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GIT Equivalence beyond the Ample Cone

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Introduction

The approach to moduli spaces (e.g., for curves of fixed genus) presented by D. Mumford in his geometric invariant theory [17] relies on his construction of quotients for actions of reductive groups G on algebraic varieties X. He introduces the notion of a G-linearized line bundle on X, and to any such bundle L he associates a G-invariant open set $X^{ss}(L) \subset X$ of semistable points. This set admits a so-called good quotient $X^{ss}(L) \to X^{ss}(L)/\!\!/G$ with a quasiprojective quotient space.

Mumford's construction, however, is in general not unique: his "GIT quotients" turn out to depend essentially on the choice of the bundle and the linearization. Therefore, it is a natural desire to describe the collection of all possible GIT quotients for a given reductive group action. For "ample GIT quotients"—that is, those arising from linearized ample line bundles—this problem has been studied by several authors; see [8; 10; 22] and [19].

A first basic step in the study of ample GIT quotients is to show that there are only finitely many of them (see [5; 10; 20; 22]). Then the subject becomes combinatorial. The situation is described by a sort of fan subdividing the so-called (open) G-ample cone: the cones of this fan correspond to the ample GIT quotients and the face relations reflect, in an order-reversing manner, the set-theoretical inclusion of the respective sets of semistable points; see [19].

However, there are interesting examples of projective GIT quotients that do not arise from linearized ample bundles (see [6]). Motivated by this observation, we study here the situation beyond the G-ample cone, and we propose a combinatorial framework for the description of the phenomena occurring there. We restrict our attention to the case of a torus action. This case is the most vivid one concerning variation of GIT quotients, and it allows an elementary treatment.

The setup is as follows: X is a normal projective variety over an algebraically closed field $\mathbb{K}$ of characteristic zero such that X has a free finitely generated divisor class group $\mathrm{Cl}(X)$ as well as a finitely generated total coordinate ring (see Section 3)

$$\mathcal{R}(X) := \bigoplus_{\mathrm{Cl}(X)} \Gamma(X, \mathcal{O}(D)).$$

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We consider the action $T \times X \to X$ of an algebraic torus $T = \mathrm{Spec}(\mathbb{K}[M])$, where M is the lattice of characters. This comprises subtorus actions on projective toric varieties [14] and, more generally, on projective spherical varieties.

In our setup, we can even do a little more than describing only the collection of the quotients arising from the possible T-linearized line bundles over X: we allow, more generally, quotients arising from T-linearized Weil divisors (see Section 1). Compared with Mumford's original approach, this has the advantage that also for singular X we obtain all good quotients with a quasiprojective quotient space; see [13].

We make use of the fact that X is a good quotient of an invariant open subset $\hat{X}$ of $\bar{X} := \mathrm{Spec}(\mathcal{R}(X))$ by the torus $H := \mathrm{Spec}(\mathbb{K}([\mathrm{Cl}(X)])$ corresponding to the divisor class group. The action of T may be lifted to the multicone $\bar{X}$ over X, and this lifting corresponds to a refinement of the grading

$$\mathcal{R}(X) = \bigoplus_{(D,w)\in \mathrm{Cl}(X)\oplus M} \Gamma(X, \mathcal{R})_{(D,w)}.$$

It turns out that the degrees $(D, w) \in \mathrm{Cl}(X) \oplus M$ of the refined grading are in one-to-one correspondence with the possible T-linearizations of the divisor classes $D \in \mathrm{Cl}(X)$ and that the sets of semistable points depend only on the classes of the T-linearized divisors.

Let us indicate how the combinatorial description runs. To any T-linearized class (D, w) having a nonempty set of semistable points, we associate what we call its *GIT bag* $\mu(D, w)$. This GIT bag is a certain pointed convex polyhedral cone living in the rational vector space associated to $\mathrm{Cl}(X) \oplus M$, and it can be directly computed from a finite set of orbit data associated to the lifted action of $H \times T$ on $\bar{X}$. The set of GIT bags is finite, and it comes with a natural partial ordering "$\le$". Here is our main result (see Theorem 4.3).

THEOREM. *Let $(D_i, w_i) \in \mathrm{Cl}(X) \oplus M$ represent two T-linearized Weil divisors on X, and let $\mu(D_i, w_i)$ denote the associated GIT bags. Then, for the associated sets of semistable points, we have*

$$X^{ss}(D_1, w_1) \subset X^{ss}(D_2, w_2) \iff \mu(D_1, w_1) \ge \mu(D_2, w_2).$$

Inside the T-ample cone, the GIT bags coincide with the cones of the fan subdivision defined by the GIT chambers of [10] and [19]. But outside the T-ample cone, not much is left from the fan properties; for example, overlappings are possible (see Section 6). Nevertheless, the GIT bags allow us to formulate answers to several questions.

For example, motivated by [5], we study *qp-maximal T-sets.* These are T-invariant open subsets $U \subset X$ admitting a good quotient $U \to U/\!\!/T$ with a quasiprojective quotient space $U/\!\!/T$ that do not occur as a saturated subset with respect to (w.r.t.) the quotient map of a properly larger $U' \subset X$ with the same properties. Any qp-maximal set is a set of semistable points of a linearized Weil divisor; in terms of GIT bags, qp-maximality is characterized as follows (see Corollary 4.5).

THEOREM. *A GIT bag describes a qp-maximal T-set if and only if its relative interior is set-theoretically minimal in the collection of the relative interiors of all GIT bags.*

Using this characterization, one can easily produce examples of qp-maximal T-sets having a noncomplete quotient space; see Example 6.3. It can as well be described in terms of GIT bags when a quotient space is projective (see Proposition 4.6), and there is a simple criterion for figuring out the ample GIT quotients (see Proposition 4.7). These two criteria are useful for discussing an "exotic orbit space" as presented in [6]; see Example 6.2.

For the case of a $\mathbb{Q}$-factorial variety X, the combinatorial description of the collection of qp-maximal T-sets allows us to extend a basic statement from the ample theory: there, one obtains as a consequence of the fan structure that any two sets of semistable points arising from ample bundles admit at most one minimal such set comprising both of them. We show the following (see Corollary 5.2).

THEOREM. *Suppose that X is $\mathbb{Q}$-factorial. Then, for any two qp-maximal T-sets $U_1, U_2 \subset X$, the collection of qp-maximal T-sets of $U \subset X$ with $(U_1 \cup U_2) \subset U$ is either empty or it contains a unique minimal element.*

The paper is organized as follows. In Section 1, we recall some basics on good quotients and the construction presented in [13]. Moreover, we introduce the group of isomorphism classes of G-linearized Weil divisors. In Section 2, we present a simple direct proof for the affine version of [19], which is needed later but also might be of independent interest. Section 3 is devoted to a combinatorial characterization of semistability, and the main results are presented in Section 4. In Section 5 we investigate the case of a $\mathbb{Q}$-factorial X, and in Section 6 we discuss some examples.

1. Good Quotients

In this section, we recall the notion of a good quotient and provide the basic facts on this concept. Moreover, we briefly recall from [13] a generalization of Mumford's construction of good quotients, using Weil divisors instead of line bundles. Finally, we introduce the group of isomorphism classes of linearized Weil divisors and the GIT equivalence.

We work over an algebraically closed field $\mathbb{K}$ of characteristic zero. The word "variety" refers to a reduced scheme of finite type over $\mathbb{K}$, and by a "point" we always mean a closed point. When we speak of an action of an algebraic group G on a variety X, we tacitly assume that this action is given by a morphism $G \times X \to X$; we then also speak of the G-variety X.

DEFINITION 1.1. Let a reductive linear algebraic group G act on a variety X.

(i) A *good quotient* for the G-action is a G-invariant affine morphism $\pi : X \to Y$ such that the canonical map $\mathcal{O}_Y \to \pi_*(\mathcal{O}_X)^G$ is an isomorphism.

(ii) A good quotient $\pi : X \to Y$ is called *geometric* if each fibre $\pi^{-1}(y)$, where $y \in Y$, consists of a single G-orbit.

The definition of a good quotient was formulated by Seshadri [21], but the concept was implicit in Mumford's book [17]. Note that good quotients are obtained by glueing classical invariant theory quotients $X \to Y$ of affine G-varieties X, which means that $Y := \operatorname{Spec}(\Gamma(X, \mathcal{O})^G)$ holds. We now list some basic properties (see e.g. [21]).

PROPOSITION 1.2. *Let $\pi: X \to Y$ be a good quotient for a G-action on a variety X.*

(i) *If $A, B \subset X$ are G-invariant closed subsets with $A \cap B = \emptyset$, then their images in Y are also closed and satisfy $\pi(A) \cap \pi(B) = \emptyset$.*
(ii) *For every $y \in Y$, there is a unique closed G-orbit $G \cdot x_0 \subset \pi^{-1}(y)$, and this orbit lies in the closure of any other orbit $G \cdot x \subset \pi^{-1}(y)$.*
(iii) *If $\varphi: X \to Z$ is a G-invariant morphism, then there is a unique morphism $\psi: Y \to Z$ with $\varphi = \psi \circ \pi$.*

The last property implies that a good quotient for a G-variety is basically unique, provided it exists. This justifies the notations $X \to X/\!\!/G$ for a good quotient and $X \to X/G$ for a geometric quotient. In general, a G-variety X need not admit a good quotient, but it may have many G-invariant open subsets $U \subset X$ with a good quotient $U \to U/\!\!/G$. For the study of such subsets, the following concept is crucial (cf. [5]).

DEFINITION 1.3. Let X be a G-variety. A G-invariant open subset $U \subset X$ is called *G-saturated in X* if $\overline{G \cdot x} \subset U$ for all $x \in U$, where $\overline{G \cdot x}$ denotes the orbit closure taken in X.

Usually, one compares invariant open subsets $V \subset U$ of a G-variety X, which means that one asks if V is G-saturated in the G-variety U. If G is reductive linear algebraic and there is a good quotient $\pi: U \to U/\!\!/G$, then, by Proposition 1.2(ii), the set V is G-saturated in U if and only if $V = \pi^{-1}(\pi(V))$ holds. In that case, $\pi(V)$ is open in $U/\!\!/G$ and the restriction $\pi|_V: V \to \pi(V)$ is a good quotient.

We now recall the construction of good quotients given in [13]. It extends Mumford's construction by taking Weil divisors instead of line bundles. The advantage of this approach is that it provides also in the singular (normal) case basically all good quotients with a quasiprojective quotient. Observe that we present here a slightly modified version that allows also nontrivial linearizations of the trivial divisor $D = 0$. However, this has no impact on the results and their proofs.

Let X be a normal G-variety, where G is a reductive linear algebraic group. To any Weil divisor D on X we associate a sheaf of $\mathcal{O}_X$-algebras, and we consider the corresponding relative spectrum with its canonical morphism:

$$\mathcal{A} := \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathcal{O}_X(nD), \quad X(D) := \operatorname{Spec}_X(\mathcal{A}), \quad q_D: X(D) \to X.$$

The $\mathbb{Z}_{\geq 0}$-grading of the sheaf of algebras $\mathcal{A}$ defines an action of the multiplicative group $\mathbb{K}^* = \operatorname{Spec}(\mathbb{K}[\mathbb{Z}])$ on $X(D)$, and the canonical morphism $q_D: X(D) \to X$ is a good quotient for this action.

Note that, near singular points of X, it is not necessary a priori for the scheme $X(D)$ to be of finite type over X; however, we need not care too much about this difficulty because it is ruled out by assumption in the cases of interest.

DEFINITION 1.4 (cf. [13]). A *G-linearization* of the divisor D is a morphical G-action on $X(D)$ that commutes with the $\mathbb{K}^*$-action on $X(D)$ and makes $q_D \colon X(D) \to X$ into a G-equivariant morphism.

Any G-linearization of the divisor D gives rise to a rational G-representation on the global sections respecting the $\mathbb{Z}_{\geq 0}$-grading, namely,

$$G \times \Gamma(X(D), \mathcal{O}) \to \Gamma(X(D), \mathcal{O}), \qquad (g \cdot f)(x) := f(g^{-1} \cdot x).$$

We now introduce a notion of semistability that is similar to that in [13]. As usual, for a section $f \in \Gamma(X, \mathcal{O}(D))$ of a Weil divisor, we denote its set of zeroes as

$$Z(f) := \operatorname{Supp}(\operatorname{div}(f) + D).$$

DEFINITION 1.5. Let D be a G-linearized Weil divisor on X. We call a point $x \in X$ *semistable* with respect to this linearization if there exist an $n \in \mathbb{Z}_{>0}$ and a section $f \in \Gamma(X, \mathcal{O}(nD))$ such that $X \setminus Z(f)$ is an affine neighborhood of x and f is invariant under the G-representation on $\Gamma(X(D), \mathcal{O})$.

We denote the set of semistable points of a G-linearized Weil divisor D by $X^{ss}(D)$, or by $X^{ss}(D, G)$ if we want to specify the group G. From [13] we infer the basic features of this construction as follows.

PROPOSITION 1.6. *Let G be a reductive linear algebraic group, and let X be a normal G-variety.*

- (i) *If D is a G-linearized Weil divisor on X, then there exists a good quotient $X^{ss}(D) \to X^{ss}(D)/\!\!/G$ with a quasiprojective quotient space.*
- (ii) *If $U \subset X$ is a G-invariant open subset having a good quotient $U \to U/\!\!/G$ with $U/\!\!/G$ quasiprojective, then U is G-saturated in some set $X^{ss}(D)$.*

In the literature, one often introduces a G-linearization of a line bundle $L \to X$ over a G-variety more geometrically as a fibrewise linear lifting of the G-action to the total space L (see e.g. [16]). This is related to our definition, as the following remark shows.

REMARK 1.7. If D is Cartier and represents the class of a line bundle $L \to X$ in $\operatorname{Pic}(X)$, then $X(D) \to X$ is the dual bundle of $L \to X$ and there is an isomorphism

$$\Gamma(X, L) \to \Gamma(X(D), \mathcal{O})_1, \qquad s \mapsto f_s, \quad \text{where} \quad f_s(z) := \langle z, s(q_D(z)) \rangle.$$

If D is G-linearized, then the G-action on $X(D)$ defines a dual, fibrewise linear action on the total space L via

$$\langle z, g \cdot y \rangle := \langle g^{-1} \cdot z, y \rangle \quad \text{for} \;\; g \in G, \; y \in L_x, \; z \in X(D)_{g \cdot x}, \; x \in X.$$

This action makes the projection equivariant, and it induces the "dual representation" of G on the space $\Gamma(X, L)$ of global sections:

$$g \cdot s(x) = g \cdot (s(g^{-1} \cdot x)).$$

With respect to this representation, the isomorphism $\Gamma(X, L) \to \Gamma(X(D), \mathcal{O})_1$ mentioned before becomes an isomorphism of G-modules.

We conclude this section with a few words about the passage to divisor classes. For any G-variety X, we have the notion of the group $\mathrm{Pic}_G(X)$ of isomorphism classes of G-linearized line bundles over X. Let us show how this concept can be extended to Weil divisors.

First of all, we must prepare the definition of the G-linearized sum of two G-linearized Weil divisors D_1 and D_2 on a normal G-variety X. For this, consider the following sheaf of bigraded $\mathcal{O}_X$-algebras and its relative spectrum:

$$\mathcal{B} := \bigoplus_{(n,m)\in\mathbb{Z}^2_{\geq 0}} \mathcal{O}(nD_1 + mD_2), \qquad X(D_1, D_2) := \mathrm{Spec}_X(\mathcal{B}).$$

Note that $X(D_1, D_2)$ comes with an action of the torus $\mathbb{T}^2 := \mathbb{K}^* \times \mathbb{K}^*$ defined by the bigrading of $\mathcal{B}$. Moreover, we have canonical morphisms $X(D_1, D_2) \to X(D_i)$ arising from the inclusions of sheaves

$$\bigoplus_{k\in\mathbb{Z}_{\geq 0}} \mathcal{O}_X(kD_i) \to \bigoplus_{(n,m)\in\mathbb{Z}^2_{\geq 0}} \mathcal{O}(nD_1 + mD_2).$$

These morphisms determine a morphism $\varphi\colon X(D_1, D_2) \to X(D_1) \times_X X(D_2)$ to the fibre product, which also comes with a canonical $\mathbb{T}^2$-action and the diagonal G-action. Here are the basic properties of this setting.

LEMMA 1.8. *For two G-linearized Weil divisors D_1, D_2 on X, let $X(D_1, D_2)$ be as just described. Then there is a commutative diagram of $\mathbb{T}^2$-equivariant morphisms*:

$$\begin{tikzcd}
& & X(D_1) \times_X X(D_2) \arrow[ddl] \arrow[dr] \arrow[ddd] & \\
X(D_1, D_2) \arrow[urr, bend left, "\varphi"] \arrow[rrr] \arrow[dr] \arrow[ddrr, bend right, "q_{D_1,D_2}"'] & & & X(D_2) \arrow[ddl] \\
& X(D_1) \arrow[dr] & & \\
& & X. &
\end{tikzcd}$$

The diagonal G-action on the fibre product lifts uniquely to $X(D_1, D_2)$ and then descends to $X(D_1 + D_2)$ within a canonical commutative diagram:

$$\begin{tikzcd}
X(D_1, D_2) \arrow[rr] \arrow[dr] & & X(D_1 + D_2) \arrow[dl] \\
& X. &
\end{tikzcd}$$

Moreover, the induced G-action on $X(D_1 + D_2)$ is a G-linearization of the Weil divisor $D_1 + D_2$.

Proof. The morphism $\varphi\colon X(D_1, D_2) \to X(D_1) \times_X X(D_2)$ is given by the universal property of the fibre product. It is an isomorphism over the set $X_{\mathrm{reg}} \subset X$ of smooth points, because there it comes from the canonical isomorphism of the corresponding sheaves of $\mathcal{O}_X$-algebras:

$$\left(\bigoplus_{m\in\mathbb{Z}_{\geq 0}} \mathcal{O}(mD_1)\right) \otimes_{\mathcal{O}_X} \left(\bigoplus_{n\in\mathbb{Z}_{\geq 0}} \mathcal{O}(nD_2)\right) \to \bigoplus_{(m,n)\in\mathbb{Z}^2_{\geq 0}} \mathcal{O}(mD_1 + nD_2).$$

Moreover, since this is a bigraded homomorphism, we can conclude that φ is $\mathbb{T}^2$-equivariant. That φ is an isomorphism over the G-invariant set $X_{\mathrm{reg}} \subset X$ allows us to shift the diagonal G-action on the fibre product over X_{reg} to a morphical action

$$\alpha\colon G \times q^{-1}_{D_1,D_2}(X_{\mathrm{reg}}) \to q^{-1}_{D_1,D_2}(X_{\mathrm{reg}}).$$

Our task is to extend this action to the whole $X(D_1, D_2)$; this is done via extending the corresponding comorphisms. Let $\beta\colon G \times X \to X$ denote the action on X. Then, by G-equivariance, we obtain the commutative diagram

$$\begin{tikzcd}
\Gamma(U \cap X_{\mathrm{reg}}, \mathcal{B}) \arrow[r, "\alpha^*"] \arrow[d, equal] & \Gamma(\beta^{-1}(U \cap X_{\mathrm{reg}}), \mathcal{O}_G \otimes \mathcal{B}) \arrow[d, equal] \\
\Gamma(U, \mathcal{B}) \arrow[r] & \Gamma(\beta^{-1}(U), \mathcal{O}_G \otimes \mathcal{B})
\end{tikzcd}$$

for any affine open subset $U \subset X$. As one can easily verify, the lower rows of these diagrams are the comorphisms of a G-action on $X(D_1, D_2)$. By construction, this extension has the desired properties and so the first part of the lemma is proved.

To see the second part, consider the antidiagonal $\mathbb{K}^*$-action on $X(D_1, D_2)$ defined by the homomorphism of tori $\mathbb{K}^* \to \mathbb{T}^2$ sending t to (t, t^{-1}). This action admits a good quotient: the morphism $X(D_1, D_2) \to X(D_1 + D_2)$ arising from the canonical injection of sheaves

$$\bigoplus_{k\in\mathbb{Z}_{\geq 0}} \mathcal{O}(kD_1 + kD_2) \to \bigoplus_{(m,n)\in\mathbb{Z}^2_{\geq 0}} \mathcal{O}(mD_1 + nD_2).$$

Since the antidiagonal $\mathbb{K}^*$-action and the G-action on $X(D_1, D_2)$ commute, it follows that the G-action descends to an action on the quotient space $X(D_1 + D_2)$. By construction, it commutes with the $\mathbb{K}^*$-action on $X(D_1 + D_2)$ and the canonical morphism $X(D_1 + D_2) \to X$ becomes G-equivariant. □

Definition 1.9. Let D_1 and D_2 be two G-linearized Weil divisors on a normal G-variety X.

(i) The *G-linearization* of the sum D_1+D_2 is the unique G-action on $X(D_1+D_2)$ provided by Lemma 1.8.
(ii) We say that D_1 and D_2 are *isomorphic* if there is a $(\mathbb{K}^* \times G)$-equivariant isomorphism $X(D_1) \to X(D_2)$ over X.

Note that, for the case of a pair of linearized Cartier divisors, our definition of the linearized sum corresponds to the usual tensor product of linearized line bundles, and the notion of isomorphism is the usual one.

PROPOSITION 1.10. *The set of isomorphism classes of G-linearized Weil divisors form a group* $\mathrm{Cl}_G(X)$. *Furthermore*:

(i) *forgetting about the linearizations gives rise to a well-defined homomorphism* $\mathrm{Cl}_G(X) \to \mathrm{Cl}(X)$; *and*
(ii) *for any G-linearized Weil divisor D on X, the set $X^{ss}(D, G)$ depends only on its class in* $\mathrm{Cl}_G(X)$.

Observe that the kernel of the forgetting homomorphism $\mathrm{Cl}_G(X) \to \mathrm{Cl}(X)$ consists precisely of the linearizations of the trivial bundle. Finally, by Proposition 1.10, we may generalize the usual concept of GIT equivalence to the setting of Weil divisors.

DEFINITION 1.11. We say that two G-linearized divisor classes in $\mathrm{Cl}_G(X)$ are *GIT equivalent* if they define the same set of semistable points.

2. The Affine Case

In this section, we study the collection of sets of semistable points arising from the possible linearizations of the trivial bundle over an affine variety with a torus action. We provide a simple direct proof for the fact that this collection is an order-reversing bijection to a (quasi)fan subdividing the weight cone of the action.

This result may be viewed as an affine version of [19]. It is well known for linear torus actions on $\mathbb{K}^n$; in this case, the describing fan is a so-called Gelfand–Kapranov–Zelevinsky decomposition (see [18]).

The precise setup is as follows: $\mathbb{K}$ is an algebraically closed field, and R is a finitely generated integral $\mathbb{K}$-algebra that is graded by a lattice $M \cong \mathbb{Z}^d$:

$$R = \bigoplus_{w \in M} R_w.$$

This grading corresponds to an action of the algebraic torus $T := \mathrm{Spec}(\mathbb{K}[M])$ on the affine variety $X := \mathrm{Spec}(R)$.

We denote by $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$ the rational vector space associated to M. Recall that the *weight cone* of the T-variety X is the convex cone in $M_{\mathbb{Q}}$ generated by all $w \in M$ admitting a nontrivial homogeneous $f \in R_w$:

$$\Omega_T(X) := \mathrm{cone}(w \in M;\ R_w \neq 0) \subset M_{\mathbb{Q}}.$$

Since the algebra R is generated by finitely many homogeneous elements, it follows that the weight cone $\Omega_T(X)$ is also finitely generated and thus is a polyhedral cone. Note that $\Omega_T(X)$ is pointed (i.e., it contains no line) if $R_0 = \mathbb{K}$ and $R^* = \mathbb{K}^*$.

DEFINITION 2.1. For a point $x \in X$, its *orbit monoid* is the semigroup consisting of all weights that admit a homogeneous function that is invertible near x:

$$S_T(x) := \{w \in M;\ \exists f \in R_w, f(x) \neq 0\}.$$

The *orbit cone* of $x \in X$ is the convex (polyhedral) cone $\omega_T(x) \subset M_{\mathbb{Q}}$ generated by the orbit monoid $S_T(x)$.

We collect some basic properties of orbit cones. A first observation is that the orbit cones are not affected by passing to the normalization.

LEMMA 2.2. *Let $\nu\colon X' \to X$ be the T-equivariant normalization. Then, for every $x' \in X'$, we have $\omega_T(x') = \omega_T(\nu(x'))$.*

Proof. The inclusion $\omega_T(\nu(x')) \subset \omega_T(x')$ is clear by equivariance. The reverse inclusion follows from considering equations of integral dependence for the homogeneous elements $f \in \mathcal{O}(X')$ with $f(x') \neq 0$. $\square$

We shall use the orbit cones to describe properties of orbit closures. The basic statement in this regard is the following one.

PROPOSITION 2.3. *For a point $x \in X$, let $T_x \subset T$ be its isotropy group and let $M_T(x) \subset M$ be the sublattice generated by the orbit monoid $S_T(x)$.*

(i) *The algebraic torus T/T_x acts with a dense free orbit on the orbit closure $\overline{T \cdot x} \subset X$.*
(ii) *$\overline{T \cdot x}$ has the affine toric variety* $\mathrm{Spec}(\mathbb{K}[\omega_T(x) \cap M_T(x)])$ *as its (T/T_x)-equivariant normalization.*

Proof. The first assertion is obvious, and the second one follows immediately from Lemma 2.2 and the fact that the algebra of global functions of $\overline{T \cdot x}$ is the semigroup algebra $\mathbb{K}[S_T(x)]$ of the weight monoid. $\square$

For two polyhedral cones ω_1 and ω_2 in a common vector space, we write $\omega_1 \preceq \omega_2$ if ω_1 is a face of ω_2. Lemma 2.2 and Proposition 2.3 have the following consequence.

COROLLARY 2.4. *Let $x \in X$. Then the T-orbits in $\overline{T \cdot x}$ correspond to the faces of $\omega_T(x)$ via $T \cdot y \mapsto \omega_T(y)$.*

The following simple observation will replace in our setup the deeper finiteness result on GIT quotients given in [10] and [22].

PROPOSITION 2.5. *The collection of orbit cones $\{\omega_T(x);\ x \in X\}$ is finite.*

Proof. Embed X equivariantly into some $\mathbb{K}^n$ on which T acts diagonally. Then the T-orbit cone of a point $x \in X$ equals its T-orbit cone w.r.t. $\mathbb{K}^n$. The T-orbit cones w.r.t. $\mathbb{K}^n$ are constant along the orbits of the standard action of $\mathbb{T}^n := (\mathbb{K}^*)^n$, because this action commutes with that of T. Since $\mathbb{T}^n$ has only finitely many orbits in $\mathbb{K}^n$, the assertion follows. $\square$

We now enter the study of the collection of sets of semistable points arising from the possible T-linearizations of the trivial bundle. First, we recall that these linearizations correspond to the characters of T.

LEMMA 2.6. *Consider a T-linearization of the trivial bundle over X. Then there is a unique $w \in M$ such that the dual T-action on $X \times \mathbb{K}$ is of the form*

$$t \cdot (x, z) := (t \cdot x, \chi^w(t) z). \tag{2.6.1}$$

Proof. The dual action is a fibrewise linear T-action on $X \times \mathbb{K}$ making $X \times \mathbb{K} \to X$ equivariant. Consequently, there is a morphism $c\colon T \times X \to \mathbb{K}^*$ such that

$$t \cdot (x, z) := (t \cdot x, c(t, x) z).$$

Clearly, we always have $c(1, x) = 1$. Thus, for fixed x, the map $t \mapsto c(t, x)$ is a homomorphism. Hence, by rigidity of tori, c does not depend on x. □

In the sequel we shall denote by $X^{ss}(w) \subset X$ the set of semistable points defined by the linearization (2.6.1). It can be explicitly described in terms of homogeneous functions and also in terms of orbit cones.

LEMMA 2.7. *The set $X^{ss}(w) \subset X$ of semistable points of the linearization* (2.6.1) *is given by*

$$\begin{aligned} X^{ss}(w) &= \bigcup_{f \in R_{nw},\, n \in \mathbb{Z}_{>0}} X_f \\ &= \{x \in X;\ w \in \omega_T(x)\}. \end{aligned}$$

In particular, the set of semistable points $X^{ss}(w)$ is nonempty if and only if $w \in \Omega_T(X) \cap M$.

Proof. As indicated in Remark 1.7, the invariant sections for the linearization (2.6.1) are precisely the functions $f \in R_{nw}$ with $n \in \mathbb{Z}_{\geq 0}$. This gives the first equality. The second one is a direct consequence of the definition of an orbit cone, and the last statement is obvious. □

As outlined in Section 1, the set $X^{ss}(w)$ is T-invariant and admits a good quotient $X^{ss}(w) \to Y(w)$ by the action of T. In fact, the quotient space $Y(w) = X^{ss}(w)/\!\!/T$ is the homogeneous spectrum of a Veronese subalgebra:

$$Y(w) = \operatorname{Proj}(R(w)), \quad \text{where } R(w) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} R_{nw} \subset R.$$

In particular, every quotient space $Y(w)$ is projective over $Y(0) = \operatorname{Spec}(R_0)$. Furthermore, if $X^{ss}(w_1) \subset X^{ss}(w_2)$ then Proposition 1.2(iii) yields the commutative diagram

$$\begin{tikzcd} X^{ss}(w_1) \arrow[rr, "\subset"] \arrow[d, "/\!\!/T"'] & & X^{ss}(w_2) \arrow[d, "/\!\!/T"] \\ Y(w_1) \arrow[rr, "\varphi_{w_2}^{w_1}"] \arrow[dr] & & Y(w_2) \arrow[dl] \\ & Y(0). & \end{tikzcd}$$

Note that the induced map $\varphi_{w_2}^{w_1} \colon Y(w_1) \to Y(w_2)$ of the quotient spaces is dominant and projective. Moreover, we have $\varphi_{w_3}^{w_1} = \varphi_{w_3}^{w_2} \circ \varphi_{w_2}^{w_1}$ whenever composition is possible.

The collection of all nonempty sets $X^{ss}(w)$, together with their good quotients $X^{ss}(w) \to Y(w)$ and the preceding diagrams, is called the *GIT system* associated to the trivial bundle on the T-variety X. Let us turn to the combinatorial description of this GIT system. We introduce a collection of convex polyhedral cones.

DEFINITION 2.8. For a weight $w \in \Omega_T(X) \cap M$, the associated *GIT cone* is the (nonempty) intersection of all orbit cones containing w:

$$\sigma_T(w) := \bigcap_{w \in \omega_T(x)} \omega_T(x).$$

Moreover, the collection of all possible GIT cones defined by the action of T on X is denoted as

$$\Sigma_T(X) := \{\sigma_T(w);\ w \in \Omega_T(X) \cap M\}.$$

Note that, for us, GIT cones are closed cones and thus are not chambers in the sense of [19]. A first important observation is that the GIT cones are in order-reversing one-to-one correspondence with the possible sets of semistable points arising from the various linearizations of the trivial bundle.

PROPOSITION 2.9. *Let $w_1, w_2 \in \Omega_T(X) \cap M$. Then*:

(i) $X^{ss}(w_1) \subset X^{ss}(w_2) \iff \sigma_T(w_1) \supset \sigma_T(w_2)$;
(ii) $X^{ss}(w_1) = X^{ss}(w_2) \iff \sigma_T(w_1) = \sigma_T(w_2)$.

Proof. This is an immediate consequence of our definition of GIT cones and the characterization of semistability in terms of orbit cones given in Lemma 2.7. □

Proposition 2.9 allows us to speak about the set of semistable points corresponding to a GIT cone $\sigma \in \Sigma_T(X)$. Set

$$X^{ss}(\sigma) := X^{ss}(w), \quad \text{where } \sigma = \sigma_T(w).$$

LEMMA 2.10. *The set of semistable points associated to a GIT cone $\sigma \in \Sigma_T(X)$ is given by*

$$X^{ss}(\sigma) = \{x \in X;\ \sigma \subset \omega_T(x)\}.$$

We now come to the main result of this section. Together with Proposition 2.9, it describes the structure of the collection of sets of semistable points associated to the linearizations of the trivial bundle as a partially ordered set.

A *quasifan* is a finite collection Σ of (not necessarily pointed) convex polyhedral cones in a common vector space such that, for $\sigma \in \Sigma$, all faces of σ belong to Σ and, for any two $\sigma, \sigma' \in \Sigma$, the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'. A quasifan is called a *fan* if it consists of pointed cones. The *support* of a quasifan is the union of its cones.

THEOREM 2.11. *The collection of all GIT cones $\Sigma_T(X)$ is a quasifan in the vector space $M_{\mathbb{Q}}$ having the weight cone $\Omega_T(X)$ as its support.*

In the proof of this result, we need a further basic property of the GIT cones (which is also needed later). For a convex polyhedral cone σ, we denote its relative interior by σ°; this means that σ° is obtained by removing all proper faces from σ.

LEMMA 2.12. *Let $w \in \Omega_T(X) \cap M$. Then the associated GIT cone $\sigma := \sigma_T(w) \in \Sigma_T(X)$ satisfies*:

$$\sigma = \bigcap_{w \in \omega_T(x)^\circ} \omega_T(x) = \bigcap_{\sigma^\circ \subset \omega_T(x)^\circ} \omega_T(x);$$

$$w \in \sigma^\circ = \bigcap_{w \in \omega_T(x)^\circ} \omega_T(x)^\circ = \bigcap_{\sigma^\circ \subset \omega_T(x)^\circ} \omega_T(x)^\circ.$$

Proof. For any orbit cone $\omega_T(x)$ with $w \in \omega_T(x)$, there is a unique minimal face $\omega \preceq \omega_T(x)$ with $w \in \omega$ that satisifies $w \in \omega^\circ$. According to Corollary 2.4, the face $\omega \preceq \omega_T(x)$ is again an orbit cone. This gives the first formula. The second one follows from an elementary observation: if the intersection of the relative interiors of a finite number of convex polyhedral cones is nonempty, then it equals the relative interior of the intersection of the cones. □

Proof of Theorem 2.11. First of all note that, by finiteness of the number of orbit cones (as shown in Proposition 2.5), the collection of all GIT cones is finite.

The remainder of the proof is split into verifications of several claims. For the sake of handy notation, we set for the moment $\Omega := \Omega_T(X)$ and $\Sigma := \Sigma_T(X)$. Moreover, we omit the subscript T when denoting orbit cones and GIT cones, and we write $X(\sigma)$ instead of $X^{ss}(\sigma)$.

Claim 1. Let $\sigma_1, \sigma_2 \in \Sigma$ with $\sigma_1 \subset \sigma_2$. Then, for every $x_1 \in X(\sigma_1)$ with $\sigma_1^\circ \subset \omega(x_1)^\circ$, there exists an $x_2 \in X(\sigma_2)$ with $\omega(x_1) \preceq \omega(x_2)$.

Let us verify the claim. By Proposition 2.9, we have $X(\sigma_2) \subset X(\sigma_1)$. Consequently, the GIT system provides a commutative diagram with a dominant, proper, and hence surjective morphism $\varphi\colon Y(\sigma_2) \to Y(\sigma_1)$ of the quotient spaces:

$$\begin{tikzcd} X(\sigma_2) \arrow[r, "\subset"] \arrow[d, "p_2"'] & X(\sigma_1) \arrow[d, "p_1"] \\ Y(\sigma_2) \arrow[r, "\varphi"'] & Y(\sigma_1). \end{tikzcd}$$

If a point $x_1 \in X(\sigma_1)$ satisfies $\sigma_1^\circ \subset \omega_T(x_1)^\circ$, then (by Lemma 2.10 and Corollary 2.4) its T-orbit is closed in $X(\sigma_1)$. Proposition 1.2(ii) thus tells us that $x_1 \in \overline{T \cdot x_2}$ holds for any point x_2 belonging to the (nonempty) intersection $X(\sigma_2) \cap p_1^{-1}(p_1(x_1))$. Using once more Corollary 2.4 now gives Claim 1.

Claim 2. Let $\sigma_1, \sigma_2 \in \Sigma$. Then $\sigma_1 \subset \sigma_2$ implies $\sigma_1 \preceq \sigma_2$.

For the verification, let $\tau_2 \preceq \sigma_2$ be the (unique) face with $\sigma_1^\circ \subset \tau_2^\circ$, and let $\omega_{1,1}, \dots, \omega_{1,r}$ be the orbit cones with $\sigma_1^\circ \subset \omega_{1,i}^\circ$. Then we obtain, using Lemma 2.12 for the second observation,

$$\tau_2^\circ \cap \omega_{1,i}^\circ \neq \emptyset, \qquad \sigma_1 = \omega_{1,1} \cap \cdots \cap \omega_{1,r}.$$

By Claim 1, we have $\omega_{1,i} \preceq \omega_{2,i}$ with orbit cones $\omega_{2,i}$ satisfying $\sigma_2 \subset \omega_{2,i}$; therefore, $\tau_2 \subset \omega_{2,i}$. The first of the displayed formulas implies $\tau_2 \subset \omega_{1,i}$, and the second one thus gives $\tau_2 = \sigma_1$. Hence, Claim 2 is verified.

Claim 3. Let $\sigma \in \Sigma$. Then every face $\sigma_0 \preceq \sigma$ belongs to Σ.

To see this, consider any $w \in \sigma_0^\circ$. Lemma 2.12 yields $w \in \sigma(w)^\circ$. By the definition of GIT cones we have $\sigma(w) \subset \sigma$, and Claim 2 gives $\sigma(w) \preceq \sigma$. Thus we have two faces of σ, σ_0 and $\sigma(w)$, having a common point w in their relative interiors. This means that $\sigma_0 = \sigma(w)$ and so Claim 3 is verified.

Claim 4. Let $\sigma_1, \sigma_2 \in \Sigma$. Then $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

Let $\tau_i \preceq \sigma_i$ be the minimal face containing $\sigma_1 \cap \sigma_2$. Choose w in the relative interior of $\sigma_1 \cap \sigma_2$, and consider the GIT cone $\sigma(w)$. By Lemma 2.12 and the definition of GIT cones, we see that

$$w \in \sigma(w)^\circ \cap \tau_i^\circ, \qquad \sigma(w) \subset \sigma_1 \cap \sigma_2 \subset \tau_i.$$

By Claim 2, the second relation implies in particular that $\sigma(w) \preceq \sigma_i$. We can therefore conclude that $\sigma(w) = \tau_i$ and so $\sigma_1 \cap \sigma_2$ is a face of both σ_i. Hence Claim 4 is verified, and the properties of a quasifan are established for $\Sigma_T(X)$. □

3. A Semistability Criterion

We present a combinatorial description of the set of semistable points associated to a linearized Weil divisor. By X we denote a normal projective variety with finitely generated free divisor class group $\mathrm{Cl}(X)$, and we consider the action $T \times X \to X$ of an algebraic torus $T = \mathrm{Spec}(\mathbb{K}[M])$ on the variety X.

The *total coordinate ring* $\mathcal{R}(X)$ of the variety X is defined as follows: Choose a subgroup $K \subset \mathrm{WDiv}(X)$ of the group of Weil divisors such that the canonical map $K \to \mathrm{Cl}(X)$ is an isomorphism, and set

$$\mathcal{R}(X) := \Gamma(X, \mathcal{R}), \quad \text{where } \mathcal{R} := \bigoplus_{D \in K} \mathcal{O}(D).$$

This ring depends only up to isomorphism on the choices made in its definition. An important property of the total coordinate ring $\mathcal{R}(X)$ is that it is a factorial ring; see [1] and [11].

Throughout this section, we assume that $\mathcal{R}(X)$ is finitely generated as a $\mathbb{K}$-algebra. We consider the following geometric objects associated to the K-graded sheaf $\mathcal{R}$ of $\mathcal{O}_X$-algebras:

$$H := \mathrm{Spec}(\mathbb{K}[K]), \quad \bar{X} := \mathrm{Spec}(\mathcal{R}(X)), \quad \hat{X} := \mathrm{Spec}_X(\mathcal{R}).$$

Thus $\hat{X}$ refers to the relative spectrum of $\mathcal{R}$. Recall that there is a canonical morphism $q_X\colon \hat{X} \to X$. We list some basic properties of this setting, which we use often in the subsequent constructions and proofs (cf. [3]).

PROPOSITION 3.1. *Let H, $\bar{X}$, $\hat{X}$, and $q_X\colon \hat{X} \to X$ be as before. Then the following statements hold.*

(i) *The K-grading of $\mathcal{R}$ defines an action of the torus H on $\hat{X}$, and $q_X\colon \hat{X} \to X$ is a good quotient for this action.*
(ii) *The K-grading of $\mathcal{R}(X)$ defines an action of the torus H on $\bar{X}$, and the canonical map $\hat{X} \to \bar{X}$ is an equivariant open embedding.*
(iii) *For $D \in K$ and $f \in \Gamma(X, \mathcal{O}(D))$ with $X \setminus Z(f)$ affine, the inverse image $q_X^{-1}(X \setminus Z(f))$ equals $\bar{X}_f$.*
(iv) *For the set $X_{\mathrm{reg}} \subset X$ of nonsingular points, the complement $\bar{X} \setminus q_X^{-1}(X_{\mathrm{reg}})$ is of codimension ≥ 2 in $\bar{X}$.*
(v) *Suppose that $H \cdot x \subset \hat{X}$ is closed; then $f \in \Gamma(\bar{X}, \mathcal{O})_D$ satisfies $f(x) = 0$ if and only if $q_X(x) \in Z(f)$ for $f \in \Gamma(X, \mathcal{O}(D))$.*
(vi) *There exists a T-action on $\bar{X}$, commuting with the H-action on $\bar{X}$, such that $\hat{X} \subset \bar{X}$ is T-invariant and $q_X\colon \hat{X} \to X$ is T-equivariant.*

Proof. We begin with a basic observation. Let $D \in K$ and $f \in \Gamma(X, \mathcal{O}(D))$ be such that $X \setminus Z(f)$ is affine. Then we have the following identities of global functions:

$$\Gamma(\bar{X}_f, \mathcal{O}) = \mathcal{R}(X)_f = \Gamma(X \setminus Z(f), \mathcal{R}) = \Gamma(q_X^{-1}(X \setminus Z(f)), \mathcal{O}).$$

Since $K = \mathrm{Cl}(X)$, it follows that the variety X is covered by such affine sets $X \setminus Z(f)$. Thus, we see in particular that $\mathcal{R}$ is locally of finite type and $\hat{X}$ is a variety.

The first assertion is then obvious. In the second, we need only explain why $\hat{X} \to \bar{X}$ is an open embedding. By the previous identities, each affine subset $q_X^{-1}(X \setminus Z(f))$ is mapped isomorphically onto $\bar{X}_f$. It follows that $\hat{X} \to \bar{X}$ is an open embedding. Moreover, assertion (iii) drops out as well.

The fourth assertion is due to an identity of global functions: it follows from the fact that $\Gamma(X_{\mathrm{reg}}, \mathcal{R})$ equals $\Gamma(X, \mathcal{R})$.

To verify assertion (v), suppose first that $q_X(x) \notin Z(f)$ holds for a section $f \in \Gamma(X, \mathcal{O}(D))$. Then f restricts to an invertible section of $\mathcal{R}$ over a suitable neighborhood $U \subset X$ of $q_X(x)$. Consequently, f is invertible as a function on $q_X^{-1}(U)$, which implies $f(x) \neq 0$.

Conversely, let $f(x) \neq 0$ for $f \in \Gamma(\bar{X}, \mathcal{O})_D$. Consider the orbit $H \cdot x$ and the zero set $B := N(f, \hat{X})$. By Proposition 1.2(i), the image $q_X(B) \subset X$ is closed and does not contain $q_X(x)$. Hence, for a suitable neighborhood $U \subset X$ of $q_X(x)$, we see that f is invertible as a function on $q_X^{-1}(U)$ and thus it is so as a section of $\mathcal{R}$ over U. This implies that $q_X(x) \notin Z(f)$.

We are therefore left with verifying the last statement. By [13], there is a T-linearization of the group $K \subset \mathrm{WDiv}(X)$ over $X_{\mathrm{reg}} \subset X$; in other words, we may lift the T-action to $q_X^{-1}(X_{\mathrm{reg}})$. By part (iv) of the proposition, the complement $\bar{X} \setminus q_X^{-1}(X_{\mathrm{reg}})$ is of codimension ≥ 2 in $\bar{X}$. Hence the lifted T-action extends to $\bar{X}$. □

For the remainder of this section, we fix a lifting of the T-action to $\bar{X}$ as in Proposition 3.1(vi). In terms of multigraded rings, this means that we have the following refinement of the K-grading:

$$\mathcal{R}(X) = \bigoplus_{(D,w)\in K\oplus M} \Gamma(X,\mathcal{R})_{(D,w)}.$$

We need a pullback construction for linearized Weil divisors. For a Weil divisor D on X, consider its restriction D_{reg} to X_{reg}, and let $\bar{D}$ denote the Weil divisor on $\bar{X}$ obtained by closing the support of $q_X^* D_{\mathrm{reg}}$. Now suppose that D is T-linearized. We then observe that $\bar{D}$ inherits, in a canonical way, an $(H \times T)$-linearization. In fact, consider the Cartesian square

$$\begin{tikzcd}
q_X^{-1}(X_{\mathrm{reg}})(q_X^* D_{\mathrm{reg}}) \arrow[r] \arrow[d] & X_{\mathrm{reg}}(D_{\mathrm{reg}}) \arrow[d, "q_D"] \\
q_X^{-1}(X_{\mathrm{reg}}) \arrow[r, "q_X"'] & X_{\mathrm{reg}}.
\end{tikzcd}$$

Viewing the upper left space as a fibre product $q_X^{-1}(X_{\mathrm{reg}}) \times_{X_{\mathrm{reg}}} X_{\mathrm{reg}}(D_{\mathrm{reg}})$, one defines an $(H \times T)$-action on it by letting H act on the first factor and letting T act diagonally. Since $\bar{X}$ is locally factorial, $\bar{X}(\bar{D}) \to \bar{X}$ is a bundle and thus, by Proposition 3.1(iv), the $(H \times T)$-action extends to the desired linearization of $\bar{D}$.

Lemma 3.2. *The set $q_X^{-1}(X^{ss}(D,T))$ is $(H \times T)$-saturated in $\bar{X}^{ss}(\bar{D}, H \times T)$ and we have the following commutative diagram, where the horizontal arrows are open embeddings*:

$$\begin{tikzcd}
\hat{X} \arrow[d, "/\!\!/H"'] & q_X^{-1}(X^{ss}(D,T)) \arrow[l] \arrow[r] \arrow[d, "/\!\!/H"] & \bar{X}^{ss}(\bar{D}, H \times T) \arrow[r] \arrow[dd, "/\!\!/H\times T"] & \bar{X} \\
X & X^{ss}(D,T) \arrow[l] \arrow[d, "/\!\!/T"] & & \\
& X^{ss}(D,T)/\!\!/T \arrow[r] & \bar{X}^{ss}(\bar{D}, H \times T)/\!\!/H \times T. &
\end{tikzcd}$$

Proof. Every T-invariant section $f \in \Gamma(X, \mathcal{O}(nD))$ defines via pullback a section of $\Gamma(\bar{X}, \mathcal{O}(n\bar{D}))$ that is $(H \times T)$-invariant. Thus, Proposition 3.1(iii) and the definition of semistability yield the desired statement. □

Lemma 3.3. *As an $(H \times T)$-linearized divisor, $\bar{D}$ is isomorphic to the trivial bundle with an $(H \times T)$-linearization, and there is a unique $w \in M$ such that the corresponding dual action is given as*

$$(h,t)\cdot(x,z) = ((h,t)\cdot x, \chi^{(D,w)}(h,t)z).$$

Moreover, the assignment $D \mapsto (D,w)$ induces an isomorphism $\mathrm{Cl}_T(X) \to K \oplus M$ from the group of T-linearized divisor classes on X to the character lattice of the torus $H \times T$.

Proof. Consider the set $X_{\mathrm{reg}} \subset X$ of nonsingular points and the restriction D_{reg} of D to X_{reg}. Then, over the sets $U_i \subset X_{\mathrm{reg}}$ of a suitably fine open cover, the sheaf $\mathcal{O}(D_{\mathrm{reg}})$ is generated by invertible elements $f_i \in \Gamma(U_i, \mathcal{O}(D))$.

The line bundle $\pi\colon L \to X_{\mathrm{reg}}$ with the transition functions $\xi_{ij} := f_j/f_i$ is the dual bundle of $X_{\mathrm{reg}}(D_{\mathrm{reg}}) \to X_{\mathrm{reg}}$; it comes with the dual T-action and with canonical trivializations

$$\pi^{-1}(U_i) \to U_i \times \mathbb{K}, \qquad v \mapsto (\pi(v), z_i(v)).$$

The pullback line bundle $q_X^*L = q_X^{-1}(X_{\mathrm{reg}}) \times_{X_{\mathrm{reg}}} L$ is dual to the line bundle $q_X^{-1}(X_{\mathrm{reg}})(q_X^* D_{\mathrm{reg}}) \to q_X^{-1}(X_{\mathrm{reg}})$ arising from the restriction of $\bar{D}$. The dual $(H \times T)$-action on q_X^*L equals the pullback linearization and is of the form

$$(h,t)\cdot(x,v) = (t\cdot h\cdot x, t\cdot v).$$

We claim that q_X^*L is H-equivariantly isomorphic to the trivial bundle, H-linearized by the character χ^D; this follows because the functions $f_i \in \Gamma(U_i, \mathcal{O}(D))$ define a global trivialization for q_X^*L, namely,

$$q_X^{-1}(X_{\mathrm{reg}}) \times_{X_{\mathrm{reg}}} L \to q_X^{-1}(X_{\mathrm{reg}}) \times \mathbb{K},$$
$$(x,v) \mapsto (x, f_i(x)z_i(v)) \quad \text{for } x \in q_X^{-1}(U_i).$$

Using Proposition 3.1(iv), we can extend this to a global trivialization of the dual bundle of $\bar{D}$, which proves the first part of the assertion.

For the second part, note first that H acts freely on $q_X^{-1}(X_{\mathrm{reg}})$ because, locally on X_{reg}, all divisors $D \in K$ are principal and hence locally any point of $q_X^{-1}(X_{\mathrm{reg}})$ has K as its weight monoid. Moreover, there is a commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & \mathrm{Pic}_{H\times T}(q_X^{-1}(X_{\mathrm{reg}})) \arrow[r] & \mathrm{Pic}_H(q_X^{-1}(X_{\mathrm{reg}})) \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[u, equal] & \mathrm{Pic}_T(X_{\mathrm{reg}}) \arrow[r] \arrow[u, "q_X^*"] & \mathrm{Pic}(X_{\mathrm{reg}}) \arrow[r] \arrow[u, "q_X^*"'] & 0,
\end{tikzcd}$$

with exact rows. Since H acts freely on $q_X^{-1}(X_{\mathrm{reg}})$, we infer from [16, Prop. 4.2] that the right-hand pullback is an isomorphism. Consequently, the pullback in the middle of the diagram must also be an isomorphism.

The assertion thus follows from the fact that we have canonical isomorphisms $\mathrm{Cl}_T(X) \cong \mathrm{Pic}_T(X_{\mathrm{reg}})$ and $\mathrm{Cl}_{H\times T}(\bar{X}) \cong \mathrm{Pic}_{H\times T}(q_X^{-1}(X_{\mathrm{reg}}))$, where the latter relies on Proposition 3.1(iv). $\square$

Via the isomorphism $[D] \mapsto (D, w)$ just established, we shall hereafter identify the T-linearized divisor classes on X with the elements on $K \oplus M$. We denote the corresponding sets of semistable points by

$$X^{ss}(D, w) := X^{ss}(D, T), \quad \text{where } \mathrm{Cl}_T(X) \ni [D] \mapsto (D, w) \in K \oplus M.$$

We are now ready to begin with the combinatorial characterization of semistability. It involves two fans: the collections of GIT cones $\Sigma_{H\times T}(\bar{X})$ and $\Sigma_H(\bar{X})$ for the actions of $H \times T$ and H on $\bar{X}$. These collections are actually fans because, by [3, Prop. 4.3], the weight cones $\Omega_{H\times T}(\bar{X})$ and $\Omega_H(\bar{X})$ are pointed.

Let $\kappa_X \in \Sigma_H(\bar{X})$ be the GIT cone corresponding to $\hat{X} \subset \bar{X}$, which (by projectivity of X) is a set of H-semistable points. Moreover, let $\Pi\colon K \oplus M \to K$ denote the projection. We consider the collections of orbit cones

$$C_T(X) := \{\omega_{H\times T}(x);\ x \in \bar{X},\ \kappa_X^\circ \subset \Pi(\omega_{H\times T}(x))^\circ\} \quad \text{and}$$

$$C_T(\sigma) := \{\omega_{H\times T}(x);\ x \in \bar{X},\ \sigma^\circ \subset \omega_{H\times T}(x)^\circ\},$$

where $\sigma \in \Sigma_{H\times T}(\bar{X})$ may be any GIT cone. The geometric meaning of these collections is that they describe the collection of closed orbits in the respective sets of semistable points as follows.

Lemma 3.4. *Let $\sigma \in \Sigma_{H\times T}(\bar{X})$ and $x \in \bar{X}$, and consider the orbit cone $\omega_{H\times T}(x)$.*

(i) *The orbit $H \cdot x$ is a closed subset of $\hat{X}$ if and only if $\omega_{H\times T}(x) \in C_T(X)$.*
(ii) *The orbit $(H \times T) \cdot x$ is a closed subset of $\bar{X}^{ss}(\sigma)$ if and only if $\omega_{H\times T}(x) \in C_T(\sigma)$.*

Proof. First note that, for any orbit cone $\omega_{H\times T}(x)$, the image $\Pi(\omega_{H\times T}(x))$ equals the orbit cone $\omega_H(x)$. Hence, the collections $C_T(X)$ and $C_T(\sigma)$ describe the orbits of H in $\hat{X} = \bar{X}^{ss}(\kappa_X)$ and $H \times T$ in $\bar{X}^{ss}(\sigma)$ having minimal orbit cones. The assertions now follow from Corollary 2.4. □

Our next result characterizes semistability in terms of the collections of orbit cones just described.

Theorem 3.5. *Let $(D, w) \in \Omega_{H\times T}(\bar{X}) \cap (K \oplus M)$ represent a T-linearized Weil divisor D on X, and consider the GIT cone $\sigma := \sigma_{H\times T}(D, w)$ in $\Sigma_{H\times T}(\bar{X})$. Then*

$$q_X^{-1}(X^{ss}(D, w)) = \{x \in \bar{X};\ \omega \preceq \omega_{H\times T}(x) \text{ for some } \omega \in C_T(X) \cap C_T(\sigma)\}.$$

Proof. As before, let $\bar{D}$ denote the divisor on $\bar{X}$ obtained by closing the support of the pullback divisor $q_X^* D_{\mathrm{reg}}$, and consider the inherited $(H \times T)$-linearization of $\bar{D}$. Then $\bar{X}^{ss}(\bar{D}, H \times T)$ is precisely the set of semistable points corresponding to the GIT cone $\sigma \in \Sigma_{H\times T}(\bar{X})$.

To verify the "$\subset$" part, consider first a closed orbit $(H \times T) \cdot x$ in $q_X^{-1}(X^{ss}(D, T))$. By Lemma 3.2, this orbit is closed in $\bar{X}^{ss}(\bar{D}, H \times T)$ and hence Lemma 3.4 yields $\sigma^\circ \subset \omega_{H\times T}(x)^\circ$. Moreover, the orbit $H \cdot x$ is closed in $(H \times T) \cdot x$ and thus also in $q_X^{-1}(X^{ss}(D, T))$; therefore, since $q_X^{-1}(X^{ss}(D, T))$ is H-saturated in $\hat{X}$, the orbit

$H \cdot x$ is also closed in $\hat{X}$. Lemma 3.4 yields $\kappa_X^\circ \subset \omega_H(x)^\circ$, so $\omega_{H\times T}(x)$ lies in $C_T(X) \cap C_T(\sigma)$.

Now, given an arbitrary point $x \in q_X^{-1}(X^{ss}(D,T))$, we may consider any point y in the $(H\times T)$-orbit closure of x having a closed $(H\times T)$-orbit in $q_X^{-1}(X^{ss}(D,T))$. According to Corollary 2.4, the orbit cone $\omega := \omega_{H\times T}(y)$ is a face of $\omega_{H\times T}(x)$ and thus, by the preceding consideration, ω belongs to $C_T(X) \cap C_T(\sigma)$.

We turn to the inclusion "$\supset$". First consider a point $x \in \bar{X}$ such that $\omega_{H\times T}(x)$ belongs to $C_T(X) \cap C_T(\sigma)$. Then we have

$$x \in \hat{X} \cap \bar{X}^{ss}(\bar{D}, H \times T).$$

Moreover, by Lemma 3.4, the orbit $H \cdot x$ is closed in $\hat{X}$ and the orbit $(H \times T) \cdot x$ is closed in $\bar{X}^{ss}(\bar{D}, H \times T)$.

By a repeated shrinking procedure, we shall now construct a neighborhood of $q_X(x) \in X$ as required in Definition 1.5. First, note that the definition of semi-stability for the linearized divisor $\bar{D}$ provides an $f \in \Gamma(\bar{X}, \mathcal{O})$, homogeneous of weight (nD, nw) with some $n \in \mathbb{Z}_{>0}$, such that

$$(H \times T) \cdot x \subset \bar{X}_f \subset \bar{X}^{ss}(\bar{D}, H \times T).$$

Consider the complement $B_1 := \bar{X}_f \setminus \hat{X}$. This is an $(H \times T)$-invariant closed subset of $\bar{X}_f$ that is disjoint from $(H \times T) \cdot x$. By Proposition 1.2(i), the good quotient

$$\bar{X}_f \to \bar{X}_f /\!\!/ (H \times T)$$

separates x and B_1. Hence we can choose an $(H \times T)$-invariant function $f_0 \in \Gamma(\bar{X}_f, \mathcal{O})$ that satisfies $f_0|_{B_1} = 0$ and has no zeroes in $(H \times T) \cdot x$.

For a suitable $k \in \mathbb{Z}_{>0}$, the product $g := f_0 f^k$ is a T-invariant element of $\Gamma(X, \mathcal{O}(knD))$. Since $H \cdot x$ is closed in $\hat{X}$, Proposition 3.1(v) yields

$$x \in q_X^{-1}(X \setminus Z(g)) \subset \bar{X}_g \subset \hat{X}.$$

Now consider the intersection $B_2 := \bar{X}_g \cap q_X^{-1}(Z(g))$; this is an $(H \times T)$-invariant closed subset of $\bar{X}_g$ that is disjoint from $(H \times T) \cdot x$. As before, we can choose an $(H \times T)$-invariant function $g_0 \in \Gamma(\bar{X}_g, \mathcal{O})$ that satisfies $g_0|_{B_2} = 0$ and has no zeroes in $(H \times T) \cdot x$.

Once more, for a suitable $l \in \mathbb{Z}_{>0}$, the product $h := g_0 g^l$ is a T-invariant element of $\Gamma(X, \mathcal{O}(lknD))$. This time we have

$$x \in q_X^{-1}(X \setminus Z(h)) \subset \bar{X}_h \subset q_X^{-1}(X \setminus Z(g)) \subset \hat{X}.$$

We make the further claim that $q_X^{-1}(X \setminus Z(h)) = \bar{X}_h$. Assume, to the contrary, that there exists a point $y \in \bar{X}_h$ with $q_X(y) \in Z(h)$. Observe that the orbit closure in $\hat{X}$ satisfies

$$\overline{H \cdot y} \subset q_X^{-1}(q_X(y)) \subset q_X^{-1}(X \setminus Z(g)).$$

Consider any $y_0 \in \overline{H \cdot y}$ such that $H \cdot y_0$ is closed in $\hat{X}$. By the preceding observation, $q_X(y_0) \notin Z(g)$ holds. Proposition 3.1(v) thus yields $g(y_0) \neq 0$. By assumption, we have

$$q_X(y_0) = q_X(y) \in Z(h).$$

Applying again Proposition 3.1(v) gives $h(y_0) = 0$, and thus $g_0(y_0) = 0$. Since g_0 is an H-invariant function, this means $g_0(y) = 0$. Hence we obtain $h(y) = 0$, which is in contradiction to $y \in \bar{X}_h$.

Having seen that $q_X^{-1}(X \setminus Z(h)) = \bar{X}_h$ holds, we easily obtain the rest: the element $h \in \Gamma(X, \mathcal{O}(lknD))$ is T-invariant and defines an affine neighborhood

$$X \setminus Z(h) = \bar{X}_h /\!\!/ H$$

of the point $q_X(x) \in X$ as required in Definition 1.5. This shows that the point x belongs to $q_X^{-1}(X^{ss}(D, T))$.

If $x \in \bar{X}$ is an arbitrary point in the set on the RHS of the equation in Theorem 3.5, then Corollary 2.4 tells us that the face $\omega \preceq \omega_{H\times T}(x)$ with $\omega \in C_T(X) \cap C_T(\sigma)$ is the orbit cone of some point y belonging to the $(H \times T)$-orbit closure of x. Given the previous consideration, we have $y \in q_X^{-1}(X^{ss}(D, T))$, and this implies that $x \in q_X^{-1}(X^{ss}(D, T))$. □

Even in the case of a trivial torus action, Theorem 3.5 is of some interest: it then provides a description of the cone of ample divisors of the variety X. See also [3, Thm. 7.3].

Corollary 3.6. *The cone of ample divisor classes on X is the relative interior $\kappa_X^\circ \subset K_{\mathbb{Q}} = \mathrm{Cl}_{\mathbb{Q}}(X)$ of the GIT cone $\kappa_X \in \Sigma_H(\bar{X})$.*

Proof. Consider the action of the trivial torus $T = \{e_T\}$ on X. Then, for any $x \in \bar{X}$, we have $\omega_{H\times T}(x) = \omega_H(x)$. Moreover, the fans $\Sigma_{H\times T}(\bar{X})$ and $\Sigma_H(\bar{X})$ coincide. Any divisor $D \in K$ is T-linearized, and $D \in K$ is ample if and only if $X^{ss}(D) = X$ holds. The latter is equivalent to $q_X^{-1}(X^{ss}(D)) = \hat{X}$; by Theorem 3.5, this holds if and only if $D \in \kappa_X^\circ$. □

Remark 3.7. The case of a trivial T-action already shows that $q_X^{-1}(X^{ss}(D, T))$ is in general properly smaller than $\hat{X} \cap \bar{X}^{ss}(\bar{D}, H \times T)$. Let D be effective but not big. Then $X^{ss}(D, T)$ is empty but $\hat{X} \cap \bar{X}^{ss}(\bar{D}, H \times T)$ is nonempty. As an explicit example, one may take $X = \mathbb{P}_1 \times \mathbb{P}_1$ and $D = \mathbb{P}_1 \times \{0\}$.

4. The General Case

In this section we present the main results of the paper. As in Section 3, X is a normal projective variety with finitely generated total coordinate ring $\mathcal{R}(X)$, and the algebraic torus $T = \mathrm{Spec}(\mathbb{K}[M])$ acts on X. We give a combinatorial description of the collection of sets of semistable points associated to the T-linearized Weil divisors on X.

Recall from Section 3 that X is a good quotient of an open subset $\hat{X}$ of the affine variety $\bar{X} = \mathrm{Spec}(\mathcal{R}(X))$ by the torus $H = \mathrm{Spec}(\mathbb{K}[K])$ corresponding to the grading lattice $K \cong \mathrm{Cl}(X)$ of $\mathcal{R}(X)$. As before, we fix a lifting of the T-action to $\bar{X}$; this corresponds to the choice of a refined grading

$$\mathcal{R}(X) = \bigoplus_{(D,w)\in K\oplus M} \Gamma(X,\mathcal{R})_{(D,w)}.$$

As observed in Lemma 3.3, the degrees $(D,w) \in K \oplus M$ describe the possible T-linearizations of the divisors $D \in K$. Again, we denote by $\kappa_X \in \Sigma_H(\bar{X})$ the GIT cone corresponding to $\hat{X} \subset \bar{X}$. Moreover, $\Pi\colon K \oplus M \to K$ denotes the projection, and we use the collection of cones

$$C_T(X) := \{\omega_{H\times T}(x);\ x \in \bar{X},\ \kappa_X^\circ \subset \Pi(\omega_{H\times T}(x))^\circ\}.$$

We must first figure out the linearized divisor classes with a nonempty set of semistable points. For that purpose, consider the set

$$C_T^\sharp(X) := \bigcup_{\omega\in C_T(X)} \omega^\circ \subset K_\mathbb{Q} \oplus M_\mathbb{Q}.$$

LEMMA 4.1. *The set $C_T^\sharp(X)$ is a convex cone in $K_\mathbb{Q}\oplus M_\mathbb{Q} = \mathrm{Cl}_T(X)_\mathbb{Q}$. For a vector $(D,w) \in K\oplus M$, the set $X^{ss}(D,w)$ is nonempty if and only if $(D,w) \in C_T^\sharp(X)$.*

Proof. That $X^{ss}(D,w)$ is nonempty if and only if $(D,w) \in C_T^\sharp(X)$ holds is a fact that follows directly from Lemma 3.2 and Theorem 3.5. Moreover, multiplying suitable invariant sections allows one to see that, for any two linearized divisor classes (D_i,w_i) with nonempty sets of semistable points $X^{ss}(D_i,w_i)$, also (D_1+D_2, w_1+w_2) admits semistable points. This gives convexity of $C_T^\sharp(X)$. $\square$

DEFINITION 4.2. Let the pair $(D,w) \in C_T^\sharp(X)\cap(K\oplus M)$ represent a T-linearized divisor on X. Then we define its associated GIT bag to be the convex polyhedral cone

$$\mu(D,w) = \bigcap_{\omega\in C_T(X);\ (D,w)\in\omega^\circ} \omega.$$

The collection of all these GIT bags is denoted by $\Lambda_T(X)$. For $\mu_1,\mu_2 \in \Lambda_T(X)$, we write $\mu_1 \le \mu_2$ if, for any $\omega_2 \in C_T(X)$ with $\mu_2^\circ \subset \omega_2^\circ$, there is a face $\omega_1 \preceq \omega_2$ with $\omega_1 \in C_T(X)$ and $\mu_1^\circ \subset \omega_1^\circ$.

Note that every GIT bag is a union of GIT cones of the GIT fan $\Sigma_{H\times T}(\bar{X})$ in $K\oplus M$ corresponding to the $(H\times T)$-action on $\bar{X}$. Moreover, the relation "$\le$" clearly is a partial ordering on $\Lambda_T(X)$. We shall now see that the partially ordered set of GIT bags describes precisely the GIT equivalence.

THEOREM 4.3. *Let $(D_i,w_i) \in C_T^\sharp(X)\cap(K\oplus M)$ represent two T-linearized Weil divisors on X. Then*

$$X^{ss}(D_1,w_1) \subset X^{ss}(D_2,w_2) \iff \mu(D_1,w_1) \ge \mu(D_2,w_2).$$

Proof. We shall make repeated use of the combinatorial characterization of semistability given in Theorem 3.5. For this, let $\sigma_1,\sigma_2 \in \Sigma_{H\times T}(\bar{X})$ denote the GIT cones associated to (D_1,w_1) and (D_2,w_2), respectively. Furthermore, set

$$W_i := q_X^{-1}(X^{ss}(D_i,w_i)) \subset \bar{X}.$$

First suppose that $X^{ss}(D_1, w_1) \subset X^{ss}(D_2, w_2)$. Consider an orbit cone $\omega_1 = \omega_{H\times T}(x) \in C_T(X)$ with $\mu(D_1, w_1) \subset \omega_1^\circ$. Then $(D_1, w_1) \in \omega_1^\circ$ and, by Lemma 2.12, we have $\sigma_1^\circ \subset \omega_1^\circ$; hence Theorem 3.5 yields $x \in W_1$. Thus, by assumption, $x \in W_2$. Again by Theorem 3.5, there exists an $\omega_2 \preceq \omega_1$ with $\omega_2 \in C_T(X)$ and $\sigma_2^\circ \subset \omega_2^\circ$. Therefore, $(D_2, w_2) \in \omega_2^\circ$ and so $\mu(D_2, w_2)^\circ \subset \omega_2^\circ$. This eventually implies that $\mu(D_1, w_1) \geq \mu(D_2, w_2)$.

Conversely, suppose $\mu(D_1, w_1) \geq \mu(D_2, w_2)$. Consider $x \in W_1$ with $(H\times T)\cdot x$ closed in W_1. By Theorem 3.5 and Corollary 2.4, the orbit cone $\omega_1 := \omega_{H\times T}(x)$ belongs to $C_T(X)$ and satisfies $\sigma_1^\circ \subset \omega_1^\circ$. The latter implies $(D_1, w_1) \in \omega_1^\circ$, so $\mu(D_1, w_1)^\circ \subset \omega_1^\circ$. By assumption, there is an $\omega_2 \preceq \omega_1$ with $\omega_2 \in C_T(X)$ and $\mu(D_2, w_2)^\circ \subset \omega_2^\circ$. The latter implies $\sigma_2^\circ \subset \omega_2^\circ$. Thus, Theorem 3.5 yields $x \in W_2$, and we can conclude that $W_1 \subset W_2$. □

We shall now use the description of the collection of sets of semistable points in terms of GIT bags in order to study basic properties of the corresponding system of quotients. The first statement is the following characterization of saturated inclusion by means of GIT bags.

THEOREM 4.4. *Let $(D_i, w_i) \in C_T^\sharp(X) \cap (K \oplus M)$ represent two T-linearized Weil divisors on X. Then $X^{ss}(D_1, w_1)$ is a T-saturated subset of $X^{ss}(D_2, w_2)$ if and only if $\mu(D_1, w_1)^\circ \supset \mu(D_2, w_2)^\circ$.*

Proof. We begin with a preparatory observation that characterizes closedness of a given T-orbit in the set $X^{ss}(D_i, w_i)$.

Claim. Consider points $x \in X^{ss}(D_i, w_i)$ and $\hat{x} \in W_i := q_X^{-1}(X^{ss}(D_i, w_i))$ such that $q_X(\hat{x}) = x$ and $H \cdot \hat{x}$ is closed in $\hat{X}$. Then $T \cdot x$ is closed in $X^{ss}(D_i, w_i)$ if and only if $(H \times T) \cdot \hat{x}$ is closed in W_i.

Let us verify the claim. The "if" part is clear, so let $T\cdot x$ be closed in $X^{ss}(D_i, w_i)$. Assume that the complement

$$Y := \overline{(H \times T) \cdot \hat{x}} \setminus (H \times T) \cdot \hat{x} \subset W_i$$

is nonempty. Then Proposition 1.2(ii) tells us that $x \notin q_X(Y)$. On the other hand, we have

$$q_X(Y) \subset q_X\big(\overline{(H \times T) \cdot \hat{x}}\big) \subset \overline{q_X(H \times T) \cdot \hat{x}} = \overline{T \cdot x} \subset X^{ss}(D_i, w_i).$$

Since $q_X(Y)$ is T-invariant and we assumed $T \cdot x$ to be closed, this is a contradiction. Thus, the claim is verified.

We come to the proof of the theorem. First, suppose that $X^{ss}(D_1, w_1)$ is T-saturated in $X^{ss}(D_2, w_2)$. We must then show that

$$\bigcap_{\omega_1 \in C_T(X);\ (D_1, w_1) \in \omega_1^\circ} \omega_1^\circ \supset \bigcap_{\omega_2 \in C_T(X);\ (D_2, w_2) \in \omega_2^\circ} \omega_2^\circ.$$

Consider $\omega_1 = \omega_{H\times T}(\hat{x}) \in C_T(X)$ as on the LHS. Then, by Theorem 3.5 and Corollary 2.4, the orbit $(H \times T) \cdot \hat{x}$ is closed in W_1 and the orbit $H \cdot \hat{x}$ is closed in $\hat{X}$. Therefore, $T \cdot x$ is closed in $X^{ss}(D_1, w_1)$. By T-saturatedness, $T \cdot x$ is closed in $X^{ss}(D_2, w_2)$. The claim just verified tells us that $(H \times T) \cdot \hat{x}$ is closed in W_2. Thus, Theorem 3.5 and Corollary 2.4 show that $(D_2, w_2) \in \omega_1^\circ$.

Conversely, suppose that $\mu(D_1, w_1)^\circ \supset \mu(D_2, w_2)^\circ$. This clearly implies that $\mu(D_1, w_1) \geq \mu(D_2, w_2)$ and so Theorem 4.3 yields $X^{ss}(D_1, w_1) \subset X^{ss}(D_2, w_2)$. We now need only show that the inclusion is T-saturated. For this, it suffices to show that every closed T-orbit in $X^{ss}(D_1, w_1)$ is also closed in $X^{ss}(D_2, w_2)$; use, for example, Proposition 1.2 and Corollary 2.4.

Consider a closed orbit $T \cdot x \subset X^{ss}(D_1, w_1)$. Choose $\hat{x} \in q_X^{-1}(X)$ such that $H \cdot \hat{x}$ is closed in $\hat{X}$. By our previous claim, $(H \times T) \cdot \hat{x}$ is closed in W_1. By Theorem 3.5 and Corollary 2.4, the orbit cone $\omega := \omega_{H\times T}(\hat{x})$ satisfies $\omega \in C_T(X)$ and $(D_1, w_1) \in \omega_1^\circ$. The assumption then gives $(D_2, w_2) \in \omega_1^\circ$. Using once more Theorem 3.5 and Corollary 2.4, we see that $(H \times T) \cdot \hat{x}$ is closed in W_2. Consequently, $T \cdot x$ is closed in $X^{ss}(D_2, w_2)$. □

As a consequence, we can describe the *qp-maximal* T-sets of X. By definition, these are open T-invariant subsets $U \subset X$ that admit a good quotient $U \to U/\!\!/T$ such that $U/\!\!/T$ is quasiprojective and U does not occur as a T-saturated subset of some properly larger $U' \subset X$ admitting a good quotient $U' \to U'/\!\!/T$ with $U'/\!\!/T$ quasiprojective.

COROLLARY 4.5. *Let $\Lambda_T^0(X) \subset \Lambda_T(X)$ consist of all GIT bags $\mu_0 \in \Lambda_T(X)$ such that μ_0° is set-theoretically minimal in $\{\mu^\circ;\ \mu \in \Lambda_T(X)\}$. Then the sets of semistable points associated to the $\mu_0 \in \Lambda_T^0(X)$ are precisely the qp-maximal T-sets of X.*

Proof. By Proposition 1.6, every qp-maximal T-set is the set of semistable points of a T-linearized Weil divisor on X. Hence, the assertion follows from Theorem 4.4. □

As shown by the examples discussed in Section 6, there may exist qp-maximal open subsets that have a noncomplete quotient even though X is assumed to be complete. For the subcollection of GIT bags defining projective quotient spaces, we obtain a simple picture as follows.

PROPOSITION 4.6. *For the subcollection of the collection $\Lambda_T(X)$ of all GIT bags*

$$\Lambda_T^{\mathrm{pr}}(X) := \{\mu \in \Lambda_T(X);\ \forall x \in \bar{X},\ \omega_{H\times T}(x)^\circ \cap \mu^\circ \neq \emptyset \Rightarrow \omega_{H\times T}(x) \in C_T(X)\},$$

the following statements hold.

(i) *A GIT bag $\mu \in \Lambda_T(X)$ belongs to $\Lambda_T^{\mathrm{pr}}(X)$ if and only if the corresponding set of semistable points has a projective quotient space.*
(ii) *For any $\mu \in \Lambda_T^{\mathrm{pr}}(X)$ we have $\mu \in \Sigma_{H\times T}(\bar{X})$, and for any two $\mu_1, \mu_2 \in \Lambda_T^{\mathrm{pr}}(X)$ we have $\mu_1 \leq \mu_2 \Leftrightarrow \mu_1 \preceq \mu_2$.*
(iii) *For any two GIT bags $\mu_1 \in \Lambda_T(X)$ and $\mu_2 \in \Lambda_T^{\mathrm{pr}}(X)$, we have $\mu_1 \leq \mu_2 \Rightarrow \mu_1 \in \Lambda_T^{\mathrm{pr}}(X)$.*

Proof. Consider a GIT bag $\mu = \mu(D, w)$ and the GIT cone $\sigma := \sigma_{H\times T}(D, w)$. If $\mu \in \Lambda_T^{\mathrm{pr}}(X)$ then the definition of GIT bags and Lemma 2.12 yield $\mu = \sigma$, which is the first part of assertion (ii). Moreover, Theorem 3.5 yields

$$q_X^{-1}(X^{ss}(D, w)) = \bar{X}^{ss}(\sigma).$$

Since $X^{ss}(D, w)/\!\!/T$ equals $q_X^{-1}(X^{ss}(D, w))/\!\!/(H \times T)$, we infer from the equation just displayed that $X^{ss}(D, w)/\!\!/T$ is projective. Thus, the "only if" part of assertion (i) is verified.

To see the "if" part of (i), suppose that the quotient space $X^{ss}(D, w)/\!\!/T$ is projective; then $q_X^{-1}(X^{ss}(D, w))/\!\!/(H \times T)$ is also projective. Thus, by Lemma 3.2, the inverse image $q_X^{-1}(X^{ss}(D, w))$ equals $\bar{X}^{ss}(\sigma)$. Consequently, Theorem 3.5 gives

$$\omega_{H\times T}(x) \in C_T(\sigma) \implies \omega_{H\times T}(x) \in C_T(X)$$

for all $x \in \bar{X}$. By the definition of a GIT bag, this shows that $\mu = \sigma$. Applying the implication once more, we obtain $\mu \in \Lambda_T^{\mathrm{pr}}(X)$. This completes the proof of assertion (i).

In order to conclude the proof of (ii), we must relate the two GIT bags $\mu_1, \mu_2 \in \Lambda_T^{\mathrm{pr}}(X)$. If $\mu_1 \leq \mu_2$ holds, then obviously we have $\mu_1 \subset \mu_2$. Since μ_1 and μ_2 are cones of the fan $\Sigma_{H\times T}(\bar{X})$, it follows that $\mu_1 \preceq \mu_2$. Conversely, $\mu_1 \preceq \mu_2$ implies $\bar{X}^{ss}(\mu_1) \supset \bar{X}^{ss}(\mu_2)$. Since $\bar{X}^{ss}(\mu_i) = q_X^{-1}(X^{ss}(D_i, w_i))$, where $\mu_i = \mu(D_i, w_i)$, we infer that $\mu_1 \leq \mu_2$ from Theorem 4.3.

The last assertion is easy to see. Let $\mu_i = \mu(D_i, w_i)$; then $\mu_1 \leq \mu_2$ implies that $X^{ss}(D_1, w_1) \supset X^{ss}(D_2, w_2)$. Since $X^{ss}(D_2, w_2)/\!\!/T$ is projective and since the induced map $X^{ss}(D_2, w_2)/\!\!/T \to X^{ss}(D_1, w_1)/\!\!/T$ is dominant, it follows that $X^{ss}(D_1, w_1)/\!\!/T$ must also be projective. This shows $\mu_1 \in \Lambda_T^{\mathrm{pr}}(X)$. □

Let us indicate how the description of the GIT equivalence for linearized ample bundle classes given in [10] and [19] fits into the present framework. For this, recall from Corollary 3.6 that $\kappa_X^\circ \subset K_{\mathbb{Q}}$ is the cone of ample divisor classes on X.

PROPOSITION 4.7. *Let $(D, w) \in C_T^\sharp(X) \cap (K \oplus M)$ represent a T-linearized divisor class on X, and consider the corresponding GIT bag $\mu(D, w)$. Then the set of semistable points $X^{ss}(D, w)$ arises from an ample T-linearized bundle if and only if $\mu(D, w) \in \Lambda_T^0(X)$ and $\Pi(\mu(D, w))^\circ \cap \kappa_X^\circ \neq \emptyset$.*

Proof. Suppose first that the set of semistable points $X^{ss}(D, w)$ arise from an ample T-linearized bundle. Then $X^{ss}(D, w)/\!\!/T$ is projective and hence $X^{ss}(D, w)$ is qp-maximal. Thus, $\mu(D, w) \in \Lambda_T^0(X)$. Moreover, according to Theorem 4.3, $\mu(D, w) = \mu(D', w')$ with some $(D', w') \in K \oplus M$ satisfying $D' \in \kappa_X^\circ$. This shows that $\Pi(\mu(D, w))^\circ \cap \kappa_X^\circ \neq \emptyset$.

Conversely, suppose $\mu(D, w) \in \Lambda_T^0(X)$ and $\Pi(\mu(D, w))^\circ \cap \kappa_X^\circ \neq \emptyset$. Then there exists a $(D', w') \in \mu(D, w)^\circ \cap (K \oplus M)$ such that $D' \in \kappa_X^\circ$. The definition of GIT bags gives $\mu(D', w')^\circ \subset \mu(D, w)^\circ$. Since $\mu(D, w) \in \Lambda_T^0(X)$, we obtain $\mu(D', w')^\circ = \mu(D, w)^\circ$. This shows that $\mu(D', w') = \mu(D, w)$ and hence Theorem 4.3 gives the assertion. □

Propositions 4.6 and 4.7 allow us to rediscover the fan structure inside the T-ample cone described in [10] and [19]. The T-ample cone is defined as the cone generated by the T-linearized ample divisor classes having a nonempty set of semistable points, and it is given by

$$C_T^+(X) = \Pi^{-1}(\kappa_X)^\circ \cap C_T^\sharp(X).$$

COROLLARY 4.8. *The ample GIT classes of the T-action on X are in order-reversing correspondence to the fan of partially open cones $\mu \cap C_T^+(X)$, where μ runs through the cones of $\Lambda_T^0(X)$ with $\kappa_X^\circ \cap \Pi(\mu)^\circ \neq \emptyset$.*

Proof. By Propositions 4.7 and 4.6(ii), the partially open cones $\mu \cap C_T^+(X)$ of the corollary are precisely the intersections $\sigma \cap C_T^+(X)$, where $\sigma \in \Sigma_{H\times T}(X)$. In particular, they form a fan. The rest is a direct consequence of Theorem 4.3 and Proposition 4.6(ii). □

5. The $\mathbb{Q}$-Factorial Case

The setup and notation in this section are the same as in Section 4. We study the partially ordered collection of qp-maximal T-sets in terms of GIT bags for the case of a $\mathbb{Q}$-factorial variety X; recall that $\mathbb{Q}$-factoriality means that X is normal and that, for every Weil divisor on X, some positive multiple is Cartier.

According to Theorem 4.3 and Proposition 4.5, the qp-maximal subsets of X are in order-reversing bijection with the GIT bags in $\Lambda_T^0(X) \subset \Lambda_T(X)$, where $\mu \in \Lambda_T(X)$ belongs to $\Lambda_T^0(X)$ if and only if μ° is set-theoretically minimal among the relative interiors of all elements of $\Lambda_T(X)$. The main result of this section is the following.

THEOREM 5.1. *Assume that X is $\mathbb{Q}$-factorial, and let $\mu_1, \mu_2 \in \Lambda_T^0(X)$.*

(i) *If $\mu_1 \le \mu_2$ holds then $\mu_1 \preceq \mu_2$ holds, and we have*

$$\operatorname{star}(\mu_1, \mu_2) := \{\mu \preceq \mu_2;\ \mu_1 \preceq \mu\} \subset \Lambda_T^0(X).$$

Moreover, $\mu_1 \le \mu_2$ implies $\mu_1 \le \mu \le \mu_2$ for any $\mu \in \operatorname{star}(\mu_1, \mu_2)$.

(ii) *If there is a $\mu_0 \in \Lambda_T^0(X)$ with $\mu_0 \le \mu_1, \mu_2$, then*

$$\mu_1 \cap \mu_2 \preceq \mu_1, \mu_2, \quad \mu_1 \cap \mu_2 \in \Lambda_T^0(X), \quad \mu_0 \le \mu_1 \cap \mu_2 \le \mu_1, \mu_2.$$

As a direct application of Theorem 5.1, we note the following statement on the structure of the collection of all qp-maximal T-sets of X as a partially ordered set.

COROLLARY 5.2. *Assume that X is $\mathbb{Q}$-factorial. Then, for any two qp-maximal T-sets $U_1, U_2 \subset X$, the collection of qp-maximal T-sets $U \subset X$ with $(U_1 \cup U_2) \subset U$ is either empty or contains a unique minimal element.*

Proof. Let U_1, U_2 arise from GIT bags $\mu_1, \mu_2 \in \Lambda_T^0(X)$, and consider the collection of all GIT bags that define sets $U \subset X$ of semistable points comprising $U_1 \cup U_2$:

$$\Gamma = \{\mu_0 \in \Lambda_T^0(X);\ \mu_0 \le \mu_1,\ \mu_0 \le \mu_2\}.$$

Suppose that $\Gamma \neq \emptyset$ holds. Then Theorem 5.1 yields $\mu_1 \cap \mu_2 \in \Gamma$ and $\mu_0 \le \mu_1 \cap \mu_2$ for any $\mu_0 \in \Gamma$. Hence, the set $X^{ss}(\mu_1 \cap \mu_2)$ is as desired. □

The key to the results in the $\mathbb{Q}$-factorial case is the following observation on the ample cone of the variety X.

PROPOSITION 5.3. *The variety X is $\mathbb{Q}$-factorial if and only if the closure $\kappa_X \subset K_{\mathbb{Q}}$ of the ample cone is of full dimension.*

Proof. The statement follows from the well-known fact that κ_X is of full dimension in the vector subspace $K_{\mathbb{Q}}^C \subset K_{\mathbb{Q}}$ generated by the Cartier divisors (see [15]). □

For the proof of Theorem 5.1, we need a couple of preparatory observations. The first one holds also for not necessarily $\mathbb{Q}$-factorial varieties X.

LEMMA 5.4. *Let $\mu_1, \mu_2 \in \Lambda_T^0(X)$ be different from each other. Then $\mu_1^\circ \cap \mu_2^\circ = \emptyset$.*

Proof. Suppose that $\mu_1^\circ \cap \mu_2^\circ \neq \emptyset$; then there exists a lattice vector $(D, w) \in \mu_1^\circ \cap \mu_2^\circ$. For the associated GIT bag, we have $\mu(D, w)^\circ \subsetneq \mu_i^\circ$. This contradicts $\mu_i \in \Lambda_T^0(X)$. □

LEMMA 5.5. *Assume that X is $\mathbb{Q}$-factorial, and consider two orbit cones $\omega := \omega_{H\times T}(x)$ and $\omega_0 := \omega_{H\times T}(x_0)$ of the $(H \times T)$-action on $\bar{X}$ satisfying $\omega_0 \preceq \omega$. Then $\omega_0 \in C_T(X)$ implies $\omega \in C_T(X)$.*

Proof. Recall that $\omega_0 \in C_T(X)$ merely means $\kappa_X^\circ \subset \Pi(\omega_0)^\circ$. The statement thus follows because κ_X is of full dimension. □

LEMMA 5.6. *Assume that X is $\mathbb{Q}$-factorial. Let $\sigma \in \Sigma_{H\times T}(\bar{X})$ and $\omega_0 \in C_T(X)$, where $\sigma \cap \omega_0^\circ \neq \emptyset$. Then there is an $\omega \in C_T(X)$ with $\omega_0 \preceq \omega$ and $\sigma^\circ \subset \omega^\circ$.*

Proof. Let $\sigma_0 \preceq \sigma$ be the face with $\sigma_0^\circ \cap \omega_0^\circ \neq \emptyset$. Then also σ_0 is a GIT cone and thus we have $\sigma_0 \subset \omega_0$.

By Lemmas 2.10 and 3.4, any $x_0 \in \bar{X}$ with $\omega_{H\times T}(x_0) = \omega_0$ belongs to $\bar{X}^{ss}(\sigma_0)$ and has a closed $(H \times T)$-orbit inside this set. So, fix such a point x_0 and consider the commutative diagram

$$\begin{tikzcd}
\bar{X}^{ss}(\sigma) \arrow[r, "\subset"] \arrow[d] & \bar{X}^{ss}(\sigma_0) \arrow[d] \\
\bar{X}^{ss}(\sigma)/\!\!/H \times T \arrow[r] & \bar{X}^{ss}(\sigma_0)/\!\!/H \times T.
\end{tikzcd}$$

Because the induced map of quotients is projective and dominant, it is surjective. Hence there is a point $x \in \bar{X}^{ss}(\sigma)$ lying in the same fibre as x_0, and we may even choose x such that its $(H \times T)$-orbit is closed in $\bar{X}^{ss}(\sigma)$.

Consider $\omega := \omega_{H\times T}(x)$. Lemma 3.4 yields $\sigma^\circ \subset \omega^\circ$. Moreover, by Proposition 1.2(ii), x_0 lies in the closure of the orbit $(H \times T)\cdot x$. Therefore, Corollary 2.4 gives $\omega_0 \preceq \omega$. Lemma 5.5 then implies that $\omega \in C_T(X)$ and so ω is as desired. $\square$

Proof of Theorem 5.1. We first verify the following three claims, and we then put them together to obtain the statements of the theorem.

Claim 1. Let $\mu_1, \mu_2 \in \Lambda_T^0(X)$ such that $\mu_1 \subset \mu_2$ holds. Then $\mu_1 \preceq \mu_2$.

To verify this claim, let $\nu_1 \preceq \mu_2$ denote the face with $\mu_1^\circ \subset \nu_1^\circ$. Since μ_2 is a union of cones of the fan $\Sigma_{H\times T}(\bar{X})$, it follows that the cones $\tau_{1,k} \in \Sigma_{H\times T}(\bar{X})$ with $\tau_{1,k}^\circ \subset \nu_1^\circ$ satisfy

$$\nu_1^\circ = \bigcup_k \tau_{1,k}^\circ.$$

Since also μ_1 is a union of some of the cones $\tau_{1,k}$, we know that at least one of them (say, $\tau_{1,0}^\circ$) satisfies $\tau_{1,0}^\circ \subset \mu_1^\circ$. We must show that all $\tau_{1,k}^\circ$ are contained in μ_1°. So, suppose that one of them (say, $\tau_{1,1}^\circ$) is not. Then there must be an orbit cone $\omega_0 = \omega_{H\times T}(x_0)$ with the following properties:

$$\omega_0 \in C_T(X), \quad \tau_{1,0}^\circ \subset \omega_0^\circ, \quad \tau_{1,1}^\circ \cap \omega_0^\circ = \emptyset.$$

Now let $\sigma_{1,0} \in \Sigma_{H\times T}(\bar{X})$ be a cone such that $\tau_{1,0} \preceq \sigma_{1,0}$ and $\sigma_{1,0}^\circ \subset \mu_2^\circ$. According to Lemma 5.6, there exists an orbit cone $\omega = \omega_{H\times T}(x)$ with the following properties:

$$\omega \in C_T(X), \quad \omega_0 \preceq \omega, \quad \sigma_{1,0}^\circ \subset \omega^\circ.$$

The last inclusion implies that $\mu_2^\circ \cap \omega^\circ \neq \emptyset$. Since μ_2 belongs to $\Lambda_T^0(X)$, we can conclude that $\mu_2^\circ \subset \omega^\circ$; otherwise, $\mu_2 \cap \omega$ would contain an element $\mu \in \Lambda_T(X)$ with $\mu^\circ \subset \mu_2^\circ$, which would be in contradiction to the definition of $\Lambda_T^0(X)$.

We have thus obtained $\mu_2 \subset \omega$. Consequently, also the face $\nu_1 \preceq \mu_2$ is contained in ω. Since $\nu_1^\circ \cap \omega_0^\circ \neq \emptyset$, we can conclude that $\nu_1^\circ \subset \omega_0^\circ$. This implies $\tau_{1,1}^\circ \subset \omega_0^\circ$, a contradiction, so Claim 1 is verified.

Claim 2. Let $\mu_1, \mu_2 \in \Lambda_T^0(X)$ with $\mu_1 \leq \mu_2$, and let $\mu \preceq \mu_2$ with $\mu_1 \preceq \mu$. Then $\mu \in \Lambda_T^0(X)$.

Let us check this claim. Choose any $(D, w) \in \mu^\circ$ and consider the associated GIT bag $\mu(D, w)$. We show that $\mu(D, w) = \mu$ and $\mu(D, w) \in \Lambda_T^0(X)$.

First, consider any $\omega_2 \in C_T(X)$ with $\mu_2^\circ \subset \omega_2^\circ$. Let $\omega_1 \preceq \omega \preceq \omega_2$ denote the faces with $\mu_1^\circ \subset \omega_1^\circ$ and $\mu^\circ \subset \omega^\circ$. Then $\mu_1 \leq \mu_2$ implies $\omega_1 \in C_T(X)$. By Lemma 5.5, this gives $\omega \in C_T(X)$. Thus, since $(D, w) \in \omega^\circ$ holds, we obtain $\mu(D, w) \subset \omega$ and hence $\mu(D, w) \subset \omega_2$.

This consideration shows $\mu(D, w) \subset \mu_2$. Because $(D, w) \in \mu(D, w)^\circ \cap \mu^\circ$ and $\mu \preceq \mu_2$, we even obtain $\mu(D, w) \subset \mu$.

To proceed, note that there exists a GIT bag $\nu \in \Lambda_T^0(X)$ such that $\nu^\circ \subset \mu(D, w)^\circ$. By Claim 1, the inclusion $\nu \subset \mu_2$ implies $\nu \preceq \mu_2$. Thus, $\nu^\circ \subset \mu(D, w)^\circ \subset \mu^\circ$ implies $\nu = \mu$ and so we have obtained $\mu \in \Lambda_T^0(X)$.

Claim 3. For any three $\mu_1, \mu_2, \mu_3 \in \Lambda_T^0(X)$ with $\mu_1 \preceq \mu_2 \preceq \mu_3$ and $\mu_1 \leq \mu_3$, we have $\mu_1 \leq \mu_2 \leq \mu_3$.

In order to verify $\mu_1 \leq \mu_2$, consider $\omega_2 \in C_T(X)$ with $\mu_2^\circ \subset \omega_2^\circ$. By Lemma 5.6, there is an $\omega_3 \in C_T(X)$ with $\omega_2 \preceq \omega_3$ and $\mu_3^\circ \subset \omega_3^\circ$. Since $\mu_1 \leq \mu_3$, the face $\omega_1 \preceq \omega_3$ with $\mu_1^\circ \subset \omega_1^\circ$ belongs to $C_T(X)$. Moreover, $\mu_1 \preceq \mu_2 \subset \omega_2$ and $\mu_1^\circ \subset \omega_1^\circ$ imply that $\omega_1 \preceq \omega_2$. This shows $\mu_1 \leq \mu_2$.

Similarly, to see $\mu_2 \leq \mu_3$, consider $\omega_3 \in C_T(X)$ with $\mu_3^\circ \subset \omega_3^\circ$. For $i = 1, 2$, let $\omega_i \preceq \omega_3$ be the faces with $\mu_i^\circ \subset \omega^\circ$. Note that $\omega_1 \preceq \omega_2$ holds. Now $\mu_1 \leq \mu_3$ implies $\omega_1 \in C_T(X)$, and by Lemma 5.5 this means $\omega_2 \in C_T(X)$. We can therefore conclude that $\mu_2 \leq \mu_3$, proving Claim 3.

We now return to the assertions of Theorem 5.1. For (i), note that $\mu_1 \leq \mu_2$ implies $\mu_1 \subset \mu_2$; thus, Claims 1, 2, and 3 yield the desired statements. In (ii) the case $\mu_1 = \mu_2$ is trivial, so we may assume that $\mu_1 \neq \mu_2$. Then Lemma 5.4 gives $\mu_1^\circ \cap \mu_2^\circ = \emptyset$. Moreover, $\mu_0 \leq \mu_i$ implies $\mu_0 \subset \mu_i$ and thus, by Claim 1, we have $\mu_0 \preceq \mu_i$.

Consider the faces $\nu_1 \preceq \mu_1$ and $\nu_2 \preceq \mu_2$ with $(\mu_1 \cap \mu_2)^\circ \subset \nu_i^\circ$; then $\mu_0 \preceq \nu_1, \nu_2$. As a result, Claim 2 yields $\nu_i \in \Lambda_T^0(X)$. Since $\nu_1^\circ \cap \nu_2^\circ \neq \emptyset$ it follows that Lemma 5.4 yields $\nu_1 = \nu_2$, and this in turn implies $\mu_1 \cap \mu_2 = \nu_1$. Thus, $\mu_1 \cap \mu_2$ is a face of μ_1 and of μ_2, and $\mu_1 \cap \mu_2 \in \Lambda_T^0(X)$. Claim 3 eventually shows that $\mu_0 \leq \mu_1 \cap \mu_2 \leq \mu_i$. □

We conclude this section with a characterization of the geometric GIT quotients in terms of their describing GIT bags in the case of a $\mathbb{Q}$-factorial variety X. We obtain it as a consequence of the following more general statement.

Proposition 5.7. *Let $(D, w) \in C_T^\sharp(X) \cap (K \oplus M)$, and consider the associated GIT bag $\mu(D, w)$. Then the following statements are equivalent.*

(i) *The morphism $X^{ss}(D, w) \to X^{ss}(D, w)/\!\!/T$ is a geometric quotient.*
(ii) *Any $\omega \in C_T(X)$ with $(D, w) \in \omega^\circ$ satisfies* $\dim(\omega) = \dim(\Pi(\omega)) + \dim(M)$.

Proof. The quotient $X^{ss}(D, w) \to X^{ss}(D, w)/\!\!/T$ is geometric if and only if all T-orbits inside $X^{ss}(D, w)$ are of full dimension. The latter holds if and only if, for all points $x \in q_X^{-1}(X^{ss}(D, w))$ with a closed $(H \times T)$-orbit, the quotient of isotropy groups $(H \times T)_x/H_x$ is finite. In terms of orbit cones, this means that

$$\dim(\omega_{H\times T}(x)) = \dim(\omega_H(x)) + \dim(T).$$

According to Theorem 3.5, the points with a closed $(H \times T)$-orbit in $q_X^{-1}(X^{ss}(D, w))$ are precisely those with an orbit cone $\omega \in C_T(X)$ satisfying $(D, w) \in \omega^\circ$. This gives the assertion. □

As an immediate consequence of Proposition 5.3, this characterization of geometric quotients breaks down in the $\mathbb{Q}$-factorial case to the following.

Corollary 5.8. *Let X be $\mathbb{Q}$-factorial and $(D, w) \in C_T^\sharp(X) \cap (K \oplus M)$. Then the quotient $X^{ss}(D, w) \to X^{ss}(D, w)/\!\!/T$ is geometric if and only if the GIT bag $\mu(D, w)$ is of full dimension.*

6. Examples

In this section we present a few examples. We first discuss a quite simple example, a $\mathbb{K}^*$-action on a Hirzebruch surface, showing that the intersection of two GIT bags need not be a GIT bag. Second, we treat an "exotic orbit space" found by Białynicki-Birula and Święcicka [6, Ex. 3]; this is a projective geometric quotient that does not arise from an ample bundle. Finally, we present a noncomplete qp-maximal quotient of a smooth projective variety.

All our examples are subtorus actions on toric varieties X. Because this setup might be of interest for further examples, we briefly explain the general procedure required to obtain the necessary data for studying the GIT equivalence. A toric variety X arises from a fan Δ in the lattice N_X of 1-parameter subgroups of the big torus $T_X \subset X$ (see [12]). As before, we suppose that X is projective and that its divisor class group $\mathrm{Cl}(X)$ is free.

The group $\mathrm{Cl}(X)$ is generated by the classes of the invariant prime divisors $D_1, \dots, D_r$ on X, which in turn correspond to the rays (i.e., the 1-dimensional cones) $\varrho_1, \dots, \varrho_r$ of Δ. By [9], the total coordinate ring $\mathcal{R}(X)$ is a polynomial ring in r indeterminates, and thus we have $\bar{X} = \mathbb{K}^r$ for the corresponding spectrum.

In terms of fans, the subset $\hat{X} \subset \bar{X}$ is obtained as follows: Let $v_1, \dots, v_r \in N_X$ denote the primitive lattice vectors generating the rays of Δ, set $F := \mathbb{Z}^r$, and consider the linear map $P\colon F \to N_X$ that sends the ith canonical base vector $e_i \in F$ to $v_i \in N_X$. The fan of $\hat{X}$ then consists of faces of the positive orthant $\delta \subset F_{\mathbb{Q}}$:

$$\Delta_{\hat{X}} = \{\hat{\sigma} \preceq \delta;\ P(\hat{\sigma}) \subset \sigma \text{ for some } \sigma \in \Delta_X\}.$$

Moreover, the torus $H = \mathrm{Spec}(\mathbb{K}[\mathrm{Cl}(X)])$ acting on $\bar{X}$ is the subtorus of $(\mathbb{K}^*)^r$ having $L := \ker(P)$ as its lattice of 1-parameter subgroups. The canonical map $q_X\colon \hat{X} \to X$ is the toric morphism corresponding to the map $P\colon F \to N_X$ of the fans $\Delta_{\hat{X}}$ and Δ_X. Observe that the map $P\colon F \to N_X$ determines a pair of exact sequences, which are mutually dual to each other:

$$0 \longrightarrow L \longrightarrow F \xrightarrow{P} N_X \longrightarrow 0,$$

$$0 \longleftarrow K \xleftarrow[Q]{} E \longleftarrow M_X \longleftarrow 0.$$

Note that the subtorus $H \subset (\mathbb{K}^*)^r$ is as well determined by its *weight map* $Q\colon E \to K$. The weight cone of the H-action is given by $\Omega_H(\bar{X}) = Q(\gamma)$, where $\gamma \subset E_{\mathbb{Q}}$ is the positive orthant. The fan $\Sigma_H(\bar{X})$ is the Gelfan–Kapranov–Zelevinsky decomposition of the cone $Q(\gamma)$, which means that it is the coarsest common refinement of all the images $Q(\gamma_0)$ for $\gamma_0 \preceq \gamma$ (cf. [18]).

The GIT cone $\kappa_X \in \Sigma_H(\bar{X})$ corresponding to $\hat{X} \subset \bar{X}$ can be calculated as follows (see [2, Thm. 10.2]). Consider the maximal cones $\delta_0 \preceq \delta$ of the fan $\Delta_{\hat{X}}$, and determine the corresponding faces $\gamma_0 = \delta_0^{\perp} \cap \gamma$ of γ. Then κ_X is the intersection over all the images $Q(\gamma_0)$. Recall from Corollary 3.6 that the relative interior of κ_X is the cone of ample divisors of X.

Now suppose that $T \subset T_X$ is a subtorus of the big torus of X. Then $T \subset T_X$ corresponds to a sublattice $N_T \subset N_X$. The lifting of the T-action to the affine multicone $\bar{X}$ corresponds to an embedding $N_X \to F$ with $N_X \cap L = 0$. By fixing a lifting of the T-action, we have thus decorated the exact sequence consisting of the map $P\colon F \to N_X$ in the following sense:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] & F \arrow[r, "P"] & N_X \arrow[r] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[u, equal] & L \times N_T \arrow[r, "P"'] \arrow[u] & N_T \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

In order to determine the $(H \times T)$-orbit cones and the fan $\Sigma_{H\times T}(\bar{X})$, we must dualize this commutative diagram. The result is:

$$\begin{tikzcd}
0 & K \arrow[l] \arrow[d, equal] & E \arrow[l, "Q"'] \arrow[d, "\hat{Q}"] & M_X \arrow[l] \arrow[d] & 0 \arrow[l] \\
0 & K \arrow[l] & K \oplus M_T \arrow[l] & M_T \arrow[l] & 0. \arrow[l]
\end{tikzcd}$$

Then the orbit cones of the $(H \times T)$-action on $\bar{X}$ are precisely the images $\hat{Q}(\gamma_0)$, where $\gamma_0 \preceq \gamma \subset E_{\mathbb{Q}}$ being the positive orthant.

Moreover, the fan $\Sigma_{H\times T}(\bar{X})$ is the coarsest common refinement of all the orbit cones $\hat{Q}(\gamma_0)$. Finally, the collections $C_T(X)$ and $C_T(\sigma)$ for $\sigma \in \Sigma_{H\times T}(\bar{X})$ can now be directly computed according to their definitions, and thus it becomes possible to determine the collection of GIT bags.

For the computation steps just outlined, it is most convenient to use suitable computer programs. We provide a (free) Maple package, `TorDiv`, that performs all the needed computations (see [4]). Therefore, in the following examples, we omit the computations and show only their results.

EXAMPLE 6.1 (A $\mathbb{K}^*$-action on a Hirzebruch surface). As a toric variety, the first Hirzebruch surface X arises from the complete fan Δ_X in $\mathbb{Z}^2$ with the four rays

$$\varrho_1 := \mathbb{Q}_{\geq 0}[1,0], \quad \varrho_2 := \mathbb{Q}_{\geq 0}[0,1], \quad \varrho_3 := \mathbb{Q}_{\geq 0}[-1,1], \quad \varrho_4 := \mathbb{Q}_{\geq 0}[0,-1].$$

We have $\bar{X} = \mathbb{K}^4$, and the action of the torus $H = \operatorname{Spec}(\mathbb{K}[K])$ is given by the weight matrix

$$Q = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Let $T := \mathbb{K}^*$ act on $\bar{X}$ with weights $1, 0, -1$, and 0. Then the weight matrix of the $(H \times T)$-action on $\bar{X}$ is given by

$$\hat{Q} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}.$$

The fans describing the corresponding GIT systems are easy to determine. Let $w_i \in \mathbb{Z}^2$ denote the ith column of Q. Then the maximal cones of $\Sigma_H(\bar{X})$ are

$$\kappa_1 := \operatorname{cone}(w_1, w_4), \qquad \kappa_2 := \operatorname{cone}(w_2, w_4).$$

Note that the first cone equals κ_X. Similarly, denoting by $\hat{w}_i$ the ith column of $\hat{Q}$, the maximal cones of $\Sigma_{H\times T}(\bar{X})$ are

$$\sigma_1 := \operatorname{cone}(\hat{w}_2, \hat{w}_3, \hat{w}_4), \quad \sigma_2 := \operatorname{cone}(\hat{w}_1, \hat{w}_2, \hat{w}_4), \quad \sigma_3 := \operatorname{cone}(\hat{w}_1, \hat{w}_3, \hat{w}_4).$$

All three of these cones are GIT bags, and they even belong to $\Lambda_T^0(X)$. Observe that σ_1 and σ_2 have a 2-dimensional face in common, but there is no element in $\Lambda_T^0(X)$ that is smaller than σ_1 and σ_2.

EXAMPLE 6.2 (Białynicki-Birula and Święcicka). Consider the smooth projective variety X obtained from $\mathbb{P}_2 \times \mathbb{P}_1$ by blowing up first the line $[z, 0, w] \times [0, 1]$ and then the proper transform of the line $[z, w, 0] \times [0, 1]$. These blowups are compatible with the action of $T := \mathbb{K}^* \times \mathbb{K}^*$ on $\mathbb{P}_2 \times \mathbb{P}_1$ given by

$$(t_1, t_2) \cdot ([x_0, x_1, x_2], [y_0, y_1]) := ([x_0, t_1x_1, t_1x_2], [y_0, t_2y_1]).$$

Hence there is an induced T-action on X, which makes the contraction map equivariant. We show that there exist precisely four different open sets admitting a geometric quotient with a projective orbit space, but only three of them are sets of semistable points of ample line bundles.

Let us verify these statements. As announced, we view X as a toric variety. It arises from the fan Δ_X in $N_X := \mathbb{Z}^3$ with the seven rays

$$v_1 := (1,0,0), \quad v_2 := (0,1,0), \quad v_3 := (-1,-1,0), \quad v_4 := (0,0,1),$$
$$v_5 := (0,0,-1), \quad v_6 := (1,0,1), \quad \text{and} \quad v_7 := (0,1,1)$$

and with ten maximal cones (we denote by C_{ijk} the cone generated by v_i, v_j, v_k):

$$C_{235},\ C_{347},\ C_{237},\ C_{135},\ C_{346},\ C_{136},\ C_{125},\ C_{467},\ C_{267},\ C_{126}.$$

The spectrum of the total coordinate ring $\mathcal{R}(X)$ is $\bar{X} = \mathbb{K}^7$, and the weight map for the action of $H = \operatorname{Spec}(\mathbb{K}[K])$ on $\bar{X}$ is given (w.r.t. the canonical bases) by

$$Q = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

By our preceding remarks, the closure $\kappa_X \subset K_{\mathbb{Q}} = \operatorname{Cl}(X)$ of the ample cone is given by

$$\kappa_X = \operatorname{cone}((1,0,0,0), (2,1,0,0), (0,1,1,1), (1,1,0,1)).$$

The sublattice $N_T \subset N_X$ describing the T-action on X is then generated by the vectors $(1,1,0)$ and $(0,0,1)$. Hence, we may work with

$$\hat{Q} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Now, one has to compute the fan $\Sigma_{H\times T}(\bar{X})$ associated to the $(H \times T)$-action on $\bar{X}$. This fan has the vectors

$$w_1 := (1,0,0,0,0,0), \quad w_2 := (0,1,0,0,0,1), \quad w_3 := (0,1,1,1,0,0,0),$$
$$w_4 := (1,0,-1,0,1,0), \quad w_5 := (1,0,0,-1,1,0), \quad w_6 := (1,0,0,0,1,0),$$
$$w_7 := (0,0,1,0,0,0), \quad \text{and} \quad w_8 := (0,0,0,1,0,0)$$

as the primitive generators of its rays, and it has exactly four full-dimensional cones:

$$\operatorname{cone}(w_1, w_2, w_3, w_4, w_5, w_6), \quad \operatorname{cone}(w_1, w_2, w_3, w_7, w_5, w_6),$$
$$\operatorname{cone}(w_1, w_2, w_3, w_4, w_8, w_6), \quad \operatorname{cone}(w_1, w_2, w_3, w_7, w_8, w_6).$$

It turns out that these cones are precisely the GIT bags of full dimension. Proposition 5.7 thus tells us that there are exactly four different geometric quotients arising from linearized bundles. Moreover, by Proposition 4.6, the associated quotient spaces are projective. Finally, Proposition 4.7 yields that the second of the listed GIT bags describes a quotient that does not arise from an ample line bundle.

Example 6.3. We present a smooth projective variety X of dimension 3 and a $\mathbb{K}^*$-action on X that has qp-maximal sets with noncomplete quotient spaces.

Our X is the toric variety arising from the fan Δ_X in $N_X := \mathbb{Z}^3$, which has the vectors

$$v_1 := (1,0,0), \quad v_2 := (0,1,0), \quad v_3 := (-1,0,1),$$
$$v_4 := (0,-1,1), \quad \text{and} \quad v_5 := (0,0,-1)$$

as the primitive generators of its rays as well as the following list of maximal cones:

$$C_1 := \operatorname{cone}(v_1, v_2, v_3), \quad C_2 := \operatorname{cone}(v_1, v_3, v_4), \quad C_3 := \operatorname{cone}(v_1, v_2, v_5),$$
$$C_4 := \operatorname{cone}(v_2, v_3, v_5), \quad C_5 := \operatorname{cone}(v_3, v_4, v_5), \quad C_6 := \operatorname{cone}(v_1, v_4, v_5).$$

We have $\bar{X} = \mathbb{K}^5$, and the action of the torus $H = \operatorname{Spec}(\mathbb{K}[K])$ on $\bar{X}$ is given by the weight map

$$Q := \begin{bmatrix} -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Moreover, the closure $\kappa_X \subset K_{\mathbb{Q}} = \operatorname{Cl}_{\mathbb{Q}}(X)$ of the ample cone of X is generated by the vectors $(1,1)$ and $(0,1)$.

Now consider $\mathbb{K}^*$-action on X corresponding to the sublattice N_T of N_X generated by $(2,-4,1)$. A lifting of this action to $\bar{X}$ is given by the weight matrix

$$\hat{Q} := \begin{bmatrix} -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 3 & -4 & 1 & 0 & 0 \end{bmatrix}.$$

The fan $\Sigma_{H\times T}(\bar{X})$ lives in the 3-dimensional lattice $K \oplus M_T$, and the cone σ generated by $(-1,1,2)$ is an element of $\Sigma_{H\times T}(\bar{X})$. It turns out that σ is a GIT bag. Hence, by Corollary 4.5, the associated set of semistable points $X(\mu)$ is qp-maximal.

However, the criterion for projectivity given in Proposition 4.6 is not fulfilled: the orbit cone

$$\omega := \operatorname{cone}((-1,0,3),(-1,0,1),(0,1,0))$$

contains σ° in its relative interior, but the image of ω° in K does not intersect κ_X°. Therefore, $X(\mu)/\!\!/T$ is not projective.

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