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Coniveau and the Grothendieck Group of Varieties

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There are two natural filtrations on the singular cohomology of a complex smooth projective variety: the coniveau filtration, which is defined geometrically; and the level filtration, which is defined Hodge theoretically. We will say that the generalized Hodge conjecture (GHC) holds for a variety X if these filtrations coincide on its cohomology. There are a number of intermediate forms of this condition, including the statement that the ordinary Hodge conjecture holds for X. We show that if the GHC (or an intermediate version of it) holds for X then it holds for any variety Y that defines the same class in a completion of a Grothendieck group of varieties. In particular, by using motivic integration we can see that this is the case if X and Y are birationally equivalent Calabi–Yau varieties or, more generally, K-equivalent varieties. This refines a result obtained in [A] by a different method.

The key point is to show that the singular cohomology with its coniveau (resp. level) filtration determines a homomorphism ν (resp. λ) from the Grothendieck group of varieties $\mathrm{K}_0(\mathcal{V}ar_{\mathbb{C}})$ to the Grothendieck group of polarizable filtered Hodge structures $\mathrm{K}_0(\mathcal{FHS})$. This is done by showing that cohomology together with these filtrations behaves appropriately under blowups. We then show that X satisfies GHC if and only if its class $[X]$ lies in the kernel of the difference $\nu - \lambda$, and the results just described follow from this.

The following conventions will be used throughout the paper. All our varieties will be defined over $\mathbb{C}$. We denote the singular cohomology of a smooth projective variety X with rational coefficients by $H^i(X)$. Our thanks to the referee for a number of helpful suggestions.

1. Filtered Hodge Structures

Let X be a smooth projective variety. Its cohomology carries a natural Hodge structure. The *coniveau* filtration on $H^i(X)$ is given by

$$N^pH^i(X) = \sum_{\operatorname{codim} S \geq p} \ker[H^i(X) \to H^i(X - S)],$$

which is a descending filtration by sub-Hodge structures. The largest rational sub-Hodge structure $\mathcal{F}^pH^i(X)$ contained in $F^pH^i(X)$ gives a second filtration, which we call the *level filtration*. We have $N^pH^i(X) \subseteq \mathcal{F}^pH^i(X)$. We will say that

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$\mathrm{GHC}(H^i(X), p)$ holds if $N^p H^i(X) = \mathcal{F}^p H^i(X)$, and we will say that the generalized Hodge conjecture holds for X if we have equality for all i and p. Note that $\mathrm{GHC}(H^{2p}(X), p)$ is just the usual Hodge conjecture.

We recall that a Hodge structure is polarizable if it admits a polarization—that is, a bilinear form satisfying the Hodge–Riemann bilinear relations. We note that the Hodge structure on the cohomology of a smooth projective variety is polarizable: once an ample line bundle is chosen, a polarization is given by taking the orthogonal direct sum of the polarizations, determined in the usual way, on primitive cohomology [W, pp. 202, 207]. Let $\mathcal{HS}$ be the category of finite direct sums of pure rational polarizable Hodge structures. The category $\mathcal{HS}$ is a semisimple abelian category with tensor products [D, Sec. 4.2.3]. Any object H can be decomposed into a sum

$$H = \bigoplus_i H^i,$$

where H^i is the largest sub-Hodge structure of weight i. We define the category $\mathcal{FHS}$ of objects that are polarizable Hodge structures with finite descending filtrations by sub-Hodge structures and whose morphisms preserve the filtration. Note that this would be a filtration by polarizable sub-Hodge structures, since a sub-Hodge structure of a polarizable Hodge structure is again polarizable.

Given an additive category $\mathcal{C}$, we can define a Grothendieck group $\mathrm{K}_0^{\mathrm{split}}(\mathcal{C})$ by generators and relations as follows. We have one generator $[X]$ for each isomorphism class of objects $X \in \mathcal{C}$, and we impose the relation $[X_3] = [X_1] + [X_2]$ whenever $X_3 \cong X_1 \oplus X_2$. When the category $\mathcal{C}$ possesses exact sequences, we can define a quotient $\mathrm{K}_0(\mathcal{C})$ by imposing this relation when X_3 is an extension of X_2 by X_1. Although it is not strictly necessary for our purposes, we will show in the Appendix that these constructions lead to the same groups when applied to $\mathcal{HS}$ and $\mathcal{FHS}$. Consequently, we will usually drop the label "split" in the sequel.

By definition, any additive invariant on $\mathcal{HS}$ or $\mathcal{FHS}$ factors through their Grothendieck groups. In particular, this remark applies to the Poincaré polynomial

$$P_H(t) = \sum_i \dim H^i t^i \in \mathbb{Z}[t, t^{-1}]$$

and a filtered version of it,

$$FP_{(H,N)}(t,u) = \sum_{i,p} \dim(N^p \cap H^i) t^i u^p \in \mathbb{Z}[t^{\pm 1}, u^{\pm 1}].$$

We can define two functors between the categories $\mathcal{HS}$ and $\mathcal{FHS}$:

$$\Gamma \colon \mathcal{HS} \longrightarrow \mathcal{FHS};\ H \longmapsto (H, \mathcal{F}^\bullet),$$

$$\Phi \colon \mathcal{FHS} \longrightarrow \mathcal{HS};\ (H, N^\bullet) \longmapsto H,$$

where $\mathcal{F}^\bullet$ is the level filtration on H (i.e., $\mathcal{F}^p H$ is the largest sub-Hodge structure of $F^p H$). These functors are clearly additive. Thus we obtain well-defined group homomorphisms γ and ϕ, respectively:

$$\gamma \colon \mathrm{K}_0(\mathcal{HS}) \longrightarrow \mathrm{K}_0(\mathcal{FHS});\ [H] \longmapsto [(H, \mathcal{F}^\bullet)],$$

$$\phi \colon \mathrm{K}_0(\mathcal{FHS}) \longrightarrow \mathrm{K}_0(\mathcal{HS});\ [(H, N^\bullet)] \longmapsto [H].$$

Let $\mathrm{K}_0(\mathcal{V}ar_{\mathbb{C}})$ denote the Grothendieck group of the category of varieties over $\mathbb{C}$ [DeL1]. A more convenient description for our purposes is provided by [Bi, Thm. 3.1]:

$$\mathrm{K}_0(\mathcal{V}ar_{\mathbb{C}}) \cong \mathrm{K}_0^{\mathrm{bl}}(\mathcal{V}ar_{\mathbb{C}}),$$

where $\mathrm{K}_0^{\mathrm{bl}}(\mathcal{V}ar_{\mathbb{C}})$ is the free abelian group generated by isomorphism classes $[X]$ of smooth projective varieties subject to the relation $[\mathrm{Bl}_Z X] - [E] = [X] - [Z]$ for every blowup $\mathrm{Bl}_Z X$ of X along a smooth closed subvariety $Z \subset X$ with exceptional divisor E.

Let X be a smooth projective variety. Set

$$[X]_{\mathcal{HS}} := \sum_i (-1)^i [H^i(X)] \in \mathrm{K}_0(\mathcal{HS}),$$

$$[X]_{\mathcal{FHS}} := \sum_i (-1)^i [(H^i(X), N^\bullet)] \in \mathrm{K}_0(\mathcal{FHS}),$$

where $N^\bullet$ is the coniveau filtration. In the next section we will show that these classes depend only on $[X] \in \mathrm{K}_0^{\mathrm{bl}}(\mathcal{V}ar_{\mathbb{C}})$.

2. Coniveau of a Blowup

We use the following notation throughout this section. Let X be a smooth projective variety and let $\sigma : \tilde{X} = \mathrm{Bl}_Z X \to X$ be the blowup of X along a smooth closed subvariety Z of X of codimension ≥ 2. The exceptional divisor E can be identified with $\mathbb{P}(N_{Z/X})$, where $N_{Z/X}$ is the normal bundle. Therefore, it has a tautological line bundle $\mathcal{O}_E(1)$. Let $r = \operatorname{codim}(Z, X) - 1 = \dim E - \dim Z$ and let $h = c_1(\mathcal{O}_E(1))$. Then we have

$$\begin{tikzcd} E \arrow[r, hook, "j"] \arrow[d, "\sigma"] & \tilde{X} \arrow[d, "\sigma"] & \tilde{X} - E \arrow[l, hook', "j_1"'] \arrow[d, "\sigma"] \\ Z \arrow[r, hook, "i"] & X & X - Z, \arrow[l, hook', "i_1"'] \end{tikzcd}$$

where i, i_1 and j, j_1 are inclusions.

Lemma 2.1.

$$N^pH^i(E) = \sigma^*(N^pH^i(Z)) \oplus (h \cup \sigma^*(N^{p-1}H^{i-2}(Z))) \oplus \cdots$$
$$\oplus (h^r \cup \sigma^*(N^{p-r}H^{i-2r}(Z))).$$

Proof. Since $h \in N^1H^2(E)$, it follows that $h^k \in N^kH^{2k}(E)$ for each $k \geq 1$. Hence, for any $\alpha_k \in \sigma^*(N^{p-k}H^{i-2k}(Z)) \subseteq N^{p-k}H^{i-2k}(E)$,

$$h^k \cup \alpha_k \in N^kH^{2k}(E) \cup N^{p-k}H^{i-2k}(E) \subseteq N^pH^i(E)$$

by [AK, Cor. 1.2] for $1 \leq k \leq r$. Hence this and $\sigma^*(N^pH^i(Z)) \subseteq N^pH^i(E)$ yield

$$N^pH^i(E) \supseteq \sigma^*(N^pH^i(Z)) \oplus (h \cup \sigma^*(N^{p-1}H^{i-2}(Z))) \oplus \cdots$$
$$\oplus (h^r \cup \sigma^*(N^{p-r}H^{i-2r}(Z))).$$

To show the converse, first note that

$$H^i(E) = \sigma^*(H^i(Z)) \oplus \left(\bigoplus_{k=1}^{r} (h^k \cup \sigma^*(H^{i-2k}(Z))) \right)$$

by [Le, Prop. 8.23]. As a consequence, for any $\alpha \in N^p H^i(E)$ we can decompose

$$\alpha = \sigma^*(\alpha_0) + (h \cup \sigma^*(\alpha_1)) + (h^2 \cup \sigma^*(\alpha_2)) + \cdots + (h^r \cup \sigma^*(\alpha_r)),$$

where $\alpha_k \in H^{i-2k}(Z)$ for each $k = 0, \ldots, r$. We will show that $\alpha_k \in N^{p-k} H^{i-2k}(Z)$ for each $k = 0, \ldots, r$ by using descending induction on k.

First note that

$$\sigma_*(h^i) = \begin{cases} 0 & \text{if } i < r, \\ 1 & \text{if } i = r. \end{cases}$$

Therefore, by the projection formula and [AK, Thm. 1.1(1)],

$$\alpha_r = \sigma_*(h^r \cup \sigma^*(\alpha_r)) = \sigma_*(\alpha) \in N^{p-r} H^{i-2r}(Z).$$

This shows the claim when $k = r$. Now suppose that $\alpha_l \in N^{p-l} H^{i-2l}(Z)$ for $k+1 \le l \le r$. We must show that $\alpha_k \in N^{p-k} H^{i-2k}(Z)$. By assumption, for $k+1 \le l \le r$ we have

$$h^l \cup \sigma^*(\alpha_l) \in N^l H^{2l}(E) \cup N^{p-l} H^{i-2l}(E) \subseteq N^p H^i(E).$$

Set

$$\beta_{k+1} = \alpha - \sum_{l=k+1}^{r} (h^l \cup \sigma^*(\alpha_l)).$$

Then

$$\beta_{k+1} = \sigma^*(\alpha_0) + (h \cup \sigma^*(\alpha_1)) + \cdots + (h^k \cup \sigma^*(\alpha_k)) \in N^p H^i(E).$$

Taking now a cup product with $h^{r-k} \in N^{r-k} H^{2(r-k)}(E)$ yields

$$\begin{aligned} h^{r-k} \cup \beta_{k+1} &= (h^{r-k} \cup \sigma^*(\alpha_0)) + (h^{r-k+1} \cup \sigma^*(\alpha_1)) + \cdots + (h^{r-k+k} \cup \sigma^*(\alpha_k)) \\ &\in N^{p+(r-k)} H^{i+2(r-k)}(E). \end{aligned}$$

Then

$$\sigma_*(h^{r-k} \cup \beta_{k+1}) = \sigma_*(h^r \cup \sigma^*(\alpha_k)) = \alpha_k \in N^{p-k} H^{i-2k}(Z)$$

as we claimed. Therefore,

$$\begin{aligned} N^p H^i(E) \subseteq \sigma^*(N^p H^i(Z)) &\oplus (h \cup \sigma^*(N^{p-1} H^{i-2}(Z))) \oplus \cdots \\ &\oplus (h^r \cup \sigma^*(N^{p-r} H^{i-2r}(Z))). \end{aligned}$$

□

COROLLARY 2.2. *If $\alpha \in N^p H^i(E)$ then we can write*

$$\alpha = \sigma^*(\alpha_0) + (h \cup \beta),$$

where $\alpha_0 \in N^p H^i(Z)$ and $\beta \in N^{p-1} H^{i-2}(E)$.

The following is a well-known result, but we include the proof here for lack of a suitable reference.

LEMMA 2.3. *There is a short exact sequence of pure Hodge structures of weight i:*

$$0 \longrightarrow H^i(X) \xrightarrow{\sigma^*+i^*} H^i(\tilde{X}) \oplus H^i(Z) \xrightarrow{-j^*+\sigma^*} H^i(E) \longrightarrow 0. \quad (1)$$

Proof. We have a commutative diagram with exact rows,

$$\begin{tikzcd} \cdots \arrow[r] & H_c^i(X-Z) \arrow[r] \arrow[d, "\cong"] & H^i(X) \arrow[r, "i^*"] \arrow[d, hook, "\sigma^*"] & H^i(Z) \arrow[r, "\partial"] \arrow[d, "\sigma^*"] & H_c^{i+1}(X-Z) \arrow[r] \arrow[d, "\cong"] & H^{i+1}(X) \arrow[r] \arrow[d, hook, "\sigma^*"] & \cdots \\ \cdots \arrow[r] & H_c^i(\tilde{X}-E) \arrow[r] & H^i(\tilde{X}) \arrow[r, "j^*"] & H^i(E) \arrow[r, "\tilde{\partial}"] & H_c^{i+1}(\tilde{X}-E) \arrow[r] & H^{i+1}(\tilde{X}) \arrow[r] & \cdots, \end{tikzcd}$$

where some of the arrows are injections or isomorphisms as indicated. The lemma now follows from a straightforward diagram chase. □

LEMMA 2.4. *The sequence of Lemma 2.3 is exact in $\mathcal{FHS}$; that is,*

$$0 \longrightarrow N^pH^i(X) \longrightarrow N^pH^i(\tilde{X}) \oplus N^pH^i(Z) \longrightarrow N^pH^i(E) \longrightarrow 0$$

is an exact sequence for all p.

Proof. Consider the short exact sequence (1) and note that

$$(\sigma^* + i^*)(N^pH^i(X)) \subseteq N^pH^i(\tilde{X}) \oplus N^pH^i(Z),$$
$$(-j^* + \sigma^*)(N^pH^i(\tilde{X}) \oplus N^pH^i(Z)) \subseteq N^pH^i(E),$$

since all maps preserve the coniveau.

We will check exactness in the middle. It suffices to show that

$$\operatorname{im}(\sigma^* + i^*)|_{N^pH^i(X)} \supseteq \ker(-j^* + \sigma^*)|_{N^pH^i(\tilde{X})\oplus N^pH^i(Z)},$$

since the reverse inclusion follows from (1). Let $(\beta, \gamma) \in N^pH^i(\tilde{X}) \oplus N^pH^i(Z)$ such that $(\beta, \gamma) \in \ker(-j^* + \sigma^*)$. Then, by the exact sequence (1), there exists an $\alpha \in H^i(X)$ such that $(\sigma^* + i^*)(\alpha) = (\beta, \gamma)$. In particular,

$$\sigma^*(\alpha) = \beta \in N^pH^i(\tilde{X}).$$

Because $\sigma : \tilde{X} \to X$ is a birational map, we have

$$\alpha = (\sigma_* \circ \sigma^*)(\alpha) = \sigma_*(\beta) \in \sigma_*(N^pH^i(\tilde{X})) \subseteq N^pH^i(X).$$

Hence $(\beta, \gamma) = (\sigma^* + i^*)(\alpha) \in (\sigma^* + i^*)(N^pH^i(X))$.

We shall now check the surjectivity of $(-j^* + \sigma^*)|_{N^pH^i(\tilde{X})\oplus N^pH^i(Z)}$. Let $\alpha \in N^pH^i(E)$. Then, by Corollary 2.2, we can decompose

$$\alpha = \sigma^*(\alpha_0) + (h \cup \beta),$$

where $\alpha_0 \in N^pH^i(Z)$ and $\beta \in N^{p-1}H^{i-2}(E)$. Note the composition

$$H^{i-2}(E) \xrightarrow{j_*} H^i(\tilde{X}) \xrightarrow{j^*} H^i(E)$$

is the same as cupping with $[E]|_E$. Since $\mathcal{O}(E)|_E = \mathcal{O}_E(-1)$ and $h = c_1(\mathcal{O}_E(1))$, it follows that

$$(j^* \circ j_*)(\beta) = [E]|_E \cup \beta = -h \cup \beta.$$

As a result,

$$\alpha = \sigma^*(\alpha_0) - j^*(j_*(\beta))$$

and we have $j_*(\beta) \in j_*(N^{p-1}H^{i-2}(E)) \subseteq N^pH^i(\tilde{X})$, and this shows that the map $(-j^* + \sigma^*)|_{N^pH^i(\tilde{X})\oplus N^pH^i(Z)}$ is surjective. □

Let $f\colon X \to Y$ be a morphism of smooth projective varieties. Then f^* preserves the coniveau and induces maps $\bar{f}^*\colon \mathrm{Gr}_N^p H^i(Y) \to \mathrm{Gr}_N^p H^i(X)$.

COROLLARY 2.5. *The following sequence is exact*:

$$0 \longrightarrow \mathrm{Gr}_N^p H^i(X) \longrightarrow \mathrm{Gr}_N^p H^i(\tilde{X}) \oplus \mathrm{Gr}_N^p H^i(Z) \longrightarrow \mathrm{Gr}_N^p H^i(E) \longrightarrow 0.$$

Proof. This follows from Lemma A.1. □

We want to construct well-defined morphisms from $\mathrm{K}_0^{\mathrm{bl}}(\mathcal{V}ar_{\mathbb{C}})$ to $\mathrm{K}_0(\mathcal{HS})$ and from $\mathrm{K}_0^{\mathrm{bl}}(\mathcal{V}ar_{\mathbb{C}})$ to $\mathrm{K}_0(\mathcal{FHS})$. First we need the following lemmas.

LEMMA 2.6. *The relation*

$$[\mathrm{Bl}_Z X]_{\mathcal{HS}} - [E]_{\mathcal{HS}} = [X]_{\mathcal{HS}} - [Z]_{\mathcal{HS}}$$

holds. Hence we have a well-defined group homomorphism,

$$\mathrm{K}_0^{\mathrm{bl}}(\mathcal{V}ar_{\mathbb{C}}) \longrightarrow \mathrm{K}_0(\mathcal{HS});\ [X] \longmapsto [X]_{\mathcal{HS}}.$$

Proof. By Lemma 2.3, we have the following short exact sequence of pure Hodge structures:

$$0 \longrightarrow H^i(X) \longrightarrow H^i(\tilde{X}) \oplus H^i(Z) \longrightarrow H^i(E) \longrightarrow 0.$$

Then, by the definition of $\mathrm{K}_0(\mathcal{HS})$,

$$[H^i(X)] + [H^i(E)] = [H^i(\tilde{X}) \oplus H^i(Z)] = [H^i(\tilde{X})] + [H^i(Z)],$$

and the result follows immediately if we take alternating sums. □

We define

$$\lambda\colon \mathrm{K}_0^{\mathrm{bl}}(\mathcal{V}ar_{\mathbb{C}}) \longrightarrow \mathrm{K}_0(\mathcal{FHS})$$

as the composition of the map constructed in Lemma 2.6 with γ.

LEMMA 2.7. *The relation*

$$[\mathrm{Bl}_Z X]_{\mathcal{FHS}} - [E]_{\mathcal{FHS}} = [X]_{\mathcal{FHS}} - [Z]_{\mathcal{FHS}}$$

holds. Hence we have a well-defined group homomorphism,

$$\nu\colon \mathrm{K}_0^{\mathrm{bl}}(\mathcal{V}ar_{\mathbb{C}}) \longrightarrow \mathrm{K}_0(\mathcal{FHS});\quad \nu([X]) = [X]_{\mathcal{FHS}}.$$

Proof. We showed in Lemma 2.4 that

$$0 \longrightarrow [(H^i(X), N^\bullet)] \longrightarrow [(H^i(\tilde{X}) \oplus H^i(Z), N^\bullet)] \longrightarrow [(H^i(E), N^\bullet)] \longrightarrow 0 \quad (2)$$

is exact in $\mathcal{FHS}$, where $N^p(H^i(\tilde{X}) \oplus H^i(Z)) = N^p H^i(\tilde{X}) \oplus N^p H^i(Z)$. The rest of the argument is exactly the same as before. □

REMARK 2.8. The proof of Lemma 2.4 shows that the sequence (2) is split exact. Therefore, as the referee pointed out, we can in particular construct the homomorphism

$$\nu : \mathrm{K}_0^{\mathrm{bl}}(\mathcal{V}ar_{\mathbb{C}}) \longrightarrow \mathrm{K}_0^{\mathrm{split}}(\mathcal{FHS})$$

directly—that is, without appealing to the results of the Appendix.

3. Main Theorem

THEOREM 3.1. *Let X be a smooth projective variety. Then the following statements are equivalent.*

1. *The GHC holds for X.*
2. $[X] \in \ker(\nu - \lambda)$.
3. *The equality $FP_{[X]_{\mathcal{FHS}}}(t,u) = FP_{\gamma([X]_{\mathcal{HS}})}(t,u)$ of filtered Poincaré polynomials holds.*

Proof. Clearly, the GHC for X implies that $[X]_{\mathcal{FHS}} = \gamma([X]_{\mathcal{HS}})$ or (equivalently) that $[X] \in \ker(\nu - \lambda)$.

Suppose $[X] \in \ker(\nu - \lambda)$. Then

$$[X]_{\mathcal{FHS}} = \gamma([X]_{\mathcal{HS}}),$$

that is,

$$\sum_i (-1)^i [(H^i(X), N^\bullet)] = \sum_i (-1)^i [(H^i(X), \mathcal{F}^\bullet)].$$

Taking the filtered Poincaré polynomial $FP(t,u)$ of both sides yields the third statement.

Assume the equality in Theorem 3.1.3. The coefficient of t^i on the left is

$$(-1)^i \sum_p \dim N^p H^i(X) u^p$$

and on the right is

$$(-1)^i \sum_p \dim \mathcal{F}^p H^i(X) u^p.$$

The equality of these expressions forces $N^p H^i(X) = \mathcal{F}^p H^i(X)$ in this case. □

REMARK 3.2. The coefficient of $t^i u^p$ in $FP_{[X]_{\mathcal{FHS}}}(t,u)$ is $(-1)^i \dim N^p H^i(X)$, and the coefficient of $t^i u^p$ in $FP_{\gamma([X]_{\mathcal{HS}})}(t,u)$ is $(-1)^i \dim \mathcal{F}^p H^i(X)$. Therefore, $\mathrm{GHC}(H^i(X), p)$ holds precisely when these coefficients coincide.

Let $\mathbb{L}' = [\mathbb{A}^1_{\mathbb{C}}] \in \mathrm{K}_0(\mathcal{V}ar_{\mathbb{C}})$ denote the Lefschetz object. Under the isomorphism

$$\mathrm{K}_0(\mathcal{V}ar_{\mathbb{C}}) \cong \mathrm{K}_0^{\mathrm{bl}}(\mathcal{V}ar_{\mathbb{C}}),$$

$\mathbb{L}'$ maps to $\mathbb{L} = [\mathbb{P}^1] - [\infty]$. Let

$$\mathbb{K} = [(\mathbb{Q}(-1), N^\bullet)],$$

where $N^\bullet$ is the filtration determined by $N^1/N^2 = \mathbb{Q}(-1)$. Then

$$\lambda(\mathbb{L}) = \nu(\mathbb{L}) = \mathbb{K}.$$

We have a product on the category $\mathcal{FHS}$ that is just the tensor product filtered by

$$N^p(H_1 \otimes H_2) = \bigoplus_{r+s=p} N^r H_1 \otimes N^s H_2.$$

This gives a commutative ring structure on $\mathrm{K}_0(\mathcal{FHS})$. The group $\mathrm{K}_0^{\mathrm{bl}}(\mathcal{V}ar_{\mathbb{C}})$ also has a commutative ring structure induced by the product of varieties. It is not clear whether λ or ν are ring homomorphisms; however, we do have the following result.

LEMMA 3.3. *For any $\eta \in \mathrm{K}_0^{\mathrm{bl}}(\mathcal{V}ar_{\mathbb{C}})$,*

$$\lambda(\mathbb{L} \cdot \eta) = \mathbb{K} \cdot \lambda(\eta),$$
$$\nu(\mathbb{L} \cdot \eta) = \mathbb{K} \cdot \nu(\eta).$$

Proof. Under the Künneth isomorphism, we have

$$N^p H^i(\mathbb{P}^1 \times X) \cong N^{p-1} H^{i-2}(X)(-1) \oplus N^p H^i(\{\infty\} \times X);$$

a similar statement holds for $\mathcal{F}^p$. The lemma is an immediate consequence. □

The element $\mathbb{K}$ is invertible, and the foregoing identities guarantee that λ or ν factors through the localization $\mathcal{M} = \mathrm{K}_0^{\mathrm{bl}}(\mathcal{V}ar_{\mathbb{C}})[\mathbb{L}^{-1}]$.

Recall [DeL1] that there is a decreasing filtration $F^\bullet\mathcal{M}$ on $\mathcal{M}$, where $F^m\mathcal{M}$ is the subgroup of $\mathcal{M}$ generated by $\{[X] \cdot \mathbb{L}^{-j} \mid \dim X - j \le -m\}$. Let $\hat{\mathcal{M}}$ be the completion of the ring $\mathcal{M}$ with respect to the filtration $F^\bullet\mathcal{M}$. A similar filtration (cf. [Lo]) can be defined on $\mathrm{K}_0(\mathcal{FHS})$ via replacement of dimension by weights. More precisely, let $L^m\mathrm{K}_0(\mathcal{FHS})$ be the subgroup generated by $\{[(H, N^\bullet)] \mid \max\{i \mid H^i \ne 0\} \le -m\}$. We denote the completion of $\mathrm{K}_0(\mathcal{FHS})$ with respect to $L^\bullet$ by $\hat{\mathcal{N}}$. Weights of the Hodge structure on the cohomology of a smooth projective variety of dimension $\le d$ are bounded by $2d$. Therefore, the induced filtrations $\lambda(F^\bullet)$ and $\nu(F^\bullet)$ are cofinal with a subfiltration of $L^\bullet$. It follows that we have a commutative diagram

$$\begin{tikzcd}
\mathrm{K}_0^{\mathrm{bl}}(\mathcal{V}ar_{\mathbb{C}}) \arrow[r, "f"] \arrow[d, "\alpha"] & \hat{\mathcal{M}} \arrow[d] \\
\mathrm{K}_0(\mathcal{FHS}) \arrow[r, "\tau"] & \hat{\mathcal{N}},
\end{tikzcd}$$

where α denotes either λ or ν and where f and τ are the canonical maps.

Lemma 3.4. *The homomorphism* $\mathrm{K}_0(\mathcal{FHS}) \to \mathbb{Z}[t^{\pm 1}, u^{\pm 1}]$ *given by the filtered Poincaré polynomial factors through the image of* τ.

Proof. Let $\eta \in \bigcap_m L^m$. Then, for each $m \in \mathbb{Z}$, it follows that η can be expressed as a linear combination of classes of filtered Hodge structures of weight at most $-m$. Thus the degree of $FP_\eta(t,u)$ in t is bounded above by $-m$ for all $m \in \mathbb{Z}$. But this is impossible unless $FP_\eta(t,u) = 0$. □

Corollary 3.5 (to theorem). *Let* X, Y *be smooth projective varieties such that* X, Y *define the same classes in* $\hat{\mathcal{M}}$. *Then the GHC holds for* X *if and only if the GHC holds for* Y.

Proof. If the images of $[X]$ and $[Y]$ coincide in $\hat{\mathcal{M}}$, then they coincide in $\hat{\mathcal{N}}$. Hence their filtered Poincaré polynomials coincide. The conclusion is now an immediate consequence of Theorem 3.1. □

Remark 3.2 now yields the following statement.

Corollary 3.6. *Let* X, Y *be smooth projective varieties such that* X, Y *define the same classes in* $\hat{\mathcal{M}}$. *Then, for each* i *and* p, $\mathrm{GHC}(H^i(X), p)$ *holds if and only if* $\mathrm{GHC}(H^i(Y), p)$ *holds.*

The proof of the next corollary depends on the motivic integration theory of Kontsevich, Denef, and Loeser. See [DeL1; DeL2; Lo] for an introduction to these ideas.

Corollary 3.7. *Let* X *and* Y *be K-equivalent smooth projective varieties. That is, there exist a smooth projective variety* Z *as well as birational maps* $\pi_1 : Z \to X$ *and* $\pi_2 : Z \to Y$ *such that* $\pi_1^* K_X = \pi_2^* K_Y$, *where* K_X (*resp.* K_Y) *is the canonical divisor on* X (*resp.* Y). *Then* $\mathrm{GHC}(H^i(X), p)$ *holds if and only if* $\mathrm{GHC}(H^i(Y), p)$ *holds.*

Proof. It is enough to show that X and Y define the same class in $\hat{\mathcal{M}}$. This follows from the K-equivalence assumption by a standard application of motivic integration theory; see [L] or [V]. For the reader's convenience we reproduce the argument here. By the change-of-variables formula [DeL1, Lemma 3.3], we have

$$\begin{aligned} f([X]) = \int_{\mathcal{L}(X)} d\mu_X &= \int_{\mathcal{L}(Z)} \mathbb{L}^{-\mathrm{ord}_t \pi_1^* \omega_X}\, d\mu_Z \\ &= \int_{\mathcal{L}(Z)} \mathbb{L}^{-\mathrm{ord}_t \pi_2^* \omega_Y}\, d\mu_Z = \int_{\mathcal{L}(Y)} d\mu_Y = f([Y]), \end{aligned}$$

where $f : \mathrm{K}_0^{\mathrm{bl}}(\mathcal{V}ar_{\mathbb{C}}) \to \hat{\mathcal{M}}$ is the canonical map, $\mathcal{L}(X), \mathcal{L}(Y)$ are the arc spaces, ω_X, ω_Y are the canonical sheaves, and $d\mu_X, d\mu_Y$ are the motivic measures. Hence the corollary follows from Corollary 3.5. □

Corollary 3.8. *If* X *and* Y *are birational Calabi–Yau varieties, then it follows that* $\mathrm{GHC}(H^i(X), p)$ *holds if and only if* $\mathrm{GHC}(H^i(Y), p)$ *holds.*

Proof. Since $K_X = K_Y = 0$, this is a consequence of Corollary 3.7. □

Appendix. Grothendieck Groups of Filtered Categories

We recall that an exact category consists of an additive category $\mathcal{C}$ together with a distinguished class of diagrams

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

which are called *exact sequences* and satisfy appropriate conditions [Q]. For example, any additive category $\mathcal{C}$ can be made exact by taking the class of exact sequences to be isomorphic to the class of split sequences:

$$0 \longrightarrow A \longrightarrow A \oplus C \longrightarrow C \longrightarrow 0.$$

An abelian category gives another example of an exact category, where exact sequences have the usual meaning. If the category is also semisimple, then this exact structure coincides with the split structure described previously. This remark applies also to $\mathcal{HS}$.

Let $\mathcal{C}$ be an abelian category, and let $\mathcal{FC}$ (resp. $\mathcal{GC}$) denote the category of filtered (resp. graded) objects in $\mathcal{C}$. The category $\mathcal{GC}$ is abelian, but $\mathcal{FC}$ is generally not. However, $\mathcal{FC}$ has a natural exact structure [BBD, Sec. 1.1.4] that may be given as follows. We have a functor

$$(H, N^\bullet) \longmapsto \bigoplus_p N^p H$$

from $\mathcal{FC}$ to $\mathcal{GC}$. We declare a sequence

$$0 \longrightarrow (H_1, N^\bullet) \longrightarrow (H_2, N^\bullet) \longrightarrow (H_3, N^\bullet) \longrightarrow 0$$

in $\mathcal{FC}$ to be *exact* if and only if its image in $\mathcal{GC}$ is exact. For the record we note the following alternative formulation, which is perhaps more common.

LEMMA A.1. *The sequence*

$$0 \longrightarrow (H_1, N^\bullet) \longrightarrow (H_2, N^\bullet) \longrightarrow (H_3, N^\bullet) \longrightarrow 0$$

is exact if and only if the following sequence is exact in $\mathcal{GC}$:

$$\begin{aligned} 0 \longrightarrow \bigoplus_p N^p H_1 / N^{p+1} H_1 &\longrightarrow \bigoplus_p N^p H_2 / N^{p+1} H_2 \\ &\longrightarrow \bigoplus_p N^p H_3 / N^{p+1} H_3 \longrightarrow 0. \end{aligned}$$

Proof. This is a straightforward application of the Snake lemma and induction. □

LEMMA A.2. *Given the notion of exact sequence just defined, the category* $\mathcal{FC}$ *is an exact category.*

The Grothendieck group $\mathrm{K}_0(\mathcal{C})$ of an exact category $\mathcal{C}$ is given by generators $[M]$, where $M \in \mathcal{C}$ and we have the relations $[M_2] = [M_1] + [M_3]$ for every exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$. Let us denote the Grothendieck group for

$\mathcal{C}$ with its split exact structure by $K_0^{\text{split}}(\mathcal{C})$. We see immediately that $K_0^{\text{split}}(\mathcal{C}) \cong K_0(\mathcal{C})$ if $\mathcal{C}$ is abelian and semisimple. This is, in particular, the case for $\mathcal{HS}$. We have a homomorphism $K_0^{\text{split}}(\mathcal{FHS}) \to K_0(\mathcal{FHS})$, which is also an isomorphism by the following lemma.

LEMMA A.3. *If $\mathcal{C}$ is semisimple abelian, then any exact sequence in $\mathcal{FC}$ is split exact.*

Proof. It is enough to check that, given the exact sequence

$$0 \longrightarrow (H_1, N^\bullet) \longrightarrow (H_2, N^\bullet) \xrightarrow{f} (H_3, N^\bullet) \longrightarrow 0 \tag{3}$$

in $\mathcal{FC}$, there is a splitting $s\colon (H_3, N^\bullet) \to (H_2, N^\bullet)$ for f. First observe that, by the semisimplicity of $\mathcal{C}$, we have the noncanonical decomposition

$$N^p H_i \cong N^{p+1} H_i \oplus \mathrm{Gr}_N^p H_i,$$

where $\mathrm{Gr}_N^p H_i = N^p H_i / N^{p+1} H_i$ for $i = 2, 3$. We define $s_p = s|_{N^p H_3}$ by descending induction on p. Note that the exact sequence (3) induces the following exact sequences in $\mathcal{C}$:

$$0 \longrightarrow N^{p+1} H_1 \longrightarrow N^{p+1} H_2 \xrightarrow{f_{p+1}} N^{p+1} H_3 \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{Gr}_N^p H_1 \longrightarrow \mathrm{Gr}_N^p H_2 \xrightarrow{\bar{f}_p} \mathrm{Gr}_N^p H_3 \longrightarrow 0,$$

where $f_{p+1} = f|_{N^{p+1} H_2}$ and where $\bar{f}_p$ is the induced map. By induction and the semisimplicity of $\mathcal{C}$, there exist splittings $s_{p+1}\colon (N^{p+1} H_3, N^\bullet \cap N^{p+1} H_3) \to (N^{p+1} H_2 N^\bullet \cap N^{p+1} H_2)$ and $t_p\colon \mathrm{Gr}_N^p H_3 \to \mathrm{Gr}_N^p H_2$ for f_{p+1} and $\bar{f}_p$, respectively. Set

$$s_p = s_{p+1} + t_p\colon N^p H_3 \longrightarrow N^p H_2.$$

Then s_p gives a well-defined splitting for f_p and hence we have a splitting $s = s_0$ for f. This completes the proof of the lemma. □

COROLLARY A.4. $K_0^{\text{split}}(\mathcal{FHS}) \cong K_0(\mathcal{FHS})$.

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