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Coincident Root Loci of Binary Forms

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1. Introduction

Coincident root loci are subvarieties of $S^d\mathbb{C}^2$—the space of binary forms of degree d—labeled by partitions of d. Given a partition λ, let X_λ be the set of forms with root multiplicity corresponding to λ. There is a natural action of $\mathrm{GL}_2(\mathbb{C})$ on $S^d\mathbb{C}^2$, and the coincident root loci are invariant under this action. We calculate their equivariant Poincaré duals, generalizing formulas of Hilbert and Kirwan. In the second part we apply these results to present the cohomology ring of the corresponding moduli spaces (in the GIT sense) by geometrically defined relations.

One of the main goals of geometric invariant theory is to calculate the cohomology ring of a geometric quotient. For the case when all semistable point are stable, several techniques have been developed. But even for very simple representations this condition is not satisfied. In this paper we study the action of $\mathrm{GL}(2)$ on the space of binary forms of degree d. In the case of d odd the methods of [JK; K1; M] can be applied, but none of these methods computes the cohomology ring of the moduli space in the case of d even. We show how equivariant Poincaré-dual calculations lead to relations for the cohomology ring in both the odd and the even case.

Closely related rings have been computed previously. The computation for $H^*_G(X^{ss})$ is well known (since Kirwan's thesis in the case of Betti numbers), and the existing procedure is independent of d being even or odd. In the d even case, rational intersection cohomology of the moduli space is also known [K2], a result we also recover in Remark 4.11.

Our Poincaré-dual (a.k.a. Thom polynomial) calculations are also interesting in their own right because they generalize formulas of Hilbert and Kirwan on coincident root loci. These calculations not only lead to explicit relations for these cohomology rings but also identify them with the equivariant Poincaré-duals of the simplest unstable coincident root loci.

Consider the dth symmetric power $S^d\mathbb{C}^2$ of the standard representation of $\mathrm{GL}_2(\mathbb{C})$—that is, the action of G on the space V_d of degree-d homogeneous polynomials in two variables x, y. For any partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of d (i.e., $\sum_j \lambda_j = d$) we define

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$$X_\lambda = \left\{ B(x,y) \in V_d \;\middle|\; B = \prod_{j=1}^{n} L_j^{\lambda_j} \text{ for some linear forms } L_j \right\},$$

which is a subvariety invariant under the group action; it is called the *coincident root loci* associated with λ and is a cone in V_d. Let $\mathbb{P}X_\lambda$ be the projectivization of X_λ in the projective space $\mathbb{P}V_d$. It is more convenient to use a different notation for partitions: $\lambda = (1^{e_1}2^{e_2}\dots r^{e_r})$ will mean the partition consisting of e_1 copies of 1, e_2 copies of 2, and so forth. Then $\sum ie_i = d$ and $\sum e_i = n$, and the complex dimension of $\mathbb{P}X_\lambda$ is exactly n.

The study of coincident root loci probably started with Cayley. For example, the very first question of this type seeks the characterization of polynomials B with a double root. The answer is the vanishing of the discriminant, which provides in this way an equation for $X_{(1^{d-2}2)}$. For higher-codimensional coincident root loci, finding the defining equations is very complicated (see [C] for recent results). However, important geometric information can be obtained about these subvarieties. For instance, the starting point of this paper is the Hilbert formula that calculates the degree of $\mathbb{P}X_\lambda \subset \mathbb{P}V_d$:

$$\deg(\mathbb{P}X_\lambda) = \frac{n!}{\prod_i (e_i!)} \prod_i i^{e_i}.$$

We can interpret this formula as follows: For a generic family of polynomials parameterized by a projective space of dimension equal to the codimension of $\mathbb{P}X_\lambda$, the number of polynomials in the family with root multiplicity λ is $\deg(\mathbb{P}X_\lambda)$.

By generalizing this approach we arrive at the theory of degeneracy loci. Suppose we have a vector bundle $E \to M$ with fiber $S^d\mathbb{C}^2$ as well as a generic section $s: M \to E$. Let $s^{-1}(X_\lambda)$ be the set of points in M where the value of s is in X_λ. Its Poincaré dual $[s^{-1}(X_\lambda)] \in H^*(M)$ measures the "size" of $s^{-1}(X_\lambda)$. It turns out that, for any $S^d\mathbb{C}^2$-bundle, $[s^{-1}(X_\lambda)]$ can be deduced from the corresponding cohomology class of the universal bundle associated with the $\mathrm{GL}_2(\mathbb{C})$-representation $S^d\mathbb{C}^2$. This universal invariant is called the $\mathrm{GL}_2(\mathbb{C})$-*equivariant Poincaré dual,* or *Thom polynomial,* of X_λ in $S^d\mathbb{C}^2$. In Section 3 we determine all these polynomials.

Calculating equivariant Poincaré duals for invariant subvarieties of representations has a long history. We can interpret many results of the nineteenth-century algebraic geometers in these terms. Beginning in the 1970s the main method, initiated by Porteous [P], was a type of resolution of the subvariety. The method requires a deep understanding of the geometry of the resolution and can be carried out only in special cases; most examples can be found in [Fu]. The first and third author designed a different method (the method of restriction equations; see [FR2]) based on ideas that came from calculating Thom polynomials in singularity theory [R]. However, the method of restriction equations works well mainly if the representation has finitely many orbits, which is usually not the case (e.g., for $S^d\mathbb{C}^2$ if $d > 3$).

In this paper we return to the technique of resolution but in a very different way. The main novelty is that our new approach requires only knowledge of some basic cohomological data. Consequently, the method is more flexible. We illustrate this method here by the coincident root loci, but the range of applications is much wider.

Parallel to our work, Kőműves [Kő] also provided a presentation of these Poincaré duals in a completely different form. He worked more in the spirit of the method of restriction equations, studying incidences of the coincident root loci with the orbits $X_{(i,d-i)}$. In a recent work, Kazarian [Ka] used multisingularity loci polynomials to compute the Thom polynomials of coincident root loci.

In Section 4 we study the cohomology ring of the moduli space of the representation $S^d\mathbb{C}^2$ (in the GIT sense). Following the paper of Atiyah and Bott [AB1], a whole theory for calculating cohomology rings of the moduli space of representations was built up by F. Kirwan; also, methods of a more algebraic nature were successfully applied by Brion [Br] and Martin [M]. However, the application of the general theorems to specific examples is often not easy. Our approach results in explicit presentations of the rational cohomology rings $H^*_G(X^{ss})$, $H^*(X^{ss}//G)$, and $H^*_G(X^s) \cong H^*(X^s//G)$ in terms of generators and relations (if d is odd then all these rings coincide, but for the even case they are different). We wish to emphasize that a main advantage of our presentation of the cohomology rings is that we attribute to the set of relations deep geometric significance: they are the universal Thom polynomials of some distinguished spaces X_λ.

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2. Review of Affine and Projective Thom Polynomials

Let the group G act on the complex vector space V, and let η be an invariant variety in V that supports a fundamental class (for more details see [FR2]). Then define the (affine) *Thom polynomial* of η as the Poincaré dual of the fundamental homology class of η in equivariant cohomology:

$$\mathrm{Tp}_\eta = \text{Poincaré dual of } [\eta] \in H^*_G(V,\mathbb{Z}).$$

The vector space V is contractible; hence the ring $H^*_G(V,\mathbb{Z})$ is naturally isomorphic to $H^*(BG,\mathbb{Z})$, the ring of G-characteristic classes. The degree of Tp_η is the real codimension $2c$ of η in V, so $\mathrm{Tp}_\eta \in H^{2c}(BG,\mathbb{Z})$. The direct geometric meaning of Tp_η is as follows.

Consider a fiber bundle ξ with fiber V and structure group G over a manifold M. Because of its invariance, the set η can be defined in each fiber; let the union of these be $\eta(\xi)$. Then consider those points where a generic section s of ξ hits $\eta(\xi)$, that is, $s^{-1}(\eta(\xi)) \subset M$. By Poincaré duality this set defines a cohomology class in M. Standard arguments show that this class equals $\mathrm{Tp}_\eta(\xi) := f_\xi^*\mathrm{Tp}_\eta$, where $f_\xi : M \to BG$ is a classifying map of ξ.

We will also use the projective version of Thom polynomials (see [FNR]) as follows. Assume that G acts on V in such a way that the scalars are in the image of $G \to \mathrm{GL}(V)$. Then the orbits of this action (different from $\{0\}$) are in bijection with the orbits of the induced action of G on $\mathbb{P}V$. Also, the corresponding orbits η and $\mathbb{P}\eta$ have the same codimension. The equivariant Poincaré dual of $\mathbb{P}\eta$ will be called the *projective Thom polynomial* of η:

$$\begin{aligned}\mathbb{P}\mathrm{Tp}_\eta &= \text{Poincaré dual of } [\mathbb{P}\eta] \in H^*_G(\mathbb{P}V, \mathbb{Z}) \\ &= H^*(BG, \mathbb{Z})[x]/(Q(x)) \quad (\deg(x) = 2);\end{aligned}$$

here $Q(x)$ is the product of all the $(x + \alpha_j)$, where $\alpha_j \in H^2(BG)$ are the weights of the representation of G on V [BT]. The projective Thom polynomial can be written as $\mathbb{P}\mathrm{Tp}_\eta = p_c + p_{c-1}x + \cdots + p_0 x^c$, where $p_i \in H^{2i}(BG)$. By [FNR, Sec. 6], $p_c = \mathrm{Tp}_\eta$ and p_0 is the degree of the variety $\mathbb{P}\eta$. The projective Thom polynomial seemingly contains more information than the "affine" one, but this is not the case; by [FNR, Thm. 6.1], $\mathbb{P}\mathrm{Tp}_\eta$ can be obtained from Tp_η via a simple substitution (although this fact will not be used in the present paper). In particular, the degree p_0 of $\mathbb{P}\eta$ itself can be obtained from Tp_η by a substitution. For this substitution in our specific case, see Remark 3.9(2).

3. Coincident Root Loci

Consider the dth symmetric power $V_d = S^d\mathbb{C}^2$ of the standard representation of $G = \mathrm{GL}_2(\mathbb{C})$ as well as the invariant subvariety X_λ associated with a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of d (cf. the Introduction). In this section we compute its Thom polynomial $\mathrm{Tp}_\lambda \in H^*(BG, \mathbb{Z})$.

Points in the projectivization $\mathbb{P}V_d$ of V_d can be identified with d-tuples of points in $\mathbb{P}^1 = \{(x : y)\}$ (counted with multiplicities). The projectivization $\mathbb{P}X_\lambda$ is then the closure of the set of d-tuples having n distinct points with multiplicities $\lambda_1, \lambda_2, \dots, \lambda_n$. The variety $\mathbb{P}X_\lambda$ is called the *coincident root locus.*

Consider also the other notation $\lambda = (1^{e_1}2^{e_2} \dots r^{e_r})$ with $\sum ie_i = d$ and $\sum e_i = n$ as described in the Introduction. Then $\mathbb{P}X_\lambda$ is the image of the map

$$\phi\colon \mathbb{P}V_{e_1} \times \mathbb{P}V_{e_2} \times \cdots \times \mathbb{P}V_{e_r} \to \mathbb{P}V_d$$

defined (via point-tuples of $\mathbb{P}^1$) by $(D_1, D_2, \dots, D_r) \mapsto \sum iD_i$. It is readily seen that ϕ is birational onto its image $\mathbb{P}X_\lambda$ (i.e., ϕ is a resolution of $\mathbb{P}X_\lambda$). In particular, $\dim(\mathbb{P}X_\lambda) = n$ and Tp_λ is of degree $d - n$ (cf. Section 2).

The map ϕ is equivariant under the action of G on the two spaces, so it makes sense to talk about the maps ϕ^* (induced by ϕ) and $\phi_!$ (the push-forward map of ϕ) in G-equivariant cohomology. The equivariant cohomology rings are as follows (cf. e.g. [BT, p. 270]):

$$H^*_G\big(\textstyle\prod_i \mathbb{P}V_{e_i}, \mathbb{Z}\big) = R[x_1, \dots, x_r]/(Q_{e_1}(x_1), \dots, Q_{e_r}(x_r))$$

and

$$H^*_G(\mathbb{P}V_d, \mathbb{Z}) = R[x]/(Q_d(x)),$$

respectively. Here

$$R = H^*(BG, \mathbb{Z}) = \mathbb{Z}[c_1, c_2] = \mathbb{Z}[u, v]^{\mathbb{Z}_2},$$

where $\mathbb{Z}_2$ permutes the roots u and v (hence $c_1 = u + v$ and $c_2 = uv$); the polynomial Q_k $(k \geq 1)$ is defined by

$$Q_k(y) = \prod_{\alpha \text{ is a weight of } S^k\mathbb{C}^2} (y+\alpha) = \prod_{j=0}^{k} (y + ju + (k-j)v).$$

The map ϕ^* is a ring homomorphism; it leaves elements of R invariant and maps x to

$$\phi^*(x) = \sum_{i=1}^{r} i x_i.$$

The foregoing rings can be described also as finite-dimensional modules over R spanned by $\prod_i x_i^{k_i}$ $(0 \le k_i \le e_i)$ and x^k $(0 \le k \le d)$, respectively. We say that a representative of an element $[f]$ (in any of these rings) is *reduced* if it is written as an R-linear combination of these monomials; we use the notation $[f]_{\text{red}}$. In this language, the value of the integration maps (along the fibers)

$$\int_{\prod \mathbb{P}V_{e_i}} : H_G^*\big(\textstyle\prod_i \mathbb{P}V_{e_i}, \mathbb{Z}\big) \to R \quad \text{and} \quad \int_{\mathbb{P}V_d} : H_G^*(\mathbb{P}V_d, \mathbb{Z}) \to R,$$

respectively, are the coefficients of the top-degree monomials in the corresponding reduced forms: the coefficient of $\mathbf{x}^e := \prod_i x_i^{e_i}$ in the first case and the coefficient of x^d in the second case.

Set

$$q(x) := (Q_d(x) - C_{d+1})/x = x^d + C_1 x^{d-1} + \cdots + C_d,$$

where $Q_d(x) = \sum_{j=0}^{d+1} C_{d+1-j}(c_1, c_2) x^j$ and $C_0 = 1$.

THEOREM 3.1. Tp_λ *equals* $\int_{\prod \mathbb{P}V_{e_i}} \phi^*(q)$.

Proof. First we prove that $\mathrm{Tp}_\lambda = \int_{\mathbb{P}V_d}(q \cdot \mathbb{P}\mathrm{Tp}_\lambda)$. Indeed, from the general theory of projective and affine Thom polynomials (cf. Section 2) we know that $\mathbb{P}\mathrm{Tp}_\lambda = p_{d-n} + p_{d-n-1}x + \cdots + p_0 x^{d-n}$, where $p_j \in \mathbb{Z}[c_1, c_2]$ and $p_{d-n} = \mathrm{Tp}_\lambda$. When we multiply $x^j p_{d-n-j}$ $(1 \le j \le d-n)$ by $q = (Q_d - C_{d+1})/x$ and reduce it modulo $Q_d(x)$, the coefficient of x^d will be 0. So the only contribution comes from $q p_{d-n}$, which is the coefficient p_{d-n} of $p_{d-n} x^d$.

Now, using the definition of $\mathbb{P}\mathrm{Tp}_\lambda$ and the fact that ϕ is birational, we have $\mathrm{Tp}_\lambda = \int_{\mathbb{P}V_d}(q \cdot \phi_!(1))$; this equals $\int_{\prod \mathbb{P}V_{e_i}} \phi^*(q)$, which we wanted to prove. □

Theorem 3.1 gives the following computational recipe: Tp_λ *is the top coefficient* (*i.e., the coefficient of* $\mathbf{x}^e$) *of* $\phi^*(q)_{\text{red}}$. Notice that any representative $[f]_{\text{red}}$ is automatically calculated by computer algebra packages (e.g. [GS]), yielding an algorithmic solution to finding the Thom polynomials (see e.g. ⟨www.unc.edu/~rimanyi/progs/rootloci.m2⟩). We can, however, give explicit formulas as well.

Formulas for Thom Polynomials

LEMMA 3.2. *Set* $f \in R[y]$ *with class* $[f]$ *modulo* $Q_e(y)$. *Then the top coefficient of the reduced representative* $[f]_{\text{red}}$ *is*

$$\int_{\mathbb{P}V_e} [f] = \frac{1}{(v-u)^e} \sum_{s=0}^{e} \frac{(-1)^s f(-(e-s)u - sv)}{s!\,(e-s)!}. \tag{1}$$

Proof. This is a simple application of the Atiyah–Bott integration formula [AB2, p. 9], but we prefer to give a direct proof as follows. The formula is linear in f, so it is enough to verify (1) for any $f(y) = y^j$ ($j \geq 0$). In this case we need A_e, where $y^j \equiv A_e y^e + A_{e-1}y^{e-1} + \cdots + A_0$ modulo the ideal $(Q_e(y))$. If we consider this congruence for $y = -eu, -(e-1)u - v, \dots, -ev$, the result is a system of equations for $A_e, \dots, A_0$ (since $Q_e(y)$ vanishes at these points). The matrix of this system is a Vandermonde matrix, so by Cramer's rule we obtain the formula. □

Corollary 3.3 (The "naive" formula). *Let* $\sum_{s_1,\dots,s_r}$ *denote the sum over* $0 \leq s_i \leq e_i$ *for each* $1 \leq i \leq r$. *Then*

$$\mathrm{Tp}_\lambda = \frac{1}{(v-u)^n}\sum_{j=n}^{d}\sum_{j_1+\cdots+j_r=j} C_{d-j}\binom{j}{j_1,\dots,j_r} \times \sum_{s_1,\dots,s_r}\prod_{i=1}^{r}\frac{(-1)^{s_i}(-i)^{j_i}((e_i-s_i)u+s_iv)^{j_i}}{s_i!\,(e_i-s_i)!}.$$

Proof. Write $\sum_{j=n}^{d} C_{d-j}\left(\sum ix_i\right)^j$ as a linear combination of monomials of type $\prod_{i=1}^{r} x_i^{j_i}$. The polynomial $Q_{e_i}(x_i)$ contains only the variable x_i. Hence, to find the top coefficient of the remainder of $\prod_i x_i^{j_i}$ we can simply multiply the top coefficient of the remainders of $x_i^{j_i}$ modulo $Q_{e_i}(x_i)$. The formula then follows from Lemma 3.2 applied to each $x_i^{j_i}$. □

One can derive a more interesting formula. First observe that $xq + C_{d+1} = Q_d$ and so $\left(\sum ix_i\right)\phi^*(q) \equiv -C_{d+1}$ modulo the ideal $\mathcal{I} \subset R[x_1,\dots,x_r]$ generated by all $Q_{e_i}(x_i)$ ($1 \leq i \leq r$). We consider the following identities regarding $1/\sum_i ix_i$. Let t be a free variable. Then

$$\frac{1}{-t+\sum_i ix_i} = \frac{1}{-t}\sum_{j\geq 0}\left(\frac{\sum_i ix_i}{t}\right)^j = \frac{1}{-t}\sum_{j\geq 0}\sum_{j_1+\cdots+j_r=j}\binom{j}{j_1,\dots,j_r}\prod_i\left(\frac{ix_i}{t}\right)^{j_i}.$$

By Lemma 3.2, the top coefficient of the last expression is

$$\frac{1}{-t}\sum_{j\geq 0}\sum_{j_1+\cdots+j_r=j}\binom{j}{j_1,\dots,j_r} \times \prod_i\sum_{s_i=0}^{e_i}\frac{(-1)^{s_i}}{(v-u)^{e_i}s_i!\,(e_i-s_i)!}((e_i-s_i)u+s_iv)^{j_i}\left(\frac{-i}{t}\right)^{j_i}$$

$$= \frac{1}{(-t)(v-u)^n}\sum_{s_1,\dots,s_r}\frac{(-1)^{\sum_i s_i}}{\prod_i s_i!\,(e_i-s_i)!}\cdot\frac{1}{1+\sum_i i((e_i-s_i)u+s_iv)/t}$$

$$= \frac{1}{(v-u)^n}\sum_{s_1,\dots,s_r}\frac{(-1)^{\sum_i s_i}}{\prod_i s_i!\,(e_i-s_i)!}\cdot\frac{1}{-t-du+\left(\sum_i is_i\right)(u-v)}.$$

Let $A(t)$ be this last expression. The foregoing identities show the following congruence (valid for generic t):

$$\left(-t+\sum_i ix_i\right)(A(t)x^e + \text{ lower-order terms}) \equiv 1 \ (\operatorname{mod} \mathcal{I}). \tag{2}$$

Evidently, this is true for $t = 0$ as well. On the other hand, notice that there is a unique *reduced* $Y \in R[x_1, \dots, x_r]$ satisfying $\left(\sum ix_i\right)Y \equiv -C_{d+1} \ (\operatorname{mod} \mathcal{I})$. Indeed, if both Y and Y' satisfy this equivalence, then $-C_{d+1}Y' \equiv Y\left(\sum ix_i\right)Y' \equiv -C_{d+1}Y$; hence $Y = Y'$. Since $\left(\sum ix_i\right)\phi^*(q) \equiv -C_{d+1}$, from (2) (with $t = 0$) we get that the top coefficient of $\phi^*(q)_{\text{red}}$ is $-C_{d+1}A(0)$. Hence we have proved the following statement.

THEOREM 3.4. *With the notation* $C_{d+1} := C_{d+1}(S^d\mathbb{C}^2) = \prod_{j=0}^d (ju + (d-j)v)$, *one has*

$$\mathrm{Tp}_\lambda = \frac{C_{d+1}}{(v-u)^n} \cdot \sum_{s_1,\dots,s_r} \frac{(-1)^{\sum_i s_i}}{\prod_i s_i!\,(e_i - s_i)!} \cdot \frac{1}{du - \left(\sum_i is_i\right)(u-v)}.$$

This can also be considered as a higher-order divided difference formula (cf. Example 3.7).

EXAMPLE 3.5. If $\lambda = i^{e_i}$ and hence $d = ie_i$, then

$$\mathrm{Tp}_\lambda = i^{e_i} \cdot \prod_{0 \le j \le d;\, i \nmid j} (ju + (d-j)v).$$

This example can be deduced from Theorem 3.4 (cf. Remark 3.6), but one also can argue as follows. Since $ix_i\phi^*(q) + C_{d+1} \equiv 0 \ (\operatorname{mod} Q_{e_i}(x_i))$, clearly $ix_i\phi^*(q)_{\text{red}} + C_{d+1} \equiv 0$ as well. Since $ix_i\phi^*(q)_{\text{red}} + C_{d+1}$ and $Q_{e_i}(x_i)$ both have degree $e_i + 1$, it follows that $ix_i\phi^*(q)_{\text{red}} + C_{d+1} = C \cdot Q_{e_i}(x_i)$ for some $C \in R$. Comparing the coefficients of $x_i^{e_i+1}$ and x_i^0, we obtain

$$i\mathrm{Tp}_\lambda = C_{d+1}(S^d\mathbb{C}^2)/C_{e_i+1}(S^{e_i}\mathbb{C}^2).$$

REMARK 3.6. Lemma 3.2 has the following consequence. For some $C \in R$ and $g \in R[y]$, we denote by $[C/g]_{\text{red}}$ (or by $\int_{\mathbb{P}V_e}[C/g]$) that reduced element satisfying $[C/g]_{\text{red}} \cdot g \equiv C \ (\operatorname{mod} Q_e(y))$ (if it exists). Then also

$$\int_{\mathbb{P}V_e} [C/g] = \frac{1}{(v-u)^e} \sum_{s=0}^{e} \frac{(-1)^s}{s!\,(e-s)!} \cdot \frac{C}{g(-(e-s)u - sv)}. \tag{3}$$

The proof is similar to that for Theorem 3.4, which is actually a multivariable version of (3) (applied to $-C_{d+1}/\sum ix_i$).

Let us consider again $\lambda = i^{e_i}$. Theorem 3.4 and (3) give that

$$\mathrm{Tp}_\lambda = \int_{\mathbb{P}V_{e_i}} [-C_{d+1}(S^d)/ix_i].$$

But $x_i(x_i^{e_i} + \cdots) + C_{e_i+1}(S^{e_i}) = Q_{e_i}$ and so $\int_{\mathbb{P}V_{e_i}}[-C_{e_i+1}(S^{e_i})/x_i] = 1$. In particular, $\mathrm{Tp}_\lambda = C_{d+1}(S^d)/iC_{e_i+1}(S^{e_i})$, as verified in Example 3.5.

Example 3.7. Assume that $\lambda = i^{e_i} j^{e_j}$ $(i \neq j)$. Consider the expression given by Theorem 3.4 for this λ, and apply to variable x_i the identity (3). Clearly $du - (is_i + js_j)(u - v) = g(-e_i u + s_i(u - v))$, where $g(x_i) := a - ix_i$ with $a := je_j u - js_j(u - v)$. Therefore,

$$\mathrm{Tp}_\lambda = \frac{C_{d+1}(S^d)}{(v-u)^{e_j}} \sum_{s_j=0}^{e_j} \frac{(-1)^{s_j}}{s_j!\,(e_j - s_j)!} \cdot \int_{\mathbb{P}V_{e_i}} [1/g(x_i)].$$

Since $Q_{e_i}(x_i) - Q_{e_i}(a/i) = (x_i - a/i)(x_i^{e_i} + \cdots)$, it follows that

$$\int_{\mathbb{P}V_{e_i}} [iQ_{e_i}(a/i)/g(x_i)] = 1.$$

Hence

$$\mathrm{Tp}_\lambda = \frac{C_{d+1}(S^d)}{(v-u)^{e_j}} \sum_{s_j=0}^{e_j} \frac{(-1)^{s_j}}{s_j!\,(e_j - s_j)!} \cdot \frac{1}{i \cdot Q_{e_i}((je_j u - js_j(u-v))/i)}.$$

For example, assume that $\lambda = i^{e_i} j$ (i.e., $e_j = 1$). Then $s_j = 0$ or 1, so

$$\mathrm{Tp}_\lambda = \frac{C_{d+1}(S^d)}{i(v-u)} \cdot \left(\frac{1}{Q_{e_i}(ju/i)} - \frac{1}{Q_{e_i}(jv/i)} \right).$$

It is convenient to express this in the language of *divided difference*: If $P(u, v)$ is a polynomial in two variables (u, v), we shall denote by $\partial(P)$ the polynomial $(P(u, v) - P(v, u))/(u - v)$. Then

$$\mathrm{Tp}_{(i^{e_i} j)} = \frac{1}{i} \cdot \partial\left(\frac{C_{d+1}(S^d)}{Q_{e_i}(jv/i)} \right) = i^{e_i} \cdot \partial\left(\prod((d-k)v + ku) \right);$$

here the product is over k with $0 \leq k \leq d$, but $k \neq is$ with $0 \leq s \leq e_i$. In particular,

$$\mathrm{Tp}_{(1^{e_1} j)} = \partial\left(\prod_{l=0}^{j-1} (lv + (e_1 + j - l)u) \right) \quad \text{for } j \geq 2, \tag{4}$$

which is equivalent with Kirwan's formula [K4, p. 902].

Example 3.8. Assume that $d = 2h$ is even, $h > 2$, and $\lambda = (1^{h-j}, j, h)$ for some $1 < j < h$. By computation and an argument similar to that used in Example 3.7,

$$\begin{aligned} \mathrm{Tp}_\lambda = \frac{C_{d+1}(S^d)}{(u-v)^2} \cdot \Bigg(& \frac{1}{Q_{h-j}(hu + ju)} - \frac{1}{Q_{h-j}(hu + jv)} \\ & - \frac{1}{Q_{h-j}(hv + ju)} + \frac{1}{Q_{h-j}(hv + jv)} \Bigg) \\ = \partial\Bigg[\frac{C_{d+1}(S^d)}{u - v} \cdot \Bigg(& \frac{1}{Q_{h-j}(hv + jv)} - \frac{1}{Q_{h-j}(hv + ju)} \Bigg) \Bigg] \\ = \partial\Bigg[D_j \cdot \prod_{l=0}^{h-1} & (lv + (d - l)u) \Bigg], \end{aligned}$$

where

$$D_j := \frac{1}{u-v} \cdot \left[\prod_{l=h-j+1}^{h} (lv + (d-l)u) - \prod_{l=0}^{j-1} (lv + (d-l)u) \right].$$

For example, if $j = 2$ then

$$\mathrm{Tp}_\lambda = h(h-1) \cdot \partial \left[(u+3v) \prod_{l=0}^{h-1} (lv + (d-l)u) \right].$$

REMARKS 3.9. (1) The Thom polynomials are connected by many interesting polynomial relations. For instance, the next section presents two situations when the ideal generated by natural families of Thom polynomials is generated only by two of them. Some of these relations can be verified easily. Assume for example that $d = 2h$ as in Example 3.8 and consider the partitions $\lambda'_0 = (1^{h-2}, 2, h)$, $\lambda_0 = (1^h, h)$, $\lambda_1 = (1^{h-1}, h+1)$, and $\lambda_2 = (1^{h-2}, h+2)$. Then (4) and Example 3.8 together imply that $\mathrm{Tp}_{\lambda_1} = hc_1 \cdot \mathrm{Tp}_{\lambda_0}$ and

$$(h-1) \cdot \mathrm{Tp}_{\lambda_2} = (h-1)(h-2)c_1 \cdot \mathrm{Tp}_{\lambda_1} + c_1 \mathrm{Tp}_{\lambda'_0}.$$

(2) Using [FNR], one can determine $\deg(\mathbb{P}X_\lambda)$ by the substitution $u = v = 1/d$ in $\mathrm{Tp}_\lambda \in \mathbb{Z}[u, v]$. The interested reader is invited to verify the compatibility of Hilbert's result (cf. the Introduction) with this section.

(3) In the sequel we will often use the following divided difference formula: For any polynomial $A \in \mathbb{Q}[u, v]$ write $A^*(u, v) := A(v, u)$; then

$$\partial(AB) = B^* \cdot \partial(A) + A \cdot \partial(B).$$

4. Thom Polynomial Description of the Cohomology Ring of the Moduli Space

In this section we apply the coincident root loci formulas in the study of the cohomology ring of the moduli space of the representation $S^d\mathbb{C}^2$ (in the GIT sense). We calculate the rational cohomology rings $H^*_G(X^{ss})$, $H^*(X^{ss}/\!/G)$, and $H^*_G(X^s) \cong H^*(X^s/\!/G)$ in terms of generators and relations. If d is odd then all these rings coincide, but for the even case they are different.

There is an extensive literature on these cohomology rings, both from the combinatorial algebraic perspective (see e.g. [Br; M]) and from geometric point of view (the Atiyah–Bott–Kirwan theory [K1]). Our approach (in the d odd case) is closest to that of Kirwan. The advantages of our approach are that (a) we treat the odd and even cases in a uniform language and (b) we provide for these cohomology rings a transparent structure, obtaining explicit presentations of them in terms of generators and relations with clear geometric meanings.

Let us sketch our approach in the odd case first (a more detailed presentation will follow). In this case the Kirwan stratification of $S^d\mathbb{C}^2$ is G-perfect because the normal (equivariant) Euler classes of the strata are not 0-divisors, and this implies that the spectral sequence of the corresponding filtration degenerates. It is not difficult to calculate all but the 0th column of the E_1-table, so by subtraction we can calculate the ranks of the 0th column: the Betti numbers of $H_G^*(X^{ss})$. Also by G-perfectness, the natural map

$$\kappa\colon H_G^*(S^d\mathbb{C}^2) \cong \mathbb{Q}[c_1, c_2] \to H_G^*(X^{ss})$$

is surjective and so we need to find relations in terms of c_1 and c_2; that is, we must find generators of $\mathrm{Ker}(\kappa)$. If $Y \cap X^{ss} = \emptyset$ for an invariant subvariety Y then clearly $[Y] \in \mathrm{Ker}(\kappa)$ (this idea was studied in [FR1]). Hence all the higher Kirwan strata provide relations. But the Kirwan strata are coincident root loci for specific partitions, and we can calculate their equivariant Poincaré duals using the first part of this paper. Our main point is that the first two Kirwan strata are enough to generate $\mathrm{Ker}(\kappa)$, which can be checked by a simple Betti number calculation.

In the d even case we shall refine the Kirwan stratification (see Discussion 4.6). The main difficulty is that for one of the strata in this refined stratification the normal (equivariant) Euler class is a 0-divisor. In order to prove G-perfectness, we use results from the first part of this paper. Namely, we show that certain elements in the E_1-table can be represented by the Poincaré dual of coincident root loci (these are not Kirwan strata!) and that they survive to E_∞; hence they could not be hit by a differential. After G-perfectness is proven, the process is the same as in the odd case. We can find coincident root loci in the null cone such that their Poincaré duals generate $\mathrm{Ker}(\kappa)$. Here we also need two coincident root loci, but one of them is not a Kirwan stratum.

In this section, all cohomologies are assumed to have rational coefficients. Let us now consider the Kirwan stratification (see [K1; K4]) of the vector space V_d:

- $X^{ss} = \{B \mid B \text{ has no root of multiplicity} > d/2\}$;
- $X_i = \{B \mid B \text{ has a root of multiplicity } i \text{ but not with multiplicity } i+1\}$, $d/2 < i \le d$;
- $X_0 = \{0\}$.

The strata are smooth open submanifolds, and the complex codimensions are 0, $i-1$, and $d+1$ in the three cases. If $F_i = \bigcup(\text{strata of complex codimension} \le i)$ then we have the following filtration of V_d:

$$\emptyset = F_{-1} \subset F_0 \subset F_1 \subset \cdots \subset F_{d+1} = S^d\mathbb{C}^2.$$

Let $E_*^{*,*}$ be the associated spectral sequence in G-equivariant cohomology with $\mathbb{Q}$-coefficients.

Proposition 4.1.

(i) $E_1^{0,*} = H_G^*(X^{ss}; \mathbb{Q})$;
(ii) $E_1^{2p,*} = H^*(BU(1); \mathbb{Q})$ *for* $p = [d/2], \dots, d-1$;
(iii) $E_1^{2(d+1),*} = H^*(BG; \mathbb{Q})$;
(iv) $E_1^{*,*} = 0$ *for all cases not covered by* (i)–(iii);
(v) *the spectral sequence converges to* $H^*(BG; \mathbb{Q})$; *and*
(vi) *the spectral sequence degenerates at* $E_1^{*,*}$ (*in particular,* $H_G^{\mathrm{odd}}(X^{ss}, \mathbb{Q}) = 0$).

Proof. By definition we have that $E_1^{2p,*} = H_G^{2p+*}(F_p, F_{p-1})$, which by Thom isomorphism is $H_G^*(F_p \setminus F_{p-1})$. This proves (i) and (iv). For $p = d+1$ we have

$E_1^{2(d+1),*} = H_G^*(\{0\}) = H^*(BG)$, which proves (iii). For $d/2 < i \le d$ we define $Y_i = \{B \in X_i : x^i | B \text{ and } \mathrm{coeff}(x^i y^{d-i}) = 1\}$. Let H be the stabilizer subgroup of Y_i, that is, the group of matrices of the form $\left(\begin{smallmatrix} \alpha_1 & \beta \\ 0 & \alpha_2 \end{smallmatrix}\right)$ with $\alpha_1^i \alpha_2^{d-i} = 1$. Since Y_i is contractible and $X_i = G \times_H Y_i$, (ii) follows from

$$H_G^*(X_i) \cong H_G^*(G \times_H Y_i) \cong H_H^*(Y_i) \cong H^*(BH) \cong H^*(BU(1)) \quad (\text{over } \mathbb{Q}).$$

The degeneracy of the spectral sequence—called G-perfectness by Atiyah and Bott in [AB1]—follows from usual arguments as follows. Let us build up V_d by gluing the strata together, one by one, in order of increasing codimension. Then at one step we have U and glue a new stratum X of complex codimension c to it. We need to prove that the first map in the diagram

$$H_G^{n-2c}(X) \cong H_G^n(U \cup X, U) \to H_G^n(U \cup X) \to H_G^n(X)$$

is injective. However, the whole composition is the multiplication with the equivariant Euler class of the stratum X. This is an injective map because it is a multiplication by a nonzero element in a polynomial ring. (For a computation of an equivariant Euler class see the proof of Proposition 4.7.) □

Since $E_\infty = E_1$, it follows that the sum of the ranks of the groups in diagonal (i.e., $p + q = r$) entries must be the rank of the appropriate cohomology group of $H^*(BG; \mathbb{Q})$. Thus we have our next result.

Corollary 4.2. *Let $h := [d/2]$. The Poincaré series of the ring $H_G^*(X^{ss}; \mathbb{Q})$ is*

$$\frac{1}{(1-t)(1-t^2)}(1 - t^{d+1}) - \frac{1}{1-t}(t^h + \cdots + t^{d-1})$$
$$= \frac{1 - t^h - t^{h+1} + t^d}{(1-t)(1-t^2)} \quad (\deg(t) = 2).$$

What we have obtained so far is basically equivalent to the Atiyah–Bott–Kirwan theory applied to our representation; see [K1, 16.2].

What can also be seen from the spectral sequence is that $H_G^*(X^{ss}) = H^*(BG)/I$, where the ideal comes from the $p > 0$ columns of the spectral sequence. Thus, among the elements of I we have those that are the images of the generators of $E_1^{2p,0}$ under the edge homomorphism. For $[d/2] \le p \le d-1$, these are exactly the Thom polynomials corresponding to the strata X_i, $i = p + 1$. We have $\mathrm{Tp}(X_i) = \mathrm{Tp}_\lambda$ with $\lambda = (1^{d-i}, i)$, since the closures of X_i and X_λ are the same. The preceding Betti number computation can be used to test whether a few of these Thom polynomials are enough to generate I.

Theorem 4.3. *Set $\lambda_1 = (1^{d-h-1}, h+1)$ and $\lambda_2 = (1^{d-h-2}, h+2)$, where $h = [d/2]$. Then I is generated by Tp_{λ_1} and Tp_{λ_2}. In particular,*

$$H_G^*(X^{ss}; \mathbb{Q}) = \mathbb{Q}[c_1, c_2]/(\mathrm{Tp}_{\lambda_1}, \mathrm{Tp}_{\lambda_2}).$$

Proof. We have already observed that the given two Thom polynomials are in I. Now we prove that the ring on the right-hand side has the same Poincaré series as the one given in Corollary 4.2.

We claim that the ideal $J := (\mathrm{Tp}_{\lambda_1}, \mathrm{Tp}_{\lambda_2})$ has the following R-resolution: $0 \leftarrow J \leftarrow (\mathrm{Tp}_{\lambda_1}) \oplus (\mathrm{Tp}_{\lambda_2}) \leftarrow U \leftarrow 0$, where U is a principal ideal generated by a $\deg d$ polynomial in $R = \mathbb{Q}[c_1, c_2]$. If $d = 2h + 1$ then for this we need only prove that Tp_{λ_1} and Tp_{λ_2} have no nontrivial common divisor D. We know that $\mathrm{Tp}_{\lambda_1} = \partial(\Pi)$ and $\mathrm{Tp}_{\lambda_2} = \partial(\Pi L)$, where $\Pi(u, v) = \prod_{l=0}^{h}(lv + (d - l)u)$ and $L(u, v) = (h + 1)v + hu$. By Remark 3.9(3), if $D | \gcd(\mathrm{Tp}_{\lambda_1}, \mathrm{Tp}_{\lambda_2})$ then $D | \Pi$ and hence $D | \gcd(\Pi, \partial(\Pi))$ as well. But $\gcd(\Pi, \Pi^*) = 1$, which ends the proof of the claim.

Hence we obtain the Poincaré series of R/J as $(1 - t^h - t^{h+1} + t^{2h+1})/(1 - t)(1 - t^2)$, which is the same as the Poincaré series of $H^*_G(X^{ss}; \mathbb{Q})$. For d even, the proof is similar. □

Discussion 4.4 (The cohomology ring of $X^{ss}//G$). Observe that if d is odd then $X^{ss} = X^s$ and all stabilizers of polynomials in X^{ss} are finite. We therefore have the ring isomorphism $H^*_G(X^{ss}; \mathbb{Q}) = H^*(X^{ss}//G; \mathbb{Q})$ with Poincaré polynomial $(1 - t^h)(1 - t^{h+1})/(1 - t)(1 - t^2)$.

If $d = 2h$ is even then $X^{ss}//G = X^s//G \cup \{p^{ss}\}$, where p^{ss} is the unique "semisimple point" of $X^{ss}//G$. The Poincaré series of $H^*_G(X^{ss})$ is infinite; namely,

$$\frac{1}{1 - t^2} + t \cdot P(t), \text{ where } P(t) \text{ is the polynomial } \frac{(1 - t^{h-1})(1 - t^h)}{(1 - t)(1 - t^2)} \quad (\deg(t) = 2). \quad (5)$$

All the stabilizers of the stable part are finite, and there is only one orbit in the strict semistable part with infinite stabilizer H^{ss}: the orbit of the partition (h, h). The stabilizer H^{ss} can be described explicitly, and one has an exact sequence $1 \to U(1) \times \mathbb{Z}_h \to H^{ss} \to \mathbb{Z}_2 \to 1$. Hence BH^{ss} is a double covering of $BU(1) \times B\mathbb{Z}_h$ with rational cohomology $H^*(BH^{ss}) = H^*(BU(1))^{\mathbb{Z}_2} = \mathbb{Q}[t]^{\mathbb{Z}_2}$ ($\deg_t = 2$). Here the $\mathbb{Z}_2$-action is $t \mapsto \pm t$, so the invariant part is $\mathbb{Q}[t^2]$ with an infinite Poincaré series $1/(1 - t^2)$. This is exactly the "infinite contribution" in the Poincaré series (5) of $H^*_G(X^{ss})$.

In fact, the map $r: H^*(BG) \to H^*(BH^{ss})$ (induced by the inclusion) is given, at the level of roots, by $u \mapsto \pm t$ and $v \mapsto \mp t$; hence it is the epimorphism $r: \mathbb{Q}[c_1, c_2] \to \mathbb{Q}[t^2]$ given by $c_1 \mapsto 0$ and $c_2 \mapsto -t^2$.

As usual, for any connected space Z we let $\tilde{H}^*(Z)$ be the kernel of $H^*(Z) \to H^*(\text{point})$ as an ideal (or subring without unit) in $H^*(Z)$. The ring $H^*(Z)$ can be reconstructed from $\tilde{H}^*(Z)$ by adding the unit: $H^*(Z) = \mathbb{Q}\langle 1\rangle \oplus \tilde{H}^*(Z)$ (with the natural multiplication).

Let o be the orbit corresponding to the partition (h, h) and consider the natural inclusion $j: o \times_G EG \to X^{ss} \times_G EG$. Obviously, $o \times_G EG$ can be identified with BH^{ss}. Moreover, $j^*: H^*_G(X^{ss}) \to H^*(BH^{ss})$ induced by j can be identified with the epimorphism $\mathbb{Q}[c_1, c_2]/(\mathrm{Tp}_{\lambda_1}, \mathrm{Tp}_{\lambda_2}) \to \mathbb{Q}[t^2]$, where $c_1 \mapsto 0$ and $c_2 \mapsto -t^2$ are induced by the epimorphism r described previously. In fact, Tp_{λ_1} and Tp_{λ_2} are both divisible by c_1 (cf. Remark 3.9(1)) and hence r sends the ideal generated by them to zero.

Finally, observe that $H^*(X^{ss} \times_G EG, BH^{ss}) = \tilde{H}^*(X^{ss} \times_G EG/BH^{ss})$ and that the natural map $r\colon X^{ss} \times_G EG/BH^{ss} \to X^{ss}//G$ induces an isomorphism at the level of rational cohomology rings. In particular, the long exact cohomology sequence of the pair $(X^{ss} \times_G EG, BH^{ss})$ transforms into the short exact sequences

$$0 \to \tilde{H}^*(X^{ss}//G) \to H^*_G(X^{ss}) \xrightarrow{j^*} H^*(BH^{ss}) \to 0. \tag{6}$$

Analyzing the kernel of j^* yields our next corollary.

Corollary 4.5. *With the notation of Theorem 4.3, one has the following ring isomorphisms*:

$$H^*(X^{ss}//G;\mathbb{Q}) = \begin{cases} \mathbb{Q}[c_1,c_2]/(\mathrm{Tp}_{\lambda_1},\mathrm{Tp}_{\lambda_2}) & \text{if } d \text{ is odd};\\ \mathbb{Q}\langle 1\rangle \oplus (c_1\mathbb{Q}[c_1,c_2])/(\mathrm{Tp}_{\lambda_1},\mathrm{Tp}_{\lambda_2}) & \text{if } d \text{ is even.}\end{cases}$$

Notice that the Poincaré series formula (5) is compatible with (6) and Corollary 4.5. In particular, if $d = 2h$ then the Poincaré polynomial of $H^*(X^{ss}//G)$ is $1 + tP(t)$.

Discussion 4.6 (The cohomology ring of $X^s//G$). Next, for the case $d = 2h$ we wish to determine the cohomology ring of the geometric quotient $X^s//G$. In the notation that follows it is convenient to assume $h > 2$ (if $h = 2$, then $X^{ss}//G = \mathbb{P}^1$ and $X^s//G = \mathbb{C}$).

We consider a similar spectral sequence but now associated with the following "refined" stratification:

- $X^s = \{B \mid B \text{ has no root of multiplicity} \geq h\}$;
- $X_i = \{B \mid B \text{ has exactly one root of multiplicity } i \text{ but no roots of multiplicity } i+1\}$, $h \leq i \leq d$;
- $o = \{\text{the orbit associated with the partition } (h,h)\}$; and
- $X_0 = \{0\}$.

In Proposition 4.1, $E_1^{0,*}$ will be replaced by $H^*_G(X^s)$. For $i > h$ the stratum X_i is the same as in the previous case, but now there are two new strata: X_h and o. Since o is an orbit with stabilizer H^{ss}, it follows that $H^*_G(o) = H^*(BH^{ss})$. The complex codimension of o in V_d is $d-2$, so this will provide an additional direct sum contribution in $E_1^{2(d-2),*}$. Hence $E_1^{2p,*} = H^*(BU(1))$ if $h \leq p \leq d-1$ but $p \neq d-2$, and $E_1^{2(d-2),*} = H^*(BU(1)) \oplus H^*(BH^{ss})$.

Finally, we compute $E_1^{2(h-1),*} = H^*_G(X_h)$. Set

$$Y_h = \{B \in X_h : B = x^h \cdot B' = x^h(y^h + a_2x^2y^{h-2} + \cdots + a_hx^h) \text{ and } B' \text{ is not an } h\text{-power}\}.$$

The stabilizer subgroup H of Y_h is the group of diagonal matrices of the form $\mathrm{diag}(\alpha_1,\alpha_2)$ with $\alpha_1^h\alpha_2^h = 1$. One can verify that $X_h = G \times_H Y_h$. Moreover, B' is not an h-power if and only if $(a_2,\ldots,a_h) \neq (0,\ldots,0)$. Hence Y_h is $\mathbb{C}^{h-1} \setminus \{0\}$ and the action of H is a diagonal torus action (modulo a finite group). In particular, $E_1^{2(h-1),*} = H^*_G(X_h)$ equals the cohomology ring of a weighted projective space of dimension $h-2$, which is $\mathbb{Q}[t]/(t^{h-1})$ $(\deg(t) = 2)$.

Proposition 4.7. *The spectral sequence converges to $H^*(BG;\mathbb{Q})$ and degenerates at $E_1^{*,*}$.*

The Euler classes of the strata are not 0-divisors except for X_h. Hence we need the following local version of the Atiyah–Bott argument.

Lemma 4.8. *Suppose that $\{X_i\}$ is a G-equivariant stratification of V and that the equivariant normal Euler class of X_i is not a 0-divisor if $\operatorname{codim}(X_i) > c$. Then all differentials of the corresponding spectral sequence $E_r^{p,q}$ starting or landing in the region $p > c$ are zero.*

Proof. Let X be the union of X_i with $\operatorname{codim}(X_i) > c$. Then the lemma is equivalent to the statement that $H_G^*(V, V\setminus X) \to H_G^*(V)$ is injective, since $\{E_r^{p,q} : p > c\}$ converges to $H_G^*(V, V\setminus X)$. Injectivity can be proved by adding the X_i one by one and then noticing that the composition

$$H_G^{n-2c}(X) \cong H_G^n(U\cup X_i, U) \to H_G^n(U\cup X_i) \to H_G^n(X_i)$$

is multiplication with the equivariant normal Euler class of the stratum X_i (where U is an open subset of V in which X_i is closed). □

Proof of Proposition 4.7. For the convenience of the reader we show how one determines the equivariant Euler class of o. Fix an element (say, $x^h y^h$) on o, let H^{ss} be its stabilizer, and consider an H^{ss} invariant normal slice N at $x^h y^h$. In fact, for N one can take the vector space spanned by $x^i y^{d-i}$, where $0 \le i \le d$ but $i \notin \{h-1, h, h+1\}$. The stabilizer H^{ss} acts on N, and our goal is the computation of the Euler class $e^{ss} \in H^*(BH^{ss})$ of $EH^{ss}\times_{H^{ss}} N \to BH^{ss}$. Consider now the subgroup $U(1)$ of H^{ss} (see Discussion 4.4). The Euler class $e \in H^*(BU(1)) = \mathbb{Q}[t]$ of $EH^{ss}\times_{U(1)} N \to BU(1)$ can be computed as follows. The eigenvalues of $\operatorname{diag}(\alpha,\bar\alpha) \in U(1)$ on N are $(\alpha^d, \alpha^{d-2}, \dots, \alpha^4, \alpha^{-4}, \dots, \alpha^{-d})$; hence $e = (dt)((d-2)t)\cdots(4t)(-4t)\cdots(-dt) = mt^{d-2}$ for some $m \neq 0$. Since d is even, this is in the invariant part $H^*(BH^{ss}) = \mathbb{Q}[t^2]$ and can be identified in this ring by e^{ss}. Therefore, $e^{ss} \neq 0$. (This type of argument does not work for the stratum X_h, since the stabilizers of its points are finite and since $H_G^*(X_h)$ has 0-divisors.)

In order to show that the differentials $d_{2h-2}^{0,q}$ (q odd and $2h-3 \le q \le 4h-7$) of the spectral sequence are trivial, we consider another spectral sequence associated with only two strata—namely, with X^s and X_h. The differential $d_{2h-2}^{0,q}$ in the two spectral sequences coincides. If we compare them by the natural maps then we obtain the exact sequence

$$0 \to I' \to H_G^*(V_d) \xrightarrow{\tau} H_G^*(X^s \cup X_h),$$

where the ideal I' is generated by all the columns $E_1^{>2h-2,*}$. In $E_\infty^{2(h-1),2(j-1)}$ we can find special elements: those represented by the Thom polynomials $\mathrm{Tp}_j \in H_G^*(V_d)$

associated with the partitions $(1^{h-j}, j, h)$, where $0 < j < h$. Hence $d_{2h-2}^{0,2j+2h-5} = 0$ if $\tau(\mathrm{Tp}_j) \neq 0$ or (equivalently) if $\mathrm{Tp}_j \notin I'$. Observe that the graded ideal I' and the graded ideal I considered in Corollary 4.2 and Theorem 4.3 are the same in the relevant degrees, so it is enough to verify that $\mathrm{Tp}_j \notin I$ for any j. But in Theorem 4.3 we verified that $I = (\mathrm{Tp}_{\lambda_1}, \mathrm{Tp}_{\lambda_2})$; we therefore need to prove

$$\mathrm{Tp}_j \notin (\mathrm{Tp}_{\lambda_1}, \mathrm{Tp}_{\lambda_2}). \tag{7}$$

Set

$$\Pi := \prod_{l=0}^{h-1} (lv + (d-l)u) \quad \text{and} \quad L = (h+1)v + (h-1)u.$$

By (4) we have $\mathrm{Tp}_{\lambda_1} = hc_1 \cdot \partial(\Pi)$ and $\mathrm{Tp}_{\lambda_2} = hc_1 \cdot \partial(\Pi L)$. In particular, by Remark 3.9(3) it follows that $\mathrm{Tp}_{\lambda_2} = hL^* c_1 \cdot \partial(\Pi) - 2hc_1\Pi$ and hence

$$(\mathrm{Tp}_{\lambda_1}, \mathrm{Tp}_{\lambda_2}) = (c_1 \cdot \partial(\Pi), c_1 \cdot \Pi) \tag{8}$$

in $\mathbb{Q}[u, v]$. Assume that (7) is not true and that we have $\mathrm{Tp}_j = Ac_1 \cdot \partial(\Pi) + Bc_1\Pi$. Since the degrees of Tp_j and Π are $h + j - 2$ and h (respectively), the degree of Ac_1 is $j-1$. From Example 3.8 and Remark 3.9(3) we obtain $\mathrm{Tp}_j = \partial(\Pi \cdot D_j) = D_j^* \cdot \partial(\Pi) + \Pi \cdot \partial(D_j)$; this means that

$$\Pi(\partial(D_j) - Bc_1) = \partial(\Pi)(Ac_1 - D_j^*). \tag{9}$$

Yet it is easy to verify that $\gcd(\Pi, \partial(\Pi)) = 1$. Indeed, if $F | \gcd(\Pi, \partial(\Pi))$ then also $F | (u-v)\partial(\Pi) = \Pi - \Pi^*$ and hence $F | \Pi^*$ as well. However, $\gcd(\Pi, \Pi^*) = 1$.

This fact, together with (9), shows that $\Pi | Ac_1 - D_j^*$; but $\deg(Ac_1 - D_j^*) = j-1 < \deg \Pi$ and hence $Ac_1 = D_j^*$. In particular, $c_1 | D_j^*$ or $u+v | D_j$. But this leads to a contradiction. Indeed, analyzing in Example 3.8 the expression for $(u-v)D_j$, one sees that the first product is divisible by $u+v$ (take $l = h$) but the second is not. Hence, (7) is true. $\square$

By arguments similar to the case for $H_G^*(X^{ss})$, for $H_G^*(X^s) = H^*(X^s//G)$ one has the following.

Corollary 4.9. $H^{\mathrm{odd}}(X^s//G, \mathbb{Q}) = 0$, *and the Poincaré series of* $H^*(X^s//G)$ *is the polynomial* $P(t)$ *introduced in* (5).

Let I'' be the ideal in $H^*(BG) = \mathbb{Q}[c_1, c_2]$ generated by the columns $E_1^{>0,*}$; then one has the ring isomorphism $H^*(X^s//G) = \mathbb{Q}[c_1, c_2]/I''$. Now we consider two special elements of I'': the Thom polynomials Tp_{λ_0} and $\mathrm{Tp}_{\lambda'_0}$, where $\lambda_0 = (1^h, h)$ and $\lambda'_0 = (1^{h-2}, 2, h)$. Their degrees are $h-1$ and h, respectively. We shall now verify that they are relative prime. Indeed, with notation as before we have $\mathrm{Tp}_{\lambda_0} = \partial(\Pi)$ (from Example 3.7). Moreover, from Example 3.8 and Remark 3.9(3) it follows that $\mathrm{Tp}_{\lambda'_0} = h(h-1)\partial((u+3v)\Pi) = h(h-1)[(v+3u)\partial\Pi - 2\Pi]$. In particular, in $\mathbb{Q}[u, v]$ we have $\gcd(\mathrm{Tp}_{\lambda_0}, \mathrm{Tp}_{\lambda'_0}) = \gcd(\Pi, \partial\Pi)$, which is 1 by the proof of Proposition 4.7. Then the usual Poincaré polynomial argument shows that $I'' = (\mathrm{Tp}_{\lambda_0}, \mathrm{Tp}_{\lambda'_0})$.

We have thus proved the following theorem.

Theorem 4.10. *For* $d = 2h$,

$$H^*(X^s//G;\mathbb{Q}) = \mathbb{Q}[c_1,c_2]/(\mathrm{Tp}_{\lambda_0},\mathrm{Tp}_{\lambda'_0}) \quad \textit{and}$$
$$H^*(X^{ss}//G;\mathbb{Q}) = \mathbb{Q}\langle 1\rangle \oplus \frac{c_1\mathbb{Q}[c_1,c_2]}{(\mathrm{Tp}_{\lambda_1},\mathrm{Tp}_{\lambda_2})},$$

and the restriction map $H^*(X^{ss}//G) \to H^*(X^s//G)$ *is induced by the identity of* $\mathbb{Q}[c_1,c_2]$.

Remark 4.11. Since the quotient $X^{ss}//G$ has a unique singular point ($d = 2h$), its intersection cohomology can be computed from the cohomology description of $X^s//G$ and $X^{ss}//G$ by using (the intersection cohomology version of) Theorem 3.5.1 in [K3]. Namely, in our case we have $IH^{\mathrm{odd}}(X^{ss}//G) = 0$ and, for i even,

$$IH^i(X^{ss}//G) = \begin{cases} H^i(X^s//G) & \text{if } i < d-3, \\ H^i(X^{ss}//G) & \text{if } i > d-3. \end{cases} \tag{10}$$

Since the Poincaré polynomials of $X^s//G$ and $X^{ss}//G$ are $P(t)$ and $1+tP(t)$, respectively (see (5)), we obtain that the Poincaré polynomial of $IH^*(X^{ss}//G)$ is

$$\frac{(1-t^{2\lfloor d/4\rfloor})(1-t^{2\lceil d/4\rceil})}{(1-t)(1-t^2)} \quad (\deg t = 2), \tag{11}$$

where $\lfloor\ \rfloor$ is the (usual, "lower") integer part function and $\lceil\ \rceil$ is the upper integer part function. The same intersection cohomology Poincaré polynomial (in a different disguise) can be computed using the method of [K2] (see e.g. [Ki]) as

$$\frac{1+t+\cdots+t^d}{1-t^2} - \sum_{d/2<r\le d}\frac{t^{r-1}}{1-t} - \frac{t^{2\lfloor d/4\rfloor}}{1-t^2}.$$

Discussion 4.12 (The cohomology ring of the link). Denote by L^{ss} the link of the unique semisimple point p^{ss} in $X^{ss}//G$; that is, $L^{ss} = \rho^{-1}(\varepsilon)$, where $\rho\colon X^{ss}//G \to [0,\infty)$ is a real analytic map with $\rho^{-1}(0) = \{p^{ss}\}$ and where ε is sufficiently small. Write CL^{ss} for the real cone over L^{ss} (i.e., $CL^{ss} = [0,1]\times L^{ss}/\{0\}\times L^{ss}$). Then $H^*(CL^{ss},L^{ss}) = H^*(X^{ss}//G, X^s//G)$ and hence $H^*(L^{ss})$ is completely determined by the restriction morphism from Theorem 4.10. In fact, L^{ss} is a rational homological manifold of real dimension $4h-7$ (with Poincaré duality). (This can also be proved as follows: The geometric quotient of the set of ordered d-points of $\mathbb{P}^1$ is smooth, and there exist only finitely many ordered semisimple points; hence L^{ss} is the quotient by a finite permutation group of a smooth $(4h-7)$-dimensional link.) Theorem 4.10, this duality, and a computation then yield our final result.

Theorem 4.13. $H^*(L^{ss},\mathbb{Q})$ *can be generated by two elements,* c_2 *of degree* 4 *and* g (*the Poincaré dual of* $c_2^{[h/2]-1}$) *of degree* $4h-4[h/2]-3$, *with relations* $c_2^{[h/2]} = 0$ *and* $g^2 = 0$. (*Notice that all the Betti numbers are* 0 *or* 1.)

Remark 4.14. Theorem 4.10 implies that the cohomology ring of the quasi-projective variety $X^s//G$ of (complex) dimension $d-3$ shares the Poincaré duality properties of a smooth projective variety of dimension $d-4$. In fact, cohomologically (over $\mathbb{Q}$), $X^s//G$ behaves like a line bundle $\mathcal{L}$ with Chern class c_1 over a smooth projective variety M with cohomology $\mathbb{Q}[c_1, c_2]/(\partial\Pi, \Pi)$, and $X^{ss}//G$ behaves like the Thom space of this line bundle (or, equivalently, like the complex cone over M associated with $\mathcal{L}$). In particular, L^{ss} has the cohomology of the S^1-bundle of $\mathcal{L}$.

REMARK 4.15. Assume that $d = 2h+1$ is odd. It is tempting to compare the moduli space $X^s//G$ with the (possibly weighted) Grassmanian $\mathrm{Gr}_2\mathbb{C}^{h+1}$, since the presentation of their cohomology rings have the same structure $\mathbb{Q}[c_1, c_2]/(\partial p_1, \partial p_2)$ (where $\deg p_1 = h+1$ and $\deg p_2 = h+2$) and share the same Betti numbers. Indeed, for the Grassmanian we can take $p_1 = u^{h+1}$ and $p_2 = u^{h+2}$. In fact, this analogy can be continued: in both cases, the set of relations are guided by some nice generating function as follows. Set $\Pi_0 := 1$ and $\Pi_j := \prod_{l=0}^{j-1}(lv + (d-l)u)$, and consider the generating function

$$\mathcal{G}(q) = \sum_{j\geq 0} \mathcal{G}_j q^j := \sum_{j\geq 0} \frac{\Pi_j q^j}{j!} = [1 + (u-v)q]^{du/(u-v)} \in \mathbb{Q}[u, v][[q]].$$

Then $H^*(X^s//G) = \mathbb{Q}[c_1, c_2]/I$, where I is generated by $\partial\mathcal{G}_j$, $j > h$.

In the Grassmanian case the same fact is true with $\mathcal{G}(q) = 1+uq+u^2q^2+\cdots = 1/(1-uq)$. However, easy computation shows that, as graded rings, these cohomology rings are *not* isomorphic (except for small d).

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