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Simply Connected Constant Mean Curvature Surfaces in $\mathbb{H}^2 \times \mathbb{R}$

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1. Introduction

In [9] Meeks proved that, if M is a properly embedded simply connected surface of constant mean curvature $H \neq 0$ in $\mathbb{R}^3$, then M is a round sphere. In particular, M cannot be topologically $\mathbb{R}^2$. More generally, he proved there is no properly embedded H-surface of finite topology in $\mathbb{R}^3$ with exactly one end. Afterwards, in [7] a different proof of Meeks's theorem was found, and in [6] it was extended to the hyperbolic space $\mathbb{H}^3$.

In this paper we consider this problem in $\mathbb{H}^2 \times \mathbb{R}$. There are properly embedded H-surfaces in $\mathbb{H}^2 \times \mathbb{R}$ that are topologically $\mathbb{R}^2$; there are entire graphs (vertical graphs over $\mathbb{H}^2$) for each H, $0 \leq H \leq 1/2$ (see [10; 11]). We will prove that such a surface cannot exist for $H > 1/\sqrt{3}$. More generally, we prove the following statement.

Theorem 1.1. *For $H > 1/\sqrt{3}$, there is no properly embedded H-surface in $\mathbb{H}^2 \times \mathbb{R}$ with finite topology and one end.*

Hsiang and Hsiang showed that any compact H-surface embedded in $\mathbb{H}^2 \times \mathbb{R}$ is a rotational sphere and has mean curvature greater than $1/2$ (see [5; 11]). Abresch and Rosenberg proved that, if the surface is simply connected, then the same result holds for compact H-surfaces immersed in $\mathbb{H}^2 \times \mathbb{R}$ (see [1]).

It is interesting to consider to what extent Theorem 1.1 holds in other homogeneous 3-manifolds (for some constant other than $1/\sqrt{3}$). In $\mathbb{S}^2 \times \mathbb{R}$ there is no properly embedded H-surface with one end. To see this, observe that an end of such a surface M would have to go up or down (but not both), since M is proper. Hence one can assume M is bounded below by height 0, say. Then Alexandrov reflection with respect to the "planes" $\mathbb{S}^2 \times \{t\}$ coming up from $t = 0$ allows us to conclude that the part of M below any $M \times \{t\}$ is a vertical graph. This contradicts the height estimates for such graphs (see [4]), so no such M exists in $\mathbb{S}^2 \times \mathbb{R}$.

The other homogeneous 3-manifolds (beside the space forms) are the Berger spheres, Heisenberg space, and $\widetilde{PSL}(2, \mathbb{R})$. Since the Berger spheres are compact, the question is interesting only for the Heisenberg space and $\widetilde{PSL}(2, \mathbb{R})$. Another

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interesting question in Heisenberg space is whether the only embedded compact H-surfaces are the rotational spheres of constant mean curvature.

Theorem 1.1 has the following straightforward consequence.

COROLLARY 1.1. *A simply connected H-surface properly embedded in $\mathbb{H}^2 \times \mathbb{R}$, $H > 1/\sqrt{3}$, is a rotational sphere.*

We remark that Theorem 1.1 and Corollary 1.1 do not hold if $H \leq 1/2$. In fact, for any $H \in (0, 1/2]$ there exists an entire rotational vertical H-graph (cf. [11]).

Theorem 1.1 is a consequence of the following fact.

THEOREM 1.2. *Let $H > 1/\sqrt{3}$ and let M be a properly embedded H-surface in $\mathbb{H}^2 \times \mathbb{R}$ with finite topology and one end. Then M is contained in a vertical cylinder of $\mathbb{H}^2 \times \mathbb{R}$.*

Our proof of Theorem 1.2 holds in $\mathbb{R}^3$, as well.

The proof of Theorem 1.2 depends on the key result of the plane separation lemma (see Section 2). The analogue of this lemma in $\mathbb{R}^3$ and $\mathbb{H}^3$ was proved in [9] and [6], respectively.

Theorem 1.2 in $\mathbb{H}^2 \times \mathbb{R}$ does not hold without the "one end" hypothesis. In fact, there are examples (see [8]) of constant mean curvature cylinders lying in the tubular neighborhood of a horizontal geodesic. We conjecture that Theorem 1.1 holds for $H > 1/2$; the bound $H > 1/\sqrt{3}$ seems due only to technical reasons.

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2. Four Key Lemmas

Let L be the stability operator for a H-surface in a Riemannian 3-manifold $\mathcal{N}^3$. For the notion of stability, see [11] and [3]. The next result is classic: we state and prove it for the sake of completeness.

LEMMA 2.1. *Let M be a compact H-surface in a Riemannian 3-manifold $\mathcal{N}^3$ that is transverse to some Killing vector field of $\mathcal{N}^3$. Then M is stable.*

Proof. Let $\langle \cdot, \cdot \rangle$ be the scalar product on the tangent space of $\mathcal{N}^3$ induced by the Riemannian structure on $\mathcal{N}^3$, and let X be the Killing vector field of $\mathcal{N}^3$. Because M is transverse to X, one can choose a unit vector field N normal to M such that $\langle N, X \rangle$ is positive on M. Then $\langle N, X \rangle$ is a positive Jacobi function on M and M is stable. □

The following lemma was proved in [11].

LEMMA 2.2 (distance lemma). *Let M be a stable H-surface in $\mathbb{H}^2 \times \mathbb{R}$ with $H > 1/\sqrt{3}$. Then, for any $p \in M$,*

$$\operatorname{dist}_M(p, \partial M) < \frac{2\pi}{\sqrt{3(3H^2 - 1)}}.$$

REMARK 2.1. If the distance lemma were true for $H > 1/2$, then Theorem 1.1 would follow for $H > 1/2$.

Rosenberg [12] generalized Lemma 2.2 to the case of an ambient manifold that is homogeneously regular.

LEMMA 2.3. *Let P be a vertical plane in $\mathbb{H}^2 \times \mathbb{R}$. Let S be a compact embedded H-surface with boundary on P and $H > 1/\sqrt{3}$. Then the distance of S from P is bounded above by the constant $c_0 = 4\pi/\sqrt{3(3H^2 - 1)}$.*

Proof. Let $p \in S$ be a furthest point from the plane P and let γ be a minimizing ambient geodesic in $\mathbb{H}^2 \times \mathbb{R}$ from the point p to the plane P. Denote by h the length of γ. Let $P(t)$ be the family of vertical planes, orthogonal to γ, obtained by translating P along γ by the isometry of $\mathbb{H}^2 \times \mathbb{R}$ that is translation along γ, parameterized such that $P(0) = P$ and $p \in P(h)$. We perform Alexandrov reflection of S with the family $P(t)$, starting at $P(h)$, and conclude that the part of S on one side of $P(h/2)$, say S^+, has no point where S is orthogonal to one of the planes $P(t)$, $h/2 \le t \le h$. Hence the Killing vector field obtained from γ is transverse to S^+, since it is orthogonal to the family of planes $P(t)$. By Lemma 2.1, S^+ is stable. Therefore, by the distance lemma, $h < c_0$. □

In [5], Hsiang and Hsiang computed the explicit form of the profile curve of a rotational H-surface. The vertical diameter and the horizontal diameter of a rotational H-surface are

$$\frac{4H}{\sqrt{4H^2-1}} \tan^{-1} \frac{1}{\sqrt{4H^2-1}} \quad \text{and} \quad 2\sinh^{-1} \frac{4H}{4H^2-1},$$

respectively. We will see that, in order to prove the plane separation lemma, the minimum distance between the two vertical planes must be the horizontal diameter of a rotational H-surface.

LEMMA 2.4 (plane separation lemma). *Let $H > 1/2$, and let P_1 and P_2 be two disjoint vertical planes in $\mathbb{H}^2 \times \mathbb{R}$. Denote by P_1^+, P_2^+ the two disjoint half-spaces determined by these planes. Let $c_1 = 2\sinh^{-1}(4H/(4H^2 - 1))$. If the distance between P_1 and P_2 is greater than c_1 then, for any properly embedded H-surface M with finite topology and one end, either $P_1^+ \cap M$ or $P_2^+ \cap M$ consists entirely of compact components.*

Proof. Assume by contradiction that both $P_1^+ \cap M$ and $P_2^+ \cap M$ contain noncompact components. Then there are two proper arcs $\alpha_1\colon [0,\infty) \to P_1^+ \cap M$ and $\alpha_2\colon [0,\infty) \to P_2^+ \cap M$.

Since M has finite topology, it follows that the end of M is topologically an annulus; hence we can assume that both $\alpha_1(t)$ and $\alpha_2(t)$ lie in the annular end of M for t sufficiently large. Set $\alpha_1(0) = p_1$ and $\alpha_2(0) = p_2$. One can choose an embedded arc β on M from p_1 to p_2 such that the arc $\delta = \alpha_1 \cup \beta \cup \alpha_2$ bounds a simply connected domain on M (see Figure 1).

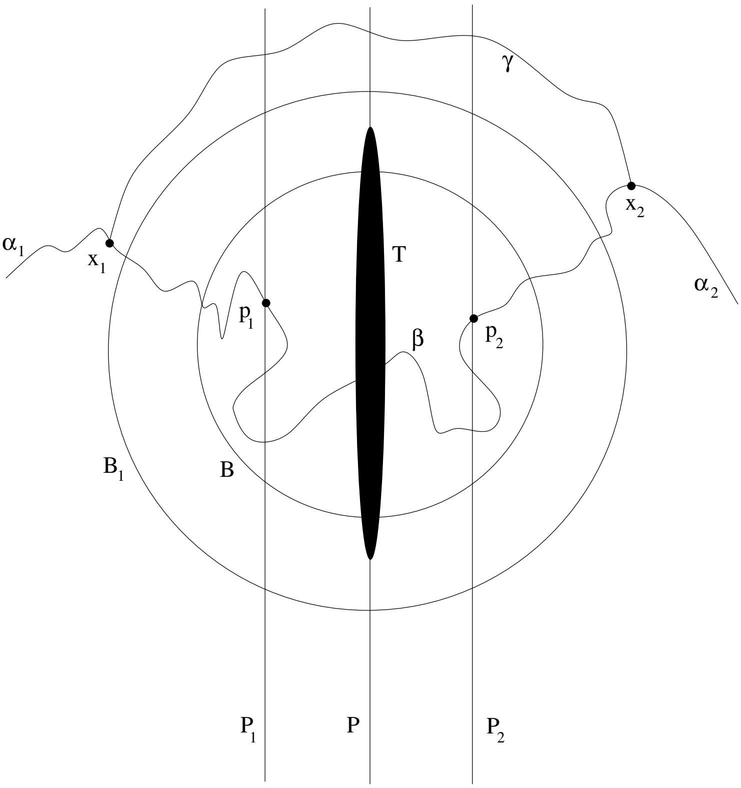

Figure 1

Let P be a vertical plane between P_1 and P_2 at equal distance from P_1 and P_2. Let B be a geodesic ball of $\mathbb{H}^2 \times \mathbb{R}$ containing β, and let C be a circle in the plane P such that:

- $C \cap B = \emptyset$, and B has nonempty intersection with the disk in P bounded by C;
- the tubular neighborhood T of C of radius $c_1/2$ is embedded, and $T \cap B = \emptyset$.

We remark that T is contained in the closed slab between P_1 and P_2 (see Figure 2).

Now let B_1 be a geodesic ball containing $B \cup T$. There exist $x_1 \in \alpha_1 \setminus B_1$, $x_2 \in \alpha_2 \setminus B_1$, and an arc γ from x_1 to x_2, embedded in M, such that the following statements hold.

(1) $\gamma \cap B_1 = \emptyset$.
(2) If we denote by ρ the subarc of δ between the points x_1 and x_2, then $\rho \cup \gamma$ is a simple closed curve with linking number ± 1 with the circle C.
(3) $\rho \cup \gamma$ bounds a compact disk U on M.

Therefore, $T \cap U$ contains a disk E such that $\partial E \subset \partial T$ and such that the linking number between ∂E and the circle C is ± 1.

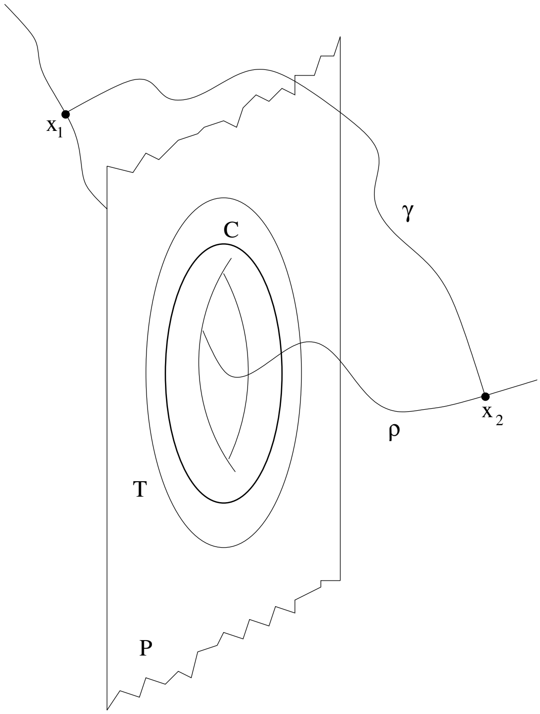

Figure 2

Next we let $\Pi: \tilde{T} \to T$ be the universal Riemannian covering space of T. The disk E lifts to a compact disk $\tilde{E} \subset \tilde{T}$. Topologically, T is $D^2 \times \mathbb{S}^1$ and E is isotopic to some $D^2 \times \{\text{point}\}$. Then $\tilde{T}$ is topologically $D^2 \times \mathbb{R}$ and $\tilde{E}$ is isotopic to some $D^2 \times \{\text{point}\}$. In particular, $\tilde{E}$ separates $\tilde{T}$ into two noncompact connected components. Call $\tilde{W}$ the *mean convex* component.

Let $\tilde{C}$ be the curve in $\tilde{T}$ that projects to C by Π. For each point $p \in C$ there is a spherical H-surface, say $S_H(p)$, that is invariant by rotation about the vertical geodesic through p. It is clear that, if the radius of C is sufficiently large, then each $S_H(p)$ is contained in T. For any point $\tilde{p} \in \tilde{C}$, denote by $\tilde{S}_H(\tilde{p})$ the compact surface that projects to $S_H(p)$ by Π, where $\Pi(\tilde{p}) = p$.

One can find a point $\tilde{q}$ on $\tilde{C}$ such that $\tilde{S}_H(\tilde{q})$ is contained in $\tilde{W}$ and is disjoint from $\tilde{E}$. Now move $\tilde{q}$ along $\tilde{C}$ toward $\tilde{E}$ until the first point $\tilde{q}_1$ at which $\tilde{S}_H(\tilde{q}_1)$ and $\tilde{E}$ are tangent. Then $\tilde{S}_H(\tilde{q}_1)$ is contained in $\tilde{W}$ and, at the tangent point, both $\tilde{E}$ and $\tilde{S}_H(\tilde{q})$ have curvature equal to H. Hence, by the maximum principle, they should coincide—a contradiction. □

Remark 2.2. The plane separation lemma holds (and the proof is the same) for horizontal planes P_1 and P_2. The distance between P_1 and P_2 must be larger than the vertical diameter of a rotational H-surface. Notice that the proof of the plane separation lemma works also in the Heisenberg space and in $\widetilde{PSL}(2, \mathbb{R})$.

3. *M* Cylindrically Bounded

We restate Theorem 1.2 here for the reader's convenience.

THEOREM 1.2. *Let $H > 1/\sqrt{3}$ and let M be a properly embedded H-surface in $\mathbb{H}^2 \times \mathbb{R}$ with finite topology and one end. Then M is contained in a vertical cylinder of $\mathbb{H}^2 \times \mathbb{R}$.*

Proof. We can assume that the point $\sigma = (\mathbf{0}, 0)$ belongs to M, where $\mathbf{0}$ is an origin of $\mathbb{H}^2$. Let $\gamma\colon [0, r] \to \mathbb{H}^2 \times \mathbb{R}$ be any horizontal geodesic starting at σ and parameterized by arc length, and denote by $P(r)$ the vertical plane passing through $\gamma(r)$ orthogonal to γ. We claim that there exists a constant c_2 (independent of the geodesic γ) such that, if $r > c_2$, then the half-space determined by $P(r)$ that does not contain the point σ is disjoint from M. This clearly implies that M is contained in the vertical cylinder with axis $\mathbf{0} \times \mathbb{R}$ and radius c_2.

Let us prove the claim. We choose $R > \max\{c_0, c_1\}$, where c_0 and c_1 are the constants given by Lemmas 2.3 and 2.4, respectively. Denote by $P(R)^+$ the half-space determined by $P(R)$ containing the point σ and by $P(2R)^+$ the half-space determined by $P(2R)$ not containing the point σ. By the plane separation lemma applied to the surface M, one of the following statements holds:

(i) $M \cap P(R)^+$ has only compact components; or
(ii) $M \cap P(2R)^+$ has only compact components.

If (i) is true then, by Lemma 2.3, the distance between the plane $P(R)$ and the point $\sigma \in M \cap P(R)^+$ must be at most c_0. This is a contradiction with our choice of R, so (ii) must be true. Then, again by Lemma 2.3, the maximum distance between $M \cap P(2R)^+$ and the plane $P(2R)$ is at most c_0; hence M is disjoint from the half-space determined by $P(2R + c_0)$ not containing the point σ.

As a result, choosing the constant $c_2 = 2\max\{c_0, c_1\} + c_0$, the claim is proved. □

We now establish our main result.

THEOREM 1.1. *For $H > 1/\sqrt{3}$, there is no properly embedded H-surface in $\mathbb{H}^2 \times \mathbb{R}$ with finite topology and one end.*

Proof. Assume by contradiction that there exists an H-surface M satisfying the hypothesis. Let $Q(t)$ be the horizontal geodesic plane at height t in $\mathbb{H}^2 \times \mathbb{R}$. By Theorem 1.2, M is contained in a vertical cylinder and, since it has only one proper end, M is bounded either above or below. We can assume that M is bounded below and that the lowest points of M lie in the plane $Q(0)$. Because reflections with respect to the planes $Q(t)$ are isometries of $\mathbb{H}^2 \times \mathbb{R}$, we can apply the Alexandrov reflection method with the planes $Q(t)$ to the surface M. Since M is contained in a cylinder it follows that no accident can occur moving $Q(0)$ up: there is no smallest t such that M becomes orthogonal to $Q(t)$ at some point; otherwise, by Alexandrov, $Q(t)$ would be a plane of symmetry for M. Hence, for any $t > 0$, the part of M below $Q(t)$ is a vertical graph and there are no points of M below $Q(t)$

where M is orthogonal to one of the planes $Q(t)$. Since $\frac{\partial}{\partial t}$ is a Killing vector field by Lemma 2.1, the part of M below the plane $Q(t)$ is stable. But one can choose t larger than the constant of the distance lemma—a contradiction. □

Added in proof. In [2] the authors prove optimal vertical height estimates for compact constant mean curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$, with boundary on a slice.

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