

A Finitely Presented Solvable Group with a Small Quasi-Isometry Group

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0. Introduction

Let \mathbf{B}_n be the upper triangular subgroup of \mathbf{SL}_n , and let $\mathbf{Ad}(\mathbf{B}_n) \leq \mathbf{PGL}_n$ be the image of \mathbf{B}_n under the map $\mathbf{Ad}: \mathbf{SL}_n \rightarrow \mathbf{PGL}_n$.

If p is a prime number, then the group $\mathbf{B}_n(\mathbb{Z}[1/p])$ is finitely presented for all $n \geq 2$. In particular it is finitely generated, so we can form its quasi-isometry group—denoted $\mathcal{QI}(\mathbf{B}_n(\mathbb{Z}[1/p]))$. In this paper we shall prove the following result.

THEOREM 1. *If $n \geq 3$, then*

$$\mathcal{QI}(\mathbf{B}_n(\mathbb{Z}[1/p])) \cong (\mathbf{Ad}(\mathbf{B}_n)(\mathbb{R}) \times \mathbf{Ad}(\mathbf{B}_n)(\mathbb{Q}_p)) \rtimes \mathbb{Z}/2\mathbb{Z}.$$

The $\mathbb{Z}/2\mathbb{Z}$ -action defining the semi-direct product in Theorem 1 is given by a \mathbb{Q} -automorphism of \mathbf{PGL}_n that stabilizes $\mathbf{Ad}(\mathbf{B}_n)$. The order-2 automorphism acts simultaneously on each factor.

For the proof of Theorem 1, the reader may advance directly to Section 1. The proof continues in Sections 2 and 3.

0.1. What's New about This Example

The group $\mathbf{B}_n(\mathbb{Z}[1/p])$ is solvable, and when $n \geq 3$ its quasi-isometry group is “small” in two senses: (i) it is virtually a product of finite-dimensional Lie groups (real and p -adic), and (ii) it is solvable. The *quasi-isometry group* of a finitely generated group Γ is the group of all quasi-isometries of Γ modulo those that have finite distance in the sup-norm to the identity.

We don't know too much about which groups can be realized as quasi-isometry groups, but the examples we do have suggest this collection could be quite diverse. A fair amount of variety is displayed even in the extremely restricted class of finitely generated groups that appear as lattices in semisimple groups. These quasi-isometry groups include the \mathbb{Q} -, \mathbb{R} -, or \mathbb{Q}_p -points of simple algebraic groups; discrete groups that are finite extensions of lattices; and infinite-dimensional groups (for a complete list of references to these results see e.g. [Fa] or [Wo]).

However, the same variety is not presently visible in the class of quasi-isometry groups for infinite finitely generated amenable groups. In fact, the quasi-isometry

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groups of all previously studied amenable groups contain infinite-dimensional subgroups. Examples include infinite finitely generated abelian groups (see e.g. [GP, 3.3.B]) along with many other groups from the abundant class of abelian-by-cyclic groups.

Indeed, using the geometric models from [FaM3] the quasi-isometry groups of finitely presented nonpolycyclic abelian-by-cyclic groups can be seen to contain infinite-dimensional subgroups of automorphism groups of trees. This generalizes the fact that $\text{Bilip}(\mathbb{Q}_m)$ is contained in the quasi-isometry group of the Baumslag–Solitar group $\langle a, b \mid aba^{-1} = b^m \rangle$, where $\text{Bilip}(\mathbb{Q}_m)$ denotes the group of all bilipschitz homeomorphisms of \mathbb{Q}_m . See Theorem 2.

For easy-to-describe polycyclic examples of abelian-by-cyclic groups, take any group of the form $\Gamma = \mathbb{Z}^m \rtimes_D \mathbb{Z}$ where $D \in \mathbf{SL}_m(\mathbb{Z})$ has a real eigenvalue $\lambda > 1$. The discrete group Γ is a cocompact subgroup of the Lie group $M = \mathbb{R}^m \rtimes_\varphi \mathbb{R}$, where $\varphi(t) = D^t$. If $L_\lambda \leq \mathbb{R}^m$ is a k -dimensional eigenspace of D corresponding to λ , then the cosets of $L_\lambda \rtimes_\varphi \mathbb{R}$ in M are the leaves of a foliation of M by totally geodesic hyperbolic spaces. Any bilipschitz map of L_λ gives a quasi-isometry of each leaf and a quasi-isometry of M . Since M is quasi-isometric to Γ , it follows that $\text{Bilip}(\mathbb{R}^k)$ is a subgroup of $\mathcal{QI}(\Gamma)$. This example generalizes the well-known fact that the quasi-isometry group of the 3-dimensional Lie group Sol contains $\text{Bilip}(\mathbb{R}) \times \text{Bilip}(\mathbb{R})$.

The evidence led Farb and Mosher to a “flexibility conjecture”: Any infinite finitely generated solvable group has an infinite-dimensional quasi-isometry group; see [FaM4; p. 124, #3]. Theorem 1 shows that this conjecture is false.

In addition, $\mathcal{QI}(\mathbf{B}_n(\mathbb{Z}[1/p]))$ is, to the best of my knowledge, the first known example of a quasi-isometry group of any infinite finitely generated group that does not contain a nonabelian free group.

0.2. Quasi-Isometries of S -Arithmetic Baumslag–Solitar Groups

The proof of Theorem 1 is motivated by the proof of the following theorem.

THEOREM 2 (Farb–Mosher).

$$\mathcal{QI}(\mathbf{B}_2(\mathbb{Z}[1/p])) \cong \text{Bilip}(\mathbb{R}) \times \text{Bilip}(\mathbb{Q}_p).$$

Notice that the Baumslag–Solitar group $\text{BS}(1, p^2)$ is an index-2 subgroup of $\mathbf{B}_2(\mathbb{Z}[1/p])$. (For those checking details, $\text{Bilip}(\mathbb{Q}_p) = \text{Bilip}(\mathbb{Q}_{p^2})$.)

Theorem 2 is proved by studying the geometry of a space that is essentially a “horosphere” in a product of a hyperbolic plane and a regular $(p+1)$ -valent tree, $\mathbb{H}^2 \times T_p$ [FaM1]. Here’s a brief account of the Farb–Mosher proof.

The group $\mathbf{SL}_2(\mathbb{R})$ acts by isometries on \mathbb{H}^2 , and $\mathbf{SL}_2(\mathbb{Q}_p)$ acts by isometries on T_p . Since the diagonal homomorphism

$$\begin{aligned} \mathbf{B}_2(\mathbb{Z}[1/p]) &\rightarrow \mathbf{SL}_2(\mathbb{R}) \times \mathbf{SL}_2(\mathbb{Q}_p), \\ M &\mapsto (M, M) \end{aligned}$$

has a discrete image, it follows that $\mathbf{B}_2(\mathbb{Z}[1/p])$ acts properly on $\mathbb{H}^2 \times T_p$. One can show that a $\mathbf{B}_2(\mathbb{Z}[1/p])$ -orbit is a finite Hausdorff distance from a quasi-isometrically embedded horosphere $\mathcal{H}_p \subseteq \mathbb{H}^2 \times T_p$. Hence, finding the quasi-isometries of $\mathbf{B}_2(\mathbb{Z}[1/p])$ amounts to finding the quasi-isometries of \mathcal{H}_p .

Farb and Mosher show that any quasi-isometry of \mathcal{H}_p is the restriction of a product of a quasi-isometry of \mathbb{H}^2 that preserves a foliation by horocycles, with a quasi-isometry of T_p that preserves an analogous “foliation by horocycles”. The group of all such products can be identified with $\text{Bilip}(\mathbb{R}) \times \text{Bilip}(\mathbb{Q}_p)$.

Note that Theorem 2 can be rephrased as stating that any quasi-isometry of the horosphere $\mathcal{H}_p \subseteq \mathbb{H}^2 \times T_p$ is the restriction of a quasi-isometry of $\mathbb{H}^2 \times T_p$ that stabilizes \mathcal{H}_p .

0.3. Relationship between Theorems 1 and 2

In the case of Theorem 1 (i.e., when $n \geq 3$), the geometry of $\mathbf{B}_n(\mathbb{Z}[1/p])$ is similar to the picture drawn by Farb–Mosher except that the rank one spaces, \mathbb{H}^2 and T_p , are replaced by higher rank analogues. In this case, $\mathbf{B}_n(\mathbb{Z}[1/p])$ is quasi-isometric to the intersection of $n - 1$ distinct horospheres in the product of a higher rank symmetric space and a higher rank Euclidean building. In comparison to the $n = 2$ case, any quasi-isometry of this intersection is the restriction of a quasi-isometry of the entire product.

The contrast between Theorem 1 and Theorem 2 is produced by the theorem of Kleiner and Leeb that any quasi-isometry of a product of a higher rank symmetric space and a Euclidean building is a bounded distance in the sup-norm from an isometry [KLe]. The group of all isometries that preserve such an intersection of horospheres up to finite Hausdorff distance is precisely

$$(\text{Ad}(\mathbf{B}_n)(\mathbb{R}) \times \text{Ad}(\mathbf{B}_n)(\mathbb{Q}_p)) \rtimes \mathbb{Z}/2\mathbb{Z}.$$

0.4. Adding Places to S -Arithmetic Baumslag–Solitar Groups

Although the theme of our proof is modeled on the theme of the Farb–Mosher proof, many of the individual techniques are different. For example, we apply a theorem of Whyte on quasi-isometries of coarse fibrations [Wh]; this theorem tells us that any quasi-isometry of our model space for $\mathbf{B}_n(\mathbb{Z}[1/p])$ preserves what can be seen of the product decomposition between real and p -adic factors.

Whyte’s theorem was used in [TWh] for the proof of the following generalization of Theorem 2.

THEOREM 3 (Taback–Whyte). *If m is composite, then*

$$\mathcal{QI}(\mathbf{B}_2(\mathbb{Z}[1/m])) \cong \text{Bilip}(\mathbb{R}) \times \prod_{p|m} \text{Bilip}(\mathbb{Q}_p).$$

0.5. Generalizing Theorems to Conjectures

In what remains of this Introduction we state some conjectures for groups similar to those from Theorems 1, 2, and 3. Two of these conjectures are new (4 and 6),

one is well-known (5), and the rest (7, 8, and 9) do not seem to be in the literature, although they are growing in popularity among those who have recognized the geometric connection between Baumslag–Solitar groups and lamplighter groups.

All of these conjectures pertain to the large-scale geometry of intersected horospheres in either a product of symmetric spaces or a product of Euclidean buildings. Whyte’s theorem cannot be used in these cases to distinguish factors, which is an obstruction to verifying the conjectures.

0.6. Polycyclic Groups

Although Theorem 1 gives an example of an infinite solvable group with a “small” quasi-isometry group, the question remains of whether a similar example exists within the class of polycyclic groups.

However, Theorem 1—along with its proof—provides evidence that such an example may exist. Indeed, although $\mathbf{B}_n(\mathbb{Z}[1/p])$ is not polycyclic, its geometry resembles the geometry of the polycyclic group $\mathbf{B}_n(\mathbb{Z}[\sqrt{2}])$. The key is to note that, if σ is the natural map derived from the nontrivial element of the Galois group of $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$, then the map

$$\mathbf{B}_n(\mathbb{Z}[\sqrt{2}]) \rightarrow \mathbf{SL}_n(\mathbb{R}) \times \mathbf{SL}_n(\mathbb{R})$$

defined by

$$M \mapsto (M, \sigma(M))$$

is an injective homomorphism with discrete image.

The polycyclic group $\mathbf{B}_n(\mathbb{Z}[\sqrt{2}])$ is quasi-isometric to an intersection of $n - 1$ horospheres in the symmetric space for $\mathbf{SL}_n(\mathbb{R}) \times \mathbf{SL}_n(\mathbb{R})$. Asking that every quasi-isometry of the intersection be the restriction of a quasi-isometry of the symmetric space is the content of our first conjecture.

CONJECTURE 4. *If $n \geq 3$, then*

$$\mathcal{QI}(\mathbf{B}_n(\mathbb{Z}[\sqrt{2}])) \cong (\mathbf{Ad}(\mathbf{B}_n)(\mathbb{R}) \times \mathbf{Ad}(\mathbf{B}_n)(\mathbb{R})) \rtimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}).$$

Observe that Theorem 1 is to Theorem 2 as Conjecture 4 is to the following well-known conjecture.

CONJECTURE 5.

$$\mathcal{QI}(\mathbf{B}_2(\mathbb{Z}[\sqrt{2}])) \cong (\mathbf{Bilip}(\mathbb{R}) \times \mathbf{Bilip}(\mathbb{R})) \rtimes \mathbb{Z}/2\mathbb{Z}.$$

This is better known as a conjecture about the 3-dimensional real Lie group \mathbf{Sol} —the geometry associated with a 3-manifold that fibers over a circle with torus fibers and Anosov monodromy. Indeed, $\mathbf{B}_2(\mathbb{Z}[\sqrt{2}])$ is a discrete cocompact subgroup of \mathbf{Sol} . (See [FaM4] for remarks concerning Conjecture 5. Since the writing of this paper, Conjecture 5 was proved by Eskin–Fisher–Whyte; see [EFWh, Thm. 2.1].)

There is almost nothing known about quasi-isometry types of polycyclic (and not virtually nilpotent) finitely generated groups, and Conjecture 5 stands as one of the more important open questions in geometric group theory in part because it

is seen as a first step toward understanding the large-scale geometry of polycyclic groups.

0.7. Geometric Comparisons between Rank One Amenable Groups

Note that Sol is quasi-isometric to a horosphere in $\mathbb{H}^2 \times \mathbb{H}^2$, and Conjecture 5 asserts that any quasi-isometry of Sol extends to a product of quasi-isometries of \mathbb{H}^2 . Each of the quasi-isometries of \mathbb{H}^2 would preserve a foliation by horocycles, and the supremum of all Hausdorff distances between horocycles and their images would be finite.

This is analogous to the known large-scale geometry of solvable Baumslag–Solitar groups described previously, and also to a conjectural picture of lamplighter groups that we discuss later. Each of these three examples of groups are rank one amenable groups.

0.8. Function-Field-Arithmetic Groups

Theorems 1, 2, and 3 are statements about S -arithmetic groups, whereas Conjectures 4 and 5 are concerned with arithmetic groups. The geometry of these groups extends easily to the geometry of function-field-arithmetic groups.

Specifically, let $\mathbb{F}_q((t))$ be a field of Laurent series with coefficients in a finite field \mathbb{F}_q , and let $\mathbb{F}_q[t, t^{-1}]$ be the ring of Laurent polynomials. Expanding on Theorem 1 and Conjecture 4 yields the following.

CONJECTURE 6. *If $n \geq 3$, then*

$$\mathcal{QI}(\mathbf{B}_n(\mathbb{F}_q[t, t^{-1}])) \cong (\mathbf{Ad}(\mathbf{B}_n)(\mathbb{F}_q((t))) \times \mathbf{Ad}(\mathbf{B}_n)(\mathbb{F}_q((t)))) \rtimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}).$$

Bux showed that the group $\mathbf{B}_n(\mathbb{F}_q[t, t^{-1}])$ is finitely generated but not finitely presented [Bux]. This group acts on a product of higher rank buildings.

Turning our attention to the case $n = 2$, the group $\mathbf{B}_2(\mathbb{F}_q[t, t^{-1}])$ acts on a horosphere in a product of $(q + 1)$ -valent trees. Theorem 2 and Conjecture 5 lead to the following conjecture.

CONJECTURE 7. (Since the writing of this paper, Conjecture 7 was proved by Eskin–Fisher–Whyte; see [EFWh, Thm. 2.3].)

$$\mathcal{QI}(\mathbf{B}_2(\mathbb{F}_q[t, t^{-1}])) \cong (\mathbf{Bilip}(\mathbb{F}_q((t))) \times \mathbf{Bilip}(\mathbb{F}_q((t)))) \rtimes \mathbb{Z}/2\mathbb{Z}.$$

The groups $\mathbf{B}_2(\mathbb{F}_q[t, t^{-1}])$ are also finitely generated and not finitely presented.

0.9. Lamplighter Groups

Conjecture 7 is closely related to the study of quasi-isometry types of lamplighter groups. Recall that if G is a finite group then the wreath product

$$G \wr \mathbb{Z} = \left(\bigoplus_{\mathbb{Z}} G \right) \rtimes \mathbb{Z}$$

is called a *lamplighter group*.

Note that the group $\mathbf{B}_2(\mathbb{F}_2[t, t^{-1}])$ is isomorphic to the wreath product $\mathbb{F}_2^2 \wr \mathbb{Z}$. This example reveals how the natural geometry of the function-field-arithmetic groups from Conjecture 7 can be formally generalized—in the same way that Baumslag–Solitar groups can be viewed as a formal generalization of the S -arithmetic groups $\mathbf{B}_2(\mathbb{Z}[1/p])$ —to act on a horosphere in a product of trees. These horospheres are examples of *Diestel–Leader graphs*.

In Section 4 we will briefly describe the geometric action of lamplighters on a horosphere in a product of trees. The analogy with solvable Baumslag–Solitar groups and with Sol leads to the following two conjectures.

CONJECTURE 8. (Since the writing of this paper, Conjecture 8 was proved by Eskin–Fisher–Whyte; see [EFWh, Thm. 2.3].) *For any nontrivial finite group G ,*

$$\mathcal{QI}(G \wr \mathbb{Z}) \cong (\text{Bilip}(G((t))) \times \text{Bilip}(G((t)))) \rtimes \mathbb{Z}/2\mathbb{Z}.$$

CONJECTURE 9. (Since the writing of this paper, Conjecture 9 was proved by Eskin–Fisher–Whyte; see [EFWh, Thm. 1.7].) *Let G and H be finite groups. Then $G \wr \mathbb{Z}$ is quasi-isometric to $H \wr \mathbb{Z}$ if and only if $|G|^k = |H|^j$ for some $k, j \in \mathbb{N}$.*

The question of when two lamplighter groups are quasi-isometric was asked by de la Harpe; see [H, IV.B.44]. The “if” implication in Conjecture 9 is well known (and easily seen) to be true, but there does not exist a single known case of the “only if” implication. For example, it is unknown even if $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ is quasi-isometric to $(\mathbb{Z}/3\mathbb{Z}) \wr \mathbb{Z}$.

Note the similarity between Conjecture 9 and the following theorem.

THEOREM 10 (Farb–Mosher). *If $\text{BS}(1, m)$ denotes the Baumslag–Solitar group $\langle a, b \mid aba^{-1} = b^m \rangle$, then $\text{BS}(1, m)$ is quasi-isometric to $\text{BS}(1, r)$ if and only if $m^k = r^j$ for some $k, j \in \mathbb{N}$.*

See [FaM1, Thm. 7.1].

With the exception of the general lamplighter groups $G \wr \mathbb{Z}$ and the general Baumslag–Solitar groups $\text{BS}(1, m)$, all of the finitely generated groups mentioned so far are discrete cocompact subgroups of solvable Lie groups over locally compact nondiscrete fields. This follows from reduction theory. As mentioned previously, one motivation for writing this paper was to add to the list of examples of groups that are known to be realized as quasi-isometry groups of finitely generated groups. It would be nice to know more examples.

NOTATION. In the remainder of this paper we let

$$G_p = (\mathbf{Ad}(\mathbf{B}_n)(\mathbb{R}) \times \mathbf{Ad}(\mathbf{B}_n)(\mathbb{Q}_p)) \rtimes \mathbb{Z}/2\mathbb{Z}.$$

0.10. Outline of the Paper

Our proof of Theorem 1 begins in Section 1, where we describe a geometric model for $\mathbf{B}_n(\mathbb{Z}[1/p])$. We use the model in Section 2 to show that G_p is included in the

quasi-isometry group of $\mathbf{B}_n(\mathbb{Z}[1/p])$, and we use the model in Section 3 to show that every quasi-isometry of $\mathbf{B}_n(\mathbb{Z}[1/p])$ is given by an element of G_p .

Section 1 does not use the assumption that $n \geq 3$, and Section 2 may also be read assuming that $n = 2$ (as long as one replaces G_p with its obvious index-2 subgroup). Even Section 3 invokes the $n \geq 3$ assumption only in efficient and isolated applications of Kleiner–Leeb’s theorem. We point all of this out because the reader may find it helpful to fall back on the case $n = 2$ at times. Indeed, it’s important to have a solid understanding of the model space while reading the first three sections, and it is possible to draw (and easiest to visualize) the model space when $n = 2$. For descriptions of the model space when $n = 2$, see for example [Ep+, 7.4] or [FaM1]; the definitions of the model space in these references are different than the one we will give, but they are easily seen to be equivalent.

Section 4 contains a sketch of the geometry of lamplighters that hints at Conjectures 8 and 9.

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1. The Model Space for $\mathbf{B}_n(\mathbb{Z}[1/p])$

Throughout, we fix $n \geq 3$. Our goal in this section is to define a metric space X that is quasi-isometric to $\mathbf{B}_n(\mathbb{Z}[1/p])$. Then $\mathcal{QI}(X) \cong \mathcal{QI}(\mathbf{B}_n(\mathbb{Z}[1/p]))$. In Sections 2 and 3, we will determine $\mathcal{QI}(X)$.

As alluded to in the Introduction, X will essentially be an intersection of $n - 1$ horospheres in a product of a symmetric space and a Euclidean building, but we will define it differently to simplify some points of our proof.

Although we are assuming $n \geq 3$ in Theorem 1, all of this section holds with the weaker assumption that $n \geq 2$.

1.1. Definitions and Descriptions

QUASI-ISOMETRIES. For constants $K \geq 1$ and $C \geq 0$, a (K, C) *quasi-isometric embedding* of a metric space Y into a metric space Z is a function $\phi: Y \rightarrow Z$ such that, for any $y_1, y_2 \in Y$,

$$\frac{1}{K}d(y_1, y_2) - C \leq d(\phi(y_1), \phi(y_2)) \leq Kd(y_1, y_2) + C.$$

We call ϕ a (K, C) *quasi-isometry* if ϕ is a (K, C) quasi-isometric embedding and if there is a number $D \geq 0$ such that every point in Z is within distance D of some point in the image of Y .

QUASI-ISOMETRY GROUPS. For a metric space Y , we define the relation \sim on the set of functions $Y \rightarrow Y$ by $\phi \sim \psi$ if

$$\sup_{y \in Y} d(\phi(y), \psi(y)) < \infty.$$

We denote the quotient space of all quasi-isometries of Y modulo \sim by $\mathcal{QI}(Y)$. We call $\mathcal{QI}(Y)$ the *quasi-isometry group* of Y because it has a natural group structure arising from function composition.

If Γ is a finitely-generated group, we use its left-invariant word metric to form the quasi-isometry group $\mathcal{QI}(\Gamma)$.

THE MODEL SPACE. The finitely presented group $\mathbf{B}_n(\mathbb{Z}[1/p])$ is a discrete subgroup of $\mathbf{SL}_n(\mathbb{R}) \times \mathbf{SL}_n(\mathbb{Q}_p)$, and thus it acts on a product of nonpositively curved spaces.

SYMMETRIC SPACE AND EUCLIDEAN BUILDING. We let X_∞ be the symmetric space for $\mathbf{SL}_n(\mathbb{R})$ and X_p the Euclidean building for $\mathbf{SL}_n(\mathbb{Q}_p)$. All of the facts we will use about X_∞ and X_p are standard and can be found, for example, in [BGS] or [Bro].

TOPOLOGICAL DESCRIPTION OF X . Let \mathbf{U}_n be the subgroup of \mathbf{B}_n consisting of matrices whose diagonal entries are all equal to 1. We define X to be the topological space $X_p \times \mathbf{U}_n(\mathbb{R})$ and let $\pi: X \rightarrow X_p$ be the projection map.

1.2. A Symmetric Space in X

Denote the diagonal subgroup of \mathbf{B}_n by \mathbf{A}_n . We let $\mathcal{A}_\mathbf{A} \subseteq X_p$ be the apartment corresponding to $\mathbf{A}_n(\mathbb{Q}_p)$ and let $F_\mathbf{A} \subseteq X_\infty$ be the flat corresponding to $\mathbf{A}_n(\mathbb{R})$. Recall that with the metric restricted from X_p and X_∞ respectively, $\mathcal{A}_\mathbf{A}$ and $F_\mathbf{A}$ are each isometric to $(n - 1)$ -dimensional Euclidean space.

The group of diagonal matrices $\mathbf{A}_n(\mathbb{Z}[1/p])$ acts on the Euclidean spaces $\mathcal{A}_\mathbf{A}$ and $F_\mathbf{A}$ as a discrete group of translations. These two isometric actions are related as follows.

LEMMA 1. *After possibly scaling the metric on X_p , there exists an $\mathbf{A}_n(\mathbb{Z}[1/p])$ -equivariant isometry*

$$f: \mathcal{A}_\mathbf{A} \rightarrow F_\mathbf{A}.$$

Proof. Choose a sector $\mathfrak{S}_{\mathbf{B}} \subseteq \mathcal{A}_{\mathbf{A}}$ that is fixed up to finite Hausdorff distance by the action of $\mathbf{B}_n(\mathbb{Q}_p)$ on X_p , and let $\mathfrak{C}_{\mathbf{B}} \subseteq F_{\mathbf{A}}$ be a Weyl chamber that is fixed up to finite Hausdorff distance by the action of $\mathbf{B}_n(\mathbb{R})$ on X_{∞} .

We denote the collection of roots of \mathbf{SL}_n with respect to \mathbf{A}_n that are positive in \mathbf{B}_n as $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$. Each α_i corresponds to a wall of $\mathfrak{S}_{\mathbf{B}}$ (resp. $\mathfrak{C}_{\mathbf{B}}$), which we denote by $L_{p,i}$ (resp. $L_{\infty,i}$) for all i , so there is a unique isometry that maps $\mathfrak{S}_{\mathbf{B}}$ onto $\mathfrak{C}_{\mathbf{B}}$ and $L_{p,i}$ onto $L_{\infty,i}$ for each i . We name this isometry $f': \mathcal{A}_{\mathbf{A}} \rightarrow F_{\mathbf{A}}$.

We denote the inverse transpose automorphism of \mathbf{SL}_n by φ^u and let φ_{∞}^u be the isometry of X_{∞} induced by φ^u . Notice that $\varphi_{\infty}^u(F_{\mathbf{A}}) = F_{\mathbf{A}}$, so the composition $\varphi_{\infty}^u \circ f'$ defines an isometry $f: \mathcal{A}_{\mathbf{A}} \rightarrow F_{\mathbf{A}}$.

For $1 \leq i \leq n-1$, we define the diagonal matrix $a_i \in \mathbf{A}_n(\mathbb{Z}[1/p])$ as

$$(a_i)_{l,k} = \begin{cases} p & \text{if } l = k = i, \\ 1/p & \text{if } l = k = i + 1, \\ \delta_{l,k} & \text{otherwise.} \end{cases}$$

Let β be any root of \mathbf{SL}_n with respect to \mathbf{A}_n , and let $\mathbf{U}_{\beta} \leq \mathbf{SL}_n$ be the corresponding root group. If a_i acts as an expanding automorphism on $\mathbf{U}_{\beta}(\mathbb{Q}_p)$, then it acts also as a contracting automorphism on $\mathbf{U}_{\beta}(\mathbb{R})$ because the p -adic norm of p is less than 1. Thus, if a_i translates $\mathcal{A}_{\mathbf{A}}$ toward the chamber at infinity corresponding to \mathbf{U}_{β} , then a_i translates $F_{\mathbf{A}}$ toward the chamber at infinity corresponding to $\mathbf{U}_{-\beta} = \varphi^u(\mathbf{U}_{\beta})$. Therefore, after rescaling X_p so that the translation lengths of each of the a_i are the same on $\mathcal{A}_{\mathbf{A}}$ as they are on $F_{\mathbf{A}}$, we have that f is $\langle a_1, a_2, \dots, a_{n-1} \rangle$ -equivariant.

Our lemma follows since the $\mathbf{A}_n(\mathbb{Z}[1/p])$ -actions factor through the group $\langle a_1, a_2, \dots, a_{n-1} \rangle$. \square

We write the connected component of the identity in $\mathbf{A}_n(\mathbb{R})$ as $\mathbf{A}_n(\mathbb{R})^{\circ}$, recalling that it acts transitively on $F_{\mathbf{A}}$ without fixed points. Thus, there is a natural bijection between $F_{\mathbf{A}}$ and $\mathbf{A}_n(\mathbb{R})^{\circ}$.

With the Weyl chamber $\mathfrak{C}_{\mathbf{B}} \subseteq F_{\mathbf{A}}$ as in the proof of Lemma 1, we can make the following statements:

- (i) every point in X_{∞} is an element of a maximal flat whose R -neighborhoods contain $\mathfrak{C}_{\mathbf{B}}$ when $R \gg 0$;
- (ii) the group $\mathbf{U}_n(\mathbb{R})$ acts transitively on the set of all maximal flats as in (i);
- (iii) $F_{\mathbf{A}}$ is a flat as in (i); and
- (iv) $\mathbf{A}_n(\mathbb{R})^{\circ}$ acts transitively on $F_{\mathbf{A}}$.

Therefore, the group $\mathbf{A}_n(\mathbb{R})^{\circ} \mathbf{U}_n(\mathbb{R})$ acts transitively on X_{∞} . It is also easy to see that this action is without fixed points. Thus, there is an isometry (up to scale) between X_{∞} and the Lie group $\mathbf{B}_n(\mathbb{R})^{\circ} = \mathbf{A}_n(\mathbb{R})^{\circ} \mathbf{U}_n(\mathbb{R})$ endowed with a left-invariant metric. This allows us to identify X_{∞} with $F_{\mathbf{A}} \times \mathbf{U}_n(\mathbb{R})$.

Observe that $\pi^{-1}(\mathcal{A}_{\mathbf{A}}) = \mathcal{A}_{\mathbf{A}} \times \mathbf{U}_n(\mathbb{R})$, so we define the diffeomorphism

$$\hat{f}: \pi^{-1}(\mathcal{A}_{\mathbf{A}}) \rightarrow X_{\infty}$$

by

$$\hat{f}(a, u) = (f(a), u)$$

for all $a \in \mathcal{A}_A$ and all $u \in \mathbf{U}_n(\mathbb{R})$.

If ω is the metric on X_∞ , then we endow $\pi^{-1}(\mathcal{A}_A)$ with the pull-back metric $\hat{f}^*(\omega)$.

1.3. A Family of Symmetric Spaces in X

Recall from the proof of Lemma 1 that we chose a sector $\mathfrak{S}_B \subseteq \mathcal{A}_A$ that is fixed up to finite Hausdorff distance by the action of $\mathbf{B}_n(\mathbb{Q}_p)$ on X_p . We call any apartment in X_p that contains a subsector of \mathfrak{S}_B a *symmetric apartment*. If $\mathcal{A} \subseteq X_p$ is a symmetric apartment, then we call $\pi^{-1}(\mathcal{A})$ a *symmetric pre-image*.

Denote the building retraction corresponding to the pair $(\mathcal{A}_A, \mathfrak{S}_B)$ by

$$\varrho_{\mathcal{A}_A, \mathfrak{S}_B} : X_p \rightarrow \mathcal{A}_A.$$

We extend $\varrho_{\mathcal{A}_A, \mathfrak{S}_B}$ to a retraction of the entire space X by defining

$$\hat{\varrho}_{\mathcal{A}_A, \mathfrak{S}_B} : X \rightarrow \pi^{-1}(\mathcal{A}_A)$$

as

$$\hat{\varrho}_{\mathcal{A}_A, \mathfrak{S}_B}(x, u) = (\varrho_{\mathcal{A}_A, \mathfrak{S}_B}(x), u)$$

for all $x \in X_p$ and $u \in \mathbf{U}_n(\mathbb{R})$.

Since $\varrho_{\mathcal{A}_A, \mathfrak{S}_B}$ maps symmetric apartments isometrically onto \mathcal{A}_A , the map $\hat{\varrho}_{\mathcal{A}_A, \mathfrak{S}_B}$ defines a diffeomorphism of any symmetric pre-image onto $\pi^{-1}(\mathcal{A}_A)$. We endow each symmetric pre-image with the pull-back metric $(f \circ \hat{\varrho}_{\mathcal{A}_A, \mathfrak{S}_B})^*(\omega)$. Given this metric, each symmetric pre-image is isometric to the symmetric space X_∞ . Also note that the metrics on any pair of symmetric pre-images agree on their intersection.

1.4. A Metric on X

Recall that any point in X_p is contained in a symmetric apartment. Therefore, every point in X is contained in a symmetric pre-image. This allows us to endow X with the path metric.

The two following lemmas illustrate how the metric space X reflects the geometry of the Euclidean building and the symmetric pre-images that defined it.

LEMMA 2. *The restricted metric from X and the metric $(f \circ \hat{\varrho}_{\mathcal{A}_A, \mathfrak{S}_B})^*(\omega)$ are equal on any symmetric pre-image.*

Proof. If γ is a path in X , then $\hat{\varrho}_{\mathcal{A}_A, \mathfrak{S}_B}(\gamma)$ is a path in X that is no longer than γ . This shows that $\pi^{-1}(\mathcal{A}_A)$ is isometric to X_∞ , and a similar argument applies to any other symmetric pre-image. \square

Let $\sigma : X_p \rightarrow X$ be the section of $\pi : X \rightarrow X_p$ defined by $\sigma(x) = (x, 1)$.

LEMMA 3. *The section $\sigma : X_p \rightarrow X$ is an isometric inclusion.*

Before proving Lemma 3, we will examine a metric property of π .

1.5. The Projection π is Distance Nonincreasing

For any symmetric apartment \mathcal{A} , let $\pi_{\mathcal{A}}$ be the restriction of π to $\pi^{-1}(\mathcal{A})$. Notice that if $\pi_{\mathcal{A}}$ is distance nonincreasing for all \mathcal{A} then π is distance nonincreasing, since any path in X can be written as a finite union of paths contained in a symmetric pre-image.

Furthermore, if $\pi_{\mathcal{A}_A}$ is distance nonincreasing, then every $\pi_{\mathcal{A}}$ is distance nonincreasing because

$$\pi_{\mathcal{A}_A} \circ \hat{\varrho}_{\mathcal{A}_A, \mathfrak{S}_B} = \varrho_{\mathcal{A}_A, \mathfrak{S}_B} \circ \pi_{\mathcal{A}}.$$

Hence, to check that π is distance nonincreasing, we need only check that $\pi_{\mathcal{A}_A}$ is. In other words, we want to show that the map

$$\begin{aligned} q: \mathbf{B}_n(\mathbb{R})^\circ &\rightarrow \mathbf{A}_n(\mathbb{R})^\circ, \\ q(au) &= a \end{aligned}$$

is distance nonincreasing.

We proceed by letting v be a tangent vector to $\mathbf{B}_n(\mathbb{R})^\circ$ at a point $au \in \mathbf{B}_n(\mathbb{R})^\circ$. Let $u' = au^{-1}a^{-1}$. We denote the derivative of a map g by g_* , and we denote the Riemannian norm of a vector by $\|\cdot\|$. Then

$$\|v\|_{au} = \|u'_*v\|_a \geq \|q_*u'_*v\|_a.$$

It is easy to check that $q_*u'_*v = q_*v$, since $u' \in \mathbf{U}_n(\mathbb{R})$. Therefore,

$$\|v\|_{au} \geq \|q_*v\|_a.$$

It follows that q , and hence π , is distance nonincreasing.

Proof of Lemma 3. Clearly σ is distance nonincreasing, so we need only show that if $x, y \in X_p$ then

$$d(x, y) \leq d(\sigma(x), \sigma(y)).$$

But if $\gamma \subseteq X$ is a path between $\sigma(x)$ and $\sigma(y)$, then $\pi(\gamma)$ is a path between x and y ; therefore,

$$d(x, y) \leq \text{length}(\pi(\gamma)) \leq \text{length}(\gamma). \quad \square$$

1.6. X Is a Model Space

We will use the Milnor-Švarc lemma to show that $\mathbf{B}_n(\mathbb{Z}[1/p])$ and X are quasi-isometric. The next three lemmas are meant to verify the hypotheses of that lemma.

LEMMA 4. *There is an isometric $\mathbf{B}_n(\mathbb{Z}[1/p])$ -action on X .*

Proof. For any $x \in X_p$, let $a_x \in \mathbf{A}_n(\mathbb{R})^\circ$ be the group element that is identified with $f \circ \varrho_{\mathcal{A}_A, \mathfrak{S}_B}(x) \in F_A$ via the action of $\mathbf{A}_n(\mathbb{R})^\circ$ on F_A . For $b \in \mathbf{B}_n(\mathbb{Z}[1/p])$, let $a_b \in \mathbf{A}_n(\mathbb{Z}[1/p])$ and $u_b \in \mathbf{U}_n(\mathbb{Z}[1/p])$ be such that $b = a_b u_b$.

Since $\mathbf{B}_n(\mathbb{R})$ acts by isometries on X_∞ , the group $\mathbf{B}_n(\mathbb{Z}[1/p])$ acts by isometries on $\pi^{-1}(\mathcal{A}_A)$. With the identifications we made earlier, this action is given by

$$b * (x, u) = (a_b x, a_x^{-1} u_b a_x u)$$

for all $x \in \mathcal{A}_A$ and all $u \in \mathbf{U}_n(\mathbb{R})$. This action is not proper, and it is of no use to us aside from motivating our desired action of $\mathbf{B}_n(\mathbb{Z}[1/p])$ on X defined by

$$b(x, u) = (bx, a_x^{-1}u_b a_x u)$$

for all $(x, u) \in X_p \times \mathbf{U}_n(\mathbb{R})$.

Any $b \in \mathbf{B}_n(\mathbb{Z}[1/p])$ restricts to an isometry $\pi^{-1}(\mathcal{A}) \rightarrow \pi^{-1}(b\mathcal{A})$ for each symmetric pre-image \mathcal{A} . It follows that $\mathbf{B}_n(\mathbb{Z}[1/p])$ acts by isometries on X , since any path in X can be written as a finite union of paths that are each contained in a symmetric pre-image. \square

LEMMA 5. *The $\mathbf{B}_n(\mathbb{Z}[1/p])$ -action on X is proper.*

Proof. From the description of the metric space X it is clear that, if $\{b_n\}_{n \in \mathbb{N}}$ is a sequence of elements in $\mathbf{B}_n(\mathbb{Z}[1/p])$ that converges to 1 in $\text{Isom}(X)$, then $b_n \rightarrow 1$ in $\text{Isom}(X_p)$ and in $\text{Isom}(X_\infty)$. Thus, $b_n \in \{\pm \text{Id}\}$ for all $n \gg 0$. \square

LEMMA 6. *The $\mathbf{B}_n(\mathbb{Z}[1/p])$ -action on X is cocompact.*

Proof. Let the vertex $v \in \mathcal{A}_A$ be the unique point in X_p that is fixed by the action of $\mathbf{SL}_n(\mathbb{Z}_p)$, and let $\mathfrak{S}_v \subseteq \mathcal{A}_A$ be the unique sector that contains v as its cone-point and such that $\mathfrak{S}_v \cap \mathfrak{S}_B$ contains a sector. Note that $\mathbf{U}_n(\mathbb{Z})$ fixes every point in \mathfrak{S}_v .

Because $\mathbf{A}_n(\mathbb{Z}[1/p])$ acts cocompactly on \mathcal{A}_A , there is a compact subset $C_p \subseteq \mathfrak{S}_v$ such that $\mathbf{A}_n(\mathbb{Z}[1/p])C_p = \mathcal{A}_A$. Since $\mathbf{U}_n(\mathbb{Z}[1/p])$ is a dense subgroup of $\mathbf{U}_n(\mathbb{Q}_p)$ and since the latter group acts transitively on the set of symmetric apartments, we have that

$$\begin{aligned} \mathbf{B}_n(\mathbb{Z}[1/p])C_p &= \mathbf{U}_n(\mathbb{Z}[1/p])\mathbf{A}_n(\mathbb{Z}[1/p])C_p \\ &= \mathbf{U}_n(\mathbb{Z}[1/p])\mathcal{A}_A \\ &= X_p. \end{aligned}$$

It is well known that the $\mathbf{U}_n(\mathbb{Z})$ -action on $\mathbf{U}_n(\mathbb{R})$ has a compact fundamental domain, which we name C_∞ . We use C_∞ to define a compact subset of X :

$$C = \bigcup_{x \in C_p} (\{x\} \times a_x^{-1}C_\infty a_x).$$

Now

$$\begin{aligned} \mathbf{B}_n(\mathbb{Z}[1/p])C &= \mathbf{B}_n(\mathbb{Z}[1/p])\mathbf{U}_n(\mathbb{Z})C \\ &= \mathbf{B}_n(\mathbb{Z}[1/p]) \bigcup_{x \in C_p} (\{x\} \times a_x^{-1}\mathbf{U}_n(\mathbb{Z})a_x a_x^{-1}C_\infty a_x) \\ &= \mathbf{B}_n(\mathbb{Z}[1/p]) \bigcup_{x \in C_p} (\{x\} \times \mathbf{U}_n(\mathbb{R})) \\ &= X. \end{aligned} \quad \square$$

Combining Lemmas 4–6 yields the following statement.

LEMMA 7. *There is an isomorphism of groups*

$$\mathcal{QI}(\mathbf{B}_n(\mathbb{Z}[1/p])) \cong \mathcal{QI}(X).$$

Proof. Apply the Milnor–Švarc lemma. □

As an aside on finiteness properties we remark that, because X is contractible, it follows that $\mathbf{B}_n(\mathbb{Z}[1/p])$ is of type F_∞ .

2. Some Quasi-Isometries of the Model Space

Recall that we set

$$G_p = (\mathbf{Ad}(\mathbf{B}_n)(\mathbb{R}) \times \mathbf{Ad}(\mathbf{B}_n)(\mathbb{Q}_p)) \rtimes \mathbb{Z}/2\mathbb{Z}.$$

The goal of this section is to show that $G_p \leq \mathcal{QI}(X)$.

For $n = 2$, one can replace all occurrences of the map φ^* (described next) with the identity to prove that the obvious index-2 subgroup of G_p is contained in $\mathcal{QI}(X)$.

NONINNER AUTOMORPHISM OF \mathbf{PGL}_n . Because $n \geq 3$, there is an order-2 automorphism of \mathbf{PGL}_n that is defined over \mathbb{Q} , stabilizes $\mathbf{Ad}(\mathbf{B}_n)$, and is not type preserving on the spherical building for $\mathbf{PGL}_n(\mathbb{Q})$. Denote this automorphism by φ^* .

2.1. Isometries Introduced from the p -adic Base

The group of isometries of X_p is identified with $\mathbf{PGL}_n(\mathbb{Q}_p) \rtimes \langle \varphi^* \rangle$. The subgroup of isometries that map a subsector of $\mathfrak{S}_{\mathbf{B}}$ to another subsector of $\mathfrak{S}_{\mathbf{B}}$ is identified with $\mathbf{Ad}(\mathbf{B}_n)(\mathbb{Q}_p) \rtimes \langle \varphi^* \rangle$.

For any $\alpha \in \mathbf{Ad}(\mathbf{B}_n)(\mathbb{Q}_p)$, we define

$$H_p(\alpha): X \rightarrow X$$

by $H_p(\alpha)(x, u) = (\alpha(x), u)$. We claim that $H_p(\alpha)$ is an isometry.

Indeed, if $l \in \mathbf{Ad}(\mathbf{B}_n)(\mathbb{Q}_p)$ fixes a subsector of $\mathfrak{S}_{\mathbf{B}}$ pointwise, then $H_p(l)$ simply permutes the symmetric pre-images via diffeomorphisms that preserve the metrics on the symmetric pre-images, as can be easily seen from the definition of the metrics.

Alternatively, if $\tau \in \mathbf{Ad}(\mathbf{B}_n)(\mathbb{Q}_p)$ restricts to a translation of $\mathcal{A}_{\mathbf{A}}$, then it follows from our identification of $\pi^{-1}(\mathcal{A}_{\mathbf{A}})$ with $\mathbf{A}_n(\mathbb{R})^\circ \mathbf{U}_n(\mathbb{R})$ that $H_p(\tau)|_{\pi^{-1}(\mathcal{A}_{\mathbf{A}})}$ is an isometry. Hence $H_p(\tau)$ is also an isometry when restricted to any other symmetric pre-image $\pi^{-1}(\mathcal{A})$; this follows because

$$H_p(\tau)|_{\pi^{-1}(\mathcal{A})} = H_p(\tau l^{-1} \tau^{-1}) \circ H_p(\tau)|_{\pi^{-1}(\mathcal{A}_{\mathbf{A}})} \circ H_p(l)|_{\pi^{-1}(\mathcal{A})},$$

where $l \in \mathbf{Ad}(\mathbf{B}_n)(\mathbb{Q}_p)$ is as in the previous paragraph with $l\mathcal{A} = \mathcal{A}_{\mathbf{A}}$. (Here we have used that $\tau l^{-1} \tau^{-1}$ fixes a subsector of $\mathfrak{S}_{\mathbf{B}}$ pointwise.) It follows that $H_p(\tau)$ is an isometry.

Our claim is substantiated since any $\alpha \in \mathbf{Ad}(\mathbf{B}_n)(\mathbb{Q}_p)$ is a composition of some l and τ as just described. Thus, we have defined a homomorphism

$$H_p: \mathbf{Ad}(\mathbf{B}_n)(\mathbb{Q}_p) \rightarrow \mathcal{QI}(X).$$

2.2. Bilipschitz Maps Introduced from the Symmetric Pre-Images

The isometry group of $\pi^{-1}(\mathcal{A}_A)$ (and of any symmetric pre-image) is identified with $\mathbf{PGL}_n(\mathbb{R}) \rtimes \langle \varphi^* \rangle$.

Let $\mathfrak{S}_B^{\text{op}} \subseteq \mathcal{A}_A$ be a sector opposite to \mathfrak{S}_B . Then $\sigma(\mathfrak{S}_B^{\text{op}})$ is a Weyl chamber in $\pi^{-1}(\mathcal{A}_A)$ and, given our definition of f from Lemma 1, it follows that $\sigma(\mathfrak{S}_B^{\text{op}})$ equals $\hat{f}^{-1}(\mathfrak{C}_B)$ up to a translation of $\sigma(\mathcal{A}_A)$. Hence, the subgroup of those isometries in $\mathbf{PGL}_n(\mathbb{R}) \rtimes \langle \varphi^* \rangle$ that stabilizes $\sigma(\mathfrak{S}_B^{\text{op}})$ up to a finite Hausdorff distance is precisely $\mathbf{Ad}(\mathbf{B}_n)(\mathbb{R}) \rtimes \langle \varphi^* \rangle$.

Any isometry $\beta \in \mathbf{Ad}(\mathbf{B}_n)(\mathbb{R})$ of $\pi^{-1}(\mathcal{A}_A)$ is a composition τl of isometries of $\pi^{-1}(\mathcal{A}_A)$, where τ acts on $\sigma(\mathcal{A}_A)$ by translations and $l \in \mathbf{U}_n(\mathbb{R})$. We fix τ and l in the following four paragraphs and define two bilipschitz maps $X \rightarrow X$, $H_\infty(\tau)$ and $H_\infty(l)$. Then we compose these two functions to define a bilipschitz map $H_\infty(\beta): X \rightarrow X$.

Right multiplication in a Lie group with a left-invariant metric is bilipschitz and a finite distance in the sup-norm from the identity. Therefore, for $u \in \mathbf{U}_n(\mathbb{R})$ and x in a given symmetric apartment \mathcal{A} , the map $(x, u) \mapsto (\tau^{-1}x, \tau u \tau^{-1})$ defines a bilipschitz map $\pi^{-1}(\mathcal{A}) \rightarrow \pi^{-1}(\mathcal{A})$ that is a bounded distance from the identity; it is precisely right multiplication by τ^{-1} . Thus, we can compose with the isometry τ to obtain the bilipschitz map $\pi^{-1}(\mathcal{A}) \rightarrow \pi^{-1}(\mathcal{A})$ given by $(x, n) \mapsto (x, \tau n \tau^{-1})$. This map is a finite distance in the sup-norm from τ .

Now we define $H_\infty(\tau): X \rightarrow X$ as the map

$$H_\infty(\tau)(x, n) = (x, \tau n \tau^{-1}).$$

From the preceding paragraph we know that $H_\infty(\tau)$ restricts to a bilipschitz map on any symmetric pre-image. Since any path in X is a union of paths in symmetric pre-images, it follows that $H_\infty(\tau)$ is bilipschitz.

It is easier to define $H_\infty(l): X \rightarrow X$ as a bilipschitz map. Simply let

$$H_\infty(l)(x, n) = (x, a_x^{-1} l a_x n).$$

Then $H_\infty(l)$ restricts to an isometry of every symmetric pre-image and thus is an isometry of X .

We define $H_\infty(\beta) = H_\infty(\tau) \circ H_\infty(l)$, where $\beta = \tau l$ as before. Note that $H_\infty(\beta)$ stabilizes each symmetric pre-image and, when restricted to any symmetric pre-image, is a finite distance in the sup-norm from the isometry β . This is enough to ensure that we have defined a homomorphism

$$H_\infty: \mathbf{Ad}(\mathbf{B}_n)(\mathbb{R}) \rightarrow \mathcal{QI}(X).$$

2.3. Coupling Real and p -adic Maps

Define the group homomorphism

$$H: \mathbf{Ad}(\mathbf{B}_n)(\mathbb{R}) \times \mathbf{Ad}(\mathbf{B}_n)(\mathbb{Q}_p) \longrightarrow \mathcal{QI}(X)$$

as $H_\infty \times H_p$.

LEMMA 8. *The group G_p is a subgroup of $\mathcal{QI}(X)$.*

Proof. First we show that the kernel of H is trivial.

Any distinct pair of symmetric pre-images have infinite Hausdorff distance. Therefore, if $H(\beta, \alpha)$ is a finite distance in the sup-norm from the identity, then $H(\beta, \alpha)$ stabilizes every symmetric pre-image. Hence, α stabilizes every symmetric apartment in X_p , and the only element in $\mathbf{Ad}(\mathbf{B}_n)(\mathbb{Q}_p)$ of this sort is the identity.

Assuming now that $\alpha = 1$, we have that $\pi^{-1}(\mathcal{A}_A)$ is stabilized by $H(\beta, \alpha)$ and thus β acts as a bilipschitz map that is equivalent to the identity. It follows that $\beta = 1$ and that H is an injective homomorphism.

The lemma follows by noting that if φ_p^* and φ_∞^* are the geometric realizations of φ^* on X_p and $\pi^{-1}(\mathcal{A}_A)$, respectively, then (i) φ_p^* permutes symmetric apartments and (ii) we may assume both that φ_p^* stabilizes \mathcal{A}_A and that $\varphi_p^*(x) = \varphi_\infty^*(\sigma(x))$ for all $x \in \mathcal{A}_A$. Thus, we can simultaneously apply φ_p^* to the X_p -factor of X and φ_∞^* to the symmetric pre-images to obtain an order-2 isometry $\varphi_X^*: X \rightarrow X$. This isometry is not type preserving on the boundary of X_p (or on the boundary of $\pi^{-1}(\mathcal{A}_A)$), so it is an infinite distance in the sup-norm from any $H(\beta, \alpha)$. \square

3. All Quasi-Isometries of the Model Space

In Section 2 we proved that $G_p \leq \mathcal{QI}(X)$; in this section we will prove that $\mathcal{QI}(X) \leq G_p$. Then we will immediately have a proof of Theorem 1, since we showed in Section 1 that

$$\mathcal{QI}(\mathbf{B}_n(\mathbb{Z}[1/p])) \cong \mathcal{QI}(X).$$

To begin, we let $\phi: X \rightarrow X$ be a (K, C) quasi-isometry. Our goal is to show that ϕ is a finite distance in the sup-norm from a quasi-isometry of the form $H(\beta, \alpha)(\varphi_X^*)^k$, where $\beta \in \mathbf{Ad}(\mathbf{B}_n)(\mathbb{R})$, $\alpha \in \mathbf{Ad}(\mathbf{B}_n)(\mathbb{Q}_p)$, and $k \in \{0, 1\}$.

3.1. An Isometry of the Base Building

Let Hd denote the Hausdorff distance between subsets of X . The following lemma identifies the distance between points in the base of X as the distance between their fibers.

LEMMA 9. *If $x, y \in X_p$, then*

$$d(x, y) = \text{Hd}(\pi^{-1}(x), \pi^{-1}(y)).$$

Proof. Let $u \in \mathbf{U}_n(\mathbb{R})$ be given, and let $u' = a_x u a_x^{-1} \in \mathbf{U}_n(\mathbb{R})$. As explained in the previous section, $H_\infty(u')$: $X \rightarrow X$ is an isometry.

By Lemma 3 there is a geodesic $\gamma \subseteq X$ from $(x, 1)$ to $(y, 1)$ with length $d(x, y)$. Therefore, $H_\infty(u')\gamma$ is a geodesic from (x, u) to $(y, a_y^{-1}u'a_y)$ in $\pi^{-1}(y)$. As result,

$$d(x, y) \geq \text{Hd}(\pi^{-1}(x), \pi^{-1}(y)).$$

For the opposite inequality, we show that the closest point in $\pi^{-1}(y)$ to $(x, 1)$ is $(y, 1)$; the lemma will follow from Lemma 3. Indeed, recall from Section 1 that π is distance nonincreasing, so for any $u \in \mathbf{U}_n(\mathbb{R})$ we have

$$d((x, 1), (y, u)) \geq d((x, 1), (y, 1)) = d(x, y). \quad \square$$

LEMMA 10. *The quasi-isometry ϕ induces an isometry $\phi_\pi : X_p \rightarrow X_p$.*

Proof. Define $\phi_\pi : X_p \rightarrow X_p$ by

$$\phi_\pi = \pi \circ \phi \circ \sigma(x).$$

Recall that π is distance nonincreasing and that σ is an isometric inclusion. Hence, for any pair $x_1, x_2 \in X_p$,

$$d(\phi_\pi(x_1), \phi_\pi(x_2)) \leq Kd(x_1, x_2) + C.$$

Now we shall establish the other inequality to show that ϕ_π is a quasi-isometry.

Recall that X is a fiber bundle and that each of the spaces X , $\mathbf{U}_n(\mathbb{R})$, and X_p are uniformly locally finite and uniformly contractible. Furthermore, $\mathbf{U}_n(\mathbb{R})$ is a manifold, and the system of apartments in X_p coarsely separates points in X_p . These are all the conditions we need to apply Whyte's theorem [Wh], which states that there are points $y_1, y_2 \in X_p$ such that

$$\text{Hd}(\phi(\pi^{-1}(x_i)), \pi^{-1}(y_i)) \leq A$$

for some constant $A = A(K, C)$.

Because π is distance nonincreasing, it follows that

$$d(\phi_\pi(x_i), y_i) \leq A.$$

Using Lemma 9 and the two preceding inequalities, we have

$$\begin{aligned} d(x_1, x_2) &= \text{Hd}(\pi^{-1}(x_1), \pi^{-1}(x_2)) \\ &\leq K \text{Hd}(\phi(\pi^{-1}(x_1)), \phi(\pi^{-1}(x_2))) + KC \\ &\leq K \text{Hd}(\pi^{-1}(y_1), \pi^{-1}(y_2)) + 2KA + KC \\ &= Kd(y_1, y_2) + 2KA + KC \\ &\leq Kd(\phi_\pi(x_1), \phi_\pi(x_2)) + 4KA + KC. \end{aligned}$$

We have shown that ϕ_π is a quasi-isometry. We may further assume that ϕ_π is an isometry by [KLe, Thm. 1.1.3]. Indeed, Kleiner–Leeb show that any quasi-isometry of a higher rank Euclidean building is a bounded distance from an isometry. \square

Because the isometry group of X_p is $\mathbf{PGL}_n(\mathbb{Q}_p) \rtimes \langle \varphi^* \rangle$, the map ϕ_π is identified with an element of $\mathbf{PGL}_n(\mathbb{Q}_p) \rtimes \langle \varphi^* \rangle$.

3.2. The Base Isometry Preserves $\mathfrak{S}_{\mathbf{B}}$

The following lemma will show that ϕ_π is identified with an element of

$$\mathbf{Ad}(\mathbf{B}_n)(\mathbb{Q}_p) \rtimes \langle \varphi^* \rangle.$$

LEMMA 11. *The sectors $\phi_\pi(\mathfrak{S}_{\mathbf{B}})$ and $\mathfrak{S}_{\mathbf{B}}$ contain a common subsector.*

Proof. Up to finite Hausdorff distance, we may assume that $\phi_\pi(\mathfrak{S}_{\mathbf{B}})$ is contained in a symmetric apartment.

After possibly replacing ϕ with an equivalent quasi-isometry, ϕ restricts to a quasi-isometry of subspaces of symmetric pre-images

$$\pi^{-1}(\mathfrak{S}_{\mathbf{B}}) \rightarrow \pi^{-1}(\phi_\pi(\mathfrak{S}_{\mathbf{B}})).$$

Applying [KLe, Thm. 1.1.3], we may further assume that this map is an isometry.

Now observe that a sector \mathfrak{S} of a symmetric apartment has a pre-image $\pi^{-1}(\mathfrak{S})$ containing a conull subset of the Furstenberg boundary if and only if \mathfrak{S} contains a subsector of $\mathfrak{S}_{\mathbf{B}}$. □

3.3. An Isometry of Symmetric Pre-Images

Let \mathcal{A} be a symmetric apartment. By Lemma 11, the map $\hat{q}_{\mathcal{A}, \mathfrak{S}_{\mathbf{B}}} \circ \phi$ restricts to a quasi-isometry $\pi^{-1}(\mathcal{A}) \rightarrow \pi^{-1}(\mathcal{A}_{\mathbf{A}})$. Then, by [KLe, Thm. 1.1.3], this restriction is a bounded distance in the sup-norm from an isometry, which we name $\phi_{\mathcal{A}}$.

In fact, all symmetric apartments determine the same element of $\mathbf{PGL}_n(\mathbb{R}) \rtimes \langle \varphi^* \rangle$, as our next lemma shows.

LEMMA 12. *For any symmetric apartment $\mathcal{A} \subseteq X_p$, we have*

$$\phi_{\mathcal{A}} = \phi_{\mathcal{A}_{\mathbf{A}}}.$$

Proof. The lemma follows from two facts: (i) $\pi^{-1}(\mathcal{A}_{\mathbf{A}})$ and $\pi^{-1}(\mathcal{A})$ intersect in an unbounded set; and (ii) isometries of X_∞ are completely determined by their restriction to any open subset of X_∞ . □

3.4. The Symmetric Pre-Image Isometry Preserves $\mathfrak{C}_{\mathbf{B}}$

Denote by $\mathfrak{S}_{\mathbf{B}}^{\text{op}}$ a sector in $\mathcal{A}_{\mathbf{A}}$ that is opposite to $\mathfrak{S}_{\mathbf{B}}$. Note that $\sigma(\mathfrak{S}_{\mathbf{B}}^{\text{op}}) \subseteq \pi^{-1}(\mathcal{A}_{\mathbf{A}})$ is the unique Weyl chamber in $\pi^{-1}(\mathfrak{S}_{\mathbf{B}}^{\text{op}})$ up to finite Hausdorff distance, as is $\hat{f}^{-1}(\mathfrak{C}_{\mathbf{B}})$.

Our final lemma identifies $\phi_{\mathcal{A}_{\mathbf{A}}}$ as an element of $\mathbf{Ad}(\mathbf{B}_n)(\mathbb{R}) \rtimes \langle \varphi^* \rangle$.

LEMMA 13. *The isometry $\phi_{\mathcal{A}_{\mathbf{A}}} : \pi^{-1}(\mathcal{A}_{\mathbf{A}}) \rightarrow \pi^{-1}(\mathcal{A}_{\mathbf{A}})$ fixes the Hausdorff equivalence class of the Weyl chamber $\hat{f}^{-1}(\mathfrak{C}_{\mathbf{B}})$.*

Proof. By Lemma 11, the isometry $\phi_{\mathcal{A}_{\mathbf{A}}}$ fixes $\pi^{-1}(\mathfrak{S}_{\mathbf{B}})$ up to finite Hausdorff distance. Thus, $\phi_{\mathcal{A}_{\mathbf{A}}}$ fixes $\pi^{-1}(\mathfrak{S}_{\mathbf{B}}^{\text{op}})$ up to finite Hausdorff distance as well. The

lemma now follows because $\sigma(\mathfrak{S}_{\mathbf{B}}^{\text{op}})$ is the only Weyl chamber in $\pi^{-1}(\mathfrak{S}_{\mathbf{B}}^{\text{op}})$ up to finite Hausdorff distance. \square

3.5. Proof of Theorem 1

By considering whether or not they preserve types in the boundary, it is clear that $\phi_{A_A} \in \mathbf{Ad}(\mathbf{B}_n)(\mathbb{R})$ if and only if $\phi_\pi \in \mathbf{Ad}(\mathbf{B}_n)(\mathbb{Q}_p)$. Therefore, ϕ is a finite distance in the sup-norm from $H(\phi_{A_A}, \phi_\pi)(\varphi_X^*)^k$, where $k \equiv 0 \pmod{2}$, if and only if $\phi_\pi \in \mathbf{Ad}(\mathbf{B}_n)(\mathbb{Q}_p)$.

4. Remarks on the Large-Scale Geometry of Lamplighter Groups

In this section we describe a geometric model for lamplighter groups that is known as a special case of a Diestel–Leader graph. We will race through its definition, leaving some small claims as exercises. Using this model, we will explain how one can arrive at Conjectures 8 and 9.

The connection between Diestel–Leader graphs and lamplighter groups is well known and has played a key role in several results (see e.g. [BaW; BrW1; BrW2; W]).

REMARK 1 (Generalizing solvable function-field-arithmetic groups). Throughout this section we are motivated by virtual lamplighter groups of the form $\mathbf{B}_2(\mathbb{F}_q[t, t^{-1}])$. They are discrete subgroups of the group

$$\mathbf{B}_2(\mathbb{F}_q((t))) \times \mathbf{B}_2(\mathbb{F}_q((t))) \leq \mathbf{SL}_2(\mathbb{F}_q((t))) \times \mathbf{SL}_2(\mathbb{F}_q((t))).$$

REMARK 2 (Laurent series with group coefficients). Let G be a finite group. The group of formal Laurent series in a variable t with coefficients in G is defined as a group of infinite formal sums:

$$G((t)) = \left\{ \sum_{i \in \mathbb{Z}} g_i t^i \mid g_i \in G \text{ for all } i \text{ and } g_i = 1 \text{ for } i \ll 0 \right\}.$$

Multiplication in $G((t))$ is given by components, so that

$$\left(\sum g_i t^i \right) \left(\sum h_i t^i \right) = \sum g_i h_i t^i.$$

REMARK 3 (Laurent polynomials with group coefficients). The group of Laurent series contains the subgroup of Laurent polynomials

$$G[t, t^{-1}] = G((t)) \cap G((t^{-1})).$$

Notice that the group $\bigoplus_{\mathbb{Z}} G$ is isomorphic to $G[t, t^{-1}]$.

The search for a space on which the lamplighter group $\bigoplus_{\mathbb{Z}} G \rtimes \mathbb{Z}$ acts geometrically begins by finding a space on which the group $G((t)) \rtimes \mathbb{Z}$ can act.

REMARK 4 (The metric for Laurent series). A bi-invariant ultrametric metric on $G((t))$ is given by the norm

$$\left| \sum g_i t^i \right| = e^{-N},$$

where N is the least integer with $g_N \neq 1$. Thus, the distance between the two group elements $a, b \in G((t))$ is $|a^{-1}b|$.

REMARK 5 (The regular tree T_G). We will now construct a regular $(|G| + 1)$ -valent tree, T_G , on which $G((t)) \rtimes \mathbb{Z}$ acts. The group of Laurent series will act as “unipotents”, and the integers will act through hyperbolic isometries.

We pick a line l_a for every $a \in G((t))$ along with a homeomorphism $\rho_a: \mathbb{R} \rightarrow l_a$, and we define the quotient

$$T_G = \left(\bigcup_{a \in G((t))} l_a \right) / \sim;$$

here $\rho_a(t) \sim \rho_b(t)$ if $t \geq \log|a^{-1}b|$. Observe that T_G naturally has the structure of a simplicial tree, with a unique end at infinity corresponding to each element of the set $G((t)) \cup \{\infty\}$. We transform T_G into a metric space by assigning length 1 to each edge.

REMARK 6 (Height function on T_G). A well-defined “height function” $h_G: T_G \rightarrow \mathbb{R}$ is given by $h_G(\rho_a(t)) = t$ for all $a \in G((t))$ and $t \in \mathbb{R}$.

REMARK 7 (Action of $G((t)) \rtimes \mathbb{Z}$ on T_G). By defining $a \cdot \rho_b(t) = \rho_{ab}(t)$, the group $G((t))$ acts by simplicial automorphisms on T_G that fix the end corresponding to ∞ . The height function is invariant under this action.

The group \mathbb{Z} acts on $G((t))$ by $n \cdot \sum g_i t^i = \sum g_i t^{i-n}$. Therefore, any $n \in \mathbb{Z}$ acts on T_G by $n \cdot \rho_a(t) = \rho_{n \cdot a}(t + n)$. Note that a positive integer $n > 0$ expands $G((t))$ and acts as a hyperbolic isometry that translates its axis, l_1 , toward the boundary point ∞ .

When combined, the actions just described yield a cocompact but nonproper action of $G((t)) \rtimes \mathbb{Z}$ on T_G by $(a, n) \cdot x = a \cdot (n \cdot x)$ for all $x \in T_G$.

REMARK 8 (Action of $G \wr \mathbb{Z}$ on $T_G \times T_G$). There is an automorphism σ of the lamplighter group $G \wr \mathbb{Z} \cong G[t, t^{-1}] \rtimes \mathbb{Z}$ that is defined by

$$\sigma \left(\sum g_i t^i, m \right) = \left(\sum g_i t^{-i}, -m \right).$$

The automorphism σ is used to include $G \wr \mathbb{Z}$ as a subgroup of $(G((t)) \rtimes \mathbb{Z}) \times (G((t)) \rtimes \mathbb{Z})$ by

$$c \mapsto (c, \sigma(c)).$$

This inclusion describes a proper action of $G \wr \mathbb{Z}$ on $T_G \times T_G$.

REMARK 9 (The horosphere of interest). We define the horosphere

$$\mathcal{H}_G = \{(x, y) \in T_G \times T_G \mid h_G(x) + h_G(y) = 0\}.$$

It is easy to check that \mathcal{H}_G is a locally finite and connected graph whose path metric is quasi-isometric to the metric restricted from $T_G \times T_G$. This graph is a *Diestel–Leader graph*.

Table 1

Γ	X_Γ	$\mathcal{QI}(X_\Gamma)_\mathcal{H}$
$\text{BS}(1, m)$	$\mathbb{H}^2 \times T_m$	$\text{Bilip}(\mathbb{R}) \times \text{Bilip}(\mathbb{Q}_m)$
Sol	$\mathbb{H}^2 \times \mathbb{H}^2$	$[\text{Bilip}(\mathbb{R}) \times \text{Bilip}(\mathbb{R})] \rtimes \mathbb{Z}/2\mathbb{Z}$
$G \wr \mathbb{Z}$	$T_G \times T_G$	$[\text{Bilip}(G((t))) \times \text{Bilip}(G((t)))] \rtimes \mathbb{Z}/2\mathbb{Z}$

The action of $G \wr \mathbb{Z}$ on $T_G \times T_G$ stabilizes \mathcal{H}_G , yielding an action of $G \wr \mathbb{Z}$ on \mathcal{H}_G ; this latter action is proper and cocompact. Therefore, $G \wr \mathbb{Z}$ is quasi-isometric to \mathcal{H}_G .

REMARK 10 (Wreathing infinite groups with \mathbb{Z}). The definition of \mathcal{H}_G and of the action of $G \wr \mathbb{Z}$ on \mathcal{H}_G does not require that G be finite. However, our assumption that G is finite is necessary for the graph \mathcal{H}_G to be locally finite and thus for $G \wr \mathbb{Z}$ to be quasi-isometric to \mathcal{H}_G .

REMARK 11 (Aside on finiteness properties). It is easy to see that \mathcal{H}_G contains loops of unbounded diameter. Thus, the topology of \mathcal{H}_G reflects that $G \wr \mathbb{Z}$ is finitely generated but not finitely presented. The former result is trivial; the latter was originally proved in much greater generality by Baumslag using combinatorial methods [Bau].

4.1. Explaining Conjecture 8

Table 1 lists three basic examples of rank one amenable groups. The first column denotes the group. Each of these groups, Γ , acts properly on a product of negatively curved rank one spaces, denoted X_Γ . The second column lists these metric spaces (T_m is an $(m + 1)$ -valent tree).

The third column lists certain subgroups of $\mathcal{QI}(X_\Gamma)$. Specifically, each group Γ is quasi-isometric to a horosphere $\mathcal{H} \subseteq X_\Gamma$, and $\mathcal{QI}(X_\Gamma)_\mathcal{H}$ denotes the subgroup of $\mathcal{QI}(X_\Gamma)$ consisting of classes of quasi-isometries that preserve \mathcal{H} up to finite Hausdorff distance.

The groups of bilipschitz homeomorphisms that appear in the third column of the table act on the boundary of a factor of X_Γ . Any map on the boundary of one of these rank one spaces that fixes a distinguished point at infinity and also restricts to the rest of the boundary points (\mathbb{R} in the case of \mathbb{H}^2 , and the m -adic numbers in the case of a $(m + 1)$ -valent tree) as a bilipschitz homeomorphism can easily be seen to extend to a quasi-isometry of the interior of the space. (Note that $G((t))$ is isometric to the $|G|$ -adic numbers $\mathbb{Q}_{|G|}$.)

It is easy to see that, in all three cases for Γ , $\mathcal{QI}(X_\Gamma)_\mathcal{H} \leq \mathcal{QI}(\Gamma)$. Theorem 2 states that the only quasi-isometries of $\Gamma = \text{BS}(1, m)$ are those accounted for in $\mathcal{QI}(X_\Gamma)_\mathcal{H}$ —in other words, that the inclusion above is an equality for the solvable Baumslag–Solitar groups. Conjectures 5 and 8 ask whether the same equality holds for the groups Sol and $G \wr \mathbb{Z}$, respectively.

4.2. Explaining Conjecture 9

It is well known and easy to see that, if G and H are finite groups and if $|G|^k = |H|^j$ for some $k, j \in \mathbb{N}$, then $G \wr \mathbb{Z}$ is quasi-isometric to $H \wr \mathbb{Z}$. Geometrically, there is a quasi-isometry $T_G \rightarrow T_{G^k}$ defined by collapsing to a point any path in T_G that bijects under the height function onto an interval of the form $[(l-1)k, lk-1]$ for $l \in \mathbb{Z}$. Applying the previous quasi-isometry to each factor defines a quasi-isometry $T_G \times T_G \rightarrow T_{G^k} \times T_{G^k}$ that maps \mathcal{H}_G within a finite Hausdorff distance of \mathcal{H}_{G^k} . Similarly, \mathcal{H}_H and \mathcal{H}_{H^j} are quasi-isometric. Then $|G|^k = |H|^j$ for some $k, j \in \mathbb{N}$ implies that $G \wr \mathbb{Z}$ is quasi-isometric to $H \wr \mathbb{Z}$, since \mathcal{H}_{G^k} and \mathcal{H}_{H^j} are isometric (given the hypothesis).

There are no known examples of a pair of lamplighter groups that are quasi-isometric but do *not* meet the hypothesis of the preceding paragraph. Given the analogy between lamplighter groups and solvable Baumslag–Solitar groups, it is possible that such nonelementary examples do not exist.

Indeed, Farb–Mosher showed that any quasi-isometry between two solvable Baumslag–Solitar groups, say $BS(1, m)$ and $BS(1, r)$, induces a bilipschitz homeomorphism between the “upper boundaries” of the respective groups—in concrete terms, a bilipschitz homeomorphism between \mathbb{Q}_m and \mathbb{Q}_r [FaM1, Thm. 6.1]. Cooper showed that such a bilipschitz map can exist only when $m^k = r^j$ for some $k, j \in \mathbb{N}$ [FaM1, Cor. 10.11]. If a full analogue held for lamplighter groups, then a quasi-isometry $G \wr \mathbb{Z} \rightarrow H \wr \mathbb{Z}$ would induce a bilipschitz homeomorphism $G(\langle t \rangle) \rightarrow H(\langle t \rangle)$, and this would imply that $|G|^k = |H|^j$ for some $k, j \in \mathbb{N}$.

4.3. A Conjecture of Diestel–Leader

Let \mathbb{F}_2 and \mathbb{F}_3 be fields with exactly two and three elements, respectively. We define the graph

$$\mathcal{H}_{2,3} = \{(x, y) \in T_{\mathbb{F}_2} \times T_{\mathbb{F}_3} \mid h_{\mathbb{F}_2}(x) + h_{\mathbb{F}_3}(y) = 0\}.$$

This graph is connected, transitive, and locally finite.

In response to a question of Woess, Diestel and Leader [DL] conjectured that $\mathcal{H}_{2,3}$ is a connected, transitive, and locally finite graph that is not quasi-isometric to a finitely generated group. (Since the writing of this paper, this conjecture was proved by Eskin–Fisher–Whyte; see [EFWh, Thm. 1.9].)

Following our previous reasoning, the quasi-isometry group of $\mathcal{H}_{2,3}$ might be $\text{Bilip}(\mathbb{Q}_2) \times \text{Bilip}(\mathbb{Q}_3)$. Assuming this to be the case, if a finitely generated group Γ were quasi-isometric to $\mathcal{H}_{2,3}$, then Γ would quasi-act on $\mathcal{H}_{2,3}$ and so define a representation

$$\Gamma \rightarrow \text{Bilip}(\mathbb{Q}_2) \times \text{Bilip}(\mathbb{Q}_3).$$

This would make the Diestel–Leader conjecture susceptible to some algebraic techniques. Perhaps techniques similar to those in [FaM2] could show that no such Γ exists.

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