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An Algebraic Version of Subelliptic Multipliers

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Introduction

Subelliptic multipliers first appeared in 1979 as a part of the regularity theory for the $\bar{\partial}$-equation [K1]. Nadel's multiplier ideal sheaf, a counterpart of subelliptic multipliers in the $\bar{\partial}$-regularity theory, appeared in [N] and has had many applications in algebraic geometry [AS; De1; De2; De3; E; EL; ELSm; L; S1; S2; S3]. The purpose of this paper is to introduce an algebraic version of one part of the work in [K1].

Subelliptic multipliers were invented by Kohn [K1] as a technique for proving subelliptic estimates for the $\bar{\partial}$-Neumann problem. See [D1] and [DK] for additional information on that subject, and see [N; E; S2] for other developments in multiplier ideal theory.

The analysis of $\bar{\partial}$ on weakly pseudoconvex domains requires dealing with singularities arising from the zeroes of the determinant of the Levi form. The determinant of the Levi form is always a subelliptic multiplier; from it, and via the process to be described here, one constructs additional subelliptic multipliers hoping eventually to obtain a nonvanishing function. The procedure, when applied in a simpler situation involving germs of holomorphic functions, leads to an interesting algorithm in commutative algebra. This algorithm provides in particular an unusual procedure for determining whether an ideal in the convergent power series ring is primary to the maximal ideal. Of course there are many algorithms for doing so, but these algorithms do not work in proving subelliptic estimates. Kohn's algorithm, although algebraic in nature, is dictated by analysis. Our generalized algorithm applies in more general commutative rings, and we establish several new results.

Kohn has expressed hope in [K2] that more precise information about subelliptic multipliers will be useful in the theory of Hölder estimates for $\bar{\partial}$. The preprint [D2] deals with issues concerning the lack of "effectiveness" in the algorithm, and perhaps the techniques in our paper will provide insight into that problem as well.

We continue the introduction by describing our results in a special case. Let O_n be the ring of germs of holomorphic functions at the origin in $\mathbb{C}^n$ and let $\mathfrak{m}$ be the maximal ideal. Given a collection of elements $f_1, \dots, f_l$, we construct an ideal (denoted by I_0) that is generated by

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$$\det\begin{pmatrix} \frac{\partial f_{j_1}}{\partial z_1} & \cdots & \frac{\partial f_{j_1}}{\partial z_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_{j_n}}{\partial z_1} & \cdots & \frac{\partial f_{j_n}}{\partial z_n} \end{pmatrix}, \tag{1}$$

where $1 \le j_1 < \cdots < j_n \le l$. Let J_0 denote the radical of I_0. The ideal J_k is obtained inductively. Let I_{k+1} be the ideal generated by

$$\det\begin{pmatrix} \frac{\partial h_1}{\partial z_1} & \cdots & \frac{\partial h_1}{\partial z_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial h_n}{\partial z_1} & \cdots & \frac{\partial h_n}{\partial z_n} \end{pmatrix} \tag{2}$$

with $h_1, \dots, h_n \in \{f_1, \dots, f_N\} \cup J_k$, and let J_{k+1} be the radical of I_{k+1}. Then we obtain an increasing sequence of ideals

$$J_0 \subseteq J_1 \subseteq \cdots \subseteq J_k \subseteq \cdots .$$

Because O_n is Noetherian, the ascending sequence stabilizes: there exists a k such that $J_k = J_{k+1} = \cdots$. The stabilized ideal J_k is called the *subelliptic multiplier ideal* for the given elements $f_1, \dots, f_l$.

In this process we take differentials of the given functions $f_1, \dots, f_l$. We do not take differentials of all the elements in the ideal generated by $f_1, \dots, f_l$; if we did, the process would be simple and uninteresting. Generally, even though two different collections of functions generate the same ideal, the process would give us different subelliptic multipliers (see Example 9). Although the process seems a bit unusual from the point of view of algebra, it is natural from the point of view of subelliptic estimates.

Kohn proved that $J_k = O_n$ for some integer k if and only if the ideal generated by $f_1, \dots, f_l$ is $\mathfrak{m}$-primary [K1, Thm. 7.13]. This situation exhibits only a small part of the general work on subelliptic estimates in [K1].

In this paper we analyze this process and reformulate it by means of the theory of differentials on a local ring. With this reformulation we understand these nonlinear operations more clearly and extend the results to more general local rings. When analyzing the process, we focus on the matrix appearing in (1) rather than $f_1, \dots, f_l \in O_n$. Each row of the matrix can be interpreted as the differential df_{j_s}, and the determinants are the coefficients of the wedge products $df_{j_1} \wedge \cdots \wedge df_{j_n}$. Therefore, we start with the O_n-module $\langle df_1, \dots, df_l \rangle$. We generalize the process (even in O_n) by allowing an arbitrary starting module of 1-forms, not by assuming that the module is generated by exact forms.

Our theory is purely algebraic. The main ideas are based on the Weierstrass preparatation theorem and the normalization theorem for the ring of formal (or convergent) power series; hence the coefficient field K can be more general than $\mathbb{C}$. Let K be an arbitrary field of characteristic 0. We study the module generated by 1-forms on the ring of formal power series with coefficients in K (or, equivalently, the ring of convergent power series with coefficients in K when K has a multiplicative valuation). We write R_n for these rings, $\mathfrak{m}$ for the maximal ideal in R_n, and $\Omega^1_{R_n}$ for the module of 1-forms on R_n.

In Definitions 2 and 3 we construct two nonlinear operations, Θ and Δ, between the set of radical ideals in R_n and the set of submodules in $\Omega^1_{R_n}$. Applying Θ and Δ alternately to an initial submodule M in $\Omega^1_{R_n}$, we obtain an increasing sequence of submodules in $\Omega^1_{R_n}$. By the Noetherian property, the sequence stabilizes after finitely many steps. The final module will be called the *subelliptic multiplier module for M* and is denoted by $\mathcal{D}(M)$. We say that M is *subelliptic* if $\mathcal{D}(M)$ equals the whole module $\Omega^1_{R_n}$ (see Section 2).

There are four main theorems in this paper. In Theorem 1 we show that for any starting module the process always stabilizes after at most n steps; Example 7 shows that the number n is sharp. In Theorem 2 we find a necessary condition for M to be subelliptic.

Theorem 3 shows that, if M satisfies property (L) (see Definition 7), then M is subelliptic. In Theorem 4 we prove that if $M = \langle df_1, \dots, df_l \rangle$ and the ideal $(f_1, \dots, f_l)$ is $\mathfrak{m}$-primary, then M is subelliptic. The analogue for q-type modules is summarized in Section 7. The complete proofs for q-type modules will appear in the author's Ph.D. thesis [C].

This paper is one of several parts of the author's thesis. I wish to thank my advisor, John D'Angelo, who encouraged me to investigate Kohn's algorithm in terms of module theory. I am grateful to Philip Griffith and Sean Sather-Wagstaff for valuable discussions about local ring theory, and I acknowledge several useful comments made by the referee.

Notation

K: a field of characteristic 0

$K[[x_1, \dots, x_n]]$: the ring of formal power series with coefficients in K

$K\{x_1, \dots, x_n\}$: the ring of convergent power series with coefficients in K with respect to a valuation v

R_n: $K[[x_1, \dots, x_n]]$ or $K\{x_1, \dots, x_n\}$

$\mathfrak{m}$: the maximal ideal of R_n

$\operatorname{rad}(I)$: the radical of an ideal I in R_n

$\operatorname{ht}\mathfrak{p}$: the height of a prime ideal $\mathfrak{p}$ of R_n

$\operatorname{ht} I$: the infimum of $\operatorname{ht}\mathfrak{p}$ with $\mathfrak{p} \supseteq I$ prime in R_n for an ideal $I \subset R_n$

$\operatorname{rank}_A(M)$: the A-rank of a module M finitely generated over an integral domain A

1. Background on 1-Forms

In this section we review 1-forms on R_n, which are useful in the study of subelliptic multipliers. For more information see [Ku, Chaps. 11–14].

DEFINITION 1. Let A be a K-algebra. The pair of a finitely generated A-module $\Omega^1_{A/K}$ and a K-derivation $d_{A/K}\colon A \to \Omega^1_{A/K}$ is said to satisfy the *universally finite condition* if, for any finitely generated module M over A and a K-derivation $d'\colon A \to M$, there exists a unique A-homomorphism $l\colon \Omega^1_{A/K} \to M$ such that $d' = l \circ d_{A/K}$ (see diagram).

$$\begin{tikzcd} A \arrow[r, "d_{A/K}"] \arrow[dr, "d'"'] & \Omega^1_{A/K} \arrow[d, "\exists!\, l"] \\ & M \end{tikzcd}$$

When there is no confusion, we simply write (Ω^1_A, d) for $(\Omega^1_{A/K}, d_{A/K})$.

REMARK. The universally finite module does not always exist: the field of quotients of $\mathcal{O}_n$ gives an example. See [Ku, p. 172].

Let $\Omega^1_{R_n} = R_n\, dx_1 \oplus \cdots \oplus R_n\, dx_n$ be the natural free R_n-module. We define a derivation $d\colon R_n \to \Omega^1_{R_n}$ by $df = \frac{\partial f}{\partial x_1}\, dx_1 + \cdots + \frac{\partial f}{\partial x_n}\, dx_n$. The pair $(\Omega^1_{R_n}, d)$ satisfies the universally finite condition but is not the universal module of Kähler differentials of R_n. Elements of $\Omega^1_{R_n}$ are called *1-forms* on R_n.

Let I be an ideal in R_n and let $A = R_n/I$. The universally finite module of K-differentials of A exists. We can construct (Ω^1_A, d_A) by the exact sequence of A-modules (called the *second fundamental exact sequence for I*)

$$I/I^2 \xrightarrow{\delta} \Omega^1_{R_n} \otimes_{R_n} A \xrightarrow{\alpha} \Omega^1_A \to 0, \tag{3}$$

where $\delta(f \bmod I^2) = df \otimes 1_A$ and $\pi: R_n \to A$ is the natural homomorphism. There exists a natural K-derivation

$$d_A\colon A \to \Omega^1_A \quad \text{by} \quad d_A(g) = \alpha(df \otimes 1_A), \tag{4}$$

where $g = \pi(f)$ for $f \in R_n$. Elements of Ω^1_A are called *1-forms on A*. If there is no risk of confusion then we will use d for d_A.

Even when Ω^1_A may have a torsion element, the A-rank of Ω^1_A is determined by the dimension of A in some cases. The following lemma is a special case of [Ku, Thm. (14.13)] combined with the fact that an analytic domain over a field of characteristic 0 is absolutely regular at its 0-ideal.

LEMMA 1. *Let $\mathfrak{p}$ be a prime ideal of R_n with* $\operatorname{ht}\mathfrak{p} = n - d$ *and $A = R_n/\mathfrak{p}$. Then*

$$\operatorname{rank}_A(\Omega^1_A) = \dim A = d.$$

2. Subelliptic Multiplier Modules

In this section we introduce the subelliptic multiplier module for a submodule in $\Omega^1_{R_n}$. Toward this end we first define two important nonlinear operations.

Let $*\colon \Omega^n_{R_n} \to R_n$ be the unique R_n-linear map for which $*(dx_1 \wedge \cdots \wedge dx_n) = 1$. With this map we identify $\Omega^n_{R_n}$ with R_n. Whereas the identification depends on the choice of a coordinate system, the ideal generated by $*(\omega_1), \dots, *(\omega_s)$ does not depend on the choice of a coordinate system.

DEFINITION 2. For a submodule M of $\Omega^1_{R_n}$, we let $\theta(M)$ be the ideal in R_n generated by $*(\phi_1 \wedge \cdots \wedge \phi_n)$ for all n collections $\phi_1, \dots, \phi_n \in M$. Define

$$\Theta(M) = \operatorname{rad}(\theta(M)). \tag{5}$$

EXAMPLE 1. Given a holomorphic mapping $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^l, 0)$, we have a submodule $M = \langle df_1, \dots, df_l\rangle$ of $\Omega^1_{O_n}$, where $f_1, \dots, f_l \in O_n$ are components of f. Then $\Theta(M)$ is the radical ideal corresponding to the germ of an analytic variety at the origin, where the Jacobian matrix of f is of rank $< n$.

DEFINITION 3. Let I be an ideal in R_n. We define

$$\Delta(I) = \langle df \mid f \in I\rangle. \tag{6}$$

REMARK. Suppose that $f_1, \dots, f_l$ generate I in R_n. It follows from the product rule that $\Delta(I) = \langle df_1, \dots, df_l\rangle + I\langle dx_1, \dots, dx_n\rangle$.

EXAMPLE 2. Let $\mathfrak{p}$ be a prime ideal in R_n with $\operatorname{ht}\mathfrak{p} \leq n-1$. It follows from Lemma 1 that $\Delta(\mathfrak{p}) \subsetneq \Omega^1_{R_n}$. Indeed, let $A = R_n/\mathfrak{p}$ and consider the exact sequence of A-modules

$$\mathfrak{p}/\mathfrak{p}^2 \xrightarrow{\delta} \Omega^1_{R_n} \otimes_{R_n} A \xrightarrow{\alpha} \Omega^1_A \to 0.$$

Then $\langle \phi \otimes 1_A \mid \phi \in \Delta(\mathfrak{p})\rangle$ equals $\ker\alpha$. Since $\operatorname{rank}_A(\Omega^1_{R_n} \otimes_{R_n} A) = n$ and since $\operatorname{rank}_A(\Omega^1_A) \leq n-1$ by Lemma 1, it follows that $\ker\alpha$ does not equal $\Omega^1_{R_n} \otimes_{R_n} A$. Therefore, $\Delta(\mathfrak{p}) \neq \Omega^1_{R_n}$.

We have now defined two operations between the set of radical ideals in R_n and the set of submodules in $\Omega^1_{R_n}$:

$$\begin{aligned} \Theta\colon \{\text{submodules in } \Omega^1_{R_n}\} &\longrightarrow \{\text{radical ideals in } R_n\},\\ M &\longmapsto \Theta(M);\\ \Delta\colon \{\text{radical ideals in } R_n\} &\longrightarrow \{\text{submodules in } \Omega^1_{R_n}\},\\ I &\longmapsto \Delta(I). \end{aligned}$$

Given an initial submodule M in $\Omega^1_{R_n}$, we combine and iterate these operations to obtain a sequence of submodules $D^l(M)$ in $\Omega^1_{R_n}$.

DEFINITION 4. Let M be a submodule in $\Omega^1_{R_n}$ and let $D^0(M) = M$. We define

$$D^1(M) = M + \Delta(\Theta(M)). \tag{7}$$

For $l \geq 0$ the module $D^{l+1}(M)$ is defined inductively:

$$D^{l+1}(M) = D^l(M) + \Delta(\Theta(D^l(M))). \tag{8}$$

Since $D^l(M) \subseteq D^{l+1}(M)$, it follows that

$$\mathcal{D}(M) = \bigcup D^l(M) \tag{9}$$

is a submodule in $\Omega^1_{R_n}$ that we shall call the *subelliptic multiplier module for M*. In the same way we define the *subelliptic multiplier ideal for M* by

$$\mathcal{I}(M) = \bigcup \Theta(D^l(M)). \tag{10}$$

DEFINITION 5. The submodule M is *subelliptic* if $\mathcal{D}(M) = \Omega^1_{R_n}$.

We obtain an ascending chain of submodules in $\Omega^1_{R_n}$,

$$M = D^0(M) \subseteq D^1(M) \subseteq \cdots \subseteq D^l(M) \subseteq \cdots .$$

We also have an ascending chain of ideals in R_n:

$$\Theta(D^0(M)) \subseteq \Theta(D^1(M)) \subseteq \cdots \subseteq \Theta(D^l(M)) \subseteq \cdots .$$

By the Noetherian property, there exists an integer l such that

$$\mathcal{D}(M) = D^l(M), \qquad \mathcal{I}(M) = \Theta(D^l(M)).$$

DEFINITION 6. Define *the stabilization number of M* by

$$\#(M) = \min\{l \mid \mathcal{D}(M) = D^l(M)\}. \tag{11}$$

We now compute $\#(M)$ in some simple cases.

EXAMPLE 3. Let M be a submodule in $\Omega^1_{R_n}$ of rank $\leq n-1$. Then $\Theta(M) = 0$ and so $D^1(M) = M = \mathcal{D}(M)$. Therefore, M is not subelliptic and $\#(M) = 0$.

EXAMPLE 4. Suppose that $\Theta(D^l(M)) = \mathfrak{m}$. Then $D^{l+1}(M) = \Omega^1_{R_n} = \mathcal{D}(M)$; hence $\#(M) = l+1$.

EXAMPLE 5. For $n = 1$ we have $\Omega^1_{R_1} = R_1\, dx_1$. Then any submodule M in Ω^1_A is identified with an ideal I in R_1 via the map $*$. As a result, $\Theta(M) = \mathrm{rad}(I)$. If I is (0) or R_1, then $\#(M) = 0$. If I is neither (0) nor R_1, then $\Theta(M) = \mathrm{rad}(I) = \mathfrak{m}$ and therefore $\#(M) = 1$ and $\mathcal{D}(M) = \Omega^1_{R_n}$.

EXAMPLE 6. Let M be a submodule of $\Omega^1_{R_2}$ generated by the differentials of x^2 and xy^2. Then $\#(M) = 2$ and $\mathcal{I}(M) = (x)$.

EXAMPLE 7. Let M be a submodule of $\Omega^1_{R_n}$ generated by $\omega_i = x_i^{m_i}\, dx_i$ $(1 \leq i \leq n)$, where $m_i \geq 1$. Then $\#(M) = n$. Indeed, we have

$$\begin{aligned}
\Theta(M) &= \Big(\prod_j x_j\Big),\\
\Theta(D^1(M)) &= \Big(\prod_{j \neq i} x_j \;\Big|\; i = 1, \dots, n\Big),\\
\Theta(D^2(M)) &= \Big(\prod_{k \neq i, j} x_k \;\Big|\; 1 \leq i < j \leq n\Big),\\
&\;\;\vdots\\
\Theta(D^n(M)) &= (1).
\end{aligned}$$

3. Stabilization Numbers

We prove Theorem 1 in this section.

LEMMA 2. *Let M be a submodule in $\Omega^1_{R_n}$. Suppose that $\mathfrak{p}$ appears in the primary decompositions of both $\Theta(M)$ and $\Theta(D^1(M))$. Then $\mathfrak{p}$ appears in the primary decomposition of $\Theta(D^2(M))$.*

Proof. Let $g_1, \dots, g_l$ generate $\mathfrak{p}$. It follows that

$$\Delta(\mathfrak{p}) = \langle dg_1, \dots, dg_l \rangle + \mathfrak{p}\langle dx_1, \dots, dx_n \rangle. \tag{12}$$

By (12) we have

$$D^1(M) = M + \Delta(\Theta(M)) \subseteq M + \langle dg_1, \dots, dg_l \rangle + \mathfrak{p}\langle dx_1, \dots, dx_n \rangle. \tag{13}$$

Since $\Theta(D^1(M)) \subseteq \mathfrak{p}$, it follows from (12) that

$$\Delta(\Theta(D^1 M)) \subseteq \langle dg_1, \dots, dg_l \rangle \mod \mathfrak{p}\langle dx_1, \dots, dx_n \rangle. \tag{14}$$

Hence, using (13) and (14) we obtain

$$\begin{aligned} D^2(M) &= D^1(M) + \Delta(\Theta(D^1(M))) \\ &\subseteq M + \langle dg_1, \dots, dg_l \rangle \mod \mathfrak{p}\langle dx_1, \dots, dx_n \rangle. \end{aligned} \tag{15}$$

Let $\Theta(M) = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_s$ be the primary decomposition of $\Theta(M)$, where $\mathfrak{p} = \mathfrak{p}_1$. We claim that there exists an $h \in R_n - \mathfrak{p}$ such that $hg_i \in \Theta(M)$ for all $i = 1, \dots, l$. Indeed, since $\mathfrak{p}_j - \mathfrak{p} \neq (0)$ for all $j = 2, \dots, s$, it follows that there exist $h_j \in \mathfrak{p}_j - \mathfrak{p}$ for all $j = 2, \dots, s$. Let $h = h_2 \cdots h_s$; then we have $h \notin \mathfrak{p}$ such that $hg_i \in \Theta(M)$ for all $i = 1, \dots, l$.

Since $d(hg_i) = h\,dg_i + g_i\,dh$ and $g_i \in \mathfrak{p}$,

$$h\,dg_i \in \Delta(\Theta(M)) \mod \mathfrak{p}\langle dx_1, \dots, dx_n \rangle \quad \text{for all } i = 1, \dots, l. \tag{16}$$

We note that $D^1(M) = M + \Delta(\Theta(M))$ and $\Theta(D^1(M)) \subset \mathfrak{p}$. Hence, it follows from (16) that, for any $k = 0, \dots, n$ and for any $\phi_1, \dots, \phi_k \in M$,

$$*(\phi_1 \wedge \dots \wedge \phi_k \wedge h\,dg_{j_1} \wedge \dots \wedge h\,dg_{j_{n-k}}) \in \mathfrak{p},$$

where the $j_1, \dots, j_{n-k}$ run over $\{1, \dots, l\}$. Since $h \notin \mathfrak{p}$, for any $k = 0, \dots, n$ and any $\phi_1, \dots, \phi_k \in M$ we have

$$*(\phi_1 \wedge \dots \wedge \phi_k \wedge dg_{j_1} \wedge \dots \wedge dg_{j_{n-k}}) \in \mathfrak{p}, \tag{17}$$

where once again the $j_1, \dots, j_{n-k}$ run over $\{1, \dots, l\}$. Therefore, (15) implies that $\Theta(D^2(M)) \subseteq \mathfrak{p}$.

Since $\Theta(M) \subseteq \Theta(D^2(M)) \subseteq \mathfrak{p}$ and since $\mathfrak{p}$ appears in the primary decomposition of $\Theta(M)$, it follows that $\mathfrak{p}$ also appears in the primary decomposition of $\Theta(D^2(M))$. □

REMARK. If $\mathfrak{p}$ satisfies the condition in Lemma 2, then $\mathfrak{p}$ appears eventually in the primary decomposition of $\mathcal{I}(M)$.

THEOREM 1. *Let M be a submodule in $\Omega^1_{R_n}$. Then*

$$\#(M) \le n. \tag{18}$$

Furthermore, the number n is sharp.

Proof. If $\Theta(M) = 0$ then $\#(M) = 0$, so we may assume that $\Theta(M) \neq 0$. Let $\Theta(M) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s$ be the primary decomposition of $\Theta(M)$; then $\operatorname{ht} \mathfrak{p}_i \ge 1$ for any $i = 1, \dots, s$.

Let $\Theta(D^1(M)) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$ be the primary decomposition of $\Theta(D^1(M))$. Since $\Theta(M) \subseteq \Theta(D^1(M))$, for each $\mathfrak{q}_j$ there are two cases:

(i) there exists a $\mathfrak{p}_i$ such that $\mathfrak{q}_j = \mathfrak{p}_i$;
(ii) there exists a $\mathfrak{p}_i$ such that $\mathfrak{q}_j \supsetneq \mathfrak{p}_i$.

In case (i), by Lemma 2 the prime ideal $\mathfrak{p}_i$ appears in the primary decomposition of $\Theta(D^k(M))$ for any $k \ge l$. Therefore, we need not consider such prime ideals.

In case (ii) we have $\operatorname{ht} \mathfrak{q}_j \gneq \operatorname{ht} \mathfrak{p}_i$ for some i; hence, $\operatorname{ht} \mathfrak{q}_j \ge 2$. Therefore, it follows that the process finishes after at most n steps.

Examples 6 and 7 show that the number n is sharp. □

4. A Necessary Condition for Subellipticity

In this section we prove Theorem 2.

THEOREM 2. *Let M be a submodule in $\Omega^1_{R_n}$, and let $\mathfrak{p}$ be a prime ideal in R_n with* $\operatorname{ht} \mathfrak{p} \le n-1$. *Suppose $M \subseteq \Delta(\mathfrak{p})$. Then*

$$\mathcal{D}(M) \subseteq \Delta(\mathfrak{p}) \subsetneq \Omega^1_{R_n}. \tag{19}$$

Therefore, M is not subelliptic.

Proof. It suffices to prove that if $M \subseteq \Delta(\mathfrak{p})$ then $\Theta(M) \subseteq \mathfrak{p}$. Indeed, if the statement is true then

$$D^1(M) = M + \Delta(\Theta(M)) \subseteq \Delta(\mathfrak{p}).$$

In the same way we deduce that if $D^l(M) \subseteq \Delta(\mathfrak{p})$ then $D^{l+1}(M) \subseteq \Delta(\mathfrak{p})$ for any $l \in \mathbb{N}$. As a result, $\mathcal{D}(M) \subseteq \Delta(\mathfrak{p})$.

We note that $\Delta(\mathfrak{p}) \neq \Omega^1_{R_n}$ by Example 2. Let $A = R_n/\mathfrak{p}$ and consider the second fundamental exact sequence

$$\mathfrak{p}/\mathfrak{p}^2 \xrightarrow{\delta} \Omega^1_{R_n} \otimes_{R_n} A \xrightarrow{\alpha} \Omega^1_A \to 0,$$

where $\delta(f \bmod \mathfrak{p}^2) = df \otimes 1_A$ for $f \in \mathfrak{p}$. It now follows from Lemma 1 that

$$\operatorname{rank}_A \Omega^1_A \ge 1. \tag{20}$$

Observe that $\Omega^1_{R_n} \otimes_{R_n} A$ is a free A-module of rank n. Hence, by (20) we have $\operatorname{rank}_A(\ker \alpha) \le n-1$.

Let $M' = \langle \phi \otimes 1_A \mid \phi \in M \rangle$. Since $M' \subseteq \ker \alpha$ by assumption, it follows that $\operatorname{rank}_A M' \le n-1$. Hence we see that, for any $\phi_1, \dots, \phi_n \in M$,

$$*((\phi_1 \otimes 1_A) \wedge \cdots \wedge (\phi_n \otimes 1_A)) = 0 \text{ in } A. \tag{21}$$

Let $\pi : R_n \to A$ be the natural homomorphism. We note that there is a natural isomorphism $\Omega^1_{R_n} \otimes_{R_n} A \simeq A\,dx_1 \oplus \cdots \oplus A\,dx_n$ defined by

$$(a_1\,dx_1 + \cdots + a_n\,dx_n) \otimes 1_A \mapsto \pi(a_1)\,dx_1 + \cdots + \pi(a_n)\,dx_n.$$

As a consequence, it follows from (21) that for any choice of $\phi_1, \ldots, \phi_n \in M$ we have $*(\phi_1 \wedge \cdots \wedge \phi_n) \in \mathfrak{p}$. Therefore, $\Theta(M) \subseteq \mathfrak{p}$. □

COROLLARY 1. *Given $f_1, \ldots, f_l \in \mathfrak{m}$ in R_n, we let $M = \langle df_1, \ldots, df_l \rangle$. Suppose that the ideal generated by $f_1, \ldots, f_l$ is not $\mathfrak{m}$-primary. Then M is not subelliptic.*

EXAMPLE 8. In Corollary 1 the subelliptic multiplier ideal need not equal the ideal generated by the given functions. Here is an example. Let

$$f_1 = x^2 - yz, \quad f_2 = y^2 - xz, \quad f_3 = z^2 - xy \in R_3.$$

Then, M is represented by the matrix

$$\begin{pmatrix} 2x & -z & -y \\ -z & 2y & -x \\ -y & -x & 2z \end{pmatrix}.$$

By calculation we have

$$\mathcal{I}(M) = (x^3 + y^3 + z^3 - 3xyz),$$
$$\#(M) = 1.$$

EXAMPLE 9. In Corollary 1 the subelliptic multiplier ideals may be different if we start with different generators. Consider the following elements in R_3:

$$f_1 = x^2 - yz, \quad f_2 = y^2 - xz, \quad f_3 = z^2 - xy, \quad f_4 = (x + y + z)(x^2 - yz).$$

Let M' be the module generated by the differentials of f_1, f_2, f_3, and f_4. Then we may calculate that

$$\mathcal{I}(M') = (x + y + z) \cap (x - y, x - z),$$
$$\#(M') = 1.$$

Therefore, even if $(f_1, f_2, f_3) = (f_1, f_2, f_3, f_4)$, we have $\mathcal{I}(M) \neq \mathcal{I}(M')$.

5. A Sufficient Condition for Subellipticity

We prove Theorem 3 in this section. Let $\mathfrak{p}$ be a prime ideal in R_n with $\operatorname{ht}\mathfrak{p} = n - d \leq n - 1$, and let $A = R_n/\mathfrak{p}$ with the surjective map $\pi : R_n \to A$. Consider the exact sequence of A-modules

$$\mathfrak{p}/\mathfrak{p}^2 \xrightarrow{\delta} \Omega^1_{R_n} \otimes_{R_n} A \xrightarrow{\alpha} \Omega^1_A \to 0.$$

DEFINITION 7. We say that a submodule M in $\Omega^1_{R_n}$ satisfies *property* (L) *for a prime ideal* $\mathfrak{p}$ in R_n with $\operatorname{ht}\mathfrak{p} = n - d \leq n - 1$ if there exist d-elements $\phi_1, \ldots, \phi_d \in M$ such that

$$\alpha(\phi_1 \otimes 1_A), \ldots, \alpha(\phi_d \otimes 1_A) \in \Omega_A^1$$

are linearly independent over A.

LEMMA 3. *Suppose that M satisfies property* (L) *for a prime ideal* $\mathfrak{p}$ *with* $\operatorname{ht}\mathfrak{p} = n - d$. *Then there exist* $\phi_1, \ldots, \phi_d \in M$ *and* $f_1, \ldots, f_{n-d} \in \mathfrak{p}$ *such that*

$$*(\phi_1 \wedge \cdots \wedge \phi_d \wedge df_1 \wedge \cdots \wedge df_{n-d}) \notin \mathfrak{p}. \tag{22}$$

Proof. Let $A = R_n/A$. Since $\operatorname{rank}_A \Omega_A^1 = d$ by Lemma 1, it follows from the second fundamental exact sequence that there exist $f_1, \ldots, f_{n-d} \in \mathfrak{p}$ such that

$$df_1 \otimes 1_A, \ldots, df_{n-d} \otimes 1_A \in \ker \alpha$$

are linearly independent over A. By assumption there exist $\phi_1, \ldots, \phi_d \in M$ such that

$$\alpha(\phi_1 \otimes 1_A), \ldots, \alpha(\phi_d \otimes 1_A) \in \Omega_A^1$$

are linearly independent over A. Hence the elements of $\Omega_{R_n}^1 \otimes_{R_n} A$,

$$\phi_1 \otimes 1_A, \ldots, \phi_d \otimes 1_A,\ df_1 \otimes 1_A, \ldots, df_{n-d} \otimes 1_A,$$

are linearly independent over A. Therefore,

$$*(\phi_1 \wedge \cdots \wedge \phi_d \wedge df_1 \wedge \cdots \wedge df_{n-d}) \notin \mathfrak{p}. \qquad \square$$

LEMMA 4. *Let I be a radical ideal in R_n and let $I \neq 0$. We assume that* $\operatorname{ht} I \leq n - 1$ *and define*

$$I_M = \operatorname{rad}(I + \theta(M + \Delta(I))).$$

Suppose that M satisfies property (L) *for any prime ideal* $\mathfrak{p}$ *with* $\operatorname{ht}\mathfrak{p} \leq n - 1$. *Then*

$$\operatorname{ht} I \lneq \operatorname{ht} I_M.$$

Proof. Let $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s$ be the primary decomposition of I. We fix $\mathfrak{p}$ from $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$. Then there exists a $g \in R_n$ such that

$$g \in \bigcap_{\mathfrak{p}_j \neq \mathfrak{p}} \mathfrak{p}_j \quad \text{and} \quad g \notin \mathfrak{p}.$$

We assume that $\operatorname{ht}\mathfrak{p} = n - d \leq n - 1$. It follows from Lemma 3 that there exist $\phi_1, \ldots, \phi_d \in M$ and $f_1, \ldots, f_{n-d} \in \mathfrak{p}$ such that

$$*(\phi_1 \wedge \cdots \wedge \phi_d \wedge df_1 \wedge \cdots \wedge df_{n-d}) \notin \mathfrak{p}. \tag{23}$$

Since $gf_i \in I$ for $i = 1, \ldots, n - d$, we have

$$d(gf_i) \equiv g df_i \mod \mathfrak{p}\langle dx_1, \ldots, dx_n \rangle. \tag{24}$$

Let

$$\Psi = \phi_1 \wedge \cdots \wedge \phi_d \wedge d(gf_1) \wedge \cdots \wedge d(gf_{n-d}).$$

Then $*(\Psi) \in \theta(M + \Delta(I))$. It follows from (24) that

$$*(\Psi) \equiv g^{n-d}(*(\phi_1 \wedge \cdots \wedge \phi_d \wedge df_1 \wedge \cdots \wedge df_{n-d})) \mod \mathfrak{p}.$$

Since $g \notin \mathfrak{p}$, by (23) we have that $*(\Psi) \notin \mathfrak{p}$. Hence there exists a prime ideal $\mathfrak{q}$ that appears in the primary decomposition of I_M and $\mathfrak{p} \subsetneq \mathfrak{q}$. Since $\mathfrak{p}$ is arbitrary in the primary decomposition of I, it follows that $\operatorname{ht} I \lneq \operatorname{ht} I_M$. □

Applying $I = \Theta(M)$ in Lemma 4 yields

$$\Theta(M)_M = \operatorname{rad}(\Theta(M) + \theta(M + \Delta(\Theta(M))) = \Theta(D^1(M)).$$

THEOREM 3. *Let M be a submodule in $\Omega^1_{R_n}$. If M satisfies property* (L) *for any prime ideal* $\mathfrak{p}$ *in R_n with* $\operatorname{ht}\mathfrak{p} \le n-1$, *then M is subelliptic. Furthermore, $\#(M) \le n$ and the number n is sharp.*

Proof. By assumption, M satisfies property (L) for the trivial ideal (0). Hence there exist $\phi_1, \dots, \phi_n \in M$ that are linearly independent over R_n, and therefore $\operatorname{ht}\Theta(M) \ge 1$.

The theorem is clear for the case $\Theta(M) = \mathfrak{m}$ or A. If $\operatorname{ht}\Theta(M) \le n-1$, then we apply Lemma 4 to the case $I = \Theta(M)$ so that $\operatorname{ht}\Theta(M) \lneq \operatorname{ht}\Theta(D^1(M))$. Thus $\operatorname{ht}\Theta(D^{n-1}(M)) \ge n$; in other words, $\Theta(D^{n-1}(M)) = \mathfrak{m}$ or R_n. Therefore, $D^n(M) = \Omega^1_{R_n}$. □

6. Subellipticity for a Module Generated by Exact Forms

In this section we prove Theorem 4.

LEMMA 5 [M, Thm. 14.14]. *Let* $\mathfrak{p}$ *be a prime ideal in R_n with* $\operatorname{ht}\mathfrak{p} = n-d$ *and let $A = R_n/\mathfrak{p}$. Let $\pi\colon R_n \to A$ be the natural ring homomorphism. Suppose that the ideal generated by $f_1, \dots, f_l$ is $\mathfrak{m}$-primary in R_n. Then there exist K-linear combinations $g_1, \dots, g_d$ of $f_1, \dots, f_l$ such that $\{\pi(g_1), \dots, \pi(g_d)\}$ is a system of parameters of A.*

THEOREM 4. *Let $f_1, \dots, f_l \in \mathfrak{m}$ in R_n and let $M = \langle df_1, \dots, df_l\rangle$. Suppose that the ideal generated by $f_1, \dots, f_l$ is $\mathfrak{m}$-primary. Then M satisfies property* (L) *for any prime ideal with height $\le n-1$. Therefore, M is subelliptic. Moreover, $\#(M) \le n$ and the number n is sharp.*

Proof. Let $\mathfrak{p}$ be an arbitrary prime ideal in R_n with $\operatorname{ht}\mathfrak{p} = n-d \le n-1$. Let $A = R_n/\mathfrak{p}$ and $\pi\colon R_n \to A$. Since $(f_1, \dots, f_l)$ is $\mathfrak{m}$-primary in R_n, by Lemma 5 there exist K-linear combinations $g_1, \dots, g_d$ of $f_1, \dots, f_l$ such that $\{\pi(g_1), \dots, \pi(g_d)\}$ is a system of parameters of A. Let $\bar{g}_i = \pi(g_i)$ and $R_d = K\{y_1, \dots, y_d\}$ or $K[[y_1, \dots, y_d]]$. By the Noetherian normalization there is an injective K-homomorphism $\beta\colon R_d \to A$ with $\beta(y_i) = \bar{g}_i$, and A is finite over R_d via β.

Seeking a contradiction, we suppose that $d\bar{g}_1, \dots, d\bar{g}_d$ are linearly dependent over A in Ω^1_A. Let B be the image of β, and fix an arbitrary element $h \in A$. Since A is finite over B, there exists a minimal monic polynomial $H(T) \in B[T]$ such that $H(h) = 0$. That is,

$$H(h) = h^m + a_{m-1}(\bar{g}_1, \dots, \bar{g}_d)h^{m-1} + \dots + a_0(\bar{g}_1, \dots, \bar{g}_d) = 0, \qquad (25)$$

where $a_j \in R_d$ for $j = 1, \ldots, m-1$. Applying the universally finite derivation $d\colon A \to \Omega^1_A$, we have

$$\frac{\partial H}{\partial T}(h)\,dh + h^{m-1}\,da_{m-1} + \cdots + da_0 = 0 \text{ in } \Omega^1_A.$$

Therefore,

$$\frac{\partial H}{\partial T}(h)\,dh \in \langle d\bar{g}_1, \ldots, d\bar{g}_d \rangle. \tag{26}$$

Since K is of characteristic 0 and since H is minimal, we have $\frac{\partial H}{\partial T}(g) \neq 0$ in A. It follows from (26) that $\operatorname{rank}_A(\Omega^1_A) \leq d-1$, in contradiction with Lemma 1. Thus $d\bar{g}_1, \ldots, d\bar{g}_d$ are A-linearly independent. Since the universally finite derivation is K-linear, we still have that $d\bar{g}_i \in \langle d\bar{f}_1, \ldots, d\bar{f}_l \rangle$. Therefore, M satisfies property (L). □

Combining Corollary 1 and Theorem 4, we immediately deduce the following result.

COROLLARY 2. *Let $f_1, \ldots, f_l \in \mathfrak{m}$ in R_n and let $M = \langle df_1, \ldots, df_l \rangle$. Then M is subelliptic if and only if $(f_1, \ldots, f_l)$ is $\mathfrak{m}$-primary. Moreover, $\#(M) \leq n$ and the number n is sharp.*

EXAMPLE 10. The stabilization number of the module $\langle df_1, \ldots, df_l \rangle$ does not depend only on the ideal $(f_1, \ldots, f_l)$. For example, let $f = \{x^2, y^2\}$ and $g = \{x^2, y^2, (x+y)x^2, (x+y)y^2\}$ be collections of elements in R_2 that generate the same ideal (x^2, y^2). Let

$$M_f = \langle x\,dx, y\,dy \rangle \quad \text{and} \quad M_g = \langle x\,dx, y\,dy, d((x+y)x^2), d((x+y)y^2) \rangle.$$

Then $\#(M_f) = 2$ and $\#(M_g) = 1$.

REMARK. The theory of this paper can apply to more general local rings. Let $(A, \mathfrak{m})$ be an equicharacteristic regular local ring of dimension n and of characteristic 0, with the coefficient field K satisfying the following condition: for a regular system of parameters $x_1, \ldots, x_n$ there exist $D_i \in \operatorname{Der}_K(A, A)$, $1 \leq i \leq n$, such that

$$D_i x_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then there exist a free A module $\Omega^1_{A/K}$ of rank n and a K-derivation $d_{A/K}\colon A \to \Omega^1_{A/K}$ satisfying the universally finite condition. Furthermore, for any submodule of $\Omega^1_{A/K}$ we can construct the subelliptic multiplier module and the results of this paper hold. The complete discussion will appear in [C].

7. The q-Type Subelliptic Multiplier Modules

In this section we summarize the q-type subelliptic multiplier module for a submodule in $\Omega^1_{R_n}$. The main difference is to take $((n-q+1) \times (n-q+1))$-minors of the matrices in (1) and (2). The similar results will be stated in this section. All the results in this section can be proved in the same way; the complete proofs will appear in [C].

DEFINITION 8. Let M be a submodule in $\Omega^1_{R_n}$, and choose an integer q with $1 \leq q \leq n$. Let $\theta_q(M)$ be the ideal generated by

$$*(\phi_1 \wedge \cdots \wedge \phi_{n-s} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_s}),$$

where s runs over $\{0, \ldots, q-1\}$, the elements $\phi_1, \ldots, \phi_{n-s+1}$ run over M, and $dx_{i_1}, \ldots, dx_{i_s}$ run over $\{dx_1, \ldots, dx_n\}$. We define

$$\Theta_q(M) = \mathrm{rad}(\theta_q(M)). \tag{27}$$

REMARK. Observe that $\Theta(M) = \Theta_1(M)$.

Definition 8 enables us to deduce the following statement.

PROPOSITION 1. *Let M be a submodule in $\Omega^1_{R_n}$; then*

$$\Theta_1(M) \subseteq \cdots \subseteq \Theta_n(M). \tag{28}$$

Given an initial submodule M in $\Omega^1_{R_n}$ and an integer q with $1 \leq q \leq n$, we combine and iterate two operations, Θ_q and Δ, to obtain a sequence of submodules in $\Omega^1_{R_n}$ as follows.

DEFINITION 9 (q-type subelliptic multiplier module). Let M be a submodule in $\Omega^1_{R_n}$ and let $D^0_q(M) = M$. Define

$$D^1_q(M) = M + \Delta(\Theta_q(M)). \tag{29}$$

For $l \geq 0$, the module $D^{l+1}_q(M)$ is defined inductively:

$$D^{l+1}_q(M) = D^l_q(M) + \Delta(\Theta_q(D^l_q(M))). \tag{30}$$

Since $D^l_q(M) \subseteq D^{l+1}_q(M)$, we may define

$$\mathcal{D}_q(M) = \bigcup D^l_q(M). \tag{31}$$

We call $\mathcal{D}_q(M)$ the *q-type subelliptic multiplier module of M*. In the same way we define the *q-type subelliptic multiplier ideal of M* by

$$\mathcal{I}_q(M) = \bigcup \Theta_q(D^l_q(M)). \tag{32}$$

DEFINITION 10 (q-type subellipticity). A submodule M of $\Omega^1_{R_n}$ is said to be *q-type subelliptic* if $\mathcal{D}_q(M) = \Omega^1_{R_n}$.

DEFINITION 11 (q-type stabilization number). We define the *q-type stabilization number of M* by

$$\#_q(M) = \min\{l \mid \mathcal{D}_q(M) = D^l_q(M)\}. \tag{33}$$

From (28) we may deduce our next proposition.

PROPOSITION 2. *Let M be a submodule in $\Omega^1_{R_n}$. Then*

$$\mathcal{D}_1(M) \subseteq \cdots \subseteq \mathcal{D}_n(M) \tag{34}$$

and

$$\mathcal{I}_1(M) \subseteq \cdots \subseteq \mathcal{I}_n(M). \tag{35}$$

In a manner similar to the case of 1-type submodules, we deduce the following theorems for q-type subelliptic multiplier modules.

THEOREM 5 (q-type stabilization number). *Let M be a submodule in $\Omega^1_{R_n}$. Then $\#_q(M) \leq n$.*

THEOREM 6 (necessary condition for q-type subellipticity). *Let M be a submodule in $\Omega^1_{R_n}$, and let $\mathfrak{p}$ be a prime ideal in A with $\operatorname{ht}\mathfrak{p} \leq n-q$. Suppose that $M \subset \Delta(\mathfrak{p})$. Then M is not q-type subelliptic.*

Let M be a submodule in $\Omega^1_{R_n}$, and let $\mathfrak{p}$ be a prime ideal in R_n with $\operatorname{ht}\mathfrak{p} = n-d \leq n-q$. Let $A = R_n/\mathfrak{p}$ and consider the second fundamental exact sequence

$$\mathfrak{p}/\mathfrak{p}^2 \to \Omega^1_{R_n} \otimes_{R_n} A \xrightarrow{\alpha} \Omega^1_A \to 0.$$

DEFINITION 12 (property (L_q)). We say that M has *property* (L_q) *for a prime ideal* $\mathfrak{p}$ with $\operatorname{ht}\mathfrak{p} = n-d \leq n-q$ if, for some integer s with $0 \leq s \leq d$, there exist $\phi_1, \ldots, \phi_{d-s} \in M$ and $dx_{i_1}, \ldots, dx_{i_s}$ such that

$$\alpha(\phi_1 \otimes 1_A), \ldots, \alpha(\phi_{d-s} \otimes 1_A),\ \alpha(dx_{i_1} \otimes 1_A), \ldots, \alpha(dx_{i_s} \otimes 1_A)$$

are A-linearly independent in Ω^1_A.

THEOREM 7 (sufficient condition for q-type subellipticity). *Let M be a submodule in $\Omega^1_{R_n}$, and suppose that M has the property* (L_q) *for any prime ideal $\mathfrak{p}$ in R_n with $\operatorname{ht}\mathfrak{p} = n-d \leq n-q$. Then M is q-type subelliptic. Moreover, $\#_q(M) \leq n$.*

THEOREM 8 (sufficient condition for q-type subellipticity for a submodule generated by exact forms). *Let $f_1, \ldots, f_l \in \mathfrak{m}$ in R_n and let $M = \langle df_1, \ldots, df_l \rangle$. Let $I = (f_1, \ldots, f_l)$ and suppose that $\operatorname{ht} I \geq n-q+1$. Then M has property* (L_q) *for any prime ideal $\mathfrak{p}$ with $\operatorname{ht}\mathfrak{p} \leq n-q$. Therefore, M is q-type subelliptic.*

Combining Theorem 6 and Theorem 8 yields our final theorem.

THEOREM A. *Let $f_1, \ldots, f_l \in \mathfrak{m}$ in R_n and let $M = \langle df_1, \ldots, df_l \rangle$. Suppose that $(f_1, \ldots, f_l) \neq (0)$. Then we have sequences of q-type subelliptic multiplier modules and ideals of M*

$$\mathcal{D}_1(M) \subseteq \mathcal{D}_2(M) \subseteq \cdots \subseteq \mathcal{D}_n(M), \tag{36}$$

$$\mathcal{I}_1(M) \subseteq \mathcal{I}_2(M) \subseteq \cdots \subseteq \mathcal{I}_n(M), \tag{37}$$

$$\#_q(M) \leq n; \tag{38}$$

there is also a unique integer q_0 with $1 \leq q_0 \leq n$ such that

$$
\begin{aligned}
\mathcal{D}_q(M) &\neq \Omega^1_{R_n} \quad \textit{if } q < q_0, \\
\mathcal{D}_q(M) &= \Omega^1_{R_n} \quad \textit{if } q \geq q_0.
\end{aligned} \tag{39}
$$

Furthermore, $\dim R_n/(f) = q_0 - 1$.

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