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Linear Symmetric Determinantal Hypersurfaces

Jens Piontkowski

The question of which equations of hypersurfaces in the complex projective space can be expressed as the determinant of a matrix whose entries are linear forms is classical. In 1844 Hesse [He] proved that a smooth plane cubic has three essentially different linear symmetric representations. Dixon [Di] showed in 1904 that, for smooth plane curves, linear symmetric determinantal representations correspond to ineffective theta-characteristics—that is, ineffective divisor classes whose double is the canonical divisor. Barth [B] proved the corresponding statement for singular plane curves. The general case for any hypersurface was treated by Catanese [C], Meyer-Brandis [M-B], and Beauville [Be].

Any plane curve has a linear symmetric determinantal representation [Be, 4.4], but every linear symmetric determinantal surface is singular. By 1865 Salmon knew that such a surface of degree n possesses in general $\binom{n+1}{3}$ nodes [S, p. 495], and Cayley [Ca] examined the position of these. Catanese [C] studied these surfaces with only nodes in a more general context. Here we are dealing mainly with the question of which combinations of singularities can occur on a linear symmetric determinantal cubic or quartic surface. For the cubics we find all their linear symmetric representations and obtain in particular the following theorem.

Theorem. *There are four types of linear symmetric determinantal cubic surfaces with isolated singularities. The combinations of their singularities are given by the subgraphs of $\tilde{E}_6$ that are obtained by removing some of the white dots in Figure 1. In addition, all nonnormal cubics* (*with the exception of the union of a smooth quadric with a transversal plane*) *are linear symmetric determinantal cubics.*

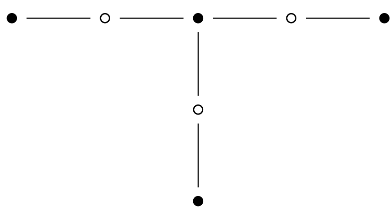

Figure 1

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The combination of isolated singularities that occur on a linear symmetric determinantal quartic can be described similarly but with more Dynkin diagrams as starting points for the splitting process.

The author's original motivation for this study was the desire to understand linear maps from a vector space V into the space of symmetric matrices, which occur for example in the examination of focal varieties (see e.g. [FPi, Sec. 2.2.4]). Such a map can be understood as a symmetric matrix M whose entries are linear forms on V, and $\det M$ describes the locus of V that is mapped to symmetric matrices of reduced rank. For $\dim V = 2$ such maps are classified up to the choice of coordinates classically [Ga, Sec. 12.6]. The case of $n = 2$ and arbitrary dimension of V is easy, and the case of $n = 3$ is treated in course of proving the foregoing Theorem. For $n = 4$ and $\dim V = 3$, the classification can be obtained with the methods used here if the linear symmetric determinantal quartic is a normal rational surface. However, if the quartic has only rational singularities, these methods are not constructive because Torelli-type theorems are used. This corresponds to the fact that, although every possible combination of rational double points on a quartic is known (by the work of Urabe [U1; U2] and Yang [Y]), equations for most of these surfaces are unknown.

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1. General Definitions and Statements

DEFINITION 1.1. Let $M \in \mathrm{Sym}(n, V^*)$ be a symmetric $n \times n$ matrix whose entries are linear forms on a vector space V over $\mathbb{C}$. If $F := \det M$ is not zero, then it determines a *linear symmetric* (*determinantal*) *hypersurface* of degree n in $\mathbb{P}(V)$. Two matrix representations M and M' of F are *equivalent* if there is a $T \in \mathrm{GL}(n, \mathbb{C})$ with $M' = T^t M T$. A matrix representation M will be called *nondegenerate* if the induced map $V \to \mathrm{Sym}(n, \mathbb{C})$, $v \mapsto M(v)$, is injective.

We note that the hypersurface F of a degenerate matrix representation M will be a cone over the kernel of the induced map.

Often M will be obtained by choosing some matrices $A_0, \dots, A_N$ and setting $M := \sum_{i=0}^N x_i A_i$, where the x_i are a basis of $(\mathbb{C}^{N+1})^*$. The representation M will be *nondegenerate* if the matrices $A_0, \dots, A_N$ are linearly independent. Choosing different generators $A'_0, \dots, A'_N$ of the space $\mathrm{span}\{A_0, \dots, A_N\}$ corresponds to a projective transformation of $\mathbb{P}^N$. Thus the hypersurface $F = \det M$ is determined up to projective equivalence by the choice of the linear space $\mathcal{A} := \mathrm{span}\{A_i\} \subseteq \mathrm{Sym}(n, \mathbb{C})$. In fact, we may view F as the intersection of $\mathbb{P}(\mathcal{A}) \subseteq \mathbb{P}(\mathrm{Sym}(n, \mathbb{C}))$ with the general determinantal hypersurface $V(\det)$, or as a cone over such a construction if we started with a degenerate representation.

One might expect that the linear symmetric hypersurfaces form a Zariski-closed subset of all hypersurfaces of degree n. However, this may be false because the map

$$\mathbb{P}(\mathrm{Sym}(n, V^*)) \dashrightarrow \mathbb{P}(\text{polynomials of degree } n), \ \ [M] \mapsto [\det M],$$

is only a rational map and is not regular for $n \geq 2$. Hence, the set of linear symmetric hypersurfaces is only constructible.

As is well known, the locus of corank-1 matrices is precisely singular along the locus of corank ≥ 2 matrices. Therefore, singularities of F appear if either $\mathbb{P}(\mathcal{A})$ intersects $V(\det)$ at a matrix of corank ≥ 2 or tangentially at a corank-1 matrix. We use this in the following definition.

Definition 1.2. A singular point $x \in F$ is called an *essential* singularity if the corank of $M(x)$ is greater or equal to 2; otherwise it is called an *accidental* singularity.

The accidental singularities are difficult to control. Luckily, we can prove that, for small sizes of the matrix M, only certain types of singularities can occur.

Proposition 1.3. *Let F be a linear symmetric determinantal hypersurface of degree n in $\mathbb{P}^N$. Then the isolated accidental singularities of F are of corank less than or equal to $n - N - 1$. (Here corank denotes the corank of the Hesse matrix of F at the singular point.)*

In particular, a linear symmetric cubic in $\mathbb{P}^3$ has no isolated accidental singularities, a quartic has only nodes, and a quintic has only A_k-singularities.

Before starting with the proof of this proposition we show the following lemma, which enables us to identify some of the nonisolated singularities of a linear symmetric hypersurface. This statement was known to Salmon [S, p. 495].

Lemma 1.4. *Let $M = (m_{ij})$ be a linear symmetric $n \times n$ matrix with $m_{11} = 0$. Then the hypersurface $F = \det M$ is singular along $V(m_{12}, \ldots, m_{1n})$.*

Proof. We expand the determinant F by the Leibniz formula. Then each summand of

$$\frac{\partial F}{\partial x_j} = \sum_{\sigma \in S(n)} \sum_{i=1}^{n} \operatorname{sgn} \sigma \cdot m_{1\sigma(1)} \cdot \ldots \frac{\partial m_{i\sigma(i)}}{\partial x_j} \ldots \cdot m_{n\sigma(n)}$$

contains $\partial m_{11}/\partial x_j = 0$, $m_{1\sigma(1)}$, or $m_{\sigma^{-1}(1)1} = m_{1\sigma^{-1}(1)}$; hence, it vanishes on $V(m_{12}, \ldots, m_{1n})$. $\square$

Proof of Proposition 1.3. Assume that we are examining the point $p = (1 : 0 : \ldots : 0)$. Since p is an accidental singularity, it follows that corank $A_0 = 1$ and we can choose coordinates on the $\mathbb{C}^r$ such that

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We set $x_0 = 1$ and write

$$M = \begin{pmatrix} f_{11} & f_{12} & f_{13} & \cdots & f_{1n} \\ f_{12} & 1+f_{22} & f_{23} & \cdots & f_{2n} \\ f_{13} & f_{23} & \ddots & & f_{3n} \\ \vdots & \vdots & & \ddots & \vdots \\ f_{1n} & f_{2n} & f_{3n} & \cdots & 1+f_{nn} \end{pmatrix} \quad \text{with } f_{ij} \in \mathbb{C}[x_1, \dots, x_N].$$

Obviously, the linear part of $F = \det M$ is f_{11}, which must vanish in order for p to be singular. Looking at

$$F = \det M = \sum_{\sigma \in S(n)} \operatorname{sgn} \sigma \cdot m_{1\sigma(1)} \cdot \ldots \cdot m_{n\sigma(n)},$$

we see that the quadratic part of F is due to the summands, where $n-2$ of the $m_{i\sigma(i)}$ are of order 0; that is, where σ is a transposition of 1 with $i \in \{2, \dots, n\}$. Hence the quadratic part of F is

$$-\sum_{i=2}^{n} (f_{1i})^2.$$

The Hessian of F in p is the associated symmetric $N \times N$ matrix S of this quadric. Our task is to show that the rank of S is at least $2N - n + 1$. By Lemma 1.4 there are N linearly independent forms among $f_{12}, \dots, f_{1n}$ because the point p is an isolated singularity. Let us assume that $f_{12}, \dots, f_{1N+1}$ are linearly independent; then the associated symmetric $N \times N$ matrix $\tilde{S}$ of the quadric $-\sum_{i=2}^{N+1} f_{1i}^2$ has rank N. The symmetric matrices $S_{N+2}, \dots, S_n$ associated to $f_{1N+2}^2, \dots, f_{1n}^2$ have rank 1 or 0. From $\tilde{S} = S + \sum_{i=N+2}^{n} S_i$ we find

$$N = \operatorname{rank} \tilde{S} \le \operatorname{rank} S + \sum_{i=N+2}^{n} \operatorname{rank} S_i \le \operatorname{rank} S + n - N - 1$$

and so $\operatorname{rank} S \ge 2N - n + 1$. □

REMARK. We will soon see that an essential singularity of a linear symmetric hypersurface can never be an A_{2k}-singularity, but an accidental singularity may as well be one. For example, the quintic given as the determinant of the matrix

$$\begin{pmatrix} 0 & x & y & z & \sqrt{-1}z \\ x & w+z & 0 & 0 & 0 \\ y & 0 & w+y & 0 & 0 \\ z & 0 & 0 & w+x & \sqrt{-1}z \\ \sqrt{-1}z & 0 & 0 & \sqrt{-1}z & w \end{pmatrix}$$

has an A_2-singularity at $(1:0:0:0)$. It seems likely that, as the size of the matrix increases, all types of singularities will occur as accidental singularities.

We turn now to examining the essential singularities. First, we will count them. The following statement was known to Salmon [S, p. 495].

Proposition 1.5. *The general linear symmetric determinantal hypersurface F of degree n has only essential singularities, and its singular locus has codimension* 2 *and degree* $\binom{n+1}{3}$.

In particular, a general linear symmetric surface $F \subset \mathbb{P}^3$ has $\binom{n+1}{3}$ essential A_1-singularities.

Proof. For the first statement we view F as $V(\det) \cap \mathbb{P}(\mathcal{A}) \subseteq \mathbb{P}(\mathrm{Sym}(n, \mathbb{C}))$. A general linear space $\mathbb{P}(\mathcal{A}) \subseteq \mathbb{P}(\mathrm{Sym}(n, \mathbb{C}))$ intersects $V(\det)$ transversally, so there are no accidental singularities. The locus of corank ≥ 2 matrices has codimension 3 in $\mathbb{P}(\mathrm{Sym}(n, \mathbb{C}))$ and degree $\binom{n+1}{3}$ [HTu]. Since its intersection with $\mathbb{P}(\mathcal{A})$ consists of the essential singularities of F, the first statement follows.

For the second statement one must show that a general essential singularity is a node. This can be done with arguments similar to those used in the proof of Proposition 1.3. □

Cossac [Co] studied these general linear symmetric surfaces in degree 4 by taking up ideas of Cayley; in particular, he pointed out their connection to Enriques surfaces. In order to examine the essential singularities further, we localize our definitions.

Definition 1.6. A *local symmetric matrix representation* of a power series $f \in \mathbb{C}[[x_1, \dots, x_N]]$ is a symmetric matrix $M \in \mathrm{Sym}(r, \mathbb{C}[[x_1, \dots, x_N]])$ with $\det M = f$. Two matrix representations M and M' are *equivalent* if there exists a $T \in \mathrm{GL}(r, \mathbb{C}[[x_1, \dots, x_N]])$ such that $M' = T^t M T$. A matrix representation M is *essential* if corank $M(0) \geq 2$ and is *reduced* if $M(0) = 0$.

If one considers the equation of the power series f only up to a choice of holomorphic coordinates, then it is convenient to extend the above definition of equivalence by allowing changes of coordinates as well. It is enough to consider only reduced matrix representations by virtue of the following well-known lemma.

Lemma 1.7. *Any local symmetric matrix representation M of a power series $f \in \mathbb{C}[[x_1, \dots, x_N]]$ is equivalent to*

$$\begin{pmatrix} \tilde{M} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

where $\tilde{M}$ is a reduced local symmetric matrix representation of f.

Not every singularity has an essential local symmetric matrix representation. For the ADE-singularities we have the following result.

Theorem 1.8. *The surface singularities A_{2k}, E_6, and E_8 have no essential local symmetric matrix representation. The reduced essential symmetric matrix representations for A_{2k+1}, D_{2k}, D_{2k+1}, and E_7 are, up to equivalence, as shown in Table 1. The symbols in parentheses in the first column denote the specific matrix*

Table 1

Singularity X	Equation	Matrix representation M	$l(X)$
A_{2k+1} ($A^\bullet_{2k+1}$)	$-x^2+y^2-z^{2k+2}$	$\begin{pmatrix} y+z^{k+1} & x \\ x & y-z^{k+1} \end{pmatrix}$	$k+1$
D_{2k} ($D^\bullet_{2k}$)	$-x^2+y^2z-z^{2k-1}$	$\begin{pmatrix} z & x \\ x & y^2-z^{2k-2} \end{pmatrix}$	2
D_{2k} ($D^\pm_{2k}$)		$\begin{pmatrix} y\pm z^{k-1} & x \\ x & z(y\mp z^{k-1}) \end{pmatrix}$	k
D_{2k+1} ($D^\bullet_{2k+1}$)	$-x^2+y^2z-z^{2k}$	$\begin{pmatrix} z & x \\ x & y^2-z^{2k-1} \end{pmatrix}$	2
E_7 ($E^\bullet_7$)	$-x^2+z^3+zy^3$	$\begin{pmatrix} z & x \\ x & z^2+y^3 \end{pmatrix}$	3

representation of the singularity from now on. The last column gives the length of the first Fitting ideal, F_1M, of the matrix representation of the singularity, which is here the ideal generated by the entries of the matrix.

Proof. Let M be a local symmetric matrix representation of an ADE-singularity that is given by the equation $f = \det M$. We set $R = \mathbb{C}[[x,y,z]]/(f)$. Then $\hat{M} = \operatorname{coker} M$ is a maximal Cohen–Macaulay module of rank 1 [Yo, Chap. 7]. Owing to the symmetry of M, we obtain a surjection $\hat{M} \twoheadrightarrow \operatorname{Hom}_R(\hat{M}, R)$. Such a module $\hat{M}$ is called a *contact module;* it determines the matrix M up to equivalence (see [KUl, Sec. 2] or [M-B, Sec. 3.34]). Over the local ring of an ADE-surface singularity there exists only a finite number of irreducible modules. This was proven by Auslander as follows: Recall that, for each of the ADE-surface singularities, there exists a group $G \subset \mathrm{GL}(2,\mathbb{C})$ such that the invariant subring $\mathbb{C}[[x,y]]^G$ is isomorphic to the local ring R of the singularity. Auslander exhibited a bijection between these irreducible modules and the irreducible representations of G [Yo, Chap. 10].

Because a contact module has rank 1, we are interested only in the irreducible rank-1 modules that are not isomorphic to R. There are k for A_k, three for D_k, two for E_6, one for E_7, and none for E_8 [Yo, p. 95]. This already proves the claim for D_{2k}, E_7, and E_8. For the other singularities one uses Auslander's bijection to work out the representation matrices in Table 2 for the irreducible modules of rank 1 (besides those that occur in Table 1).

Kleiman and Ulrich [KUl, 2.2] showed that, if an R-module of rank 1 represented by an $r \times r$ matrix M is a contact module, then there exists a matrix $T \in \mathrm{GL}(r, \mathbb{C}[[x,y,z]])$ such that TM is symmetric. Since we are dealing only with 2×2 matrices, this condition is easy to check. Let

Table 2

Singularity	Standard equation	Representation matrix
A_k	$-xy + z^{k+1}$	$M_i = \begin{pmatrix} z^i & y \\ x & z^{k+1-i} \end{pmatrix}$ for $1 \le i \le k$
D_{2k+1}	$-x^2 + zy^2 + z^{2k}$	$M_\pm = \begin{pmatrix} z^k \pm x & yz \\ y & z^k \mp x \end{pmatrix}$
E_6	$x^2 - y^3 - z^4$	$M_\pm = \begin{pmatrix} x \pm z^2 & y^2 \\ y & x \mp z^2 \end{pmatrix}$

$$T = \begin{pmatrix} g_1 & g_2 \\ f_2 & f_1 \end{pmatrix} \quad \text{with } g_1 f_1 - g_2 f_2 \in \mathbb{C}[[x, y, z]]^*.$$

For A_k, the symmetry of TM_i is equivalent to

$$xf_1 + z^i f_2 = yg_1 + z^{k+1-i} g_2.$$

Clearly we have $f_1(0) = g_1(0) = 0$, which implies that $f_2(0) \cdot g_2(0) \neq 0$ for T to be invertible. Therefore, $i = k + 1 - i$; that is, $k + 1$ is even and $i = (k + 1)/2$. Hence, there can be no contact module for A_{2k} and only one for A_{2k+1}. Completely analogous arguments work for the matrices $M_\pm$ for D_{2k+1} and E_6.

The computation of the length of the Fitting ideals is simple. Denoting $S := \mathbb{C}[[x, y, z]]$, we have:

$$\begin{aligned} l(A^\bullet_{2k+1}) &= \dim S/(x, y + z^{k+1}, y - z^{k+1}) = \dim S/(x, y, z^{k+1}) = k + 1, \\ l(D^\bullet_{2k}) &= \dim S/(x, z, y^2 - z^{2k-2}) = \dim S/(x, z, y^2) = 2, \\ l(D^\pm_{2k}) &= \dim S/(x, y \pm z^{k-1}, z(y \mp z^{k-1})) = \dim S/(x, z^k, y \pm z^{k-1}) = k, \\ l(D^\bullet_{2k+1}) &= \dim S/(x, z, y^2 - z^{2k-1}) = \dim S/(x, z, y^2) = 2, \\ l(E^\bullet_7) &= \dim S/(x, z, z^2 + y^3) = \dim S/(x, z, y^3) = 3. \end{aligned}$$ □

Often it does not make much sense to distinguish between the representations D^+_{2k} and D^-_{2k} because the automorphism of the local ring of the singularity induced by $x \mapsto -x$, $y \mapsto -y$ swaps them. Theorem 1.8 severely restricts the possible combinations of essential singularities on a linear symmetric surface, as follows.

Corollary 1.9. *Let F be a linear symmetric determinantal surface of degree n in $\mathbb{P}^3$ whose essential singularities $X_1, \dots, X_t$ are ADE-singularities. Then $X_i \in \{A^\bullet_{2k+1}, D^\bullet_{2k+1}, D^\bullet_{2k}, D^\pm_{2k}, E^\bullet_7\}$ and*

$$\sum_{i=1}^{t} l(X_i) = \binom{n+1}{3}.$$

Proof. This follows in a manner similar to Proposition 1.5. We view F as $\mathbb{P}(\mathcal{A}) \cap V(\det) \subset \mathbb{P}(\mathrm{Sym}(n, \mathbb{C}))$. Let I_i be the vanishing ideal of symmetric matrices of corank $\geq i$. Then we find the essential singularities as the intersection of $\mathbb{P}(\mathcal{A})$ and $V(I_2)$, so the sum of their intersection multiplicity is $\deg V(I_2) = \binom{n+1}{3}$. By Theorem 1.8 we see that the essential ADE-singularities appear only at corank-2 matrices and never at matrices of higher corank. Since $V(I_2)$ is smooth outside $V(I_3)$, the local intersection multiplicities of $\mathbb{P}(\mathcal{A})$ and $V(I_2)$ can be found by computing locally the length of the sum of the ideal I_2 and the vanishing ideal of $\mathbb{P}(\mathcal{A})$—that is, by computing locally the length of the first Fitting ideal of matrix representation. This was done for the various singularities in Theorem 1.8. □

REMARK. We will see later that linear symmetric cubics and quartics cannot have D_{2k+1}-singularities. However, here is a quintic with an essential D_5-singularity at $x = y = z = 0$, showing that essential D_{2k+1}-singularities are in fact possible:

$$\begin{pmatrix} 0 & 665x & -2y+z & 3y+z & 2y+4z \\ 665x & 2y & -2771x & 6606x & 7138x \\ -2y+z & -2771x & 26y-6z & 0 & 4z+w \\ 3y+z & 6606x & 0 & w & 0 \\ 2y+4z & 7138x & 4z+w & 0 & 224y+136z \end{pmatrix}.$$

A linear symmetric representation of $F \subset \mathbb{P}^3$ is closely related to the contact surfaces of F (a surface G is a *contact surface* if the intersection $G \cap F$ is twice a curve C). These partially classical ideas, which are connected with the Hilbert–Burch theorem, were recently refined by Beauville [Be], Catanese [C], Eisenbud [E1], Kleiman and Ulrich [KUl], and Meyer-Brandis [M-B]. The next few pages are devoted to extending Catanese's results for even sets of nodes to sets of ADE-singularities.

While studying contact surfaces, one also encounters nonlinear symmetric matrices; hence the following definition will be useful.

DEFINITION 1.10. A symmetric matrix $M = (m_{ij}) \in \mathrm{Sym}(r, \mathbb{C}[x_0, \dots, x_N])$ is *homogeneous* if all its entries are homogeneous polynomials and if $\deg m_{ii} + \deg m_{jj} = 2 \deg m_{ij}$ for all $i, j = 1, \dots, r$. The degree of M is $\deg M := (d_1, d_2, \dots, d_r)$, where $d_i := \deg m_{ii}$. By permutation of the rows and columns, one obtains $d_1 \leq d_2 \leq \cdots \leq d_r$. For a homogeneous matrix M, the determinant $F = \det M$ is a homogeneous polynomial of degree $n = \sum_{i=1}^{r} d_i$. Such an F is called a *symmetric* (*determinantal*) *hypersurface*.

A matrix M is linear if and only if $d_1 = \cdots = d_r = 1$. A consequence of the homogeneity of M is that adj M, the adjoint matrix of M, is also homogeneous. From the adjoint matrix one obtains contact surfaces. Various versions of the following well-known lemma have appeared in the literature; we repeat the proof for the reader's convenience.

LEMMA 1.11. *Let $F = \det M$ be a symmetric surface in $\mathbb{P}^3$ and let m^{ii} be a diagonal entry of* adj M*. Assume that no component of* $\mathrm{div}_F(m^{ii})$ *is contained in the essential singular locus of F. Then* $\mathrm{div}_F(m^{ii}) = 2C$*, where C is a Cartier divisor outside the essential singularities of F.*

Proof. The proof is based on the Laplace identity (see [KUl, Sec. 2.4] or [C, (1.3)]) stating that

$$Fm^{ik,jl} = m^{kj}m^{il} - m^{kl}m^{ij},$$

where the m^{ij} are entries of the adjoint matrix and $m^{ik,jl}$ is $(-1)^{i+j+k+l}$ times the determinant of the matrix M with the rows i, k and columns j, l deleted. Setting $k = j$ and $l = i$ yields $m^{ii}m^{jj} = (m^{ij})^2$ modulo F, so

$$\mathrm{div}_F(m^{ii}) + \mathrm{div}_F(m^{jj}) = 2\,\mathrm{div}_F(m^{ij}). \tag{$*$}$$

This formula also implies that, at the zero locus of $m^{11} = \cdots = m^{rr} = 0$ on F, all m^{ij} (and with them adj M) vanish. Therefore, the divisors $\mathrm{div}_F(m^{ii})$ cannot have a common component outside the essential singular locus and hence $(*)$ shows that all components in $\mathrm{div}_F(m^{ii})$ occur with even multiplicity. Finally, $m^{ii}m^{jj} = (m^{ij})^2 \bmod F$ shows that C is Cartier outside the essential singularities. □

If instead of only M one uses all equivalent matrix representations of F, then one obtains a whole system of contact surfaces [M-B, Sec. 2.1]. From now on we restrict our attention to symmetric surfaces whose essential singularities are ADE-singularities. To understand their contact surfaces it is important to examine the local symmetric ADE-singularities as described in Theorem 1.8.

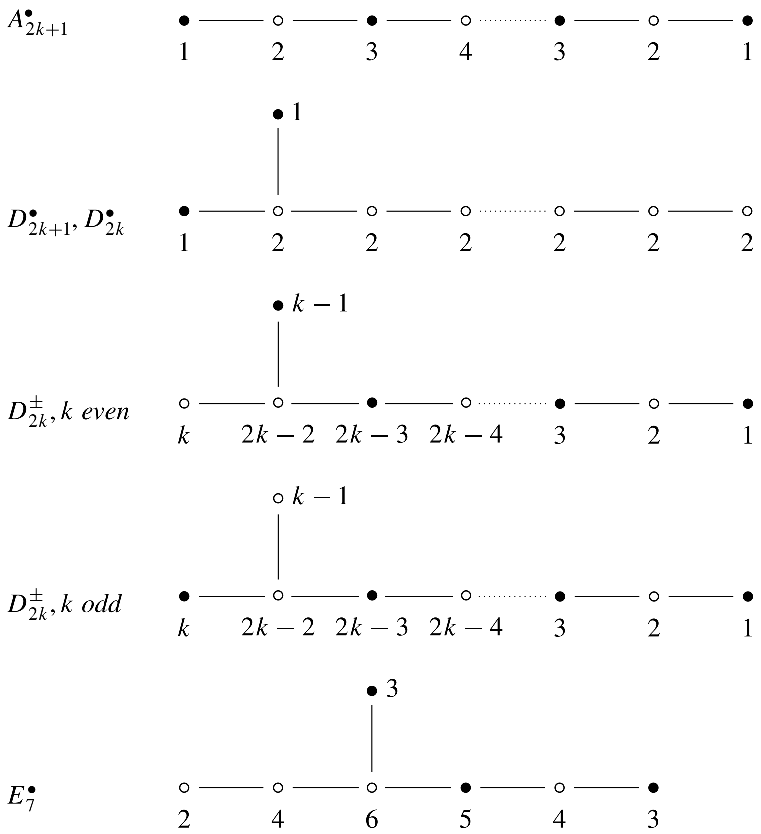
DEFINITION 1.12. Let $X \in \{A^\bullet_{2k+1}, D^\bullet_k, D^\pm_{2k}, E^\bullet_7\}$ be one of the essential symmetric surface singularities with equation $f = \det M$. The *Fitting cycle* of X on the minimal resolution $\pi : \tilde{X} \to X$ is defined as

$$Z_X := \gcd\{\mathrm{div}_{\tilde{X}}(\pi^*g) \mid \text{for all } g \in F_1 M\}.$$

Let g be a local contact surface induced by M (e.g., one of the main corank-1 minors). The *parity diagram* of X is the minimal resolution graph G_X of X where the vertices are marked as follows: a vertex of G is drawn as $\bullet$ if the corresponding curve occurs with odd multiplicity in the total transform π^*g of g and is drawn as $\circ$ otherwise.

The generalized Laplace identity [M-B, Sec. 2.2] implies that the parity diagrams are the same for equivalent matrix representations and are thus well-defined. Let us compute them.

PROPOSITION 1.13. *The essential symmetric surface ADE-singularities have the following parity diagrams and Fitting cycles, where the multiplicity of an exceptional rational curve in the Fitting cycle is noted near the vertex representing this curve in the Dynkin diagram.*



In particular, the number of $\bullet$-vertices in the parity diagram is the length of the first Fitting ideal of the matrix representation, and the self–intersection number of the Fitting cycle is -2 times the length of the Fitting ideal; *that is,* $(Z_X)^2 = -2l(X)$. *Furthermore,* $Z_X \cdot E \leq 0$ *for any exceptional curve* E.

Proof. By Theorem 1.8 we need only resolve the singularities while keeping track of the divisors given by the matrix entries. Such a task is traditionally left to the interested reader. □

We return now to the global situation.

DEFINITION 1.14. Let $F \subset \mathbb{P}^3$ be a surface and let $P = \{p_1, \ldots, p_t\} \subset F$ be a set of singular points of type A_{2k+1}, D_k, or E_7 on F. To each of these singularities assign an essential symmetric surface ADE-singularity symbol of the same underlying type. That is, for A_{2k+1}, D_{2k+1}, and E_7 one uses $A^\bullet_{2k+1}$, $D^\bullet_{2k+1}$, and $E^\bullet_7$ (respectively), but for D_{2k} one may choose between $D^\bullet_{2k}$ and $D^\pm_{2k}$. Let $X = \{X_1, \ldots, X_t\}$ be the resulting set. Further, let $\pi\colon \tilde{F} \to F$ be the minimal resolution of F in these points and let H be the pull-back of a hyperplane divisor of F to $\tilde{F}$.

The set X is said to be *even* if, for some $\delta \in \{0,1\}$, the divisor $\delta H + \sum_{i=1}^{t} Z_{X_i}$ is divisible by 2 in $\mathrm{Pic}(\tilde{F})$. The set X is *strictly even* if $\delta = 0$ and is *weakly even* otherwise.

The set X is called (*linearly*) *symmetric* if there is a (linearly) homogeneous symmetric matrix M with $F = \det M$ such that X is precisely the set of essential symmetric singularities of F.

Note that, in the case of a symmetric set of ADE-singularities, the pull-backs of the entries of the adjoint matrix of M define the cycle $\sum_{i=1}^{t} Z_{X_i}$ scheme-theoretically by the definition of the Fitting cycles.

Proposition 1.15. *A symmetric set of ADE-singularities is even.*

Proof. Let $F = \det M$ be the surface that has X as essential singularities, and let G be a contact surface given by a main corank-1 minor of M. Set $l = \deg G$ and $C = \frac{1}{2}\operatorname{div}_F(G)$. Pulling G back to the minimal resolution $\pi\colon \tilde{F} \to F$ yields $\pi^*G = 2\tilde{C} + D$, where $\tilde{C}$ is the strict transform of C and D is a divisor supported on the exceptional set. By the definition of the Fitting cycle, $D - \sum_{i=1}^{t} Z_{X_i}$ is effective as well. Moreover, by Proposition 1.13 we see that, for all singularities, the parity of the multiplicity of the exceptional rational curves in the Fitting cycle is the same as the one in the pull-back of a contact surface; hence $D - \sum_{i=1}^{t} Z_{X_i}$ is divisible by 2, say $D - \sum_{i=1}^{t} Z_{X_i} = 2B$. Altogether, with $\delta = l - 2\lfloor l/2 \rfloor$ we have

$$\operatorname{div}_{\tilde{F}} \pi^*G = lH = \sum_{i=1}^{t} Z_{X_i} + 2(\tilde{C} + B)$$
$$\implies \sum_{i=1}^{t} Z_{X_i} + \delta H = 2(\lceil l/2 \rceil H - \tilde{C} - B);$$

that is, $\sum_{i=1}^{t} Z_{X_i} + \delta H$ is divisible by 2 in $\mathrm{Pic}(\tilde{F})$. □

We want to ensure the existence of contact surfaces for an even set of ADE-singularities with the same properties as G in the foregoing proof.

Proposition 1.16. *Let X be an even set of ADE-singularities on a surface $F \subset \mathbb{P}^3$, and let $\pi\colon \tilde{F} \to F$ be the minimal resolution of F in these singular points. Then there exists a surface $G \subset \mathbb{P}^3$ such that* (a) *its pull-back divisor* $\operatorname{div}_{\tilde{F}} \pi^*G$ *on $\tilde{F}$ contains the Fitting cycles Z_{X_i} for $X_i \in X$ and* (b) *the effective divisor* $\operatorname{div}_{\tilde{F}} \pi^*G - \sum_{i=1}^{t} Z_{X_i} \in \mathrm{Pic}(\tilde{F})$ *is divisible by* 2.

Proof. The proof is the same as the second half of [C, Sec. 3.6]; we repeat it here because it is short and helps to explain the rest of this section. Let L be a divisor such that $2L = \sum_{i=1}^{t} Z_{X_i} + \delta H$, and choose l such that $lH - L$ is linearly equivalent to an effective divisor $\tilde{C}$. Then $(2l - \delta)H = 2\tilde{C} + \sum_{i=1}^{t} Z_{X_i}$; hence, there exists a surface of degree $2l - \delta$ with the required properties. □

From now on the theory of the even sets of ADE-singularities is the same as Catanese's theory of even nodes [C, Secs. 2.16–2.23]. We shall repeat the statements but leave out the proofs if they are identical to those for the node case.

DEFINITION 1.17. Let X be an even set of ADE-singularities on F. The *order* of X is the smallest degree of a surface with the same properties as G in Proposition 1.16.

Let S be the graded ring $\mathbb{C}[x, y, z, w]$, and let $L \in \operatorname{Pic}(\tilde{F})$ be such that $2L = \delta H + \sum_{i=1}^{t} Z_{X_i}$. Then the associated graded S-module of X is

$$R^- = \bigoplus_{l=0}^{\infty} H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH - L)) = \bigoplus_{l=0}^{\infty} H^0(F, (\pi_* \mathcal{O}_{\tilde{F}}(-L))(l)).$$

Note that if $w \in H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH - L))$ and $w' \in H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(l'H - L))$, then $ww' \in H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}((l + l')H - 2L)) = H^0(F, \mathcal{O}_F(l + l' - \delta))$. In particular, if l is the smallest number for which $R_l^- \neq 0$ then, by Proposition 1.16, X has order $2l - \delta$.

LEMMA 1.18. $H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH - L)) \cong H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}((l - \delta)H + L))$.

Proof (cf. [C, Sec. 2.1.5]). Given the long exact cohomology sequence associated to

$$0 \to \mathcal{O}_{\tilde{F}}(lH - L) \to \mathcal{O}_{\tilde{F}}((l - \delta)H + L) \to \bigoplus_{i=0}^{t} \mathcal{O}_{Z_{X_i}}(L) \to 0,$$

it is enough to show that the cohomology group $H^0(Z_{X_i}, \mathcal{O}_{Z_{X_i}}(L))$ vanishes. If there exists a section $s \in H^0(Z_X, \mathcal{O}_{Z_{X_i}}(L))$ then $s^2 \in H^0(Z_{X_i}, \mathcal{O}_{Z_{X_i}}(2L)) = H^0\big(Z_{X_i}, \mathcal{O}_{Z_{X_i}}\big(Z_{X_i} + \delta H + \sum_{j \neq i} Z_{X_j}\big)\big)$, but the last homology group is zero by [R, Ex. 4.14]. □

THEOREM 1.19. *If X is a symmetric set of ADE-singularities on a reduced surface $F = \det M$, then the associated module R^- is a Cohen–Macaulay S-module.*

More precisely: *if* $\deg M = (d_1, \ldots, d_r)$ *then set* $k_i = (n + \delta - d_i)/2$ *and* $l_j = (n + \delta + d_j)/2$, *where* $n = \deg F$ *and* $\delta = n - d_i \bmod 2$. *Then there exists a minimal set of generators* $w_1, \ldots, w_r$ *of* R^- *of degrees* $k_1, \ldots, k_r$ *such that* $w_i w_j = m^{ij}$, *where* $(m^{ij}) = \operatorname{adj} M$. *Moreover* R^- *admits the minimal free resolution*

$$0 \to \bigoplus_{j=1}^{r} S[-l_j] \xrightarrow{(m_{ij})} \bigoplus_{i=1}^{r} S[-k_i] \xrightarrow{(w_j)} R^- \to 0.$$

The order of X is $n - \max\{d_i\}$.

THEOREM 1.20. *Let F be an irreducible and reduced surface of degree n and let X be an even set of ADE-singularities on F. Then the following conditions are equivalent.*

1. *X is symmetric.*
2. *Let $w_1, \ldots, w_r$ be a minimal set of homogeneous generators for the S-module R^-, and set $m^{ij} = w_i w_j \in \bigoplus_{l=0}^{\infty} H(F, \mathcal{O}(l)) = S/(F)$. Then* $\det(m^{ij})$ *is a nonzero polynomial of degree $n(r - 1)$.*
3. *R^- is a Cohen–Macaulay S-module.*
4. *$H^1(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH - L)) = 0$ for all $l \in \mathbb{Z}$.*

Catanese's construction of the symmetric matrix is such that none of the matrix entries is a nonzero constant, because the set of generators of R^- was chosen to be minimal.

PROPOSITION 1.21. *Let F be a surface of degree n with an even set X of ADE-singularities.*

If $l(X) := \sum_{i=1}^{t} l(X_i) \le \binom{n+1}{3}$, then X has order $\le n-1$.

If $l(X) = \binom{n+1}{3}$ and $n = \delta \bmod 2$, then n is divisible by 8.

Proof (following [C, Sec. 2.21]). By the remark that follows Definition 1.17, it suffices to show that $h^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH - L)) \neq 0$ for $2l - \delta \ge n-1$ or even only for $2l \ge n-1$, observing that the order of X is an element of $2l - \delta + 2\mathbb{N}$. By Serre duality and Lemma 1.18,

$$\begin{aligned} h^2(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH - L)) &= h^0(\tilde{F}, \mathcal{O}_{\tilde{F}}((n-4-l)H + L)) \\ &= h^0(\tilde{F}, \mathcal{O}_{\tilde{F}}((n-4-l+\delta)H - L)). \end{aligned}$$

Since $n - 4 - l + \delta \le l - 3 + \delta < l$ it follows that $h^2(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH - L)) \le h^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH - L))$, and it is enough to show $\chi(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH - L)) > 0$. Using $\left(\sum Z_{X_i}\right)^2 = -2l(X)$ from Proposition 1.13, the results of Riemann–Roch yield

$$\begin{aligned} &\chi(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH - L)) \\ &\quad = \chi(\tilde{F}, \mathcal{O}_{\tilde{F}}) + \frac{1}{2}(lH - L)(lH - L - (n-4)H) \\ &\quad = \chi(F, \mathcal{O}_F) \\ &\qquad + \frac{1}{2}\left(\left(l - \frac{\delta}{2}\right)H - \frac{1}{2}\sum Z_{X_i}\right)\left(\left(l - n + 4 - \frac{\delta}{2}\right)H - \frac{1}{2}\sum Z_{X_i}\right) \\ &\quad = 1 + \binom{n-1}{3} + \frac{1}{2}\left(\left(l - \frac{\delta}{2}\right)\left(l - n + 4 - \frac{\delta}{2}\right)n - \frac{1}{2}l(X)\right). \end{aligned}$$

It is not hard to see that this term is positive for $2r \ge n-1$ and $l(X) \le \binom{n+1}{3}$. For further reference we note that, for $l(X) = \binom{n+1}{3}$ and $n - 1 = \delta \bmod 2$,

$$\chi(\tilde{F}, \mathcal{O}_{\tilde{F}}(\lfloor n/2 \rfloor H - L)) = n \quad \text{and} \quad \chi(\tilde{F}, \mathcal{O}_{\tilde{F}}((\lfloor n/2 \rfloor - 1)H - L)) = 0.$$

For $l(X) = \binom{n+1}{3}$ and $n = \delta \bmod 2$, we have

$$\chi(\tilde{F}, \mathcal{O}_{\tilde{F}}((\lceil n/2 \rceil - 1)H - L)) = 3n/8 \in \mathbb{Z},$$

showing that n is divisible by 8. $\square$

THEOREM 1.22. *Let X be an even set of ADE-singularities on a reduced surface $F \subset \mathbb{P}^3$ of degree n. Then X is linearly symmetric if and only if X has length $\binom{n+1}{3}$ and order $n-1$.*

Proof. If X is linearly symmetric, then its order is $n-1$ (by Theorem 1.19) and its length was computed in Corollary 1.9. Alternatively, one can compute the length with the arguments in the proof of Proposition 1.21 and using Theorem 1.20. For the nontrivial reverse implication of the theorem, see Catanese's proof of [C, Sec. 2.23]. □

Catanese showed by example that, in general, the hypothesis on the order of X cannot be dropped.

2. Cubics

Before studying the determinantal cubics, we recall the following beautiful theorem about cubics in $\mathbb{P}^3$; see Bruce and Wall [BrW] or Looijenga [L].

THEOREM 2.1. *The combinations of singularities that can occur on a normal cubic surface in $\mathbb{P}^3$ are precisely the subgraphs of $\tilde{E}_6$ that are obtained by removing some of the points in Figure 2. The nonnormal cubics are the cones over plane cubic curves, the reducible cubics, and two special irreducible types.*

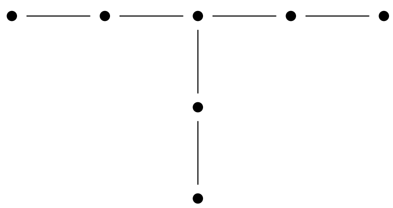


Figure 2

We want to prove a similar statement for linear symmetric cubics. Since all plane cubics have a linear symmetric matrix representation (see Section A), we focus first on nondegenerate linear symmetric representations of the cubic surfaces.

THEOREM 2.2. *There are three nondegenerate linear symmetric determinantal cubics with isolated singularities. Their combinations of the singularities are given by the subgraphs of $\tilde{E}_6$ that are obtained by removing some—but at least one—of the white dots in Figure 3. They all have unique matrix representations up to equivalence.*

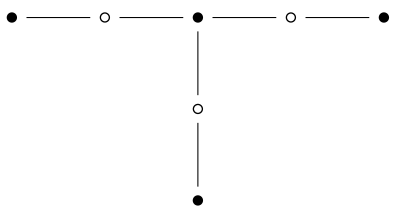


Figure 3

Moreover: of the nonnormal cubics, both special irreducible types as well as the smooth quadric with a tangent plane, the quadric cone with a transversal plane, and the double plane with an additional plane are all nondegenerate linear symmetric cubics; and all their nondegenerate linear symmetric matrix representations are unique.

Including the degenerate matrix representations and with them all cubic cones, we immediately obtain the following corollary.

COROLLARY 2.3. *There are four types of linear symmetric determinantal cubics with isolated singularities. Their combinations of the singularities are given by the subgraphs of Figure 3 that are obtained by removing some of the white dots. The cubics with an elliptic singularity have three matrix representations up to equivalence, the other cubics only one.*

In addition, all nonnormal cubics (with the exception of the smooth quadric with a transversal plane) are linear symmetric cubics.

Proof of Theorem 2.2 (following the outline in [CoD, Prop. 0.5.5], where only the normal cubics are considered). A nondegenerate linear symmetric representation is determined up to equivalence by choosing a 4-dimensional linear subspace $\mathcal{A} \subset \mathrm{Sym}(3,\mathbb{C})$; see the discussion near Definition 1.1. Now there is a nondegenerate symmetric bilinear form on $\mathrm{Sym}(r,\mathbb{C})$ given by

$$\langle\cdot,\cdot\rangle\colon \mathrm{Sym}(r,\mathbb{C}) \times \mathrm{Sym}(r,\mathbb{C}) \to \mathbb{C}, (A,B) \mapsto \mathrm{tr}(A\cdot B) = \sum_{i,j=1}^{r} a_{ij}b_{ij},$$

where tr denotes the trace. Therefore, instead of choosing a 4-dimensional linear subspace $\mathcal{A} \subset \mathrm{Sym}(3,\mathbb{C})$, we may choose dually a 2-dimensional linear subspace $\mathcal{A}^\perp \subset \mathrm{Sym}(3,\mathbb{C})$. There is only a finite number of these pencils of $\mathcal{A}^\perp$; this can be extracted from [Ga, Sec. 12.6], where these pencils together with a choice of basis are classified. However, using the identification of a symmetric 3×3 matrix modulo $\mathbb{C}^*$ with a quadric in $\mathbb{P}^2$, we can also view $\mathcal{A}^\perp \subset \mathrm{Sym}(3,\mathbb{C})$ as a pencil of quadrics in $\mathbb{P}^2$. Then one can see that prescribing the intersection type of two general members of this pencil determines the pencil up to a choice of coordinates. From there one can compute the corresponding determinantal cubic. We will give one example of this and summarize the remaining cases in a table.

Let us assume that two quadrics of the pencil intersect with multiplicities 1, 1, and 2. We choose coordinates such that $(0:0:1)$ and $(0:1:0)$ are the simple intersection points and $(1:0:0)$ is the point where the quadrics intersect with multiplicity 2; that is, they have a common tangent. This tangent cannot pass through $(0:0:1)$ or $(0:1:0)$, because otherwise it would intersect every quadric of the pencil with multiplicity $2+1=3$: it would (by Bezout's theorem) thus be a component of every quadric. Hence, by a further change of coordinates, we may assume that the tangent is spanned by $(1:0:0)$ and $(1:1:1)$. Let $(r:s:t)$ be the coordinates on $\mathbb{P}^2$. Then a quadric q passing through $(1:0:0)$, $(0:1:0)$, and $(0:0:1)$ has the form $ars+brt+cst$. Its tangent in the point $(1:0:0)$ is given by $\operatorname{grad}_{(1,0,0)} q = (0, a, b)$, so passing through $(1:1:1)$ implies $a = -b$. Therefore, the pencil of quadrics is spanned by $2r(s-t)$ and $2st$. These correspond to the symmetric matrices

$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(respectively), which therefore span $\mathcal{A}^{\perp}$. From this, a basis of $\mathcal{A}$ can easily be computed as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and the equation of the cubic is

$$F = \det \begin{pmatrix} w & z & z \\ z & x & 0 \\ z & 0 & y \end{pmatrix} = wxy - xz^2 - yz^2.$$

It is easy to see that the singularities of F are the two A_1-singularities at $(0:1:0:0)$ and $(0:0:1:0)$ and an A_3-singularity at $(1:0:0:0)$.

We summarize all cases in Table 3, whose first column describes the pencil of quadrics. If it contains only numbers, we consider the pencil whose general member is smooth and where two of those intersect with multiplicities given by the numbers. □

One may observe in Table 3 that, whenever the cubic F has only isolated singularities, these singularities are precisely the singularities of the quartic Q that is the union of the two smooth members of the pencil of the quadrics given by $\mathcal{A}$. We shall explain this amazing fact.

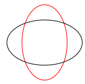
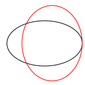
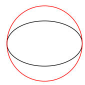
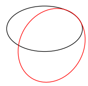
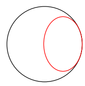
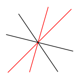
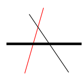
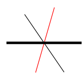
Recall that a plane A_{2k+1}-singularity is defined to be the intersection of two smooth branches intersecting with multiplicity k. Thus knowing the singularities of the quartic Q, which is the union of the two smooth quadrics C_1 and C_2, is the same as knowing the intersection multiplicities of $C_1 \cap C_2$, whose sum is 4. Now we embed $\mathbb{P}^2$ via the Veronese embedding

$$v \colon \mathbb{P}^2 \to \mathcal{V} \subset \mathbb{P}^5 = \mathbb{P}(\mathrm{Sym}(3, \mathbb{C})), \; [x] \mapsto [x \cdot x^t]$$

as the Veronese surface $\mathcal{V}$ into $\mathbb{P}^5$. Then the quadrics C_1 and C_2 are pull-backs of two hyperplanes H_1 and H_2 of $\mathbb{P}^5$. By the projection formula, the intersection multiplicities of $C_1 \cap C_2$ are the same as the intersection multiplicities of the curves $H_1 \cap \mathcal{V}$ and $H_2 \cap \mathcal{V}$ on the Veronese surface $\mathcal{V}$. These are also the intersection multiplicities of the Veronese surface $\mathcal{V}$ and the 3-plane $H_1 \cap H_2 = \mathbb{P}(\mathcal{A})$. Denoting the affine coordinate ring of $\mathrm{Sym}^2(3, \mathbb{C})$ by $\mathbb{C}[x_0, \dots, x_5]$, these multiplicities can be computed as the vector space dimensions of the ring

$$\mathbb{C}[x_0, \dots, x_5]/(I(\mathcal{V}) + I(H_1) + I(H_2))$$

Table 3

Description of pencil	Pencil of quadrics	Two general members	F	Description of cubic
$(1,1,1,1)$	$r(s-t)$ $t(r-s)$		$wxy+wxz$ $+wyz+xyz$	cubic with $4A_1$
$(2,1,1)$	$r(s-t)$ st		$wxy-xz^2-yz^2$	cubic with $2A_1+A_3$
$(2,2)$	r^2 st		$-xy^2-wz^2$	irreducible type I
$(3,1)$	$2r^2-2st$ rt		$wxz-z^3-xy^2$	cubic with A_5+A_1
(4)	$2s^2-2rt$ r^2		$-w^3+2wxy$ $-x^2z$	irreducible type II
All quadrics singular; no fixed line	r^2 s^2		$-x(wx-2yz)$	nondeg. quadric + tangent plane
Fixed line; pencil with center outside line	rt st		$y(wx-z^2)$	quadric cone + transversal plane
Fixed line; pencil with center on line	r^2 rs		$-wy^2$	double plane + plane

localized at the corresponding points of $\mathbb{P}(\mathrm{Sym}(3,\mathbb{C}))$. Since H_1 and H_2 are linear and since the ideal of the Veronese surface is given by the 2×2 minors of the general symmetric matrix, it follows that the ring just described is isomorphic to

$$\mathbb{C}[w,x,y,z]/(2\times 2 \text{ minors of } M),$$

where M is the matrix representation of $F\in\mathbb{C}[w,x,y,z]$.

To determine the singularities of $F=\det M$, we project F from a general smooth point of F. Then it is classically known that the singularities of F are stably equivalent to the singularities of the branch curve of the projection. Let us recall the proof. If

$$F(w,x,y,z)=w^2g_1(x,y,z)+wg_2(x,y,z)+g_3(x,y,w)$$

with $\deg g_i=i$ and $g_1\neq 0$, then $(1:0:0:0)$ is a smooth point of F and the branch curve G of the projection is

$$G = g_2^2 - 4g_1g_3.$$

The stable equivalence between the points of F and G can be seen from

$$\frac{F}{g_1} = \left(w + \frac{g_2}{2g_1}\right)^2 - \frac{1}{4g_1^2}G.$$

Now we apply this to our $F = \det M$. We know a priori that F has at least four A_1-singularities or worse in terms of the sum of the Milnor numbers; thus the branch curve G has these singularities as well and so will be a reducible quartic. We will show that G is the union of two quadrics. We choose coordinates such that the general projection point is $(1:0:0:0)$ and

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

then

$$M = \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & w + f_{22} & f_{23} \\ f_{13} & f_{23} & w + f_{33} \end{pmatrix} \quad \text{with } f_{ij} \in \mathbb{C}[x, y, z] \text{ linear.}$$

We denote the adjoint matrix of M by $\operatorname{adj} M = (m^{ij})$. Then

$$F = w^2 g_1 + w g_2 + g_3 \quad \text{with}$$
$$g_1 = f_{11}, \quad g_2 = m^{22}_{w=0} + m^{33}_{w=0}, \quad g_3 = \det M_{w=0},$$

where the index $w = 0$ stands for setting w equal to zero in the polynomial (or matrix, as applies). By the determinantal formula of Laplace (see [KUl, Sec. 2.4]),

$$F \cdot f_{11} = m^{22}m^{33} - (m^{23})^2 \implies g_1 g_3 = m^{22}_{w=0} m^{33}_{w=0} - (m^{23}_{w=0})^2$$

and

$$\begin{aligned} G = g_2^2 - 4g_1g_3 &= (m^{22}_{w=0} + m^{33}_{w=0})^2 - 4m^{22}_{w=0}m^{33}_{w=0} + 4(m^{23}_{w=0})^2 \\ &= \big(m^{22} - m^{33} + 2\sqrt{-1}m^{23}\big)\big(m^{22} - m^{33} - 2\sqrt{-1}m^{23}\big); \end{aligned}$$

here we have used $m^{22}_{w=0} - m^{33}_{w=0} = m^{22} - m^{33}$ and $m^{23}_{w=0} = m^{23}$. Hence, G is the union of the quadric cones $\tilde{C}_1 = V\big(m^{22} - m^{33} + 2\sqrt{-1}m^{23}\big)$ and $\tilde{C}_2 = V\big(m^{22} - m^{33} - 2\sqrt{-1}m^{23}\big)$ with vertex $(1:0:0:0)$. We consider them as plane curves and compute their intersection multiplicities, which are given by the vector space dimensions of the ring

$$\mathbb{C}[x, y, z]/\big(m^{22} - m^{33} \pm 2\sqrt{-1}m^{23}\big) = \mathbb{C}[x, y, z]/(m^{22} - m^{33}, m^{23})$$

localized at the appropriate points. Since

$$(m^{22} - m^{33}, m^{23}) + (m^{11} = g_1 w + m^{11}_{w=0}) \subseteq (2 \times 2 \text{ minors of } M)$$

and since the sum of the intersection multiplicities is 4 in all cases, it follows that the intersection multiplicities of $C_1 \cap C_2$, $V \cap \mathbb{P}(\mathcal{A})$, and $\tilde{C}_1 \cap \tilde{C}_2$ are equal at corresponding points! And by our previous remarks, the intersection multiplicities of $\tilde{C}_1 \cap \tilde{C}_2$ would determine the singularities of the branch curve if we knew that $\tilde{C}_1$

and $\tilde{C}_2$ were smooth, which we do not. However, the singularities will only get worse if $\tilde{C}_1$ or $\tilde{C}_2$ are singular, and we can at least conclude that the singularities of the branch curve, which are also the singularities of the cubic F, are equal to or worse than the singularities of the quartic $Q = C_1 + C_2$ that we started with. But the combination of the singularities for $C_1 + C_2$ are $4A_1$, $2A_1 + A_3$, $2A_3$, $A_5 + A_1$, and A_7. By the classification of cubics [BrW], these combinations are all extremal combinations of isolated singularities on a normal cubic (with the exception of $2A_3$, which is impossible). Therefore, the singularities of F are in fact the singularities of $C_1 + C_2$ if F is normal.

3. Quartics

The methods of studying a normal quartic in $\mathbb{P}^3$ depend on whether its resolution is a $K3$ surface or a rational surface. If the quartic has only rational double points then its resolution is a $K3$ surface; for this case, Urabe and Yang used Torelli-type theorems for $K3$ surfaces to list all possible combinations of rational singularities. If the normal quartic surface possesses a nonrational double point or a triple point, then the quartic is rational and can be examined by studying the projection of the quartic from this singular point. Degtyarev [De] used this fact to list all possible combinations of singularities for this case. The proof also yields a method for producing equations of the quartics. In contrast to this, for quartics with only rational singularities there is in general no obvious way of constructing an equation of the quartic for a given possible combination of singularities, and thus most equations are unknown. In Sections 3.1–3.3 we will adapt all this to the case of linear symmetric quartics.

A quartic surface with a quadruple point is a cone over a plane curve. Since any plane curve can be represented by a linear symmetric matrix [Be, 4.4], the same holds for any such quartic surface, and we will not discuss this case further.

3.1. Linear Symmetric Quartics with Only Rational Double Points

Urabe and Yang [U1; U2; Y] examined the question of which combinations of rational double points can even occur on a quartic. The general idea is to study not the quartic in $\mathbb{P}^3$ directly but rather its minimal desingularization Y, which is a $K3$ surface. For general facts about $K3$ surfaces see [BPV, Sec. VIII]; we recall only the following: For all $K3$ surfaces, the second cohomology group $H^2(Y, \mathbb{Z})$ is a free abelian group of rank 22. Together with the intersection form, this group is the unique unimodular even lattice of signature $(3, 19)$, which is $Q(-E_8) \oplus Q(-E_8) \oplus \mathbb{H} \oplus \mathbb{H} \oplus \mathbb{H}$. Here, $\oplus$ denotes the orthogonal direct sum, $Q(-E_8)$ the rank-8 lattice whose bilinear form is given by the Dynkin graph E_8 with sign-reversed weights, and $\mathbb{H}$ the hyperbolic plane $\mathbb{H} = \mathbb{Z}u + \mathbb{Z}v$ where (writing the symmetric bilinear form as multiplication) $u^2 = v^2 = 0$ and $u \cdot v = 1$. Because $H^1(Y, \mathcal{O}) = 0$, the Picard group $\mathrm{Pic}(Y)$ injects into $H^2(Y, \mathbb{Z})$ and is, in fact, a primitive subgroup there; that is, $H^2(Y, \mathbb{Z})/\mathrm{Pic}(Y)$ is torsion free.

Using Torelli-type theorems for $K3$ surfaces and the work of Saint-Donat, Urabe proved the following.

THEOREM 3.1 [U1, Thm. 1.15]. *Let $G = \sum a_k A_k + \sum b_l D_l + \sum c_m E_m$ be a Dynkin graph with components of type A, D, or E only. Then the following conditions are equivalent.*

1. *There is a quartic surface in $\mathbb{P}^3$ with only rational double points as singularities: the combination of singularities corresponding to G.*
2. *Let $Q = Q(G)$ be the lattice of type G, and let $\Lambda := Q(-E_8) \oplus Q(-E_8) \oplus \mathbb{H} \oplus \mathbb{H} \oplus \mathbb{H}$ denote the unimodular even lattice with signature* $(3, 19)$. *The lattice $S = \mathbb{Z}H \oplus Q$ ($H^2 = 4$, orthogonal direct sum) has an embedding $S \subseteq \Lambda$ satisfying the following conditions* (a) *and* (b). *Let $\tilde{S} = \{x \in \Lambda \mid mx \in S$ for some $m \in \mathbb{Z} \setminus \{0\}\}$ denote the primitive hull of S in Λ.*
   (a) *If $\eta \in \tilde{S}$, $\eta \cdot H = 0$, and $\eta^2 = -2$, then $\eta \in Q$.*
   (b) *$\tilde{S}$ does not contain any element u with $u^2 = 0$ and $u \cdot H = 2$.*

The sum $\mu := \sum a_k k + \sum b_l l + \sum c_m m$ is called the *Milnor number* of G or X. For quartic surfaces, one always has $\mu \leq 19$.

Condition (a) ensures that there exist only the expected singularities G on the quartic; condition (b) ensures that the linear system given by H induces an embedding into $\mathbb{P}^3$.

By this theorem, Urabe reduced the question of the existence of a quartic with a given combination of singularities to a purely lattice-theoretic problem. We want a similar theorem for our situation, and we start by providing the Dynkin graph with additional information.

DEFINITION 3.2. A *parity Dynkin graph G* is a formal sum of the following marked Dynkin diagrams.

- The essential parity Dynkin diagrams, which are the marked Dynkin diagrams of Proposition 1.13. (We do not distinguish between D^+_{2k} and D^-_{2k} or between $D^\bullet_4$ and $D^\pm_4$.)
- The accidental parity Dynkin diagrams, which are the Dynkin diagrams of A_k, D_k, E_6, E_7, and E_8 with every vertex drawn as $\circ$; they are denoted (respectively) by A°_k, D°_k, E°_6, E°_7, and E°_8.

The number of vertices of G is the *Milnor number* $\mu(G)$ of G, and the number of $\bullet$-vertices is the *length* $l(G)$ of G.

To a linear symmetric surface with only rational singularities we assign a parity Dynkin diagram whose components correspond to the singularities in the obvious way: for the essential singularities we use the correspondence of Proposition 1.13, and to the accidental singularities we assign the corresponding accidental Dynkin diagrams.

Every parity Dynkin diagram comes with a special divisor in a corresponding lattice, as described in our next definition.

DEFINITION 3.3. The lattice $Q(G)$ of a parity Dynkin graph has a canonical basis given by the vertices of the graph G. The *parity divisor D_G* is the sum of the $\bullet$-vertices.

Now we can state the extension of Urabe's theorem for linear symmetric quartics.

THEOREM 3.4. *Let G be a parity Dynkin graph. Then the following conditions are equivalent.*

1. *There is a linear symmetric quartic in $\mathbb{P}^3$ with only rational double points as singularities: the combination of singularities corresponding to G.*
2. *Let G satisfy condition 2 of Theorem 3.1 and, in addition:*
   (c) *The length of G is* 10 *and $\frac{1}{2}H + \frac{1}{2}D_G \in \tilde{S}$, where D_G is the parity divisor of G.*

Proof. In Urabe's correspondence between the lattices and the quartics, the primitive lattices $\tilde{S}$ correspond to the Picard group of the minimal resolution $\tilde{F}$ of the quartic. Now, on a linear symmetric quartic F, the essential singularities form an even set X of ADE-singularities of length 10 (by Proposition 1.15 and Theorem 1.22). Let G be the parity Dynkin graph of F. Clearly, by the definitions we have $l(G) = l(X)$ and that $H + D_G$ is divisible by 2 in $\operatorname{Pic}(\tilde{F}) = \tilde{S}$ precisely if $H + \sum Z_{X_i}$ is. Therefore, condition (c) holds.

Starting with a parity Dynkin graph with properties (a)–(c), Urabe's theorem yields a quartic F with an even set of ADE-singularities of length 10. Let $\tilde{F}$ be the minimal resolution of F (here for all singularities of F, but this makes no difference for the statements of Section 1). By Theorem 1.22, the quartic F is linearly symmetric if the order of X is 3. By Proposition 1.21, X is weakly even and so we need only show that the order of X is not 1. Setting $L = \frac{1}{2}\big(H + \sum Z_{X_i}\big) \in \operatorname{Pic}(\tilde{F})$ and using the remark after Definition 1.17, this is equivalent to $H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(H-L)) = 0$; that is, we need to show that $H - L = \frac{1}{2}\big(H - \sum Z_{X_i}\big) \in \operatorname{Pic}(\tilde{F})$ is not effective.

Assume that $H - L$ is effective. Then we can decompose it into $\sum_{j=1}^{s} C_j + \sum_k B_k$, where the C_j, B_k are irreducible curves with $H \cdot C_j > 0$ and $H \cdot B_k = 0$. Recall that, for any curve C on a $K3$ surface, $C^2 \geq -2$ and C^2 is divisible by 2 [BPV, Sec. VIII, (3.6)]. Because Q is a negative definite lattice and $B_k \in \mathbb{Q} \otimes Q$, we get $B_k^2 = -2$ and $B_k \in Q$ by condition (a). We claim that there are at most two curves C_j, that is, $s \leq 2$. Write $C_j = a_j H + \tilde{C}_j \in \mathbb{Q}H \oplus \mathbb{Q} \otimes Q$; then $H \cdot C_j = 4a_j \in \mathbb{N}$ and so $a_j = n_j/4$ for some $n_j \in \mathbb{N}$. From $\sum C_j = \frac{1}{2}H$ modulo $\mathbb{Q} \otimes Q$ we find either $s = 1$ and $a_1 = \frac{1}{2}$ or $s = 2$ and $a_1 = a_2 = \frac{1}{4}$. It is not difficult to obtain contradictions for $s = 1$ or $s = 2$ and $C_1 \neq C_2$ by completely elementary calculations with divisors, but the $C_1 = C_2$ case seems inaccessible by these simple methods. Hence, we recall more lattice theory.

The primitive hull $\tilde{S}$ of S will always lie in $S^* = \operatorname{Hom}(S, \mathbb{Z}) \subset \mathbb{Q} \otimes S$, so $\tilde{S}/S \subseteq S^*/S$. The finite group S^*/S is well known. If $G = \sum X_i$ is the decomposition of the parity Dynkin graph G into the parity Dynkin diagrams, then $S^*/S = \mathbb{Z}/4\mathbb{Z} \oplus \bigoplus_i Q(-X_i)^*/Q(-X_i)$, where the first summand is generated by $H/4$ and where $Q(-X_i)^*/Q(-X_i)$ depends only on the underlying Dynkin diagram and is isomorphic to $\mathbb{Z}/(k+1)\mathbb{Z}$ for A_k, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ for D_{2k}, $\mathbb{Z}/4\mathbb{Z}$ for D_{2k+1}, and $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, 0 for E_6, E_7, E_8, respectively [U3, Sec. 1.3]. For $D \in Q^*$ define

$$m(D) := \max\{(D + B)^2 \mid B \in Q\}.$$

Because the intersection form is negative definite, $m(D) < 0$ for $D \notin Q$. These numbers were computed by Urabe [U3, Sec. 1.3]. In particular, he found that $m\big(\frac{1}{2}D_{X_i}\big) = -\frac{1}{2}l(X_i)$ for the parity divisors of the singularities X_i. Since $\mathbb{Q} \otimes Q$ is the orthogonal sum of the $\mathbb{Q} \otimes Q(X_i)$, we obtain $m\big(\frac{1}{2}D_G\big) = \sum m\big(\frac{1}{2}D_{X_i}\big) = -\frac{1}{2}\sum l(X_i) = -5$.

Now, if $s = 1$ then $C_1 = H - L - \sum B_k = \frac{1}{2}H + \frac{1}{2}D_G + B$ for some $B \in Q$ and

$$\begin{aligned} C_1^2 &= \big(\tfrac{1}{2}H + \tfrac{1}{2}D_G + B\big)^2 \\ &= \big(\tfrac{1}{2}H\big)^2 + \big(\tfrac{1}{2}D_G + B\big)^2 \le 1 + m\big(\tfrac{1}{2}D_G\big) = 1 - 5 = -4, \end{aligned}$$

contradicting $C_1^2 \ge -2$.

If $s = 2$ then write

$$C_j = \frac{1}{4}H + \sum_i C_{j,X_i} \quad \text{with } C_{j,X_i} \in \mathbb{Q} \otimes Q(X_i).$$

We see from $C_1 + C_2 = \frac{1}{2}H + \frac{1}{2}D_G \bmod Q$ that $C_{1,X_i} + C_{2,X_i} = \frac{1}{2}D_{X_i} \bmod Q$. Further, we find the estimates

$$C_j^2 \le \left(\frac{1}{4}H\right)^2 + \sum_i m(C_{j,X_i}) = \frac{1}{4} + \sum_i m(C_{j,X_i}),$$

$$C_1^2 + C_2^2 \le \frac{1}{2} + \sum_i (m(C_{1,X_i}) + m(C_{2,X_i})).$$

A small computation using Urabe's values for the function m shows that

$$m(C_{1,X_i}) + m(C_{2,X_i}) \le m\big(\tfrac{1}{2}D_{X_i}\big) = -\tfrac{1}{2}l(X_i)$$

for any C_{j,X_i} with $C_{1,X_i} + C_{2,X_i} = \frac{1}{2}D_{X_i} \bmod Q(X_i)$. This implies $C_1^2 + C_2^2 \le \frac{1}{2} - 5 = -4\frac{1}{2}$ and hence that $C_1^2 < -2$ or $C_2^2 < -2$, which yields the required contradiction. □

From Urabe's theorem it follows immediately that if there exists a quartic with Dynkin graph G then one can find a quartic for any complete subgraph $G' \subset G$. For linear symmetric quartics, a similar statement holds as follows.

DEFINITION 3.5. A parity Dynkin subgraph G' of a parity Dynkin graph G is a complete subgraph $G' \subset G$ that contains all the $\bullet$-vertices of G; that is, $l(G') = l(G)$.

COROLLARY 3.6 (Parity splitting principle). *If there exists a linear symmetric quartic with parity Dynkin graph G, then there exists a linear symmetric quartic for any parity Dynkin subgraph G' of G.*

Proof. Because $D_{G'} = D_G \in \mathbb{Q}H \oplus \mathbb{Q} \otimes Q(G')$, we can use $\mathbb{Z}H \oplus Q(G') \subseteq \mathbb{Z}L \oplus Q(G) \subseteq \Lambda$ for the embedding required in the theorem. □

This parity splitting principle has amazing consequences, which we state in the following summarizing theorem.

THEOREM 3.7. *Let G be the parity Dynkin graph of a linear symmetric quartic with only rational double points. Then the following statements hold.*

1. $10 \leq \mu(G) \leq 19$ *and* $l(G) = 10$.
2. *G is a union of the parity Dynkin diagrams $A^{\bullet}_{2k+1}$, $D^{\pm}_{2k}$, and A°_{1}.*

In particular, the parity Dynkin graph G is determined by its underlying Dynkin graph.

Proof. $l(G) = 10$ was stated in Theorem 3.4, and $\mu(G) \leq 19$ holds for any quartic. By Proposition 1.3, the only possible accidental singularity on a linear symmetric quartic is an A_1-singularity. Proposition 1.13 has shown that there are no essential A_{2k}, E_6, or E_8 singularities. Further, for the parity Dynkin diagrams $D^{\bullet}_{2k+1}$ and $D^{\bullet}_{2k}$ (except for $D^{\bullet}_4 = D^{\pm}_4$), there exist parity splittings that have an accidental A°_l ($l \geq 2$) parity Dynkin diagram as a component, contradicting the parity splitting principle. □

Urabe used his theorem to give a short list of so-called basic Dynkin graphs and to define two kinds of transformations for Dynkin graphs such that, after applying two transformations to a basic Dynkin graph, the resulting graph is a possible combination of rational singularities on a quartic [U1; U2]. This produced a long list of possible combinations of singularities on a quartic. Unfortunately, these operations are not compatible with our new condition (c). Yet this long list of combinations of singularities was not complete, as Urabe himself noted [U2, Sec. 3]. There he also remarked that each Dynkin graph G can be checked individually by a tedious computation using the lattice theory of Nikulin [N]. A computer program that does precisely this was written by Yang [Y], who was kind enough to make this program available to the author. The modification to incorporate condition (c) is not difficult, and the output of the program can be summarized as follows.

THEOREM 3.8. *For linear symmetric quartics with only rational double points, only the following parity Dynkin graphs or their parity splittings occur*:

$D_{18} + A_1$,	$D_{14} + A_5$,	$D_{14} + A_3 + 2A_1$,
$D_{12} + D_6 + A_1$,	$D_{12} + A_5 + 2A_1$,	$D_{10} + D_8 + A_1$,
$D_{10} + D_6 + A_3$,	$D_{10} + A_9$,	$D_{10} + A_7 + 2A_1$,
$D_{10} + A_5 + A_3 + A_1$,	$2D_8 + 3A_1$,	$D_8 + D_6 + D_4 + A_1$
$D_8 + D_6 + A_5$,	$D_8 + D_6 + A_3 + 2A_1$,	$D_8 + D_4 + A_5 + 2A_1$
$D_8 + A_9 + 2A_1$,	$D_8 + A_5 + A_3 + 3A_1$,	$3D_6 + A_1$,
$2D_6 + D_4 + 3A_1$,	$2D_6 + A_7$,	$2D_6 + A_5 + 2A_1$,
$D_6 + 2D_4 + A_3 + 2A_1$,	$D_6 + D_4 + A_5 + A_3 + A_1$,	$D_6 + A_{13}$,
$D_6 + A_9 + A_3 + A_1$,	$D_6 + A_7 + A_5 + A_1$,	$4D_4 + 3A_1$,
$D_4 + A_9 + A_5 + A_1$,	$D_4 + 2A_5 + A_3 + 2A_1$,	A_{19},
$A_{17} + 2A_1$,	$A_{15} + A_3 + A_1$,	$A_{13} + A_5 + A_1$,
$A_{11} + A_7 + A_1$,	$A_{11} + A_5 + 3A_1$,	$A_{11} + 2A_3 + 2A_1$,
$2A_9 + A_1$,	$A_9 + A_7 + 3A_1$,	$2A_7 + A_3 + 2A_1$,
$3A_5 + 4A_1$,	$4D_4 + 2A_1$,	$16A_1$.

(*Only the underlying Dynkin graphs are listed because, by Theorem 3.7, they determine the parity Dynkin diagrams.*)

This theorem shows that the possible combinations of singularities on a linear symmetric determinantal quartic are far less than the one of a general quartic, where one has 27 pages of combinations for the Milnor numbers 19, 18, and 17 alone, where most combinations are possible for the Milnor numbers 16 and 15, and where below that all combinations are possible [Y]. However, one might have hoped for even fewer possible combinations in the case of Theorem 3.8.

Without the use of the program, it is not clear why one needs only the parity diagrams of Milnor number 19 as well as $4D_4^{\pm} + 2A_1^{\bullet}$ and $10A_1^{\bullet} + 6A_1^{\circ}$ as starting points for the parity splitting process.

EXAMPLE. In general it is difficult to find an explicit matrix representation for the combinations of rational singularities listed in Theorem 3.8. However, with some tricks and enough computing power one can find the matrix representation

$$\begin{pmatrix} x & iy & iy/2 & y/2 - iz \\ iy & x & y/2 + iz & iy/2 \\ iy/2 & y/2 + iz & w + ix + 3iy/2 & i(-2x + y - 4z)/4 \\ y/2 - iz & iy/2 & i(-2x + y - 4z)/4 & w - ix - 3iy/2 \end{pmatrix}$$

of the unique quartic with an A_{19}-singularity reported by Kato and Naruki [KaN].

In the following sections, when the quartic has nonsimple singular point we will study the quartic by projecting it from a singular point. The referee suggested that this might also be possible in the case of Theorem 3.8 owing to a result of Cayley [Ca, Sec. 2.4.3]: we project the quartic from one of its simple singular double points. Let D be the branch locus of this projection and C the projection of the tangent cone of this singular point. Cayley proved that, for a linear symmetric quartic, the sextic curve B is a union of two cubics and has contact with the conic C. Enumerating all possible intersection configurations of the cubics and the conic, especially with respect to their singularities, will lead to the foregoing classification. This might even yield equations for the quartics.

3.2. Linear Symmetric Quartics with a Nonsimple Double Point

As soon as the normal quartic acquires a nonsimple double point, it is no longer a $K3$ surface but instead a rational surface. Hence the techniques of the last section cannot be used to study this case. Degtyarev [De] studied these quartics by projecting them from their worst singularity onto a plane. We will use his extensive study of quartic equations to obtain the following result.

THEOREM 3.9. *Only the following combinations of double points occur on a rational linear symmetric quartic with at most double points*:

$X_{1,0} + A_3 + \{X_{1,0}, D_6 + A_1, D_4 + 3A_1, A_7, 2A_3 + A_1, 6A_1\}$,

$X_{1,2} + A_3 + \{D_6, D_4 + 2A_1\}$, $\{X_{1,2} + A_1, X_{1,4}\} + \{D_6 + A_1, A_7, 2A_3 + A_1\}$,

$$X_{1,4} + A_3 + \{A_3 + A_1, A_3\},\ X_{1,6} + A_3 + A_1,\ X_{1,8} + A_3,$$
$$Y^1_{2,2} + A_5 + A_1,\ Y^1_{2,2} + 2A_1 + \{D_4, 4A_1\},\ Y^1_{2,4} + 2A_1 + \{2A_1, A_1\},$$
$$Y^1_{4,4} + 2A_1 + \{A_1, \emptyset\},\ Y^1_{2,6} + 2A_1.$$

Here, one must choose one element out of the sets in order to obtain a valid expression, and the A_{2k+1}- and D_{2k}-singularities may be split in the same manner as $A^\bullet_{2k+1}$ and $D^\pm_{2k}$ in Section 3.1.

Proof. To apply the results of Degtyarev, we need explicit equations. Let us assume that $M = wA_0 + xA_1 + yA_2 + zA_3$, that $F = \det M$, and that the worst singular point is at $p = (1 : 0 : 0 : 0)$. The rank of A_0 is 2 by (Proposition 1.3 and) the obvious fact that the multiplicity of F at p is equal to or higher than the corank of A_0. We can choose a basis of $\mathbb{C}^4$ such that

$$A_0 = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{E}_2 \end{pmatrix} \quad \text{with } \tilde{E}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If we use a 2×2 blocking for M,

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M^t_{12} & M_{22} \end{pmatrix},$$

then the quadric part of F in p is given by $-\det M_{11}$. Since we are still free to choose an arbitrary basis in $\operatorname{span}\{e_1, e_2\}$ or $\operatorname{span}\{x, y, z\}$, we may think of M_{11} as given by a linear subspace in $\mathbb{P}(\mathrm{Sym}(2, \mathbb{C})) \cong \mathbb{P}^2$. The matrices of rank 1 form a smooth conic C in this $\mathbb{P}^2$, and the linear spaces inside this $\mathbb{P}^2$ are characterized by their intersection with this conic [Ha, Sec. 10]. We obtain the list shown as Table 4.

In the last two cases of Table 4, we have nonisolated singularities by Lemma 1.4. In the first two cases, we have A_k-singularities by the the classification of singularities [AGV, Sec. 16.2]. We want to show that only D_k-singularities occur in the third case. We write

$$M = \begin{pmatrix} 0 & x & f_{13} & f_{14} \\ x & y & f_{23} & f_{24} \\ f_{13} & f_{23} & f_{33} & w + f_{34} \\ f_{14} & f_{24} & w + f_{34} & f_{44} \end{pmatrix} \quad \text{with } f_{ij} \in \mathbb{C}[x, y, z].$$

After a base change of the type $e_3 \mapsto e_3 - \lambda_3 e_1 - \mu_3 e_2$ and $e_4 \mapsto e_4 - \lambda_4 e_1 - \mu_4 e_2$, we may assume that $f_{13}, f_{14}, f_{23}, f_{24} \in \mathbb{C}[y, z]$. Setting $w = 1$, computing the determinant, and performing the substitution $x \to x - x f_{34} + f_{14} f_{23} + f_{13} f_{24}$, we see that the equation of F starts with

$$x^2 + 2y f_{13} f_{14} + \cdots .$$

By Lemma 1.4, the linear polynomials f_{13} and f_{14} are linear independent; thus, F has a D_k-singularity in p [AGV, Sec. 16.2].

As a result, the fourth case is the only one in which nonsimple double points may occur. We have

Table 4

Subspace	Normal form of M_{11}	$\det M_{11}$
$\mathbb{P}^2$	$\begin{pmatrix} x & z \\ z & y \end{pmatrix}$	$xy - z^2$
Secant of C	$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$	xy
Tangent to C	$\begin{pmatrix} 0 & x \\ x & y \end{pmatrix}$	$-x^2$
Point outside C	$\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}$	$-x^2$
Point on C	$\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$	0
$\emptyset$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	0

$$M = \begin{pmatrix} 0 & x & f_{13} & f_{14} \\ x & 0 & f_{23} & f_{24} \\ f_{13} & f_{23} & f_{33} & w + f_{34} \\ f_{14} & f_{24} & w + f_{34} & f_{44} \end{pmatrix} \quad \text{with } f_{ij} \in \mathbb{C}[x, y, z].$$

The surface F is given as $F = w^2x^2 + wxP + Q$, where

$$P = 2xf_{34} - 2(f_{13}f_{24} + f_{14}f_{23}) \quad \text{and}$$

$$Q = x^2(f_{34}^2 - f_{33}f_{44}) + 2x(f_{13}f_{23}f_{44} + f_{14}f_{24}f_{33} - f_{13}f_{24}f_{34} - f_{14}f_{23}f_{34}) + (f_{13}f_{24} - f_{14}f_{23})^2.$$

The branch curve of the canonical projection of F from p is (besides the x^2-factor)

$$D = P^2 - 4Q = 4(xf_{33} - 2f_{13}f_{23})(xf_{44} - 2f_{14}f_{24}) = 4C_1C_2,$$

a union of two conics C_1, C_2. Note that there is no restriction on the equation of the conics, since we can choose the f_{ij} arbitrary so far. Further, let the line L be the projected tangent cone $V(x)$ of F in p.

According to Degtyarev [De, Sec. 2], p will be an isolated singularity only if D is smooth at $L \cap Q$. Note that D cannot contain L owing to the linear independence of the linear forms x, f_{13}, f_{14} and x, f_{23}, f_{24} (cf. Lemma 1.4). This excludes singularities of the type N [De, Sec. 3]. Now F has the following singularities.

- To each singular point of D not lying on L there corresponds a singular point of F that is stably equivalent to it. In particular, the curve D cannot have a multiple component for a normal surface F.

Table 5

$D \cap L$	q	Singularity
$(1,1,1,1)$	—	$X_{1,0}$
$(2,1,1)$	q	$X_{1,q}$
$(2,2)$	q	$Y_{1,q}$
$(2,2)$	$(2,2)$	$Y^1_{2,2}$
$(A_k,1,1)$	—	$X_{1,k+1}$
$(A_k,2)$	q	$Y^1_{k+1,q}$
(A_k,A_l)	—	$Y^1_{k+1,l+1}$

- To each s-fold point of $Q \cap L$ not on D there corresponds an exceptional singular point of F of type A_{s-1}.
- The type of the singularity of F at p can be read off Table 5. The first column describes the intersection configuration of L and D, where 1 (resp. 2) stands for a transversal (resp. tangential) intersection at a smooth point of D and where A_k stands for a transversal intersection at an A_k-singularity of D. If $D \cap L$ and $Q \cap L$ have a common multiple point then its multiplicity in $Q \cap L$ is denoted by q; otherwise, we set $q = 1$. The case of two common double points is written loosely as $q = (2,2)$.

We now apply this to our case, yielding

$$D \cap L = V(f_{13}f_{14}f_{23}f_{24}, x).$$

By Lemma 1.4, the linear forms x, f_{13}, f_{14} and x, f_{23}, f_{24} are linear independent, so D and L may intersect only in the configurations listed in Table 5. Note that

$$Q \cap L = V((f_{13}f_{24} - f_{14}f_{23})^2, x);$$

thus, the exceptional singularities are of type A_1 or A_3 if the corresponding multiple points do not lie on D. We treat separately the cases of the different intersection configurations of D and L.

Case 1: $(1,1,1,1)$. Our main singularity is an $X_{1,0}$. Since f_{13}, f_{14}, f_{23}, f_{24} have pairwise distinct zeros on L, it follows that $Q \cap L$ and $D \cap L$ cannot have a common multiple zero; thus we can have either an A_3 or two A_1s as exceptional singularities. Further, we can change (f_{14}, f_{24}) to $(\lambda f_{14}, \lambda^{-1} f_{24})$ with $\lambda \in \mathbb{C}^*$ without changing the equation of D, but $Q \cap L$ changes to $V((f_{13}f_{24} - \lambda^2 f_{14}f_{23})^2, x)$. This restricted pencil for $\lambda^2 \in \mathbb{C}^*$ contains a quadruple point, because the complete pencil with $\lambda^2 \in \mathbb{P}^1$ does and we can exclude $\lambda = 0, \infty$. Therefore, we can always have two A_1 as well as an A_3 as exceptional singularities.

Finally, in Figure 4 we sketch all singularities that can occur on a quartic D that is the union of two conics and also list which singularities (apart from the exceptional ones) the surface F has in the corresponding case. In each case, the fat line represents L.

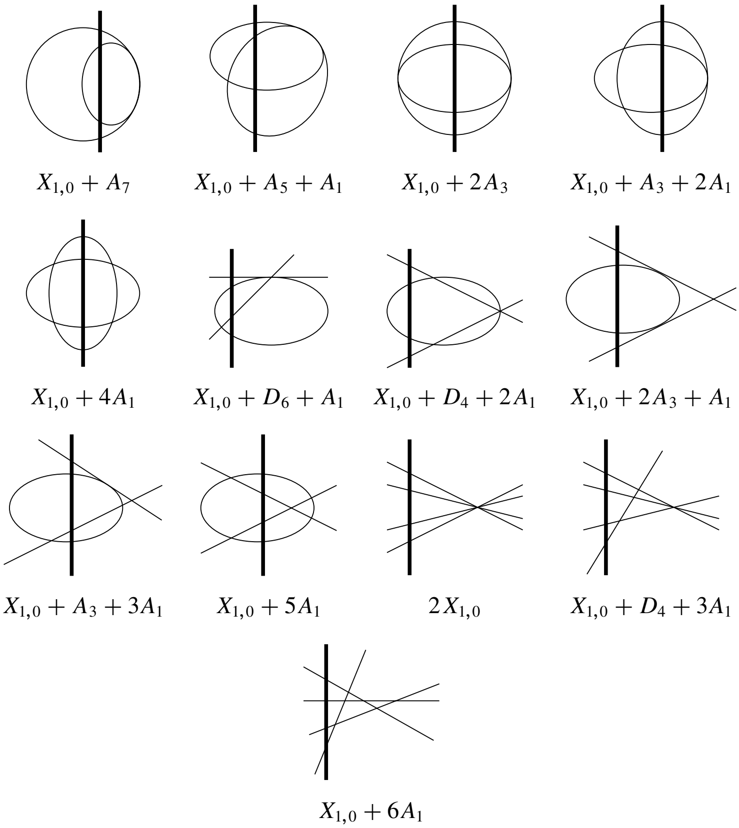


Figure 4

Case 2: $(2, 1, 1)$. Because L intersects D tangentially, it follows that one of the two conics C_1, C_2 (say, C_1) must be smooth and that f_{13} and f_{23} are proportional modulo x. In other words, there exist $\alpha \in \mathbb{C}^*$ and $\beta \in \mathbb{C}$ with $f_{23} = \alpha f_{13} + \beta x$. Therefore,

$$D \cap L = V(f_{13}^2 f_{14} f_{24}, x) \quad \text{and} \quad Q \cap L = V(f_{13}^2 (f_{24} - \alpha f_{14})^2, x)$$

have a common double point. This point may become a quadruple point of $Q \cap L$ and thus the main singularity is either an $X_{1,4}$ or an $X_{1,2}$; in the latter case, there exists an exceptional singularity of type A_1. With a similar argument as used for Case 2, the condition that $Q \cap L$ has a quadruple point is seen to be independent of the equation for D. Figure 5 shows the possible singularities of F (apart from $X_{1,4}$ or $X_{1,2} + A_1$) in dependence of the shape of D.

Case 3: $(2, 2)$. Because of the two tangential intersections of L and D, it follows that both conics C_1, C_2 must be smooth and that f_{13} and f_{23} as well as f_{14} and f_{24} are proportional modulo x; hence

$$D \cap L = V(f_{13}^2 f_{14}^2, x) \quad \text{and} \quad Q \cap L = V(f_{13}^2 f_{14}^2, x)$$

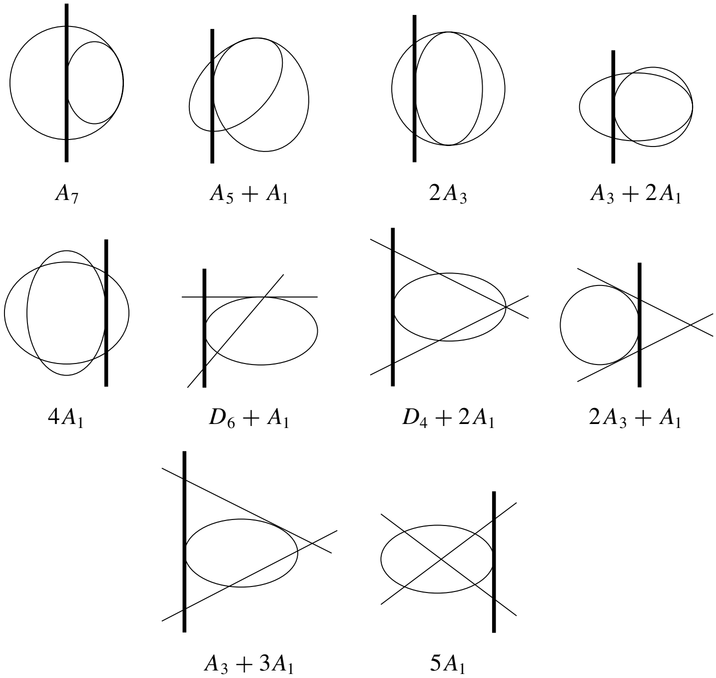


Figure 5

are the same divisor with two double points. Therefore, our main singularity is a $Y^1_{2,2}$ and there are no exceptional singularities. The singularities of F in dependence of the shape of D are shown in Figure 6.

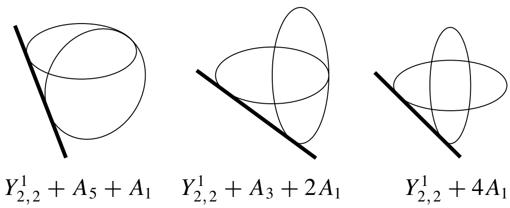


Figure 6

Case 4: $(A_k, 1, 1)$. Here D has an A_k-singularity on L, where k is necessarily odd; its two branches belong to C_1 and C_2. In particular, if both branches belong to C_1 (i.e., if f_{13} and f_{23} are proportional modulo x), then the singular point of D would belong to $Q \cap L$ and F would have a nonisolated singularity. Remembering that f_{13}, f_{14} and f_{23}, f_{24} are also not proportional modulo x, we find that $f_{24} = \alpha f_{13} + \beta x$ for some $\alpha \in \mathbb{C}^*$ and $\beta \in \mathbb{C}$ (or the same with the indices 1 and 2 exchanged) and that

$$D \cap L = V(f_{13}^2 f_{14} f_{23}, x) \quad \text{and} \quad Q \cap L = V((\alpha f_{13}^2 - f_{14} f_{23})^2, x).$$

Thus $D \cap L$ and $Q \cap L$ cannot have a common multiple point. Our usual argument that $Q \cap L$ may have two double points as well as a quadruple point without changing the equation of D can be adapted to this case also. Thus we can always have one A_3 and two A_1 as exceptional singularities. It remains to list all the singularities of F apart from the exceptional ones that depend on the shape of D; see Figure 7.

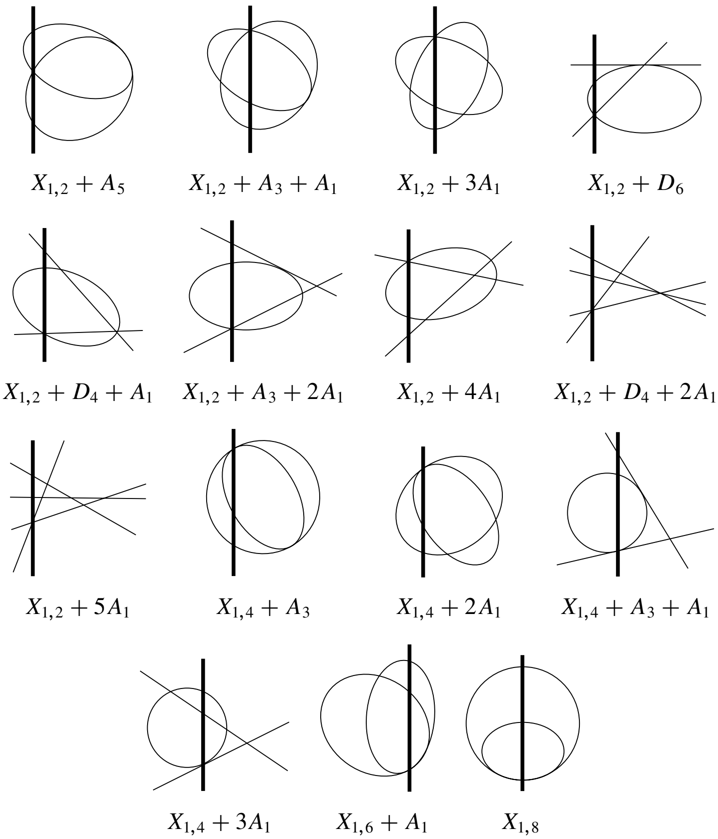


Figure 7

Case 5: $(A_k, 2)$ *and* (A_k, A_l). We have already seen that an A_k-singularity of D on L can occur only as the intersection of both C_1 and C_2. Hence, no further tangential intersection of L and C_1 or C_2 is possible; that is, the case $(A_k, 2)$ does not occur. For (A_k, A_l) we obtain $f_{23} = \alpha_1 f_{14} + \beta_1 x$ and $f_{24} = \alpha_2 f_{13} + \beta_2 x$ for some $\alpha_1, \alpha_2 \in \mathbb{C}^*$ and $\beta_1, \beta_2 \in \mathbb{C}$; thus,

$$D \cap L = V(f_{13}^2 f_{14}^2, x) \quad \text{and}$$

$$Q \cap L = V\big(\big(\sqrt{\alpha_2}\, f_{13} + \sqrt{\alpha_1}\, f_{14}\big)^2 \big(\sqrt{\alpha_2}\, f_{13} - \sqrt{\alpha_1}\, f_{14}\big)^2, x\big).$$

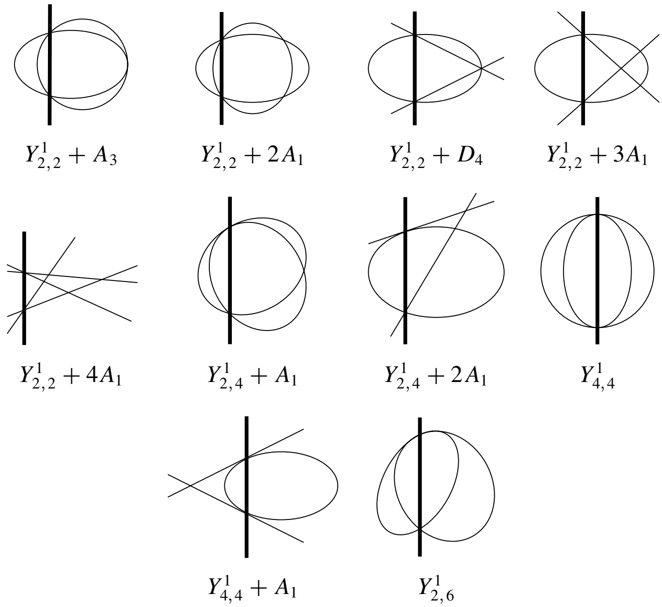


Figure 8

Consequently, $Q \cap L$ always has two double points outside $D \cap L$; that is, we have two exceptional A_1-singularities. Figure 8 shows the remaining singularities of F according to the shape of D. □

3.3. Linear Symmetric Quartics with a Triple Point

Similar to the case of Section 3.2, we obtain the following theorem.

THEOREM 3.10. *Only the following combinations of singularities occur on a normal linear symmetric quartic with a triple point*:

$$T_{3,3,3} + A_{11},\ T_{3,3,5} + A_9,\ T_{3,3,7} + A_7,\ T_{3,3,9} + A_5,\ T_{3,3,11} + A_3,\ T_{3,3,13} + A_1,$$
$$T_{3,3,15},\ T_{3,5,5} + A_5 + A_1,\ T_{3,5,7} + A_5,\ T_{3,5,9} + 2A_1,\ T_{3,5,11} + A_1,\ T_{3,7,7} + A_3,$$
$$T_{3,7,9} + A_1,\ T_{3,7,11},\ T_{5,5,5} + 3A_1,\ T_{5,5,7} + 2A_1,\ T_{5,7,7} + A_1,\ T_{7,7,7},$$
$$T_{3,3,4} + A_{11},\ T_{3,4,4} + A_3 + A_7,\ T_{4,4,4} + 3A_3,$$
$$Q_{11} + A_9,\ S_{1,0} + A_5 + A_1,\ S^{\#}_{1,2} + A_5,\ S^{\#}_{1,4} + 2A_1,\ S^{\#}_{1,6} + A_1.$$

Here, the A_{2k+1}-singularities can be split in the same manner as $A^{\bullet}_{2k+1}$ was split before.

Proof. Let $p = (1 : 0 : 0)$ be the triple point of the quartic F. From the first part of the proof of Theorem 3.9 it follows that the rank of A_0 is 1; hence, we choose a basis of $\mathbb{C}^4$ such that

Table 6

Singularities of P	Triple point of F
—	$P_8 = T_{3,3,3}$
A_1	$P_{5+q_1} = T_{3,3,q_1}$, where $q_i = \max\{4, 3+r_i\}$
$2A_1$	$R_{q_1,q_2} = T_{3,q_1,q_2}$
$3A_1$	T_{q_1,q_2,q_3}
A_2	Q_{10}, Q_{11}, Q_{12} for $r_1 = 0, 2, 3$ (respectively)
A_3	S_{11}, S_{12}, $S_{1,0}$ for $r_1 = 0, 2, 4$ (respectively)
	$S^{\#}_{1,r_1-4}$ for $r_1 > 4$.

$$A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

in a $(3,1)$ blocking. Then the expansion of $F = \det(wA_0 + xA_1 + yA_2 + zA_3)$ with respect to w is $F = wP + Q$, where Q is the determinant of the matrix $A_{123} = xA_1 + yA_2 + zA_3$ and P is the upper left 3×3 minor of the same matrix. We therefore consider A_{123} as a matrix representation of the curve Q, and P is one of the contact curves of Q (i.e., all intersection multiplicities are even). In order to determine the singularities of F, we quote the following results of Degtyarev [De, Sec. 4]:

- The point p is an isolated singularity of F only if P and Q have no common singularities.
- Apart from the triple point, the normal surface F has only A_{r-1}-singularities that are in one-to-one correspondence with points of the r-fold intersection of P and Q at smooth points of P.
- The type of the double point of F is determined as follows: for a singular point S_i of P, let r_i be the intersection multiplicity of P and Q in S_i; then we have the correspondences shown in Table 6.

Now we have to analyze our linear symmetric quartic F for the different possibilities of P. For the case of a smooth cubic P, we can use abstract arguments involving the Jacobian of P and the theory of contact curves; for singular P, we must analyze the equations using the determinantal representations of P (see Section A).

Case 1: P smooth cubic. Since P and Q have contact, the results of Degtyarev say that F has a $P_8 = T_{3,3,3}$-singularity and a combination of A_k-singularities, which is a splitting of A_{11} in the usual way. To show that any such splitting is possible we have to prove that, for any partition $\sum m_i = 6$ ($m_i \in \mathbb{N}$) of 6, there exists a quartic Q that intersects P with the multiplicities $2m_i$. We compute in the Jacobian of the smooth cubic P, which is isomorphic to P. We think of the Jacobian as a complex torus given as the wraparound of a parallelogram inside $\mathbb{C}$ determined by the numbers $1, \tau \in \mathbb{C} \setminus \mathbb{R}$. Clearly, we can find pairwise distinct points q_i in the interior of the parallelogram given by $\frac{1}{2}$ and $\frac{\tau}{2}$ such that $\sum m_i q_i = \frac{\tau+1}{2}$ in $\mathbb{C}$. Then $\sum 2m_i q_i = 0$ in the Jacobian of P; that is, $\sum 2m_i q_i$ is a principal divisor. Owing to our choice of the q_i, all proper nontrivial subcombinations $\sum n_i q_i$ $(0 \leq n_i \leq m_i)$ are not principal. Since a plane cubic is projectively normal and since $\deg \sum 2m_i q_i = 3 \cdot 4$, there exists a quartic Q with $P \cap Q = \sum 2m_i q_i$. Now it remains to show that there is a linear symmetric matrix representation of Q such that the top 3×3 minor of this matrix is P, because we can obtain a matrix representation of F from this matrix by adding w to the bottom right entry. Since P is smooth, we find a self-linked ideal I with respect to (P, Q), that is, $(P, Q) : I = I$ [M-B, Prop. 4.3]. Note that I does not contain a quadric polynomial. Namely, if $G \in I$ with $\deg G = 2$ then $G^2 \in (P, Q)$; that is, $G^2 = \lambda Q + LP$ with $\lambda \in \mathbb{C}^*$ and $L \in \mathbb{C}[x, y, z]_1$. We would thus have $2G \cap P = Q \cap P$ and so $G \cap P$ would be a principle subdivisor of $Q \cap P$ on P, which is impossible by the construction of Q. By [Be, Sec. 2.4] or [M-B, Sec. 4], such a self-linked ideal induces a linear symmetric matrix representation of Q with a contact cubic P. After a change of basis we may assume that P is the upper left 3×3 minor of this matrix. In fact, knowing that such a matrix exists, it can be easily constructed using Dixon's method [Di].

Case 2: P nodal cubic. From Section A we know that, up to a choice of basis, there are only two different linear symmetric matrix representations of a nodal cubic $P = x^3 + y^3 + xyz$. In other words, for a matrix M with $F = \det M$ we may assume that

$$M = \begin{pmatrix} -y & 0 & x & a_y y + a_z z \\ 0 & -x & y & b_y y + b_z z \\ x & y & z & c_y y + c_z z \\ a_y y + a_z z & b_y y + b_z z & c_y y + c_z z & w + f \end{pmatrix} \quad \text{or}$$

$$M' = \begin{pmatrix} -y & \frac{1}{2}z & x & a_y y + a_z z \\ \frac{1}{2}z & -x & y & b_y y + b_z z \\ x & y & 0 & c_y y + c_z z \\ a_y y + a_z z & b_y y + b_z z & c_y y + c_z z & w + f \end{pmatrix},$$

where $f \in \mathbb{C}[x, y, z]$. The variable x was eliminated from the last column by adding suitable multiples of the first three columns. Now we can compute Q and Q' and obtain that $(a_z, b_z) \neq 0$ (resp. $c_z \neq 0$) for Q (resp. Q') to be smooth at the singular point $(0 : 0 : 1)$ of P. To compute the intersection multiplicities of P and the quartics, we choose the parameterization $(s : t) \mapsto (-s^2 t : st^2 : (t - s)(s^2 + st + t^2))$ of P that maps $(1 : 0)$ and $(0 : 1)$ to the singular point of P. Plugging it into the quartics yields, respectively,

$$st(a_z s^5 - c_z s^4 t + (b_z - a_y)s^3 t^2 + (c_y - a_z)s^2 t^3 + (c_z - b_y)st^4 - b_z t^5)^2$$

and

$$\bigl(-\tfrac{1}{2}c_z s^6 + b_z s^5 t + \bigl(a_z + \tfrac{1}{2}c_y\bigr)s^4t^2 \\ - b_y s^3t^3 - (b_z + a_y)s^2t^4 - \bigl(a_z - \tfrac{1}{2}c_y\bigr)st^5 + \tfrac{1}{2}c_z t^6\bigr)^2.$$

The first polynomial is of degree 5 and is arbitrary apart from $(a_z, b_z) \neq 0$; we can distribute its zeros arbitrarily with the exception that we cannot have zeros at $(1 : 0)$ and $(0 : 1)$ at the same time. The second, sextic polynomial can also have any combination of multiple zeros, but none of the zeros can be at the points that map to the singular point of P because $c_z \neq 0$. Therefore, by Degtyarev's results we obtain the following possible combinations of singularities together with the usual splitting of the A-singularities:

$$T_{3,3,5+2k} + A_{9-2k} \quad \text{for } k \in \{0, \dots, 4\},$$
$$T_{3,3,15}, \qquad T_{3,3,4} + A_{11}.$$

Case 3: P smooth quadric + secant. Again there are two linear symmetric matrix representations of $P = x(x^2 + yz)$, so we may assume that

$$M = \begin{pmatrix} y & 0 & x & a_y y + a_z z \\ 0 & -x & 0 & b_y y + b_z z \\ x & 0 & -z & c_y y + c_z z \\ a_y y + a_z z & b_y y + b_z z & c_y y + c_z z & w + f \end{pmatrix} \quad \text{or}$$

$$M' = \begin{pmatrix} 0 & y & x & a_y y + a_z z \\ y & -x & \frac{1}{2}z & b_y y + b_z z \\ x & \frac{1}{2}z & 0 & c_y y + c_z z \\ a_y y + a_z z & b_y y + b_z z & c_y y + c_z z & w + f \end{pmatrix}.$$

The singularities of P are $(0 : 1 : 0)$ and $(0 : 0 : 1)$. Because Q (resp. Q') must be smooth at these points, we find that $(a_z, b_z) \neq 0$ and $(b_y, c_y) \neq 0$ (resp. $a_z \neq 0$ and $c_y \neq 0$). We parameterize the secant of P by $(s : t) \mapsto (0 : s : t)$ and the quadric by $(s : t) \mapsto (st : -s^2 : t^2)$; thus, in either case $(1 : 0)$ and $(0 : 1)$ map to the singular points of P. In order to compute the intersection multiplicities of P and the quartics Q and Q', we pull the quartics back via these parameterizations to obtain

$$st(b_y s + b_z t)^2 \quad \text{and} \quad -st(c_y s^3 + a_y s^2 t - c_z st^2 - a_z t^3)^2$$

for Q and

$$\bigl(c_y s^2 + \bigl(c_z - \tfrac{1}{2}a_y\bigr)st - \tfrac{1}{2}a_z t^2\bigr)^2 \quad \text{and}$$
$$\bigl(c_y s^4 + b_y s^3 t - \bigl(c_z + \tfrac{1}{2}a_y\bigr)s^2t^2 - b_z st^3 + \tfrac{1}{2}a_z t^4\bigr)^2$$

for Q'. In the first case we can distribute the zeros arbitrarily—with the exception that the linear and the cubic polynomial cannot both have zeros at points that are mapped to the same singular point of P. In the second case we cannot have zeros at the points that are mapped to the singular points of P, yet any combination of multiple zeros can occur; hence F can have the following combinations of singularities with the usual splitting of the A-singularities:

$$T_{3,5,5}+A_5+A_1,\ T_{3,5,7}+A_5,\ T_{3,5,9}+2A_1,\ T_{3,5,11}+A_1,\ T_{3,7,7}+A_3,$$
$$T_{3,7,9}+A_1,\ T_{3,7,11},\ T_{3,4,4}+A_3+A_7.$$

Case 4: P three noncongruent lines. This is the last case where there are two nonequivalent linear symmetric matrix representations of the cubic. We take P as xyz and may assume that

$$M=\begin{pmatrix} x & 0 & 0 & a_y y+a_z z\\ 0 & y & 0 & b_x x+b_z z\\ 0 & 0 & z & c_x x+c_y y\\ a_y y+a_z z & b_x x+b_z z & c_x x+c_y y & w+f\end{pmatrix}\quad\text{or}$$

$$M'=\begin{pmatrix} 0 & x & \frac{1}{2}y & a_z z\\ x & 0 & z & b_y y+b_z z\\ \frac{1}{2}y & z & 0 & c_x x+c_y y+c_z z\\ a_z z & b_y y+b_z z & c_x x+c_y y+c_z z & w+f\end{pmatrix}.$$

In order for Q (resp. Q') to be smooth at the singular points of P, we must have $(a_z,b_z)\neq 0$, $(a_y,c_y)\neq 0$, and $(b_x,c_x)\neq 0$ (resp. $a_z\neq 0$, $b_y\neq 0$, and $c_x\neq 0$). We use the parameterizations $(s:t)\mapsto(0:s:t)$, $(s:t)\mapsto(t:0:s)$, and $(s:t)\mapsto(s:t:0)$, which map $(1:0)$ and $(0:1)$ to the singular points of P. Pulling Q and Q' back via these mappings gives

$$-st(a_y s+a_z t)^2,\ -st(b_z s+b_x t)^2,\ -st(c_x s+c_y t)^2$$

for Q and

$$\left(\tfrac{1}{2}b_y s^2+\tfrac{1}{2}b_z st-a_z t^2\right)^2,\ (a_z s^2-c_z st-c_x t^2)^2,\ \left(c_x s^2+c_y st-\tfrac{1}{2}b_y t^2\right)^2$$

for Q'. The preceding inequalities imply (in the first case) that the linear forms can contribute to the intersection multiplicities of $P\cap Q$ only at different singular points of P and (in the second case) that the quadrics cannot have zeros at the points that map to the singular points of P. Therefore, F can have the following combinations of singularities with the usual splitting of the A-singularities:

$$T_{5,5,5}+3A_1,\ T_{5,5,7}+2A_1,\ T_{5,7,7}+A_1,\ T_{7,7,7},\ T_{4,4,4}+3A_3.$$

Case 5: P cuspidal cubic. Up to a choice of coordinates, there is only one representation of $P=x^3+yz^2$ as a linear symmetric determinant; hence we may assume that

$$M=\begin{pmatrix} -y & 0 & x & a_y y+a_z z\\ 0 & -x & z & b_y y+b_z z\\ x & z & 0 & c_y y+c_z z\\ a_y y+a_z z & b_y y+b_z z & c_y y+c_z z & w+f\end{pmatrix}.$$

The condition that Q has no singular point at the singular point $(0:1:0)$ of P turns out to be $c_y\neq 0$. Plugging the parameterization $(s:t)\mapsto(t^2s:-s^3:t^3)$ of P into Q gives

$$t^2(c_y s^5+b_y s^4 t-a_y s^3t^2-c_z s^2t^3-b_z st^4+a_z t^5)^2.$$

Since $c_y \neq 0$, the intersection multiplicity of P and Q is always 2 in the singular point of P; otherwise, the quintic is arbitrary. Hence F can have $Q_{11} + A_9$ as singularities as well as any combination of singularities obtained by splitting A_9 in the usual way.

Case 6: P smooth quadric + tangent. Because there is only one linear symmetric matrix representation of $P = z(x^2 + yz)$, we may assume that

$$M = \begin{pmatrix} y & x & 0 & a_y y + a_z z \\ x & -z & 0 & b_y y + b_z z \\ 0 & 0 & -z & c_x x + c_y y \\ a_y y + a_z z & b_y y + b_z z & c_x x + c_y y & w + f \end{pmatrix}.$$

The meeting point of the quartic and the tangent is the singular point $(0 : 1 : 0)$ of P. For Q to be nonsingular at this point means $b_y^2 + c_y^2 \neq 0$. In order to compute the intersection multiplicities of P and Q, we parameterize the line and the quadric of P by $(s : t) \mapsto (t : s : 0)$ and $(s : t) \mapsto (st : s^2 : -t^2)$; both parameterizations map $(1 : 0)$ to the singular point of P. Pulling Q back via these parameterizations yields

$$t^2(c_y s + c_x t)^2 \quad \text{and} \quad -t^2(b_y s^3 - a_y s^2 t - b_z s t^2 + a_z t^3)^2.$$

Since $b_y^2 + c_y^2 \neq 0$, both terms cannot simultaneously contribute further to the intersection multiplicities of P and Q at the singular point of F. Therefore, F can have the following combinations of singularities with the usual splitting of the A-singularities:

$$S_{1,0} + A_5 + A_1, \; S_{1,2}^{\#} + A_5, \; S_{1,4}^{\#} + 2A_1, \; S_{1,6}^{\#} + A_1.$$

Case 7: P three congruent lines, double line + line, triple line, empty set. All these cases lead to nonnormal quartics. Since a linear symmetric matrix representation $\tilde{M}$ of three congruent lines involves only the variables x and y, the rank of a 4×4 matrix M with $\tilde{M}$ in the upper left corner is only 2 along the line $\{x = y = 0\}$. Therefore, F is singular along this line. For the remaining cases one can apply Lemma 1.4 after a reshuffling of coordinates. □

A. Appendix: Linear Symmetric Matrix Representations of Plane Cubics

Finding linear symmetric matrix representations of plane cubics is a classical problem. The three representations of a smooth cubic were found by Hesse [He]. The matrix representations of the singular cubics are scattered throughout the literature; most of them were computed by Barth [B] and Meyer-Brandis [M-B]. The case of the empty cubic is a special case of Atkinson [At]. For the complete list given as Table 7, the representation matrices of the singular cubics were computed using the straightforward method of Barth or Taussky [T]. The remarkable fact is that a reduced singular cubic has two nonequivalent representations if it has only A_1-singularities but only one representation if it has another singularity. The

Table 7

Cubic	Equation	Representation	Number of rank-1 parameter ranges
Smooth	$x^3+y^3+z^3-\lambda xyz$	$\frac{-1}{\mu}\begin{pmatrix}\mu x & z & y\\ z & \mu y & x\\ y & x & \mu z\end{pmatrix}$	0 $\mu^2+2\mu^{-1}=\lambda$
Nodal	x^3+y^3+xyz	$\begin{pmatrix}-y & \frac{1}{2}z & x\\ \frac{1}{2}z & -x & y\\ x & y & 0\end{pmatrix}$	0
		$\begin{pmatrix}-y & 0 & x\\ 0 & -x & y\\ x & y & z\end{pmatrix}$	1
Quadric + secant	$x(x^2+yz)$	$\begin{pmatrix}0 & y & x\\ y & -x & \frac{1}{2}z\\ x & \frac{1}{2}z & 0\end{pmatrix}$	0
		$\begin{pmatrix}y & 0 & x\\ 0 & -x & 0\\ x & 0 & -z\end{pmatrix}$	2
3 lines	xyz	$\begin{pmatrix}0 & x & \frac{1}{2}y\\ x & 0 & z\\ \frac{1}{2}y & z & 0\end{pmatrix}$	0
		$\begin{pmatrix}x & 0 & 0\\ 0 & y & 0\\ 0 & 0 & z\end{pmatrix}$	3
Cuspidal	x^3+yz^2	$\begin{pmatrix}-y & 0 & x\\ 0 & -x & z\\ x & z & 0\end{pmatrix}$	1
Quadric + tangent	$z(x^2+yz)$	$\begin{pmatrix}y & x & 0\\ x & -z & 0\\ 0 & 0 & -z\end{pmatrix}$	1
3 congruent lines	$x(x^2+y^2)$	$\begin{pmatrix}0 & y & x\\ y & -x & \frac{1}{2}y\\ x & \frac{1}{2}y & 0\end{pmatrix}$	1
Double line + line	x^2y	$\begin{pmatrix}az & x & bz\\ x & 0 & 0\\ bz & 0 & -y\end{pmatrix}$	1 or line $a,b\in\mathbb{C}$
Triple line	x^3	$\begin{pmatrix}az & by & x\\ by & -x & 0\\ x & 0 & 0\end{pmatrix}$	0 $a,b\in\{0,1\}$
Empty cubic	$\emptyset$	$\begin{pmatrix}* & * & 0\\ * & * & 0\\ 0 & 0 & 0\end{pmatrix}$	—
Empty cubic	$\emptyset$	$\begin{pmatrix}* & * & *\\ * & 0 & 0\\ * & 0 & 0\end{pmatrix}$	—

table's last column gives the number of accidental singularities—that is, the number of points of $\mathbb{P}^2$ where the matrix has only rank 1; this number distinguishes the two representations of the cubics with A_1-singularities.

References

[AGV] V. Arnold, S. Gusein-Zade, and A. Varchenko, *Singularities of differentiable maps,* vol. I, Monogr. Math., 82, Birkhäuser, Boston, 1985.

[At] M. Atkinson, *Primitive space of matrices of bounded rank, II,* J. Austral. Math. Soc. Ser. A 34 (1983), 306–315.

[B] W. Barth, *Moduli of vector bundles on the projective plane,* Invent. Math. 42 (1977), 63–91.

[BPV] W. Barth, C. Peters, and A. Van de Ven, *Compact complex surfaces,* Ergeb. Math. Grenzgeb. (3), 4, Springer-Verlag, Berlin, 1984.

[Be] A. Beauville, *Determinantal hypersurfaces,* Michigan Math. J. 48 (2000), 39–64.

[BrW] J. Bruce and C. Wall, *On the classification of cubic surfaces,* J. London Math. Soc. (2) 19 (1979), 245–256.

[C] F. Catanese, *Babbage's conjecture, contact of surfaces, symmetric determinantal varieties and applications,* Invent. Math. 63 (1981), 433–465.

[Ca] A. Cayley, *A memoir on quartic surfaces,* Proc. London Math. Soc. 3 (1869–1871), 19–69.

[Co] F. Cossec, *Reye congruences,* Trans. Amer. Math. Soc. 280 (1983), 737–751.

[CoD] F. Cossec and I. Dolgachev, *Enriques surfaces I,* Progr. Math., 76, Birkhäuser, Boston, 1989.

[De] A. Degtyarev, *Classification of quartic surfaces that have a nonsimple singular point,* Math. USSR-Izv. 35 (1990), 607–627.

[Di] A. Dixon, *Note on the reduction of a ternary quartic to a symmetrical determinant,* Proc. Cambridge Phil. Soc. 11 (1900–1902), 350–351.

[Eb] W. Ebeling, *Funktionentheorie, Differentialtopologie und Singularitäten,* Vieweg, Braunschweig, 2001.

[E1] D. Eisenbud, *Homological algebra on a complete intersection with an application to group representations,* Trans. Amer. Math. Soc. 260 (1980), 35–64.

[E2] ———, *Commutative algebra. With a view toward algebraic geometry,* Grad. Texts in Math., 150, Springer-Verlag, New York, 1995.

[FPi] G. Fischer and J. Piontkowski, *Ruled varieties,* Adv. Lectures Math., Vieweg, Braunschweig, 2001.

[Ga] F. Gantmacher, *Matrizentheorie,* Springer-Verlag, Berlin, 1986.

[Ha] J. Harris, *Algebraic geometry,* Grad. Texts in Math., 133, Springer-Verlag, New York, 1992.

[HTu] H. Harris and L. Tu, *On symmetric and skew-symmetric determinantal varieties,* Topology 30 (1984), 71–84.

[He] L. Hesse, *Über die Elimination der Variabeln aus drei algebraischen Gleichungen von zweiten Grad mit zwei Variablen,* J. Reine Angew. Math. 28 (1844), 68–96.

[KaN] M. Kato and I. Naruki, *Depth of rational double points on quartic surfaces,* Proc. Japan Acad. Ser. A Math. Sci. 58 (1982), 72–75.

[KUl] S. Kleiman and B. Ulrich, *Gorenstein algebras, symmetric matrices, self-linked ideals, and symbolic powers,* Trans. Amer. Math. Soc. 349 (1997), 4973–5000.

[L] E. Looijenga, *On the semi-universal deformation of a simple elliptic hypersurface singularity. Part II: The discriminant,* Topology 17 (1978), 23–40.

[M-B] T. Meyer-Brandis, *Berührungssysteme und symmetrische Darstellungen ebener Kurven,* Diplomarbeit, Mainz, 1998.

[N] V. Nikulin, *Integer symmetric bilinear forms and some of their applications,* Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), 111–177.

[R] M. Reid, *Chapters on algebraic surfaces,* Complex algebraic geometry (Park City, 1993), IAS/Park City Math. Ser., 3, pp. 3–159, Amer. Math. Soc., Providence, RI, 1997.

[S] G. Salmon, *Solid geometry,* 2nd ed., 1865.

[U1] T. Urabe, *Elementary transformations of Dynkin graphs and singularities on quartic surfaces,* Invent. Math. 87 (1987), 549–573.

[U2] ———, *Tie transformations of Dynkin graphs and singularities on quartic surfaces,* Invent. Math. 100 (1990), 207–230.

[U3] ———, *Dynkin graphs and quadrilateral singularities,* Lecture Notes in Math., 1548, Springer-Verlag, Berlin, 1993.

[T] O. Taussky, *Nonsingular cubic curves as determinantal loci,* J. Math. Phys. Sci. 21 (1987), 665–678.

[Y] J.-G. Yang, *Enumeration of combinations of rational double points on quartic surfaces,* Singularities and complex geometry (Beijing, 1994), AMS/IP Stud. Adv. Math., 5, pp. 275–312, Amer. Math. Soc., Providence, RI, 1997.

[Yo] Y. Yoshino, *Cohen–Macaulay modules over Cohen–Macaulay rings,* London Math. Soc. Lecture Note Ser., 146, Cambridge Univ. Press, 1990.

Johannes Gutenberg Universität Mainz
Fachbereich Mathematik Staudingerweg 9
55099 Mainz
Germany

piontkow@mathematik.uni-mainz.de