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Function Theory on the Neil Parabola

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1. Introduction

Distances on a complex space X that are invariant under biholomorphic maps have played an important role in the geometric approach to complex analysis. One of the oldest such distances is the the Carathéodory pseudodistance c_X ("pseudo" because the distance between two points can be zero). It was introduced by Carathéodory in 1926 and is extremely simple to define. The distance between two points x and y is defined to be the largest distance (using the Poincaré hyperbolic distance) that can occur between $f(x)$ and $f(y)$ under a holomorphic map f from X to the unit disk $\mathbb{D} \subset \mathbb{C}$. The Kobayashi pseudodistance k_X, introduced by Kobayashi in 1967, is defined in the opposite direction: the "distance" between two points x and y is now the infimum of the (hyperbolic) distance that can occur between two points $a, b \in \mathbb{D}$ for which there is a holomorphic map f from the disk to X mapping a to x and b to y. (Actually, there is a small technicality here; see Section 4 for the true definition.) A consequence of the Schwarz–Pick lemma on the disk (which says holomorphic self-maps of the disk are distance decreasing in the hyperbolic distance) is the fact that $c_X \leq k_X$.

For the purposes of motivating this paper, let us indulge in a short tangent. An interesting question—because of its geometric implications (including the existence of 1-dimensional analytic retracts)—is: For which complex spaces do we have $c_X = k_X$? The most important contribution to this question is by Lempert [11]. Lempert's theorem proves the Carathéodory and Kobayashi distances agree on a convex domain. This theorem came as a surprise for a couple of reasons: first, convexity is not a biholomorphic invariant; and second, which is our main point here, *there were not many explicit examples available at the time*. (The plot thickens on this problem: There is a domain—namely, the symmetrized bidisc—in $\mathbb{C}^2$ for which the two distances agree, yet this domain is not biholomorphically equivalent to a convex domain; see [9] for a summary of these results.) Although we cannot remedy the problem of a lack of examples in the past, we can attempt to add to the current selection of explicit examples. Many theorems about invariant metrics can be proved in the generality of complex spaces (see e.g. [10]) yet curiously there do not seem to be any nontrivial, explicit examples of the Carathéodory distance for a complex space *with a singularity*. Perhaps the

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simplest complex space with a singularity is the variety contained in the bidisk given by

$$N = \{(z,w) \in \mathbb{D}^2 : z^2 = w^3\}.$$

Following [9], we shall call this the *Neil parabola*. (The real curve $y^2 = x^3$ and its variations are referred to as Neil's semi-cubical parabola; named after William Neil, a student of John Wallis, it was the first algebraic curve to have its arc length computed via proto-calculus techniques [15].) In their recent follow-up [9] to their monograph [8], Jarnicki and Pflug pose the following problem:

> Find an effective formula for the Carathéodory distance on the Neil parabola N.

In this paper, we give an answer to this problem (see Theorem 2.3). In addition, we compute the infinitesimal Carathéodory pseudodistance for the Neil parabola (Theorem 2.4). As applications, we prove a mixed Carathéodory–Pick interpolation result for which known interpolation theorems do not apply (Theorem 2.7) as well as a result on extending bounded holomorphic functions on the Neil parabola to the entire bidisk (Theorem 2.9).

The general layout of the rest of the paper is as follows. Motivation and background for the two previously mentioned applications are presented in the balance of Section 1. In Section 2, precise statements of definitions and results are given along with a subsection on preliminary facts about complex analysis on the Neil parabola. The rest of the paper is devoted to proofs. (The locations of specific proofs are given near the corresponding theorem statements in Section 2.)

1.1. A Mixed Carathéodory–Pick Problem

Given n points in the unit disk z_i and n target values w_i also in the unit disk, the well-known theorem of Pick [13] states exactly when there exists a holomorphic $F\colon \mathbb{D} \to \mathbb{D}$ satisfying $F(z_i) = w_i$ (this problem was studied independently by Nevanlinna [12]). In fact, the Schwarz–Pick lemma is just the version of this for two points: z_1, z_2 can be interpolated to w_1, w_2 if and only if

$$\left|\frac{w_1 - w_2}{1 - \bar{w}_1 w_2}\right| \leq \left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right|.$$

Similarly, given n complex numbers $a_0, a_1, \dots, a_{n-1}$, a well-known theorem of Carathéodory and Fejér [3] states when there exists a holomorphic function $F\colon \mathbb{D} \to \bar{\mathbb{D}}$ with $a_0, a_1, \dots, a_{n-1}$ as the first n Taylor coefficients of F. (Using $\bar{\mathbb{D}}$ instead of $\mathbb{D}$ is just a trick used to include the constant unimodular-valued functions, because we are really talking about functions in the closed unit ball of $H^\infty(\mathbb{D})$; the same idea applies later on to $\mathcal{O}(\mathbb{D}, \bar{\mathbb{D}})$ (though this notation has not yet been introduced).) For $n = 2$, this is given again by the (infinitesimal) Schwarz–Pick lemma: a_0 and a_1 can be the first two Taylor coefficients exactly when

$$|a_0|^2 + |a_1| \leq 1.$$

The first kind of interpolation problem is called Nevanlinna–Pick interpolation and the second is called Carathéodory–Fejér interpolation. More modern proofs of these theorems, using ideas from operator theory like the commutant lifting theorem of Sz.-Nagy and Foiaş and reproducing kernel Hilbert spaces (see [4] and [1]), make it possible to study *mixed Carathéodory–Pick problems* wherein the idea is to specify several Taylor coefficients at several points in the disk and determine whether there exists a holomorphic function from the disk to the disk with those properties. However, a restriction imposed in all of the usual mixed Carathéodory–Pick problems is that the Taylor coefficients must be specified sequentially (i.e., one cannot ask to specify the first and third Taylor coefficients at a point without specifying the second as well). For example, these problems do not address an interpolation problem of the following form: given $z_1, z_2, z_3, w_1, w_2 \in \mathbb{D}$, when is there a holomorphic function $F\colon \mathbb{D} \to \mathbb{D}$ satisfying the equalities (1.1)?

$$\begin{aligned} F(z_1) &= w_1, \\ F(z_2) &= w_2, \\ F'(z_3) &= 0. \end{aligned} \tag{1.1}$$

In fact, as we shall see, solving (1.1) amounts to computing the Carathéodory distance for the Neil parabola. See Theorem 2.7 for the exact statement of our result.

1.2. Extension of Bounded Holomorphic Functions on the Neil Parabola

The following result is a special case of the work of Cartan on Stein varieties; see [6, p. 99]. (In fact, we are stating it in almost as little generality as possible.)

THEOREM 1.2 (Cartan). *Every holomorphic function on a subvariety V of $\mathbb{D}^2$ is the restriction of a holomorphic function on all of $\mathbb{D}^2$.*

A vast improvement on this theorem (again stated in simple terms) was given by Polyakov and Khenkin [14]. They used the methods of integral formulas to prove that any subvariety V of $\mathbb{D}^2$ satisfying a certain transversality condition has the property that any bounded holomorphic function on V can be extended to a bounded holomorphic function on all of $\mathbb{D}^2$. In fact, there is a bounded linear operator $T\colon H^\infty(V) \to H^\infty(\mathbb{D}^2)$ with $Tf|_V = f$; in other words, there is some constant C such that, for any $f \in H^\infty(V)$,

$$\|Tf\|_\infty \le C\|f\|_\infty. \tag{1.3}$$

The previously mentioned “transversality condition” applies to the Neil parabola, so any bounded holomorphic function on N can be extended to a bounded holomorphic function on the bidisk.

Related to these ideas is a paper of Agler and McCarthy [2], which gives a description of varieties in the bidisk with the property that bounded holomorphic functions can be extended to the bidisk without increasing their H^∞ norm. The Neil parabola is not such a variety, as their results show. This can be seen relatively easily from the fact that the Carathéodory pseudodistance on the Neil parabola is not the restriction of the Carathéodory pseudodistance on the bidisk. In other words, there is some holomorphic function from N to $\mathbb{D}$ that separates two points of N farther than a function from the bidisk to the disk could. Hence, such a function could not be extended to the bidisk without increasing its norm.

This suggests that extremal functions on the Neil parabola for the Carathéodory pseudodistance might be good candidates for functions that extend "badly" to the bidisk. Indeed, this allows us to give a lower bound of 5/4 on the constant C in (1.3) for the Neil parabola. In addition to this we present a simple proof using Agler's Nevanlinna–Pick interpolation theorem for the bidisk that any bounded holomorphic function on the Neil parabola can be extended to a bounded holomorphic function on the bidisk with norm increasing by at most a factor of $\sqrt{2}$ if the function vanishes at the origin and by a factor of $2\sqrt{2}+1$ otherwise. This does not exactly reprove Polyakov and Khenkin's result in our context, since we are not claiming that the extension can be given by a linear operator. Nevertheless, it is certainly relevant to their result, is much easier to prove, and provides an explicit bound (see Theorem 2.9).

2. Definitions and Statements of Results

Let us define several important notions for this paper. We shall use $\mathcal{O}(X, Y)$ to denote the set of holomorphic maps from X to Y and $\mathcal{O}(X)$ to denote the set of holomorphic functions from X to $\mathbb{C}$, where X and Y are complex spaces possibly containing singularities (this holds for X hereafter).

- Frequent use will be made of the family of holomorphic automorphisms ϕ_α of the unit disk $\mathbb{D} \subset \mathbb{C}$ given by

$$\phi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}, \tag{2.1}$$

  where $\alpha \in \mathbb{D}$. Note that ϕ_α is its own inverse function. Sometimes we allow α to be in $\partial\mathbb{D}$, but keep in mind that the resulting ϕ_α is no longer an automorphism of the disk and is instead the constant function α.
- The *pseudo-hyperbolic distance* on $\mathbb{D}$ is defined to be

$$m(a, b) = \left| \frac{a - b}{1 - \bar{a} b} \right|.$$

  The *Poincaré distance* on $\mathbb{D}$ is given by $\rho = \tanh^{-1} m$.
- The *Poincaré metric* on the disk, which we shall also denote by ρ, is defined to be

$$\rho(z; v) = \frac{|v|}{1 - |z|^2}$$

  for $z \in \mathbb{D}$ and $v \in \mathbb{C}$.
- The *Carathéodory pseudodistance* on X is denoted by c_X and is defined by

$$c_X(x, y) := \sup\{\rho(f(x), f(y)) : f \in \mathcal{O}(X, \mathbb{D})\}.$$

Replacing ρ with m in this expression yields what Jarnicki and Pflug call the *Möbius pseudodistance*:

$$c_X^*(x,y) := \sup\{m(f(x), f(y)) : f \in \mathcal{O}(X,\mathbb{D})\}.$$

Because of the simple formula for m and the relation $c_X = \tanh^{-1} c_X^*$, the Möbius pseudodistance is more computationally useful for our purposes and hence will be used exclusively in all proofs.

- The *Carathéodory pseudometric* C_X is defined to be

$$C_X(x;v) = \sup\{\rho(f(x); df_x(v)) : f \in \mathcal{O}(X,\mathbb{D})\}$$

for $x \in X$ and $v \in T_x X$, the tangent space of X at x. The Carathéodory pseudometric is often referred to as the *infinitesimal Carathéodory pseudodistance*.

- Finally, the *Lempert function* for X is denoted $\tilde{k}_X$ and is defined by

$$\tilde{k}_X(x,y) = \inf\{\rho(a,b) : \exists f \in \mathcal{O}(\mathbb{D},X) \text{ with } f(a) = x,\ f(b) = y\},$$

where $\tilde{k}_X$ is defined to equal ∞ if the set over which the infimum is taken is empty. The *Kobayashi pseudodistance* k_X is then defined to be largest pseudodistance bounded by $\tilde{k}_X$.

For more information on and examples of these definitions see [7; 8; 9; 10].

Recall from Section 1 that the Neil parabola is the set

$$N = \{(z,w) \in \mathbb{D}^2 : z^2 = w^3\}.$$

The set N is a 1-dimensional connected analytic variety in $\mathbb{D}^2$ with a singularity at $(0,0)$. Furthermore, N has a bijective holomorphic parameterization $p\colon \mathbb{D} \to N$ given by

$$p(\lambda) := (\lambda^3, \lambda^2). \tag{2.2}$$

The function $q := p^{-1}$ is continuous on N and holomorphic on $N \setminus \{(0,0)\}$, and it can be given by $q(z,w) = z/w$ when $(z,w) \neq (0,0)$ (and $q(0,0) = 0$). For the benefit of those readers unfamiliar with holomorphic functions on a variety with a singularity, we include a discussion of these ideas in the concrete context of the Neil parabola in Section 2.1. It is known that the Kobayashi pseudodistance k_N and the Lempert function $\tilde{k}_N$ for N are as simple as possible (see [9]):

$$k_N((a,b),(z,w)) = \tilde{k}_N((a,b),(z,w)) = \rho(q(a,b), q(z,w)).$$

On the other hand (and to reiterate our goal in this paper), in [9] the authors lament that, despite the simplicity of N, an effective formula for the Carathéodory pseudodistance c_N is not known. We propose the following as an effective formula for c_N.

THEOREM 2.3 (Carathéodory pseudodistance formula). *Given nonzero* $\lambda, \delta \in \mathbb{D}$, *let*

$$\alpha_0 = \frac{1}{2}\left(\frac{1}{\bar{\lambda}} + \lambda + \frac{1}{\bar{\delta}} + \delta\right).$$

Then

$$c_N(p(\lambda), p(\delta)) = \begin{cases} \rho(\lambda^2, \delta^2) & \text{if } |\alpha_0| \geq 1, \\ \rho(\lambda^2\phi_{\alpha_0}(\lambda), \delta^2\phi_{\alpha_0}(\delta)) & \text{if } |\alpha_0| < 1. \end{cases}$$

Also, $c_N(p(0), p(\lambda)) = \rho(0, \lambda^2) = \tanh^{-1}|\lambda|^2$.

In particular it should be noted that, if λ and δ have an acute angle between them (i.e., if $\mathrm{Re}(\lambda\bar{\delta}) > 0$), then $|\alpha_0| > 1$ and so the first formula gives the distance between $p(\lambda)$ and $p(\delta)$. Also, the theorem shows that $k_N \neq c_N$ as one might suspect.

In Section 3 we shall reduce this problem to a maximization problem on the closed unit disk, and in Section 4 we solve the maximization problem to yield Theorem 2.3. In addition, a slightly nicer form of the preceding formula will be presented as Proposition 4.14.

As will be explained in Section 2.1, the tangent spaces of N can be identified with subspaces of the tangent spaces of $\mathbb{D}^2$. In particular, for $x = (a, b) \neq (0, 0)$, $T_x N$ is simply the span of the vector $(3a, 2b)$; the tangent space at the origin of N is 2-dimensional and thus equal to all of $\mathbb{C}^2 = T_{(0,0)}\mathbb{D}^2$. We can now present our formula for the Carathéodory pseudometric of N (this is proved in Section 5).

THEOREM 2.4 (Carathéodory pseudometric formula). *For* $v = (v_1, v_2) \in \mathbb{C}^2$ *we have*

$$C_N((0,0); v) = \begin{cases} |v_2| & \text{if } |v_2| \geq 2|v_1|, \\ \frac{4|v_1|^2+|v_2|^2}{4|v_1|} & \text{if } |v_2| < 2|v_1|, \end{cases} \tag{2.5}$$

and for $(a, b) \in N$ *nonzero and* $z \in \mathbb{C}$ *we have*

$$C_N((a,b); z(3a, 2b)) = \frac{2|b|}{1-|b|^2}|z|. \tag{2.6}$$

As mentioned in Section 1.1, a direct consequence of Theorem 2.3 is the following atypical mixed Carathéodory–Pick interpolation result (see Section 6 for the proof).

THEOREM 2.7 (Mixed interpolation problem). *Given distinct* $z_1, z_2, z_3 \in \mathbb{D}$ *and* $w_1, w_2 \in \mathbb{D}$, *there exists an* $F \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ *with*

$$F(z_i) = w_i \quad (i = 1, 2) \quad \text{and}$$
$$F'(z_3) = 0$$

if and only if

$$\rho(w_1, w_2) \leq c_N(p(\phi_{z_3}(z_1)), p(\phi_{z_3}(z_2))). \tag{2.8}$$

Moreover, if the problem is extremal (*i.e., if there is equality in* (2.8)) *then the solution is unique and is a Blaschke product of order* 2 *or* 3.

The significance of the theorem (which, as now worded, practically follows from definitions) is of course that c_N is directly computable by Theorem 2.3. (So, inequality (2.8) is easy to check.)

Finally, in Section 7 we prove the following result on extending bounded holomorphic functions from the Neil parabola to the bidisk.

THEOREM 2.9 (Bounded analytic extension). *For any* $f \in \mathcal{O}(N, \mathbb{D})$ *with* $f(0,0) = 0$, *there exists an extension of* f *to a function in* $\mathcal{O}(\mathbb{D}^2, \sqrt{2}\mathbb{D})$. *If* $f(0,0) \neq 0$, *then* f *can be extended to* $\mathcal{O}(\mathbb{D}^2, (2\sqrt{2}+1)\mathbb{D})$. *In addition, there exists a function in* $\mathcal{O}(N, \mathbb{D})$ *that cannot be extended to a function in* $\mathcal{O}(\mathbb{D}^2, r\mathbb{D})$ *for* $r < 5/4$.

Here $r\mathbb{D}$ simply denotes the disk of radius r.

2.1. Complex Analysis on the Neil Parabola

In this section we discuss how to perform complex analysis on a variety with a singularity in the concrete setting of the Neil parabola. This is adapted from [9] and [5, pp. 18–20, Chap. B], and nothing in this section is by any means new. The most important facts of this section are summarized in the two "observations" 2.10 and 2.11.

A function f on N is defined to be holomorphic if at each point $x \in N$ there is a holomorphic function F on a neighborhood U of x in the bidisk that agrees with f on $U \cap N$. Fortunately, we can give a more concrete description of the set of holomorphic functions on N. Given $f \in \mathcal{O}(N)$, the function $h := f \circ p$ (recall p from (2.2)) is an element of $\mathcal{O}(\mathbb{D})$ satisfying $h'(0) = 0$. The reason is that, if an extension F of f is holomorphic on a neighborhood of $(0,0)$ in $\mathbb{D}^2$, then $h = F \circ p$ is holomorphic on a neighborhood of 0 in $\mathbb{D}$. Hence, the derivative $h'(\lambda) = dF_{p(\lambda)}(3\lambda^2, 2\lambda)$ and so $h'(0) = 0$.

Conversely, suppose $h \in \mathcal{O}(\mathbb{D})$ satisfies $h'(0) = 0$. Then, $f := h \circ q$ (recall $q := p^{-1}$) is holomorphic on $N \setminus \{(0,0)\}$ because $F(z,w) = h(z/w)$ is holomorphic on the set $\{(z,w) \in \mathbb{D}^2 : |z| < |w|\}$, which is an open neighborhood of $N \setminus \{(0,0)\}$. To prove f is holomorphic at $(0,0)$, observe first of all that h can be written as an absolutely convergent power series $h(\lambda) = a_0 + a_2\lambda^2 + a_3\lambda^3 + \cdots$ in some (or any) closed disk centered at the origin of radius, say, $r < 1$. Then, for (z,w) with $|z| < 1$ and $|w| < r^3$,

$$F(z,w) := a_0 + a_2 w + a_3 z + a_4 w^2 + a_5 zw + a_6 w^3 + \cdots$$

converges absolutely and extends f. (Here we are choosing to extend $(z/w)^k$ to a monomial of the form zw^m or w^m—that is, we want the power of w to be as large as possible.)

Let us emphasize the correspondence just proved as follows.

OBSERVATION 2.10. The map given by $f \mapsto f \circ p$ is a bijection from $\mathcal{O}(N)$ to $\{h \in \mathcal{O}(\mathbb{D}) : h'(0) = 0\}$ with inverse given by $h \mapsto h \circ q$, where $p \in \mathcal{O}(\mathbb{D}, N)$ is $p(\lambda) = (\lambda^3, \lambda^2)$ and $q = p^{-1}$.

Next, we discuss the complex tangent spaces of N. We can define $T_x N$ as a subset of $T_x\mathbb{D}^2 \cong \mathbb{C}^2$ in the following way. If $v \in \mathbb{C}^2$, then $v \in T_x N$ if and only if $dG_x v = 0$ for every holomorphic function G defined in a neighborhood U (in $\mathbb{D}^2$) of x with G identically zero restricted to $U \cap M$. Notice that this formulation is designed to make it easy to define the differential of a function $g \in \mathcal{O}(N)$.

If $x = p(\lambda) = (a,b) \neq (0,0)$, then $T_x N$ is the span of the vector $(3a, 2b)$ because, given G as in the previous paragraph, the function $g := G \circ p$ is identically zero and so $0 = g'(\lambda) = dG_x(3\lambda^2, 2\lambda)$. Hence, $dG_x(3a, 2b) = 0$. On the other hand, $h(z, w) = z^2 - w^3$ vanishes on N and $dh_x v = 0$ if and only if v is a multiple of $(3a, 2b)$.

At the origin $x = (0, 0)$, we have $T_x N = \mathbb{C}^2$ because, given G as before, we have $dG_{(0,0)} = (0, 0)$. This is because the partial derivatives of G at $(0, 0)$ are the coefficients of λ^3 and λ^2 in the identically zero power series for $G(\lambda^3, \lambda^2)$. Let us emphasize these facts as follows.

OBSERVATION 2.11. The tangent space $T_{(a,b)}N$ at the point $(a, b) \in N$ with $(a, b) \neq (0, 0)$ can be identified with $\{\zeta(3a, 2b) : \zeta \in \mathbb{C}\} \subset \mathbb{C}^2$. The tangent space $T_{(0,0)}N$ can be identified with $\mathbb{C}^2$.

3. Reduction of Theorem 2.3 to a max Problem on $\bar{\mathbb{D}}$

As mentioned previously, we shall compute a formula for c_N^* (which of course gives a formula for c_N).

By Observation 2.10, we immediately have

$$c_N^*(p(\lambda), p(\delta)) = \sup\{m(h(\lambda), h(\delta)) : h \in \mathcal{O}(\mathbb{D}, \mathbb{D}),\ h'(0) = 0\}. \tag{3.1}$$

Because m is invariant under automorphisms of the disk, we may assume that $h(0) = 0$ by applying appropriate automorphisms of the disk, since the condition $h'(0) = 0$ is preserved by (post) composition. Then, by the Schwarz lemma, h may be written as $h(\zeta) = \zeta^2 g(\zeta)$ for some $g \in \mathcal{O}(\mathbb{D}, \bar{\mathbb{D}})$. At this stage it is clear that $c_M^*(p(0), p(\lambda)) = |\lambda|^2$. Since g varies over all of $\mathcal{O}(\mathbb{D}, \bar{\mathbb{D}})$, the set of pairs $(g(\lambda), g(\delta))$ is just the set of all (A, B) satisfying $m(A, B) \leq m(\lambda, \delta)$. Hence

$$c_N^*(p(\lambda), p(\delta)) = \sup\{m(\lambda^2 A, \delta^2 B) : A, B \in \mathbb{D} \text{ with } m(A, B) \leq m(\lambda, \delta)\}.$$

Because $m(\lambda^2 A, \delta^2 B)$ is the modulus of a holomorphic function in A, the maximum principle allows us to take this supremum over all (A, B) with $m(A, B) = m(\lambda, \delta)$. We may safely multiply both A and B by a unimodular constant and leave $m(\lambda^2 A, \delta^2 B)$ unchanged. Thus, we can assume there is some $\alpha \in \mathbb{D}$ such that $A = \phi_\alpha(\lambda)$ and $B = \phi_\alpha(\delta)$ (recall ϕ_α from (2.1)).

This discussion yields the following formula for c_N^*, which gives the desired reduction to a maximization problem.

PROPOSITION 3.2.

$$c_N^*(p(\lambda), p(\delta)) = \max_{\alpha \in \bar{\mathbb{D}}} m(\lambda^2 \phi_\alpha(\lambda), \delta^2 \phi_\alpha(\delta)).$$

In particular, the supremum in (3.1) is attained by some function of the form $h(\zeta) = \zeta^2 \phi_\alpha(\zeta)$, where $\alpha \in \bar{\mathbb{D}}$. Moreover, if h attains the supremum in (3.1) and if $h(0) = 0$, then h is of the same form (i.e., $h = \zeta^2 \phi_\alpha$) up to multiplication by a unimodular constant. As we shall see, the supremum will be obtained either with a unique $\alpha \in \mathbb{D}$ or with any $\alpha \in \partial\mathbb{D}$.

4. Proof of Theorem 2.3

To begin, we shall keep λ and δ fixed throughout the section and define a continuous function $F : \bar{\mathbb{D}} \to [0, 1)$, smooth except possibly where it is zero, by

$$F(\alpha) := m(\lambda^2\phi_\alpha(\lambda), \delta^2\phi_\alpha(\delta)). \tag{4.1}$$

Note that

$$F(\alpha) < m(\lambda, \delta) \quad \text{for all } \alpha \in \bar{\mathbb{D}}$$

and

$$F(\alpha) = m(\lambda^2, \delta^2) \quad \text{for all } \alpha \in \partial\mathbb{D}. \tag{4.2}$$

As in the statement of Theorem 2.3, we let

$$\alpha_0 := \frac{1}{2}\left(\frac{1}{\bar{\lambda}} + \lambda + \frac{1}{\bar{\delta}} + \delta\right).$$

By Proposition 3.2, the following two claims (to be given as Lemmas 4.6 and 4.11) yield Theorem 2.3. First, F has no local maximum in $\mathbb{D}$ except possibly α_0. Second, if $|\alpha_0| < 1$ then $F(\alpha) \leq F(\alpha_0)$ for all α with $|\alpha| = 1$. Before we prove these facts, let us mention a couple of useful formulas for F whose proofs we defer to the end of the section.

CLAIM 4.3.

$$F(\alpha) = m(\lambda, \delta)\left|\frac{(\lambda+\delta)(\alpha+\lambda\delta\bar{\alpha}-\lambda-\delta)+\lambda\delta(1-|\alpha|^2)}{(1+\lambda\bar{\delta})(1+\lambda\bar{\delta}-\bar{\alpha}\lambda-\alpha\bar{\delta})-\lambda\bar{\delta}(1-|\alpha|^2)}\right| \tag{4.4}$$

$$= m(\lambda, \delta)\left|\frac{1-(\bar{\alpha}-\bar{\alpha}_0-\bar{\beta}_2)(\alpha-\alpha_0+\beta_2)}{1-(\bar{\alpha}-\bar{\alpha}_0-\bar{\beta}_1)(\alpha-\alpha_0+\beta_1)}\right|, \tag{4.5}$$

where

$$\beta_1 := \frac{1}{2}\left(\frac{1}{\bar{\lambda}} - \lambda - \frac{1}{\bar{\delta}} + \delta\right) \quad \textit{and}$$

$$\beta_2 := \frac{1}{2}\left(\frac{1}{\bar{\lambda}} - \lambda + \frac{1}{\bar{\delta}} - \delta\right).$$

LEMMA 4.6. *The function F has no local maximum in $\mathbb{D}$ except possibly at α_0.*

Proof. Using formula (4.5), it suffices to prove that the function given by

$$G(z) = \left(\frac{F(z+\alpha_0)}{m(\lambda,\delta)}\right)^2 = \left|\frac{1-(\bar{z}-\bar{\beta}_2)(z+\beta_2)}{1-(\bar{z}-\bar{\beta}_1)(z+\beta_1)}\right|^2 \tag{4.7}$$

has no local maximum for $|z + \alpha_0| < 1$ except possibly at $z = 0$. Some omitted computations show that G can be written as G_2/G_1, where

$$G_k(z) = 1 + 2|\beta_k|^2 - 2|z|^2 + |z^2 - \beta_k^2|^2 \tag{4.8}$$

for $k = 1, 2$.

Throughout the following, suppose that z is a local maximum satisfying $0 < |z + \alpha_0| < 1$. This implies several statements:

- $0 < G(z) < 1$;
- z is a critical point for G;
- $\Delta \log G(z) \leq 0$; and
- $\det \operatorname{Hess}(\log G) \geq 0$ at z.

Here Hess denotes the matrix of second partial derivatives. We will prove that all of these conditions cannot be satisfied.

First, compute all of the derivatives of G_1 and G_2 up to second order. (Luckily we can examine G_1 and G_2 simultaneously.) Writing $z = x + iy$ then yields

$$\begin{aligned}
\partial_z G_k &= -2\bar{z} + 2z(\bar{z}^2 - \bar{\beta}_k^2),\\
\partial_x G_k &= -4x + 4\operatorname{Re}[z(\bar{z}^2 - \bar{\beta}_k^2)],\\
\partial_y G_k &= -4y - 4\operatorname{Im}[z(\bar{z}^2 - \bar{\beta}_k^2)],\\
\partial_{xx}^2 G_k &= -4 + 4|z|^2 + 8x^2 - 4\operatorname{Re}\beta_k^2,\\
\partial_{yy}^2 G_k &= -4 + 4|z|^2 + 8y^2 + 4\operatorname{Re}\beta_k^2,\\
\partial_{xy}^2 G_k &= 8xy - 4\operatorname{Im}\beta_k^2.
\end{aligned}$$

Because z is a critical point for G, we have $G_1\partial_z G_2 - G_2\partial_z G_1 = 0$ at z. Neither G_1 nor G_2 vanish at z, so if $\partial_z G_1 = 0$ then $\partial_z G_2 = 0$. But that $\partial_z G_1$ and $\partial_z G_2$ vanish simultaneously only at 0—

$$\partial_z G_k = -2\bar{z} + 2z(\bar{z}^2 - \bar{\beta}_k^2) = 0$$

for $k = 1, 2$—implies $\bar{z}(\beta_1^2 - \beta_2^2) = 0$, which can happen only if $z = 0$ (because $\beta_1^2 - \beta_2^2 = -(1 - |\lambda|^2)(1 - |\delta|^2)/(\bar{\lambda}\bar{\delta}) \neq 0$). Therefore, at z we have

$$\frac{G_2}{G_1} = \frac{\partial_z G_2}{\partial_z G_1}, \quad \frac{\partial_x G_1}{G_1} = \frac{\partial_x G_2}{G_2}, \quad \frac{\partial_y G_1}{G_1} = \frac{\partial_y G_2}{G_2}. \tag{4.9}$$

A fact derived from the first of these equations is that

$$\left(\frac{\bar{\beta}_1^2}{G_1} - \frac{\bar{\beta}_2^2}{G_2}\right) z^2 = |z|^2(1 - |z|^2)\left(\frac{1}{G_2} - \frac{1}{G_1}\right); \tag{4.10}$$

in particular, the expression on the left is real.

Next, the last two equations in (4.9) show that, at the critical point z, the following equations hold:

$$\begin{aligned}
\partial_{xx}^2 \log G &= \frac{\partial_{xx}^2 G_2}{G_2} - \frac{\partial_{xx}^2 G_1}{G_1}\\
&= (-4 + 4|z|^2 + 8x^2)\left(\frac{1}{G_2} - \frac{1}{G_1}\right) + 4\operatorname{Re}\left(\frac{\bar{\beta}_1^2}{G_1} - \frac{\bar{\beta}_2^2}{G_2}\right)\\
&= -4\left[(1 - |z|^2)\left(1 - \operatorname{Re}\left(\frac{z^2}{|z|^2}\right)\right) - 2x^2\right]\left(\frac{1}{G_2} - \frac{1}{G_1}\right),
\end{aligned}$$

where the last equality follows from (4.10). Similarly,

$$\begin{aligned}
\partial_{yy}^2 \log G &= -4\left[(1 - |z|^2)\left(1 + \operatorname{Re}\left(\frac{z^2}{|z|^2}\right)\right) - 2y^2\right]\left(\frac{1}{G_2} - \frac{1}{G_1}\right),\\
\partial_{xy}^2 \log G &= 4\left[2xy + (1 - |z|^2)\operatorname{Im}\left(\frac{z^2}{|z|^2}\right)\right]\left(\frac{1}{G_2} - \frac{1}{G_1}\right).
\end{aligned}$$

Therefore,

$$\Delta \log G = -8(1-3|z|^2)\left(\frac{1}{G_2}-\frac{1}{G_1}\right);$$

since this must be less than or equal to zero at z, we see that $|z|^2 \le 1/3$.

Finally, we can show that $\det \operatorname{Hess}(\log G) < 0$, contradicting our assumption that z is a local maximum. The determinant of the Hessian of the logarithm of G (with the positive factor $16(1/G_2 - 1/G_1)^2$ omitted) is

$$\begin{aligned}(1-|z|^2)^2&(1-(\operatorname{Re}(z^2/|z|^2))^2) + 4x^2y^2 - 2|z|^2(1-|z|^2)\\ &+ 2(y^2-x^2)(1-|z|^2)\operatorname{Re}(z^2/|z|^2)\\ &- 4x^2y^2 - 4xy(1-|z|^2)\operatorname{Im}(z^2/|z|^2) - (1-|z|^2)^2(\operatorname{Im}(z^2/|z|^2))^2.\end{aligned}$$

Canceling the positive factor $(1-|z|^2)$ and simplifying yields $-4|z|^2$, which is indeed negative as promised. □

LEMMA 4.11. *If* $|\alpha_0| < 1$, *then* $F(\alpha) \le F(\alpha_0)$ *for all* α *with* $|\alpha| = 1$.

Proof. Recall from (4.2) that, on the boundary of $\bar{\mathbb{D}}$, F is constant and equal to $m(\lambda^2, \delta^2)$. By equation (4.4), it suffices to prove the inequality

$$\left|\frac{\lambda+\delta}{1+\bar\lambda\delta}\right|^2 \le \left|\frac{(\lambda+\delta)(\alpha_0+\lambda\delta\bar\alpha_0-\lambda-\delta)+\lambda\delta(1-|\alpha_0|^2)}{(1+\lambda\bar\delta)(1+\lambda\bar\delta-\bar\alpha_0\lambda-\alpha_0\bar\delta)-\lambda\bar\delta(1-|\alpha_0|^2)}\right|^2.$$

Assuming the left-hand side is nonzero (which we can), it suffices to prove

$$\begin{aligned}\left|(\alpha_0+\lambda\delta\bar\alpha_0-\lambda-\delta)+\lambda\delta\frac{(1-|\alpha_0|^2)}{\lambda+\delta}\right|^2&\\ -\left|(1+\lambda\bar\delta-\bar\alpha_0\lambda-\alpha_0\bar\delta)-\lambda\bar\delta\frac{(1-|\alpha_0|^2)}{1+\lambda\bar\delta}\right|^2 &\ge 0. \qquad (4.12)\end{aligned}$$

If we think of the left-hand side as

$$|A+B|^2-|C+D|^2 = |A|^2-|C|^2+2\operatorname{Re}(A\bar B - C\bar D)+|B|^2-|D|^2,$$

then first of all $|A|^2-|C|^2$ equals

$$|\alpha_0+\lambda\delta\bar\alpha_0-\lambda-\delta|^2-|1+\lambda\bar\delta-\bar\alpha_0\lambda-\alpha_0\bar\delta|^2 = -(1-|\alpha_0|^2)(1-|\lambda|^2)(1-|\delta|^2).$$

Using the identities

$$\alpha_0+\lambda\delta\bar\alpha_0-(\lambda+\delta) = \frac{\bar\lambda+\bar\delta}{2\bar\lambda\bar\delta}(1+|\lambda\delta|^2) \quad \text{and}$$

$$1+\lambda\bar\delta-\bar\alpha_0\lambda-\alpha_0\bar\delta = -\frac{1+\bar\lambda\delta}{2\bar\lambda\delta}(|\lambda|^2+|\delta|^2),$$

we get $2\operatorname{Re}(A\bar B - C\bar D) = (1-|\alpha_0|^2)(1-|\lambda|^2)(1-|\delta|^2)$. Also, using the identity

$$|1+a\bar b|^2-|a+b|^2 = (1-|a|^2)(1-|b|^2), \qquad (4.13)$$

we see that $|B|^2-|D|^2$ equals

$$|\lambda\delta|^2(1-|\alpha_0|^2)^2\frac{(1-|\lambda|^2)(1-|\delta|^2)}{|\lambda+\delta|^2|1+\lambda\bar\delta|^2}.$$

Summing this all up, we see that proving (4.12) amounts to showing

$$|\lambda\delta|^2(1-|\alpha_0|^2)^2\frac{(1-|\lambda|^2)(1-|\delta|^2)}{|\lambda+\delta|^2|1+\lambda\bar{\delta}|^2}\geq 0,$$

which is certainly true. □

This concludes the proof of Theorem 2.3. As promised, here is a slightly nicer formula for $c_N^*(p(\lambda),p(\delta))$.

PROPOSITION 4.14. *If* $\lambda,\delta\in\mathbb{D}$ *are nonzero, then*

$$c_N^*(p(\lambda),p(\delta))=\begin{cases} m(\lambda^2,\delta^2) & \text{if } |\alpha_0|\geq 1,\\ m(\lambda,\delta)\frac{1+|\beta_2|^2}{1+|\beta_1|^2} & \text{if } |\alpha_0|<1.\end{cases}$$

This follows from Proposition 3.2, the definition of F (viz., equation (4.1)), formula (4.5) for F, and Lemmas 4.6 and 4.11.

We conclude this section with the proof of Claim 4.3.

Proof of Claim 4.3. We start from equation (4.1). Observe that

$$\begin{aligned}
F(\alpha)&=\left|\frac{\lambda^2\frac{\alpha-\lambda}{1-\bar{\alpha}\lambda}-\delta^2\frac{\alpha-\delta}{1-\bar{\alpha}\delta}}{1-\lambda^2\bar{\delta}^2\frac{\alpha-\lambda}{1-\bar{\alpha}\lambda}\frac{\bar{\alpha}-\bar{\delta}}{1-\alpha\bar{\delta}}}\right|\\
&=\left|\frac{\lambda^2(\alpha-\lambda)(1-\bar{\alpha}\delta)-\delta^2(\alpha-\delta)(1-\bar{\alpha}\lambda)}{(1-\bar{\alpha}\lambda)(1-\alpha\bar{\delta})-\lambda^2\bar{\delta}^2(\alpha-\lambda)(\bar{\alpha}-\bar{\delta})}\right|\\
&=\left|\frac{\alpha(\lambda^2-\delta^2)-(\lambda^3-\delta^3)-|\alpha|^2\lambda\delta(\lambda-\delta)+\lambda\delta(\lambda^2-\delta^2)\bar{\alpha}}{1-\lambda^3\bar{\delta}^3-\bar{\alpha}\lambda(1-\lambda^2\bar{\delta}^2)-\alpha\bar{\delta}(1-\lambda^2\bar{\delta}^2)+|\alpha|^2\lambda\bar{\delta}(1-\lambda\bar{\delta})}\right|\\
&=m(\lambda,\delta)\left|\frac{\alpha(\lambda+\delta)-(\lambda^2+\lambda\delta+\delta^2)-|\alpha|^2\lambda\delta+\lambda\delta(\lambda+\delta)\bar{\alpha}}{1+\lambda\bar{\delta}+\lambda^2\bar{\delta}^2-\bar{\alpha}\lambda(1+\lambda\bar{\delta})-\alpha\bar{\delta}(1+\lambda\bar{\delta})+|\alpha|^2\lambda\bar{\delta}}\right| \qquad (4.15)\\
&=m(\lambda,\delta)\left|\frac{\alpha(\lambda+\delta)+\lambda\delta(\lambda+\delta)\bar{\alpha}-(\lambda+\delta)^2+\lambda\delta(1-|\alpha|^2)}{(1+\lambda\bar{\delta})^2-\bar{\alpha}\lambda(1+\lambda\bar{\delta})-\alpha\bar{\delta}(1+\lambda\bar{\delta})-(1-|\alpha|^2)\lambda\bar{\delta}}\right|;
\end{aligned}$$

from here it is easy to derive (4.4).

Second, to prove (4.5) we start from (4.15):

$$\begin{aligned}
F(\alpha)&=m(\lambda,\delta)\left|\frac{\lambda\delta-(\lambda\delta\bar{\alpha}-(\lambda+\delta))(\alpha-(\lambda+\delta))}{\lambda\bar{\delta}-(\bar{\alpha}\lambda-(1+\lambda\bar{\delta}))(\alpha\bar{\delta}-(1+\lambda\bar{\delta}))}\right|\\
&=m(\lambda,\delta)\left|\frac{1-(\bar{\alpha}-(1/\delta+1/\lambda))(\alpha-(\lambda+\delta))}{1-(\bar{\alpha}-(1/\lambda+\bar{\delta}))(\alpha-(1/\bar{\delta}+\lambda))}\right|;
\end{aligned}$$

this equals (4.5) as a consequence of the following identities:

$$\begin{aligned}
\bar{\alpha}_0+\bar{\beta}_2&=\frac{1}{\lambda}+\frac{1}{\delta},\\
\alpha_0-\beta_2&=\lambda+\delta,
\end{aligned}$$

$$\bar{\alpha}_0 + \bar{\beta}_1 = \frac{1}{\lambda} + \bar{\delta},$$

$$\alpha_0 - \beta_1 = \lambda + \frac{1}{\bar{\delta}}. \qquad \square$$

5. The Infinitesimal Carathéodory Pseudodistance

In this section we prove Theorem 2.4, our formula for the Carathéodory pseudometric.

The Carathéodory pseudometric at the origin with respect to a vector $v = (v_1, v_2) \in \mathbb{C}^2$ is

$$C_N((0,0); v) = \sup\{|df_{(0,0)}v| : f \in \mathcal{O}(N, \mathbb{D}) \text{ and } f(0,0) = 0\}.$$

Any such f satisfies $f(\lambda^3, \lambda^2) = \lambda^2 g(\lambda)$ for some $g \in \mathcal{O}(\mathbb{D}, \bar{\mathbb{D}})$ (see the beginning of Section 3). Also, the partial derivative of f with respect to the first variable at the origin is just $g'(0)$, and the partial derivative of f with respect to the second variable at the origin is $g(0)$ (see Section 2.1). Therefore,

$$C_N((0,0); v) = \sup\{|v_1 g'(0) + v_2 g(0)| : g \in \mathcal{O}(\mathbb{D}, \bar{\mathbb{D}})\}.$$

The set of pairs $(g'(0), g(0))$ as g varies over $\mathcal{O}(\mathbb{D}, \bar{\mathbb{D}})$ is really just the set of pairs (A, B) where $|A| + |B|^2 \le 1$ (by the Schwarz–Pick lemma). With suitable choices for the arguments of A and B, we can reduce the problem to maximizing $|v_1|s + |v_2|t$ over all $s, t \in [0, 1]$ that satisfy $s + t^2 \le 1$. The function we are maximizing is linear, so the maximum occurs on the boundary. Therefore, the problem is just a matter of finding the maximum of $|v_1|(1 - t^2) + |v_2|t$ for $0 \le t \le 1$. By calculus, we have

$$C_N((0,0); v) = \begin{cases} |v_2| & \text{if } |v_2| \ge 2|v_1|, \\ \frac{4|v_1|^2 + |v_2|^2}{4|v_1|} & \text{if } |v_2| < 2|v_1|, \end{cases}$$

as desired.

Next, let $x = (a, b) \in N \setminus \{(0,0)\}$ and define $v = (3a, 2b)$. The Carathéodory pseudometric at (a, b) is

$$C_N(x; v) = \sup\left\{\frac{|df_x v|}{1 - |f(x)|^2} : f \in \mathcal{O}(N, \mathbb{D})\right\}.$$

If we set $\lambda = a/b$ and $h = f \circ p$, then $v = \lambda(3\lambda^2, 2\lambda)$ and, since $df_x(3\lambda^2, 2\lambda) = h'(\lambda)$, we see that

$$C_N(x; v) = |\lambda| \sup\{\rho(h(\lambda); h'(\lambda)) : h \in \mathcal{O}(\mathbb{D}, \mathbb{D}) \text{ and } h'(0) = 0\}.$$

By (post) composing h with an automorphism of the unit disk (which is allowed by invariance properties of ρ), we can assume $h(0) = 0$ and thus h has the form $h(\zeta) = \zeta^2 g(\zeta)$ for some $g \in \mathcal{O}(\mathbb{D}, \bar{\mathbb{D}})$. Hence,

$$C_N(x; v) = |\lambda| \sup\left\{\frac{|\lambda^2 g'(\lambda) + 2\lambda g(\lambda)|}{1 - |\lambda|^4 |g(\lambda)|^2} : g \in \mathcal{O}(\mathbb{D}, \bar{\mathbb{D}})\right\}.$$

As before, $(g'(\lambda), g(\lambda))$ varies over all pairs (A, B) that satisfy $|A|(1 - |\lambda|^2) \le 1 - |B|^2$. This reduces the problem to maximizing

$$\frac{|\lambda|^2 s + 2|\lambda| t}{1 - |\lambda|^4 t^2}$$

over the set of nonnegative s, t satisfying $t^2 + s(1 - |\lambda|^2) \leq 1$. It is easy to check that the maximum always occurs when $t = 1$ and $s = 0$. Since $\lambda^2 = b$, we see that

$$C_N(x; v) = \frac{2|b|}{1 - |b|^2}.$$

6. Proof of Theorem 2.7

By precomposing all functions with ϕ_{z_3}, we may assume that $z_3 = 0$ in Theorem 2.7. Then, all functions of interest will correspond to functions in $\mathcal{O}(N, \mathbb{D})$. It is therefore clear that, if there is a function $h \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ satisfying both $h'(0) = 0$ and $h(z_i) = w_i$ for $i = 1, 2$, then inequality (2.8) holds (by Theorem 2.3 and the definition of Carathéodory pseudodistance).

On the other hand, if inequality (2.8) holds (again with $z_3 = 0$), then we can pick a function $f \in \mathcal{O}(N, \mathbb{D})$ with

$$\rho(f(p(z_1)), f(p(z_2))) = c_N(p(z_1), p(z_2))$$

(we know such a function exists by the formula for c_N) and then set $h := f \circ p \in \mathcal{O}(\mathbb{D}, \mathbb{D})$. The function h satisfies $\rho(w_1, w_2) \leq \rho(h(z_1), h(z_2))$ and, by composing h with an appropriate function, we can find a function $F \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ with $F(z_1) = w_1$, $F(z_2) = w_2$, and $F'(0) = 0$.

To prove the last part of Theorem 2.7, suppose F satisfies the interpolation problem and suppose there is equality in (2.8). Then $h := \phi_{F(0)} \circ F$ satisfies equality as well. Hence, if

$$\alpha_0 := \frac{1}{2}\left(\frac{1}{\bar{z}_1} + z_1 + \frac{1}{\bar{z}_2} + z_2\right)$$

is in the disk then $h(\lambda)$ is of the form $\mu\lambda^2\phi_{\alpha_0}(\lambda)$, where μ is a unimodular constant, and if $\alpha_0 \notin \mathbb{D}$ then $h(\lambda)$ is of the form $\mu\lambda^2$ (again with $\mu \in \partial\mathbb{D}$). But μ and $F(0)$ are uniquely determined by the fact that $w_i = \phi_{F(0)}(h(z_i))$ for $i = 1, 2$, since $h(z_1)$ and $h(z_2)$ must be distinct. So there exists a unique automorphism of the disk ψ such that

$$F(\lambda) = \begin{cases} \psi(\lambda^2\phi_{\alpha_0}(\lambda)) & \text{if } \alpha_0 \in \mathbb{D}, \\ \psi(\lambda^2) & \text{if } \alpha_0 \notin \mathbb{D}. \end{cases}$$

In the first case, F is a Blaschke product of order 3 and in the second a Blaschke product of order 2.

7. Proof of Extension Theorem

In this section we prove Theorem 2.9.

First, we need to define a few basic notions. Let X be a set. A self-adjoint function $F \colon X \times X \to \mathbb{C}$ (i.e., $F(x, y) = \overline{F(y, x)}$) is *positive semidefinite* if, for every positive integer n and every finite subset $\{x_1, x_2, \ldots, x_n\} \subset X$, the $n \times n$ matrix with entries $F(x_i, x_j)$ is positive semidefinite. For example, by the Pick interpolation theorem the function $F\colon \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ given by

$$F(\lambda, \delta) = \frac{1 - g(\lambda)\overline{g(\delta)}}{1 - \lambda\bar{\delta}}$$

is positive semidefinite for any $g \in \mathcal{O}(\mathbb{D}, \bar{\mathbb{D}})$.

The Pick interpolation theorem on the bidisk (see [1, p. 180]) can be stated as a theorem about extensions of bounded analytic functions in the following way. Given a subset X of the bidisk and a function $\psi\colon X \to \mathbb{D}$, there exists a $\Psi \in \mathcal{O}(\mathbb{D}^2, \mathbb{D})$ with $\Psi|_X = \psi$ if and only if there exist positive semidefinite functions Δ and Γ on $X \times X$ such that, for each $z = (z_1, z_2) \in X$ and $w = (w_1, w_2) \in X$,

$$1 - \psi(z)\overline{\psi(w)} = \Gamma(z, w)(1 - z_1\bar{w}_1) + \Delta(z, w)(1 - z_2\bar{w}_2).$$

We should mention that the portion of this theorem that we shall use (i.e., sufficiency) has a quite simple proof—it is an application of the "lurking isometry" technique.

To prove Theorem 2.9, suppose $f \in \mathcal{O}(N, \mathbb{D})$ and $f(0,0) = 0$. Then, as in earlier arguments, $(f \circ p)(\lambda) = f(\lambda^3, \lambda^2) = \lambda^2 g(\lambda)$ for some $g \in \mathcal{O}(\mathbb{D}, \bar{\mathbb{D}})$. For any $\delta, \lambda \in \mathbb{D}$ we have

$$\begin{aligned} &2 - f(p(\lambda))\overline{f(p(\delta))} \\ &\quad = (1 - \lambda^3\bar{\delta}^3) + \left(1 + \lambda^2\bar{\delta}^2 \frac{1 - g(\lambda)\overline{g(\delta)}}{1 - \lambda\bar{\delta}} + \frac{\lambda^3\bar{\delta}^3 g(\lambda)\overline{g(\delta)}}{1 - \lambda^2\bar{\delta}^2}\right)(1 - \lambda^2\bar{\delta}^2). \end{aligned}$$

Hence, for $z = (z_1, z_2) \in N$ and $w = (w_1, w_2) \in N$ we have

$$2 - f(z)\overline{f(w)} = \Gamma(z, w)(1 - z_1\bar{w}_1) + \Delta(z, w)(1 - z_2\bar{w}_2), \tag{7.1}$$

where $\Gamma(z, w) = 1$ and

$$\Delta(z, w) = 1 + z_1\bar{w}_1 \frac{1 - g(q(z))\overline{g(q(w))}}{1 - q(z)\overline{q(w)}} + \frac{z_2\bar{w}_2 g(q(z))\overline{g(q(w))}}{1 - z_1\bar{w}_1}$$

(recall $q(z) = z_1/z_2$ for $z \neq (0,0)$ and $q(0,0) = 0$). Now Γ is clearly positive semidefinite, and Δ is positive semidefinite because positive semidefinite functions are closed under addition and multiplication (by the Schur product theorem) and by the Pick interpolation theorem on the disk (applied to g). This proves that f has an extension to the bidisk with supremum norm at most $\sqrt{2}$ (by dividing through (7.1) by 2).

In order to prove that any holomorphic function $f \in \mathcal{O}(N, \mathbb{D})$ (regardless of its value at the origin) can be extended to the bidisk with supremum norm at most $2\sqrt{2} + 1$, simply apply the result just proved to $(f - f(0))/2$.

Finally, the function

$$h(\lambda) = \lambda^2 \frac{0.5 - \lambda}{1 - 0.5\lambda}$$

corresponds to a function $f \in \mathcal{O}(N, \mathbb{D})$ with $f(\lambda^3, \lambda^2) = h(\lambda)$. The partial derivatives of f at $(0,0)$ are just the coefficients of λ^3 and λ^2 in the power series for h: -0.75 and 0.5. Suppose F is a bounded holomorphic extension of f to the bidisk with sup norm R. Then, by the Schwarz lemma on the bidisk,

$$0.75/R + 0.5/R \leq 1;$$

this implies $R \geq 5/4$, as desired.

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