square$

We are now in a position to prove Theorem 1.

Proof of Theorem 1. Set $\Lambda_{\lambda,m} = \{\zeta \in \mathbb{D} : \mathrm{Re}(\lambda\bar{\zeta}) \geq \cos(\pi/m)|\lambda\zeta|\}$ with $\lambda \in \mathbb{D}$ and $m \in \mathbb{N}$. Recall again that $\Lambda_{\lambda,m}$ is an angular sector around λ.

As a first step we shall prove that if $\lambda \in \mathbb{D}$ and $\mu \in \Lambda_{\lambda,m}$, then

$$c^i_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) = c_{\mathbb{D}}(\lambda^m, \mu^m).$$

Since

$$c^i_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) \geq c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) \geq c_{\mathbb{D}}(\lambda^m, \mu^m), \tag{1}$$

we need only prove the opposite inequality. After rotation, we may assume that $\lambda \in [0,1)$; by continuity, we may also assume that $\lambda, \mu \neq 0$ and $\arg(\mu) \in (-\pi/m, \pi/m)$. Then the geodesic for $c^i_{\mathbb{D}}(\lambda^m, \mu^m)$ does not intersect the segment $(-1, 0]$. Denote by α this geodesic and by α_m its mth root ($1^{1/m} = 1$). Observe that if $\zeta, \eta \in A^*_{m,n} := A_{m,n} \setminus \{0\}$, then

$$c^i_{A_{m,n}}(\zeta,\eta) = \inf\left\{\int_0^1 \gamma_{A_{m,n}}(\alpha(t); \alpha'(t))\, dt : \alpha : [0,1] \to A^*_{m,n} \right.$$
$$\left. \text{is a } C^1\text{-curve connecting } \zeta, \eta\right\}$$

(see [5, Thm. 4.2.7]). It follows from Theorem 3 that

$$\begin{aligned}
c^i_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) &\leq \int_0^1 \gamma_{A_{m,n}}(p_{m,n} \circ \alpha_m(t); (p_{m,n} \circ \alpha_m)'(t))\, dt \\
&= \int_0^1 \frac{m|(\alpha_m(t))|^{m-1}|\alpha'_m(t)|}{1 - |\alpha_m(t)|^{2m}}\, dt = \int_0^1 \frac{|\alpha'(t)|}{1 - |\alpha(t)|^2}\, dt \\
&= c^i_{\mathbb{D}}(\lambda^m, \mu^m) = c_{\mathbb{D}}(\lambda^m, \mu^m).
\end{aligned}$$

It remains to prove that if $\mu \notin \Lambda_{\lambda,m}$ then

$$c^i_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) = c^i_{A_{m,n}}(p_{m,n}(\lambda), 0) + c^i_{A_{m,n}}(0, p_{m,n}(\mu)).$$

By the triangle inequality, we need only prove that

$$c^i_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) \geq c^i_{A_{m,n}}(p_{m,n}(\lambda), 0) + c^i_{A_{m,n}}(0, p_{m,n}(\mu)). \tag{2}$$

Take an arbitrary C^1-curve $\alpha\colon [0,1] \to A^*_{m,n}$ with $\alpha(0) = p_{m,n}(\lambda)$ and $\alpha(1) = p_{m,n}(\mu)$. Let $t_0 \in (0,1)$ be the smallest number such that $\lambda_0 := q_{m,n}(\alpha(t_0)) \in \partial\Lambda_{\lambda,m}$. Then

$$\begin{aligned}
&\int_0^1 \gamma_{A_{m,n}}(\alpha(t); \alpha'(t))\, dt \\
&\quad = \int_0^{t_0} \gamma_{A_{m,n}}(\alpha(t); \alpha'(t))\, dt + \int_{t_0}^1 \gamma_{A_{m,n}}(\alpha(t); \alpha'(t))\, dt \\
&\quad \geq c^i_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\lambda_0)) + c^i_{A_{m,n}}(p_{m,n}(\lambda_0), p_{m,n}(\mu)) \\
&\quad \geq c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\lambda_0)) + c_{A_{m,n}}(p_{m,n}(\lambda_0), p_{m,n}(\mu)) \\
&\quad \geq c_{\mathbb{D}}(\lambda^m, \lambda_0^m) + c_{\mathbb{D}}(\lambda_0^m, \mu^m) \\
&\quad = c_{\mathbb{D}}(\lambda^m, 0) + c_{\mathbb{D}}(0, \lambda_0^m) + c_{\mathbb{D}}(\lambda_0^m, \mu^m) \quad (\text{since } \lambda_0^m \in (-1, 0)) \\
&\quad \geq c_{\mathbb{D}}(\lambda^m, 0) + c_{\mathbb{D}}(0, \mu^m).
\end{aligned}$$

Now, (2) follows by taking the infimum over all curves under consideration. □

Proof of Corollary 2. Part (a) follows by Theorem 1 and inequality (1).

(b) The inequalities

$$\begin{aligned} c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) &\le \max\{c_{\mathbb{D}}(\lambda^m f(\lambda), \mu^m f(\mu)) : f \in \mathcal{O}(\mathbb{D}, \bar{\mathbb{D}})\} \\ &\le \max\{c_{\mathbb{D}}(\lambda^m f(\lambda), 0) + c_{\mathbb{D}}(0, \mu^m f(\mu)) : f \in \mathcal{O}(\mathbb{D}, \bar{\mathbb{D}})\} \\ &\le c_{\mathbb{D}}(\lambda^m, 0) + c_{\mathbb{D}}(0, \mu^m) = c^i_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) \end{aligned}$$

show that

$$c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) = c^i_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))$$

if and only if $\lambda^m f(\lambda)$ and $\mu^m f(\mu)$ lie on opposite rays and $|f(\lambda)| = |f(\mu)| = 1$ for some $f \in \mathcal{O}(\mathbb{D}, \bar{\mathbb{D}})$—that is, f is a unimodular constant and $(\lambda\bar{\mu})^m < 0$.

Part (c) of Corollary 2 follows because $c_{\mathbb{D}}(z, 0) + c_{\mathbb{D}}(0, w) = c_{\mathbb{D}}(z, w)$ if and only if $z\bar{w} \le 0$. □

REMARKS. (a) For $m \in \mathbb{N}$, consider the following distance on $\mathbb{D}$:

$$\rho^{(m)}(\lambda, \mu) := \max\{\rho_{\mathbb{D}}(\lambda^m h(\lambda), \mu^m h(\mu)) : h \in \mathcal{O}(\mathbb{D}, \bar{\mathbb{D}})\}.$$

Note that

$$\begin{aligned} \lim_{\varepsilon\to 0, \varepsilon\ne 0} \frac{\rho^{(m)}(\lambda, \lambda+\varepsilon)}{|\varepsilon|} &= |\lambda|^{m-1} \max\left\{\frac{|mh(\lambda) + \lambda h'(\lambda)|}{1 - |\lambda^m h(\lambda)|^2} : h \in \mathcal{O}(\mathbb{D}, \bar{\mathbb{D}})\right\} \\ &= \gamma_{A_{m,n}}(p_{m,n}(\lambda); p'_{m,n}(\lambda)) \end{aligned}$$

by the proof of Theorem 3. It follows that the associated inner distance of $\rho^{(m)}$ equals $c^i_{A_{m,n}}(p_{m,n}(\cdot), p_{m,n}(\cdot))$. Then

$$\begin{aligned} c^i_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) &\ge \rho^{(m)}(\lambda, \mu) \\ &\ge c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) \ge \rho(\lambda^m, \mu^m). \end{aligned}$$

Moreover, the proof of Corollary 2 shows that the following conditions are equivalent:

- $c^i_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) = \rho^{(m)}(\lambda, \mu)$;
- $c^i_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) = c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))$;
- $c^i_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) = \rho(\lambda^m, \mu^m)$;
- $\operatorname{Re}(\lambda\bar{\mu}) \ge \cos(\pi/m)|\lambda\mu|$ or $(\lambda\bar{\mu})^m < 0$.

As an application of these observations we obtain a simple proof (without calculations) of Lemma 14 in [6]:

If $a, b \in [0, 1)$, $s \in (0, 1]$, *and* $\theta \in [-\pi, \pi]$, *then* $\rho(a, be^{i\theta}) \le \rho(a^s, b^s e^{is\theta})$.

In fact, we may assume that $s \in \mathbb{Q}$. If $s = p/q$ $(1 \le p \le q)$, $\lambda = a^{1/q}$, and $\mu = b^{1/q}e^{i\theta/q}$, then we have to prove that $\rho(\lambda^q, \mu^q) \le \rho(\lambda^p, \mu^p)$. But the angle between λ and μ does not exceed $\pi/q \le \pi/p$, so

$$\rho(\lambda^p, \mu^p) = \rho^{(p)}(\lambda, \mu) \ge \rho(\lambda^q, \mu^q)$$

(the last inequality holds for any $\lambda, \mu \in \mathbb{D}$ and $q \ge p$).

(b) Recall that

$$c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))$$
$$= \max\{\rho_{\mathbb{D}}(\lambda^m h(\lambda), \mu^m h(\mu)) : h \in \mathcal{O}(\mathbb{D}, \bar{\mathbb{D}}),\ h^{(j)}(0) = 0,\ j + m \in S_{m,n}\}.$$

If $m = 1$ or $(m,n) = (2,3)$ then $\rho^{(m)}(\lambda, \mu) = c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))$, because $S_{1,n} = \emptyset$ and $S_{2,3} = \{1\}$.

On the other hand, if $m \neq 1$ and $m \neq n-1$, then the following conditions are equivalent:

- $\rho^{(m)}(\lambda, \mu) = \rho(\lambda^m, \mu^m)$;
- $\rho^{(m)}(\lambda, \mu) = c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))$.

It is clear that the first condition implies the second one. For the converse, observe that as h varies over $\mathcal{O}(\mathbb{D}, \mathbb{D})$, the pair $(h(\lambda), h(\mu))$ varies over all $(z, w) \in \mathbb{D}^2$ with $m_{\mathbb{D}}(z, w) \leq m_{\mathbb{D}}(\lambda, \mu)$. Thus,

$$\rho^{(m)}(\lambda, \mu)$$
$$= \max\{\rho_{\mathbb{D}}(\lambda^m z, \mu^m w) : z, w \in \mathbb{D} \text{ with } m_{\mathbb{D}}(z, w) \leq m_{\mathbb{D}}(\lambda, \mu) \text{ or } z = w \in \partial\mathbb{D}\}.$$

It follows by the maximum principle for the continuous plurisubharmonic function $m_{\mathbb{D}}(\lambda^m \cdot, \mu^m w)$ that if $\rho^{(m)}(\lambda, \mu) = \rho_{\mathbb{D}}(\lambda^m z, \mu^m w)$, then either $z = w \in \partial\mathbb{D}$ or $m_{\mathbb{D}}(z, w) = m_{\mathbb{D}}(\lambda, \mu)$. Assuming that $\rho^{(m)}(\lambda, \mu) \neq \rho(\lambda^m, \mu^m)$ excludes the first possibility. Then any extremal function h for $\rho^{(m)}(\lambda, \mu)$ satisfies

$$m_{\mathbb{D}}(h(\lambda), h(\mu)) = m_{\mathbb{D}}(\lambda, \mu);$$

that is, $h \in \operatorname{Aut}(\mathbb{D})$. Because any such function should be also extremal for $c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))$, it follows that either $h^{(j)} \neq 0$ $(j \in \mathbb{N})$ or h is a rotation. In particular, $m + 1 \notin S_{m,n}$; that is, $m = 1$ or $m = n - 1$—a contradiction.

Let $m \geq 3$. Then $m + 2 \notin S_{m,m+1}$ and hence h must be a rotation. Thus, the following conditions are equivalent:

- $\rho^{(m)}(\lambda, \mu) = \max\{\rho(\lambda^m, \mu^m), \rho(\lambda^{m+1}, \mu^{m+1})\}$;
- $\rho^{(m)}(\lambda, \mu) = c_{A_{m,m+1}}(p_{m,n}(\lambda), p_{m,n}(\mu))$.

(c) Concerning the first condition just listed, we point out that if $m > 1$ then, by Corollary 5, there are points $\lambda, \mu \in \mathbb{D}$ such that

$$\begin{aligned}\rho^{(m)}(\lambda, \mu) &\geq c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))(\lambda, \mu)\\ &> \max\{\rho(\lambda^m, \mu^m), \rho(\lambda^{m+1}, \mu^{m+1})\}.\end{aligned}$$

On the other hand, we have $\rho^{(2m)}(\lambda, -\lambda) = \rho(\lambda^{2m+1}, -\lambda^{2m+1})$ because

$$\begin{aligned}&m_{\mathbb{D}}(\lambda^{2m}\Phi_\alpha(\lambda), \lambda^{2m}\Phi_\alpha(-\lambda))\\ &\quad= \frac{2(1 - |\alpha|^2)|\lambda|^{2m+1}}{|1 + |\lambda|^{4m+2} - |\alpha|^2(|\lambda|^2 + |\lambda|^{4m}) + (1 - |\lambda|^{4m})(\alpha\bar{\lambda} - \bar{\alpha}\lambda)|}\\ &\quad\leq \frac{2(1 - |\alpha|^2)|\lambda|^{2m+1}}{1 + |\lambda|^{4m+2} - |\alpha|^2(|\lambda|^2 + |\lambda|^{4m})} \leq \frac{2|\lambda|^{2m+1}}{1 + |\lambda|^{4m+2}}\end{aligned}$$

(use that $1 + |\lambda|^{4m+2} > |\lambda|^2 + |\lambda|^{4m}$).

Proof of Proposition 4. Observe that there is a constant $c > 0$ with

$$c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) \geq \max\{\rho(\lambda^m, \mu^m), \rho(\lambda^{m+1}, \mu^{m+1})\} \overset{\text{near } 0}{\geq} c|X_{\lambda,\mu}|$$

and

$$\max\{|\lambda|^{k-n}, |\mu|^{k-n}|\}|X_{\lambda,\mu}| \geq c|\lambda^k - \mu^k| \quad \text{for any } k > n,\ \lambda, \mu \in \mathbb{D}.$$

Let $h_{\lambda,\mu}$ be an extremal function for $c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))$. Then

$$h_{\lambda,\mu}(\zeta) = \sum_{j=1}^{[n/m]} a_{jm,\lambda,\mu}\zeta^{jm} + a_{n,\lambda,\mu}\zeta^n + \sum_{j>n,\, j\in S_{m,n}} a_{j,\lambda,\mu}\zeta^j.$$

Since $|a_{j,\lambda,\mu}| \leq 1$, it follows that

$$|h_{\lambda,\mu}(\lambda) - h_{\lambda,\mu}(\mu)| \leq H(\lambda,\mu) := |a_{m,\lambda,\mu}(\lambda^m - \mu^m) + a_{n,\lambda,\mu}(\lambda^n - \mu^n)| + \sum_{j=2}^{[n/m]} |\lambda^{jm} - \mu^{jm}| + \sum_{j=n+1}^{\infty} |\lambda^j - \mu^j|.$$

Thus,

$$\begin{aligned}
1 &\leq \liminf_{\lambda,\mu\to 0,\lambda\neq\mu} \frac{H(\lambda,\mu)}{|h_{\lambda,\mu}(\lambda) - h_{\lambda,\mu}(\mu)|} = \liminf_{\lambda,\mu\to 0,\lambda\neq\mu} \frac{H(\lambda,\mu)}{c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))} \\
&\leq \liminf_{\lambda,\mu\to 0,\lambda\neq\mu} \frac{|a_{m,\lambda,\mu}(\lambda^m - \mu^m) + a_{n,\lambda,\mu}(\lambda^n - \mu^n)|}{c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))} \\
&\quad + \limsup_{\lambda,\mu\to 0,\lambda\neq\mu} \frac{\sum_{j=2}^{[n/m]}|\lambda^{jm} - \mu^{jm}| + \sum_{j=n+1}^{\infty}|\lambda^j - \mu^j|}{c|X_{\lambda,\mu}|} \\
&= \liminf_{\lambda,\mu\to 0,\lambda\neq\mu} \frac{|a_{m,\lambda,\mu}(\lambda^m - \mu^m) + a_{n,\lambda,\mu}(\lambda^n - \mu^n)|}{c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))} \\
&\leq \liminf_{\lambda,\mu\to 0,\lambda\neq\mu} \frac{\gamma_{A_{m,n}}(0; X_{\lambda,\mu})}{c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))};
\end{aligned}$$

this follows because

$$\gamma_{A_{m,n}}(0; X) = \max\left\{\left|X_1\frac{h^{(n)}(0)}{n!} + X_2\frac{h^{(m)}(0)}{m!}\right| : h \in \mathcal{O}_{m,n}(\mathbb{D}), h(0) = 0\right\}.$$

The opposite inequality

$$\limsup_{\lambda,\mu\to 0,\lambda\neq\mu} \frac{\gamma_{A_{m,n}}(0; X_{\lambda,\mu})}{c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))} \leq 1$$

can be proven in a similar way; we omit the details. □

Proof of Corollary 5. Observe that for any neighborhood U of 0 we may find points $\lambda, \mu \in U$ such that $\lambda^m - \mu^m = \lambda^n - \mu^n \neq 0$. Then, by Proposition 4, it is enough to show that

$$\gamma_{A_{m,n}}(0; X_0) > 1, \quad \text{where } X_0 := (1,1).$$

Since

$$\begin{aligned}&\gamma_{A_{m,n}}(0;X_0)\\&\quad=\max\left\{\left|\frac{h^{(n-m)}(0)}{(n-m)!}+h(0)\right|:h\in\mathcal{O}(\mathbb{D},\bar{\mathbb{D}}),\ h^{(j)}(0)=0,\ j+m\in S_{m,n}\right\}\end{aligned}$$

and since $\max_{s\in S_{m,n}}s=nm-m-n$, it follows that

$$\gamma_{A_{m,n}}(0;X_0)\geq\max\{|a+b|:(a,b)\in T_{n-m}\},$$

where T_{n-m} is the set of all pairs $(a,b)\in\mathbb{C}^2$ for which there is a function $h\in\mathcal{O}(\mathbb{D},\bar{\mathbb{D}})$ of the form $h(z)=a+bz^{n-m}+o(z^{nm-2m-n})$.

Let $k\in\mathbb{N}$ be such that $k(n-m)\geq nm-2m-n$. We shall show that there is a function $f\in\mathcal{O}(\mathbb{D},\bar{\mathbb{D}})$ of the form $f(z)=a+bz+o(z^k)$ such that $a,b>0$ and $a+b>1$, which will imply that $\gamma_{A_{m,n}}(0;X_0)>1$.

Note that by Schur's theorem (cf. [1]) such a function f exists if and only if

$$(1-|a|^2)X_1^2+(1-|a|^2-|b|^2)\sum_{j=2}^{n}X_j^2\geq 2|ab|\sum_{j=2}^{n}X_{j-1}X_j,\quad X\in\mathbb{R}^n.\quad(3)$$

Since $\cos\frac{\pi}{n+1}$ is the maximal eigenvalue of the quadratic form $\sum_{j=2}^{n}X_{j-1}X_j$, we have

$$\cos\frac{\pi}{n+1}\sum_{j=1}^{n}X_j^2\geq\sum_{j=2}^{n}X_{j-1}X_j,\quad X\in\mathbb{R}^n.$$

Then (3) is satisfied by all pairs $(a,b)\in\mathbb{C}^2$ for which

$$2\cos\frac{\pi}{n+1}|ab|\leq 1-|a|^2-|b|^2.$$

In particular, we may choose $a,b>0$ such that $2ab>1-a^2-b^2$; that is, $a+b>1$. □

We now turn to a discussion of the Carathéodory–Reiffen pseudometric on the $(3,4)$-parabola.

Proof of Proposition 6. Recall that

$$\gamma_{A_{3,4}}(0;X)=\max\{|X_1h'(0)+X_2h(0)|:h\in\mathcal{O}(\mathbb{D},\bar{\mathbb{D}}),\ h''(0)=0\}.$$

So, we need to describe the pairs $(a_0,a_1)\in\mathbb{C}^2$ for which there is a function $h\in\mathcal{O}(\mathbb{D},\bar{\mathbb{D}})$ of the form $h(\zeta)=a_0+a_1\zeta+o(\zeta^2)$. Let I_3 be the 3×3 unit matrix and let

$$M=\begin{bmatrix}a_0&a_1&0\\0&a_0&a_1\\0&0&a_0\end{bmatrix}.$$

It follows by Schur's theorem (cf. [1]) that such an h exists if and only if I_3-M^*M is a semipositive matrix. It is easy to check that this condition means that the pair $(|a_0|^2,|a_1|^2)$ belongs to the set

$$C := \{(a,b) \in \mathbb{R}^2_+ : a + \sqrt{b} \le 1,\ ab(1-a) \le ((1-a)^2 - b)(1-a-b)\}.$$

The second inequality can be written as

$$b \le (1-a)(1-\sqrt{a}) \quad \text{or} \quad b \ge (1-a)(1+\sqrt{a}).$$

Hence $C = \{(a,b) \in \mathbb{R}^2_+ : b \le (1-a)(1-\sqrt{a}),\ a \le 1\}$. Therefore,

$$\begin{aligned}\gamma_{A_{3,4}}(0; X) &= \max\{|X_1|\sqrt{b} + |X_2|\sqrt{a} : (a,b) \in C\}\\ &= \max\{t \in [0;1] : |X_1|(1-t)\sqrt{1+t} + |X_2|t\}.\end{aligned}$$

Straightforward calculations show that this last maximum is equal to

$$\begin{cases} |X_1| & \text{if } |X_1| \ge 2|X_2|,\\ |X_2| & \text{if } |X_2| \ge \sqrt{2}|X_1|,\\ |X_1|\frac{c^3-18c+(c^2+24)^{3/2}}{108} & \text{if } 1 < c := 2\frac{|X_2|}{|X_1|} < 2\sqrt{2}. \end{cases}$$ □

Proof of Proposition 7. This proposition holds trivially for $k = 0$, so let $k \ge 1$. Recall that

$$\begin{aligned}&m_{A_{2,2k+1}}(p_{2,2k+1}(\lambda), p_{2,2k+1}(\mu))\\ &\quad = \max\{m_{\mathbb{D}}(f(\lambda), f(\mu)) : f \in \mathcal{O}(\mathbb{D},\mathbb{D}),\ f^{(2j-1)}(0) = 0,\ j = 1,\dots,k\}\\ &\quad = \max\{m_{\mathbb{D}}(\lambda^2 h(\lambda), \mu^2 h(\mu)) : h \in \mathcal{O}(\mathbb{D},\bar{\mathbb{D}}),\ h^{(2j-1)}(0) = 0,\ j = 1,\dots,k-1\}.\end{aligned}$$

Then we may take $\zeta \to \zeta^{2k+1}$ as a competitor for $m_{A_{2,2k+1}}$ to derive that

$$m_{A_{2,2k+1}}(p_{2,2k+1}(\lambda), p_{2,2k+1}(-\lambda)) \ge \frac{2|\lambda|^{2k+1}}{1+|\lambda|^{4k+2}}.$$

Moreover, it follows that

$$\begin{aligned}&m_{A_{2,2k+1}}(p_{2,2k+1}(\lambda), p_{2,2k+1}(\mu))\\ &\qquad = \sup\{m_{\mathbb{D}}(\lambda^2 z, \mu^2 w) : m_{\mathbb{D}}(z,w) \le m_{A_{2,2k-1}}(p_{2,2k-1}(\lambda), p_{2,2k-1}(\mu))\}.\end{aligned}$$

Then Proposition 7 will follow by induction on $k \in \mathbb{N}$ if we show that

$$m_{\mathbb{D}}(z,w) \le \frac{2|\lambda|^{2k-1}}{1+|\lambda|^{4k-2}} \implies m_{\mathbb{D}}(\lambda^2 z, \lambda^2 w) \le \frac{2|\lambda|^{2k+1}}{1+|\lambda|^{4k+2}}.$$

Since $2|\lambda|^{2k-1}/(1+|\lambda|^{4k-2}) = m_{\mathbb{D}}(\lambda^{2k-1}, -\lambda^{2k-1})$, we may assume as in Remark (b) that $z = \Phi_\alpha(\lambda^{2k-1})$ and $w = \Phi_\alpha(-\lambda^{2k-1})$ for some $\alpha \in \mathbb{D}$. Then

$$\begin{aligned}&m_{\mathbb{D}}(\lambda^2 z, \lambda^2 w)\\ &\quad = \frac{2(1-|\alpha|^2)|\lambda|^{2k+1}}{|1+|\lambda|^{4k+2} - |\alpha|^2(|\lambda|^4 + |\lambda|^{4k-2}) + (1-|\lambda|^4)(\alpha\bar{\lambda}^{2k-1} - \bar{\alpha}\lambda^{2k-1})|}\\ &\quad \le \frac{2(1-|\alpha|^2)|\lambda|^{2k+1}}{1+|\lambda|^{4k+2} - |\alpha|^2(|\lambda|^4 + |\lambda|^{4k-2})} \le \frac{2|\lambda|^{2k+1}}{1+|\lambda|^{4k+2}}\end{aligned}$$

because $1 + |\lambda|^{4k+2} > |\lambda|^4 + |\lambda|^{4k-2}$. □

REMARK. The foregoing allows us to deduce an interpolation result as follows. For given $k \in \mathbb{N}$ and $\lambda, \eta, \zeta \in \mathbb{D}$, the following conditions are equivalent:

(i) $m_{\mathbb{D}}(\eta, \zeta) \le m_{\mathbb{D}}(\lambda^{2k+1}, -\lambda^{2k-1})$;
(ii) $\exists_{f \in \mathcal{O}(\mathbb{D},\mathbb{D})} : f(\lambda^{2k+1}) = \eta,\ f(-\lambda^{2k+1}) = \zeta$;
(iii) $\exists_{f \in \mathcal{O}(\mathbb{D},\mathbb{D})} : f(\lambda) = \eta,\ f(-\lambda) = \zeta,\ f^{(j)}(0) = 0,\ j = 1, \dots, 2k$;
(iv) $\exists_{f \in \mathcal{O}(\mathbb{D},\mathbb{D})} : f(\lambda) = \eta,\ f(-\lambda) = \zeta,\ f^{(2j-1)}(0) = 0,\ j = 1, \dots, k$.

Indeed, it is trivial that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv), and the implication (iv) $\Rightarrow$ (i) follows by the equalities

$$m_{A_{2,2k+1}}(p_{2,2k+1}(\lambda), p_{2,2k+1}(-\lambda)) = \frac{2|\lambda|^{2k+1}}{1 + |\lambda|^{4k+2}} = m_{\mathbb{D}}(\lambda^{2k+1}, -\lambda^{2k-1}).$$

Finally, we discuss the proof for the Kobayashi distance and metric.

Proof of Proposition 8. The proof of the formula for $\tilde{k}_{A_{m,n}}$ follows the one for the case $(m, n) = (2, 3)$ (see [4]). For the reader's convenience we include it here.

First, $\tilde{k}_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) \le \rho(\lambda, \mu)$ because $p_{m,n}$ is holomorphic. Second, since m and n are relatively prime, it is easy to see that $\mathcal{O}(\mathbb{D}, A_{m,n}) = \{p_{m,n} \circ \psi : \psi \in \mathcal{O}(\mathbb{D},\mathbb{D})\}$. Then any $\varphi \in \mathcal{O}(\mathbb{D}, A_{m,n})$ with $\varphi(\tilde{\lambda}) = p_{m,n}(\lambda)$ and $\varphi(\tilde{\mu}) = p_{m,n}(\mu)$ corresponds to some $\psi \in \mathcal{O}(\mathbb{D},\mathbb{D})$ with $\psi(\tilde{\lambda}) = \lambda$ and $\psi(\tilde{\mu}) = \mu$. Thus, $\rho(\lambda, \mu) \le \rho(\tilde{\lambda}, \tilde{\mu})$ and hence $\rho(\lambda, \mu) \le \tilde{k}_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))$. Therefore, $\tilde{k}_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) = \rho(\lambda, \mu)$; in particular, $\tilde{k}_{A_{m,n}}$ is a distance and so $\tilde{k}_{A_{m,n}} = k_{A_{m,n}}$.

The formulas for $\kappa_{A_{m,n}}$ can be proven in a similar way; we omit the details. □

We conclude this paper by mentioning the simplest example of a reducible variety.

REMARK. Put $A_{2,2} := \{(z, w) \in \mathbb{D}^2 : z^2 = w^2\}$. Here $A_{2,2}$ is reducible and clearly is biholomorphically equivalent to the coordinate cross $V := \{(z, w) \in \mathbb{D}^2 : zw = 0\}$. Therefore, we discuss V instead of $A_{2,2}$.

It is evident that $c_V((z_1, 0), (z_2, 0)) = \tilde{k}_V((z_1, 0), (z_2, 0)) = \rho(z_1, z_2)$ and that

$$\tilde{k}_V((z, 0), (0, w)) = \infty \quad (zw \neq 0),$$

$$k_V((z, 0), (0, w)) = \tilde{k}_V((z, 0), (0, 0)) + \tilde{k}_V((0, 0), (0, w)) = \rho(|z|, -|w|).$$

Moreover, $\gamma_V((z, 0); (1, 0)) = \kappa_V((z, 0); (1, 0)) = 1/(1 - |z|^2)$ and so

$$\kappa_V(0; X) = \begin{cases} |X| & \text{if } X_1 X_2 = 0, \\ \infty & \text{otherwise.} \end{cases}$$

Recall now that

$$\mathcal{O}(V, \mathbb{D}) = \{f + g - f(0) : f \in \mathcal{O}(\mathbb{D} \times \{0\}, \mathbb{D}),\ g \in \mathcal{O}(\{0\} \times \mathbb{D}, \mathbb{D}),\ f(0) = g(0)\}.$$

Then obviously $\gamma_V(0; X) = |X_1| + |X_2|$.

Finally, since $z + w \in \mathcal{O}(V, \mathbb{D})$, it follows that

$$c_V((z, 0), (0, w)) = c_V((|z|, 0), (-|w|, 0)) \ge \rho(|z|, -|w|).$$

Thus, $c_V = k_V$; in particular, $c_V = c_V^i$.

References

[1] J. B. Garnett, *Bounded analytic functions,* Pure Appl. Math., 96, Academic Press, New York, 1981.
[2] M. Jarnicki and P. Pflug, *Invariant distances and metrics in complex analysis,* de Gruyter Exp. Math., 9, de Gruyter, Berlin, 1993.
[3] ———, *Invariant distances and metrics in complex analysis—Revisited,* Dissertationes Math. (Rozprawy Mat.) 430 (2005), 1–192.
[4] G. Knese, *Function theory on the Neil parabola,* Michigan Math. J. 55 (2007), 139–154.
[5] S. Kobayashi, *Hyperbolic complex spaces,* Grundlehren Math. Wiss., 318, Springer-Verlag, Berlin, 1998.
[6] P. Pflug and W. Zwonek *Effective formulas for invariant functions—Case of elementary Reinhardt domains,* Ann. Polon. Math. 69 (1998), 175–196 (erratum: ibid., 301).

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