Michigan Math. J. 54 (2006)

Complexes of Nonseparating Curves and Mapping Class Groups

Elmas Irmak

1. Introduction

Let R be a compact, connected, orientable surface of genus g with p boundary components. The mapping class group Mod_R of R is the group of isotopy classes of orientation-preserving homeomorphisms of R. The extended mapping class group Mod_R^* of R is the group of isotopy classes of all (including orientation-reversing) homeomorphisms of R. Let $\mathcal{A}$ denote the set of isotopy classes of nontrivial simple closed curves on R. The *complex of curves* $\mathcal{C}(R)$ on R is an abstract simplicial complex, introduced by Harvey [H], with vertex set $\mathcal{A}$ such that a set of n vertices $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ forms an $(n-1)$-simplex if and only if $\alpha_1, \alpha_2, \dots, \alpha_n$ have pairwise disjoint representatives.

Definition. A simplicial map $\lambda \colon \mathcal{C}(R) \to \mathcal{C}(R)$ is called *superinjective* if the following condition holds: If α and β are two vertices in $\mathcal{C}(R)$ such that the geometric intersection number $i(\alpha, \beta)$ of α and β is not equal to zero, then $i(\lambda(\alpha), \lambda(\beta))$ is not equal to zero.

The combinatorial structure of curve complexes on surfaces are studied in order to derive information about the algebraic structure of the mapping class groups. In [Iv1], Ivanov proved that if $g \geq 2$ then every automorphism of $\mathcal{C}(R)$ is induced by a homeomorphism of R. He proved that $\mathrm{Aut}(\mathcal{C}(R)) \cong \mathrm{Mod}_R^*$ for most surfaces, and as an application he gave a complete description of isomorphisms between finite index subgroups of Mod_R^*. Ivanov proved that every such isomorphism is induced by a homeomorphism of R; that is, it is of the form $k \to hkh^{-1}$ for some $h \in \mathrm{Mod}_R^*$ for most surfaces. These theorems were extended to most of the surfaces of genus 0 and 1 by Korkmaz in [K] and independently by Luo in [L2]. Luo gave a proof by using a multiplicative structure on the set of isotopy classes of nonseparating simple closed curves on R, a structure introduced by him in [L1].

Ivanov and McCarthy [IvM] gave a complete description of injective homomorphisms between mapping class groups of surfaces Mod_R and $\mathrm{Mod}_{R'}$ when the maxima of ranks of abelian subgroups of Mod_R and $\mathrm{Mod}_{R'}$ differ by at most 1. In particular they showed that, for most surfaces, an injective homomorphism of Mod_R to itself is of the form $k \to hkh^{-1}$ for some $h \in \mathrm{Mod}_R^*$.

Received July 12, 2004. Revision received September 28, 2005.
Supported by a Rackham Faculty Fellowship, Rackham Graduate School, University of Michigan.

Superinjective simplicial maps were first defined and used by the author in [I1] to find a complete description of injective homomorphisms from finite index subgroups of Mod_R^* to Mod_R^*. This was motivated by the work of Ivanov and Ivanov–McCarthy. The main results from [I1] are as follows: if R is closed and $g \geq 3$, then an injective homomorphism f from a finite index subgroup to Mod_R^* induces a superinjective map λ on $\mathcal{C}(R)$; any superinjective simplicial map of $\mathcal{C}(R)$ is induced by a homeomorphism; and f is induced by the homeomorphism that induces λ (i.e., it is of the form $k \to hkh^{-1}$ for some $h \in \mathrm{Mod}_R^*$). In [I2] these theorems were extended to the cases where $g = 2$ and $p \geq 2$ or where $g \geq 3$ and $p \geq 0$. These results generalize those of Ivanov, since an automorphism of $\mathcal{C}(R)$ is a superinjective map of $\mathcal{C}(R)$; they also generalize Ivanov–McCarthy's results mentioned previously.

This paper contains two parts. In the first part, we complete the generalization of Ivanov's results by proving the cited theorems for $g = 2$ and $p \leq 1$. We see that an exceptional case appears for injective homomorphisms from finite index subgroups when R is a closed surface of genus 2. In this case, our result is similar to McCarthy's explicit description of automorphisms of Mod_R^* for a closed surface of genus 2 as given in [M1].

The main results in the first part are the following two theorems.

THEOREM 1.1. *Suppose $g = 2$ and $p \leq 1$. A simplicial map $\lambda\colon \mathcal{C}(R) \to \mathcal{C}(R)$ is superinjective if and only if λ is induced by a homeomorphism of R.*

THEOREM 1.2. *Let K be a finite index subgroup of Mod_R^* and let f be an injective homomorphism $f\colon K \to \mathrm{Mod}_R^*$. If $g = 2$ and $p = 1$, then f has the form $k \to hkh^{-1}$ for some $h \in \mathrm{Mod}_R^*$ and f has a unique extension to an automorphism of Mod_R^*. If R is a closed surface of genus 2 then f has the form $k \to hkh^{-1}i^{m(k)}$ for some $h \in \mathrm{Mod}_R^*$, where m is a homomorphism $K \to \mathbb{Z}_2$ and i is the hyperelliptic involution on R.*

In the second part of this paper, we consider the *complex of nonseparating curves* $\mathcal{N}(R)$, which is the subcomplex of $\mathcal{C}(R)$ where the vertices are the isotopy classes of nonseparating curves and a set of n vertices $\{\beta_1, \beta_2, \ldots, \beta_n\}$ forms an $(n-1)$-simplex if and only if it forms an $(n-1)$-simplex in $\mathcal{C}(R)$. Before stating our theorems, we mention some known results of a similar nature.

After Ivanov's work in [Iv1], the extended mapping class group Mod_R^* was viewed as the automorphism group of various geometric objects on R. Schaller considered the graph $\mathcal{G}(R)$: the vertex set of $\mathcal{G}(R)$ is the set of isotopy classes of nonseparating simple closed curves on R. Two vertices are connected by an edge if and only if their geometric intersection number is 1. His main result in [S] is the following theorem, though we state only as much as we need in this paper (Schaller defines the graph $\mathcal{G}(R)$ for surfaces of genus 0 and 1 as well): If $g \geq 2$ and if R is not a closed surface of genus 2, then $\mathrm{Aut}(\mathcal{G}(R)) \cong \mathrm{Mod}_R^*$. In [Ma], Margalit considered the complex of pants decompositions $\mathcal{P}(R)$. His main result is that $\mathrm{Aut}(\mathcal{P}(R)) \cong \mathrm{Mod}_R^*$ for most closed surfaces.

Farb and Ivanov [FIv] proved that every automorphism of the Torelli geometry $\mathcal{TG}$ is induced by a homeomorphism of R if $g \geq 5$ in order to show that the automorphisms of Torelli subgroup $\mathcal{T}$, the subgroup of Mod_R^* consisting of elements that act trivially on $H_1(R, \mathbb{Z})$, are induced by homeomorphisms of the surface. In [Iv1] it was proved that $\mathrm{Aut}(\mathcal{TG}) \cong \mathrm{Mod}_R^*$ and $\mathrm{Aut}(\mathcal{T}) \cong \mathrm{Mod}_R^*$. McCarthy and Vautaw extended the $\mathrm{Aut}(\mathcal{T}) \cong \mathrm{Mod}_R^*$ result to $g \geq 3$ in [MV].

Brendle and Margalit [BrMa] have proved that if $g \geq 5$ then a superinjective simplicial map of separating curve complex $\mathcal{S}(R)$ (the subcomplex of $\mathcal{C}(R)$ spanned by the vertices corresponding to separating curves) is induced by a homeomorphism; they extended the simplicial map to a superinjective map of $\mathcal{C}(R)$ and used the results in [I1]. This, in turn, was used to prove that an injection from a finite index subgroup of K to the Torelli group (where K is the subgroup of Mod_R^* generated by Dehn twists about separating curves) is induced by a homeomorphism when $g \geq 5$. Brendle and Margalit also proved that if $g \geq 5$ then $\mathrm{Aut}(\mathcal{S}(R)) \cong \mathrm{Mod}_R^*$.

Irmak and Korkmaz [IK] have proved a theorem about automorphism groups of the Hatcher–Thurston complex $\mathrm{Aut}(\mathcal{HT}(R))$. The main result is that if $g > 0$ then $\mathrm{Aut}(\mathcal{HT}(R)) \cong \mathrm{Mod}_R^*/\mathcal{C}(\mathrm{Mod}_R^*)$. Superinjective simplicial maps and automorphisms of some geometric objects on surfaces are important because they help us to understand the mapping class groups better. The second part of this paper gives such results about the complex $\mathcal{N}(R)$ of nonseparating curves. Since mapping class groups are generated by Dehn twists about nonseparating curves, it is natural to consider $\mathcal{N}(R)$. As mentioned before, some structures and complexes on the set of isotopy classes of nonseparating curves were examined in [L1; L2; S]. We note that our Theorem 1.4 concerning $\mathcal{N}(R)$ has the following application with respect to $\mathcal{HT}(R)$: the main result in [IK] about the automorphism group of the Hatcher–Thurston complex $\mathcal{HT}(R)$ also follows from our Theorem 1.4 about $\mathcal{N}(R)$ and Proposition 9 in [IK] for closed surfaces of genus ≥ 2.

Now we state the main results of the second part of this paper. We note that a simplicial map $\lambda\colon \mathcal{N}(R) \to \mathcal{N}(R)$ is called superinjective if it preserves nondisjointness as in the $\mathcal{C}(R)$ case; that is, if α, β are two vertices in $\mathcal{N}(R)$ such that $i(\alpha, \beta) \neq 0$, then $i(\lambda(\alpha), \lambda(\beta)) \neq 0$.

THEOREM 1.3. *Suppose that $g \geq 2$ and that R has at most $g - 1$ boundary components. Then a simplicial map $\lambda\colon \mathcal{N}(R) \to \mathcal{N}(R)$ is superinjective if and only if λ is induced by a homeomorphism of R.*

THEOREM 1.4. *Suppose that $g \geq 2$. If R is a closed surface of genus 2, then $\mathrm{Aut}(\mathcal{N}(R)) \cong \mathrm{Mod}_R^*/\mathcal{C}(\mathrm{Mod}_R^*)$. If R is not a closed surface of genus 2, then $\mathrm{Aut}(\mathcal{N}(R)) \cong \mathrm{Mod}_R^*$.*

In Section 2 we give some properties of the superinjective simplicial maps of $\mathcal{N}(R)$. In Section 3, we use some methods from Section 2 to give some properties of the superinjective simplicial maps of $\mathcal{C}(R)$; then we prove Theorem 1.1 and

Theorem 1.2. In Section 4 we prove that, if $g \geq 2$ and if R has at most $g-1$ boundary components, then a superinjective simplicial map $\lambda\colon \mathcal{N}(R) \to \mathcal{N}(R)$ extends to a superinjective simplicial map on $\mathcal{C}(R)$; then we prove Theorem 1.3 by using the theorems from [I1; I2] that superinjective maps of $\mathcal{C}(R)$ are induced by homeomorphisms. Theorem 1.4 follows from Theorem 1.3 when $g \geq 2$ and R has at most $g-1$ boundary components. For the remaining cases, we prove Theorem 1.4 by using Schaller's result [S] that $\mathrm{Aut}(\mathcal{G}(R)) \cong \mathrm{Mod}_R^*$.

2. Properties of Superinjective Simplicial Maps of $\mathcal{N}(R)$

A *circle* on R is a properly embedded image of an embedding $S^1 \to R$. A circle on R is said to be *nontrivial* (or *essential*) if it doesn't bound a disk and is not homotopic to a boundary component of R. Let C be a collection of pairwise disjoint circles on R. The surface obtained from R by cutting along C is denoted by R_C. A nontrivial circle a on R is called *nonseparating* if the surface R_a is connected. Let α and β be two vertices in $\mathcal{N}(R)$. The *geometric intersection number* $i(\alpha, \beta)$ is defined to be the minimum number of points of $a \cap b$ where $a \in \alpha$ and $b \in \beta$.

LEMMA 2.1. *Suppose $g \geq 2$ and $p \geq 0$. A superinjective simplicial map λ: $\mathcal{N}(R) \to \mathcal{N}(R)$ is injective.*

Proof. Let α and β be two distinct vertices in $\mathcal{N}(R)$. If $i(\alpha, \beta) \neq 0$, then we have $i(\lambda(\alpha), \lambda(\beta)) \neq 0$ because λ preserves nondisjointness. Hence $\lambda(\alpha) \neq \lambda(\beta)$. If $i(\alpha, \beta) = 0$ then, since $g \geq 2$ and $p \geq 0$, we can choose a vertex γ of $\mathcal{N}(R)$ such that $i(\gamma, \alpha) = 0$ and $i(\gamma, \beta) \neq 0$. Then $i(\lambda(\gamma), \lambda(\alpha)) = 0$ and $i(\lambda(\gamma), \lambda(\beta)) \neq 0$. Therefore, $\lambda(\alpha) \neq \lambda(\beta)$ and so λ is injective. □

Let P be a set of pairwise disjoint circles on R. We call P a *pair of pants decomposition* of R if R_P is a disjoint union of genus-0 surfaces with three boundary components (pairs of pants). A pair of pants of a pants decomposition is the image of one of these connected components under the quotient map $q\colon R_P \to R$. The image of the boundary of this component is called the *boundary of the pair of pants.* A pair of pants is called *embedded* if the restriction of q to the corresponding component of R_P is an embedding. An ordered set $(a_1, \dots, a_{3g-3+p})$ is called an *ordered pair of pants decomposition* of R if $\{a_1, \dots, a_{3g-3+p}\}$ is a pair of pants decomposition of R.

LEMMA 2.2. *Suppose $g \geq 2$ and $p \geq 0$. Let $\lambda\colon \mathcal{N}(R) \to \mathcal{N}(R)$ be a superinjective simplicial map, and let P be a pair of pants decomposition consisting of nonseparating circles on R. Then λ maps the set of isotopy classes of elements of P to the set of isotopy classes of elements of a pair of pants decomposition P' of R.*

Proof. The set of isotopy classes of elements of P forms a top-dimensional simplex $\triangle$ in $\mathcal{N}(R)$. Since λ is injective, it maps $\triangle$ to a top-dimensional simplex $\triangle'$

in $\mathcal{N}(R)$. A set of pairwise disjoint representatives of the vertices of $\triangle'$ is a pair of pants decomposition P' of R. □

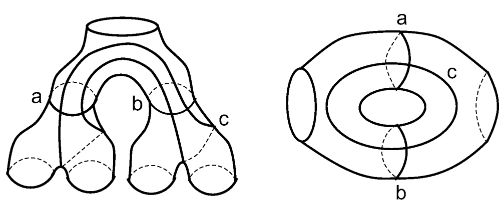
Let P be a pair of pants decomposition of R, and let a and b be two distinct elements in P. Then a is called *adjacent* to b with respect to P if and only if there exists a pair of pants in P that has a and b on its boundary.

Remark. Let P be a pair of pants decomposition of R. Let $[P]$ be the set of isotopy classes of elements of P, and let $\alpha, \beta \in [P]$. We say that α is adjacent to β with respect to $[P]$ if the representatives of α and β in P are adjacent w.r.t. P. By Lemma 2.2, λ gives a correspondence on the isotopy classes of elements of pair of pants decompositions consisting of nonseparating circles on R. Under this correspondence, $\lambda([P])$ is the set of isotopy classes of elements of a pair of pants decomposition that corresponds to P.

Lemma 2.3. *Suppose $g \geq 2$ and $p \geq 0$. Let $\lambda\colon \mathcal{N}(R) \to \mathcal{N}(R)$ be a superinjective simplicial map, and let P be a pair of pants decomposition consisting of nonseparating circles on R. Then λ preserves the adjacency relation for two circles in P; that is, if $a, b \in P$, if a is adjacent to b w.r.t. P, and if $[a] = \alpha$ and $[b] = \beta$, then $\lambda(\alpha)$ is adjacent to $\lambda(\beta)$ w.r.t. $\lambda([P])$.*

Proof. Let P be a pair of pants decomposition consisting of nonseparating circles on R. If $g = 2$ and $p \leq 1$, then every element in $\lambda([P])$ is adjacent to any other element in $\lambda([P])$ and so the lemma is clear. For the other cases, let a, b be two adjacent circles in P and let $[a] = \alpha$ and $[b] = \beta$. By Lemma 2.2, we can choose a pair of pants decomposition P' such that $\lambda([P]) = [P']$. Let P_o be a pair of pants of P having a and b on its boundary; then P_o is an embedded pair of pants. There are two possible cases for P_o, depending on whether or not a and b are the boundary components of another pair of pants. For each case, we show how to choose a circle c that essentially intersects a and b and does not intersect any other circle in P; see Figure 1.

Let $\gamma = [c]$, and assume that $\lambda(\alpha)$ and $\lambda(\beta)$ do not have adjacent representatives. Since $i(\gamma, \alpha) \neq 0$ and $i(\gamma, \beta) \neq 0$, it follows that $i(\lambda(\gamma), \lambda(\alpha)) \neq 0$ and



(i) (ii)

Figure 1 Two possible cases for P_o

$i(\lambda(\gamma), \lambda(\beta)) \neq 0$ by superinjectivity. Since $i(\gamma, [e]) = 0$ for all e in $P \setminus \{a, b\}$, we have $i(\lambda(\gamma), \lambda([e])) = 0$ for all e in $P \setminus \{a, b\}$. But this is not possible because $\lambda(\gamma)$ must intersect geometrically essentially with some isotopy class other than $\lambda(\alpha)$ and $\lambda(\beta)$ in the image pair of pants decomposition in order to make essential intersections with $\lambda(\alpha)$ and $\lambda(\beta)$. This is a contradiction to the assumption that $\lambda(\alpha)$ and $\lambda(\beta)$ do not have adjacent representatives. □

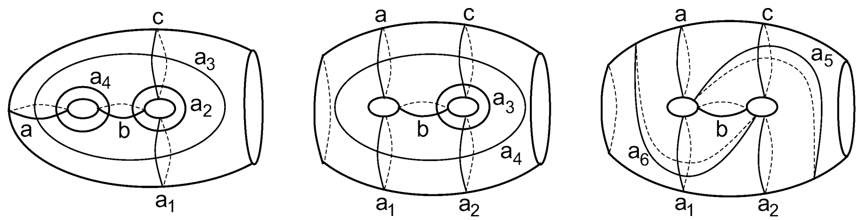
Let P be a pair of pants decomposition of R. A curve $x \in P$ is called a *4-curve* in P if there exist four distinct circles in P that are adjacent to x w.r.t. P. Note that, in a pants decomposition, every curve is adjacent to at most four curves.

LEMMA 2.4. *Suppose $g \geq 2$ and $p \geq 0$. Let $\lambda \colon \mathcal{N}(R) \to \mathcal{N}(R)$ be a superinjective simplicial map, and let α, β, γ be distinct vertices in $\mathcal{N}(R)$ having pairwise disjoint representatives that bound a pair of pants in R. Then $\lambda(\alpha), \lambda(\beta), \lambda(\gamma)$ are distinct vertices in $\mathcal{N}(R)$ having pairwise disjoint representatives that bound a pair of pants in R.*

Proof. Let a, b, c be pairwise disjoint representatives of α, β, γ, respectively. If R is a closed surface of genus 2 then the statement is obvious. Consider the case when $g = 2$ and $p = 1$. We complete $\{a, b, c\}$ to a pair of pants decomposition $P = \{a, b, c, a_1\}$ consisting of nonseparating circles, as shown in Figure 2(i). Let P' be a pair of pants decomposition of R such that $\lambda([P]) = [P']$. Let a', b', c', a_1' be (respectively) the representatives of $\lambda([a]), \lambda([b]), \lambda([c]), \lambda([a_1])$ in P'. Since a is adjacent to b w.r.t. P, it follows that a' is adjacent to b' w.r.t. P'. Then there exists a pair of pants Q_1 in P' having a' and b' on its boundary. Let x be the other boundary component of Q_1. If $x = c'$ then we are done, so assume that $x \neq c'$. Then x cannot be a' because otherwise b' would be a separating circle, which is a contradiction; similarly, x cannot be b'. Then x is either a_1' or the boundary component of R.

Assume x is a boundary component of R. Then, since a is adjacent to c, a_1 w.r.t. P, it follows that a' is adjacent to c', a_1' w.r.t. P' and so there exists a pair of pants Q_2 in P' having a' and c', a_1' on its boundary. Similarly, since b is adjacent to c, a_1 w.r.t. P, it follows that b' is adjacent to c', a_1' w.r.t. P' and so there exists a pair of pants Q_3 in P' having b' and c', a_1' on its boundary. Then it is clear that $R = Q_1 \cup Q_2 \cup Q_3$. Now we consider the circles a_2, a_3, as shown in Figure 2(i). Since a_2 intersects b essentially and since a_2 is disjoint from a, we can choose a representative a_2' of $\lambda([a_2])$ such that there exists an essential arc w of a_2' in Q_1 that starts and ends on b' and that does not intersect $a' \cup \partial R$. Since a_3 is disjoint from b and a_2, there exists a representative a_3' of $\lambda([a_3])$ such that a_3' is disjoint from $b' \cup w$. But then a_3' could be isotoped so that it is disjoint from a', since a' is a boundary component of a regular neighborhood of $b' \cup w$ in Q_1. This is a contradiction because $i([a], [a_3]) \neq 0$ and so $i(\lambda([a]), \lambda([a_3])) \neq 0$. Therefore, x cannot be a boundary component of R.

Assume that $x = a_1'$. Then, since a is adjacent to c w.r.t. P, it follows that a' is adjacent to c' w.r.t. P' and so there exists a pair of pants Q_2 in P' having a' and c' on its boundary. Let y be the other boundary component of Q_2. If $y = c'$ then a'



(i) (ii) (iii)

Figure 2 Curve configurations

would be a separating circle, which is a contradiction. If y is the boundary component of R then, since c is adjacent to b, a_1 w.r.t. P, it follows that c' is adjacent to b', a_1' w.r.t. P' and so there exists a pair of pants Q_3 in P' having b', c', a_1' on its boundary. Then it is clear that $R = Q_1 \cup Q_2 \cup Q_3$. Now we consider the circles a_2, a_4 as shown in Figure 2(i). Since a_4 intersects a essentially and since a_4 is disjoint from c, we can choose a representative a_4' of $\lambda([a_4])$ such that there exists an essential arc w of a_4' in Q_2 that starts and ends on a' and that does not intersect $c' \cup \partial R$. Next we consider a_2, which is disjoint from a and a_4. Then there exists a representative a_2' of $\lambda([a_2])$ such that a_2' is disjoint from $a' \cup w$. But then a_2' could be isotoped so that it is disjoint from c', since c' is a boundary component of a regular neighborhood of $a' \cup w$ in Q_2. This is a contradiction because $i([c],[a_2]) \neq 0$ and so $i(\lambda([c]), \lambda([a_2])) \neq 0$. Therefore, y cannot be the boundary component of R.

If $y = a_1'$, then there exists a pair of pants Q_3 having b', c' and the boundary component of R as its boundary components. Then we have $R = Q_1 \cup Q_2 \cup Q_3$. Now consider the circles a_3 and a_4. Since a_3 intersects c essentially and since a_3 is disjoint from b, we can choose a representative a_3' of $\lambda([a_3])$ such that there exists an essential arc w of a_3' in Q_3 that starts and ends on c' and that does not intersect $b' \cup \partial R$. Next we consider a_4, which is disjoint from c and a_3. Then there exists a representative a_4' of $\lambda([a_4])$ such that a_4' is disjoint from $c' \cup w$. But then a_4' could be isotoped so that it is disjoint from b', since b' is a boundary component of a regular neighborhood of $c' \cup w$ in Q_3. This is contradiction, since $i([a_4],[b]) \neq 0$ and so $i(\lambda([a_4]), \lambda([b])) \neq 0$. Hence, we conclude that $y = b'$. Then a', b', c' bound a pair of pants, which completes the proof for the case of $g = 2$ and $p = 1$.

For the remaining cases we take a subsurface N of genus 2 with two boundary components of R containing nonisotopic circles a, b, c, a_1, a_2 as shown in Figure 2(ii). Then we complete $\{a, b, c, a_1, a_2\}$ to a pair of pants decomposition P consisting of nonseparating circles on R in any way we want. Let P' be a pair of pants decomposition of R such that $\lambda([P]) = [P']$, and let a', b', c', a_1', a_2' be (respectively) the representatives of $\lambda([a]), \lambda([b]), \lambda([c]), \lambda([a_1]), \lambda([a_2])$ in P'. Since b is adjacent to a, c, a_1, a_2 w.r.t. P, it follows that b' is adjacent to a', c', a_1', a_2' w.r.t. P'. Then there are two pairs of pants Q_1, Q_2 having b' as boundary components, and $Q_1 \cup Q_2$ has a', c', a_1', a_2' on its boundary. Assume without loss of generality that Q_1 has a', b' on its boundary. Let x be the other boundary component of Q_1.

If $x = c'$ then we are done, so assume that $x \neq c'$. Then x is either a'_1 or a'_2. Assume that $x = a'_1$. Since a_3 intersects b essentially and since a_3 is disjoint from $a \cup a_1$, we can choose a representative a'_3 of $\lambda([a_3])$ such that there exists an essential arc w of a'_3 in Q_1 that starts and ends on b' and that does not intersect $a' \cup a'_1$. Next we consider a_4, which is disjoint from b and a_3. Then there exists a representative a'_4 of $\lambda([a_4])$ such that a'_4 is disjoint from $b' \cup a'_3$. But then a'_4 could be isotoped so that it is disjoint from a', since a' is a boundary component of a regular neighborhood of $b' \cup w$ in Q_1. This is a contradiction because $i([a_4],[a]) \neq 0$ and so $i(\lambda([a_4]), \lambda([a])) \neq 0$. Hence $x \neq a'_1$, so now assume that $x = a'_2$. We consider the curves a_5, a_6 on N as shown in Figure 2(iii). Since a_5 intersects c essentially and since a_5 is disjoint from a_1 and b, we can choose a representative a'_5 of $\lambda([a_5])$ such that there exists an essential arc w of a'_5 in Q_2 that starts and ends on c' and that does not intersect $a'_1 \cup b'$. The circle a_6 is disjoint from c and a_5, so there exists a representative a'_6 of $\lambda([a_6])$ such that a'_6 is disjoint from $c' \cup a'_5$. But then a'_6 could be isotoped so that it is disjoint from a'_1, since a'_1 is a boundary component of a regular neighborhood of $c' \cup w$ in Q_2. This is a contradiction because $i([a_6],[a_1]) \neq 0$ and so $i(\lambda([a_6]), \lambda([a_1])) \neq 0$. This completes the proof of the lemma. □

Let α and β be two distinct vertices in $\mathcal{N}(R)$. We call (α, β) a *peripheral pair* in $\mathcal{N}(R)$ if they have disjoint representatives x, y (respectively) such that x, y and a boundary component of R bound a pair of pants in R.

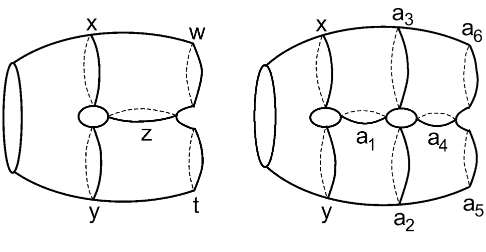
LEMMA 2.5. *Suppose $g \geq 2$ and $p \geq 1$. Let $\lambda\colon \mathcal{N}(R) \to \mathcal{N}(R)$ be a superinjective simplicial map and let (α, β) be a peripheral pair in $\mathcal{N}(R)$. It then follows that $(\lambda(\alpha), \lambda(\beta))$ is a peripheral pair in $\mathcal{N}(R)$.*

Proof. Let x, y be disjoint representatives of α, β (respectively) such that x, y and a boundary component of R bound a pair of pants in R. If $g = 2$ and $p = 1$, then we complete x, y to a pair of pants decomposition P consisting of nonseparating circles a, b on R. Then there exist two distinct pairs of pants in P: one has a, b, x on its boundary and the other has a, b, y on its boundary. Let P' be a pair of pants decomposition of R such that $\lambda([P]) = [P']$, and let x', y', a', b' (respectively) be the representatives of $\lambda([x]), \lambda([y]), \lambda([a]), \lambda([b])$ in P'. By Lemma 2.4, there exist two pairs of pants Q_1, Q_2 in P' such that Q_1 has a', b', x' on its boundary and Q_2 has a', b', y' on its boundary. Then it is clear that x', y' and the boundary component of R bound a pair of pants, which proves the lemma for this case. For $g \geq 3$ and $p = 1$, the proof is similar.

If $g = 2$ and $p = 2$, then it is easy to see that we can complete x, y to a pair of pants decomposition P consisting of nonseparating circles a, b, z on R such that $([y],[z])$ is a peripheral pair, a, b, x bound a pair of pants in P, and also a, b, z bound a pair of pants in P. Let P' be a pair of pants decomposition of R such that $\lambda([P]) = [P']$. Let x', y', z', a', b' be the representatives of $\lambda([x])$, $\lambda([y]), \lambda([z]), \lambda([a]), \lambda([b])$ in P', respectively. By Lemma 2.4, there exist two pairs of pants Q_1, Q_2 in P' such that Q_1 has a', b', x' on its boundary and Q_2 has a', b', z' on its boundary. Then, since x is adjacent to y w.r.t. P, it follows that

x' is adjacent to y' w.r.t. P'. Hence there exists a pair of pants Q_3 in P' having x', y' on its boundary. Let w be the third boundary component of Q_3. It is easy to see that $Q_1 \cup Q_2 \cup Q_3$ is a genus-1 surface with three boundary components: w, y', z'. If $w = z'$ then y' would be a separating curve, which is a contradiction. If $w = y'$ then z' would be a separating curve—also a contradiction. Hence it is clear that w must be be a boundary component of R, which proves the lemma for this case.

Assume now that $g = 2$ and $p \geq 3$. We choose distinct essential circles z, t, w as shown in Figure 3(i). Then we complete x, y, z, t, w to a pair of pants decomposition P consisting of nonseparating circles in any way we like. Let P' be a pair of pants decomposition of R such that $\lambda([P]) = [P']$, and let x', y', z', t', w' be (respectively) the representatives of $\lambda([x]), \lambda([y]), \lambda([z]), \lambda([t]), \lambda([w])$ in P'. Since z is a 4-curve in P, we know that z' is a 4-curve in P'. Since x, z, w are the boundary components of a pair of pants in P, by Lemma 2.4 it follows that x', z', w' are the boundary components of a pair of pants Q_1 in P'. Similarly, since z, y, t are the boundary components of a pair of pants in P, by Lemma 2.4 we have that z', y', t' are the boundary components of a pair of pants Q_2 in P'. Since x is adjacent to y in P, x' is adjacent to y' in P'. Then there exists a pair of pants Q_3 such that Q_3 has x', y' on its boundary. Let r be the boundary component of Q_3 that is different from x' and y'. Suppose that r is not a boundary component of R, in which case r is an essential circle. Then $Q_1 \cup Q_2 \cup Q_3$ is a genus-1 subsurface with three boundary components r, w', t' that are nonseparating circles in R. Any two of x', w', t' can be connected by an arc in the complement of $Q_1 \cup Q_2 \cup Q_3$ in R, but this would be possible only if the genus of R were at least 3. Given our assumption that $g = 2$, we have a contradiction. Therefore, r must be a boundary component of R; that is, $(\lambda(\alpha), \lambda(\beta))$ is a peripheral pair in $\mathcal{N}(R)$.



(i) (ii)

Figure 3 Pants decompositions

Next assume that $g = 3$ and $p \geq 2$. We shall choose distinct essential circles $a_1, \ldots, a_6$ as shown in Figure 3(ii). Then we complete $x, y, a_1, \ldots, a_6$ to a pair of pants decomposition P consisting of nonseparating circles in any way we like. Let P' be a pair of pants decomposition of R such that $\lambda([P]) = [P']$. Let $x', y', a'_1, \ldots, a'_6$ be (respectively) the representatives of $\lambda([x]), \lambda([y]), \lambda([a_1]), \ldots, \lambda([a_6])$ in P'. Since a_1 is a 4-curve in P, it follows that a'_1 is a 4-curve in P'. Since

x, a_1, a_3 are the boundary components of a pair of pants in P, by Lemma 2.4 we have that x', a'_1, a'_3 are the boundary components of a pair of pants Q_1 in P'. Similarly, since y, a_1, a_2 are the boundary components of a pair of pants in P, by Lemma 2.4 (again) y', a'_1, a'_2 are the boundary components of a pair of pants Q_2 in P'. Similar arguments show that a'_4 is a 4-curve, a'_3, a'_4, a'_6 are the boundary components of a pair of pants Q_3 in P', and a'_2, a'_4, a'_5 are the boundary components of a pair of pants Q_4 in P'. Since x is adjacent to y in P, x' is adjacent to y' in P'. Then there exists a pair of pants Q_5 such that Q_5 has x', y' on its boundary. Let r be the boundary component of Q_5 that is different from x' and y'. Suppose that r is not a boundary component of R, in which case r is an essential circle. Then $Q_1 \cup Q_2 \cup Q_3 \cup Q_4 \cup Q_5$ is a genus-2 subsurface with three boundary components r, a'_5, a'_6 that are nonseparating circles in R. Any two of r, a'_5, a'_6 can be connected by an arc in the complement of $Q_1 \cup Q_2 \cup Q_3 \cup Q_4 \cup Q_5$ in R, but this would be possible only if the genus of R were at least 4. Because we assumed that $g = 3$, we obtain a contradiction. Hence, r must be a boundary component of R; that is, $(\lambda(\alpha), \lambda(\beta))$ is a peripheral pair in $\mathcal{N}(R)$. The proof for $g \geq 4$ and $p \geq 2$ is similar. □

LEMMA 2.6. *Suppose $g \geq 2$ and $p \geq 0$, and let $\lambda\colon \mathcal{N}(R) \to \mathcal{N}(R)$ be a superinjective simplicial map. Then λ preserves topological equivalence of ordered pairs of pants decompositions consisting of nonseparating circles on R. In other words, for a given ordered pair of pants decomposition $P = (c_1, c_2, \ldots, c_{3g-3+p})$ of R where $[c_i] \in \mathcal{N}(R)$ and for a corresponding ordered pair of pants decomposition $P' = (c'_1, c'_2, \ldots, c'_{3g-3+p})$ of R (where $[c'_i] = \lambda([c_i])$ for all $i = 1, 2, \ldots, 3g-3+p$), there exists a homeomorphism $H\colon R \to R$ such that $H(c_i) = c'_i$ for all $i = 1, 2, \ldots, 3g-3+p$.*

Proof. Let $P = (c_1, c_2, \ldots, c_{3g-3+p})$ be an ordered pair of pants decomposition consisting of nonseparating circles on R, and let $c'_i \in \lambda([c_i])$ be such that the elements of $\{c'_1, c'_2, \ldots, c'_{3g-3+p}\}$ are pairwise disjoint. Then $P' = (c'_1, c'_2, \ldots, c'_{3g-3+p})$ is an ordered pair of pants decomposition of R. Let $(B_1, B_2, \ldots, B_m)$ be an ordered set of all the pairs of pants in P. By Lemma 2.4 and Lemma 2.5, there is a corresponding or "image" ordered collection of pairs of pants $(B'_1, B'_2, \ldots, B'_m)$. Thus, a pair of pants having three essential curves on its boundary corresponds to a pair of pants having three essential curves on its boundary, and a pair of pants having an inessential boundary component corresponds to a pair of pants having an inessential boundary component. Then, by the classification of surfaces, there exists an orientation-preserving homeomorphism $h_i\colon B_i \to B'_i$ for all $i = 1, \ldots, m$. We can compose each h_i with an orientation-preserving homeomorphism r_i that switches the boundary components (if necessary) in order to induce $h'_i = r_i \circ h_i$ to agree with the correspondence given by λ on the boundary components. (That is: for each essential boundary component a of B_i, $i = 1, \ldots, m$, we have $\lambda([q(a)]) = [q'(h'_i(a))]$ for $q\colon R_P \to R$ and $q'\colon R_{P'} \to R$ the natural quotient maps.) Then, for two pairs of pants with a common boundary component, we can glue the homeomorphisms by isotoping the homeomorphism of one pair so that it agrees with the homeomorphism of the other pair on the common boundary component. Adjusting these homeomorphisms on the boundary components and gluing them yields a homeomorphism $h\colon R \to R$ such that $h(c_i) = c_i'$ for all $i = 1, 2, \ldots, 3g - 3 + p$. □

LEMMA 2.7. *Suppose $g \geq 2$ and $p \geq 0$, and let α_1 and α_2 be two vertices in $\mathcal{N}(R)$. Then $i(\alpha_1, \alpha_2) = 1$ if and only if there are isotopy classes $\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$ in $\mathcal{N}(R)$ such that the following properties hold.*

(i) *$i(\alpha_i, \alpha_j) = 0$ if and only if the ith and jth circles in Figure 4 are disjoint.*

(ii) *$\alpha_1, \alpha_3, \alpha_5, \alpha_6$ have pairwise disjoint representatives a_1, a_3, a_5, a_6 (respectively) such that $a_5 \cup a_6$ divides R into two pieces, one of which is a torus T with two holes and containing some representatives of the isotopy classes α_1, α_2; moreover, a_1, a_3, a_5 bound a pair of pants in T and a_1, a_3, a_6 bound a pair of pants in T.*

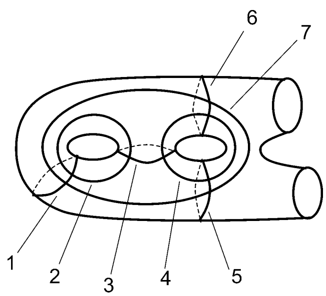


Figure 4 Circles intersecting once

Proof. Let $i(\alpha_1, \alpha_2) = 1$. Let a_1, a_2 be representatives of α_1, α_2 (respectively) such that a_1 intersects a_2 transversely once. Let N be a regular neighborhood of $a_1 \cup a_2$. Then it is easy to see that N is a genus-1 surface with one boundary component and that, since R is a surface of genus at least 2, there exist simple closed curves (as shown in Figure 4) whose isotopy classes $\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$ satisfy properties (i) and (ii). Suppose that there exist isotopy classes $\alpha_1, \alpha_2, \ldots, \alpha_7$ in $\mathcal{N}(R)$ satisfying (i) and (ii). Then we have a_1, a_3, a_5, a_6 as pairwise disjoint representatives of $\alpha_1, \alpha_3, \alpha_5, \alpha_6$ (respectively) such that $a_5 \cup a_6$ divides R into two pieces, one of which is a torus T with two holes and containing some representatives of the isotopy classes α_1, α_2; also, a_1, a_3, a_5 (resp. a_1, a_3, a_6) bound a pair of pants P (resp. Q) in T. Let a_2, a_4, a_7 be representatives of $\alpha_2, \alpha_4, \alpha_7$ such that all the curves a_i $(i = 1, \ldots, 7)$ have minimal intersection with each other. Then $a_4 \cap a_1 = \emptyset$, $a_2 \cap a_6 = \emptyset$, and $a_7 \cap a_3 = \emptyset$. Because $i(\alpha_4, \alpha_1) = 0$, $i(\alpha_4, \alpha_3) \neq 0$, and $i(\alpha_4, \alpha_6) \neq 0$, we know that $a_4 \cap Q$ has an arc that connects a_3 to a_6. Since $i(\alpha_7, \alpha_3) = 0$, $i(\alpha_7, \alpha_1) \neq 0$, and $i(\alpha_7, \alpha_6) \neq 0$, it follows that $a_7 \cap Q$ has an arc connecting a_1 to a_6. Similarly, since $i(\alpha_2, \alpha_6) = 0$, $i(\alpha_2, \alpha_1) \neq 0$, and $i(\alpha_2, \alpha_3) \neq 0$, $a_2 \cap Q$ has an arc that connects a_1 to a_3. Then, since a_2, a_4, a_7 are pairwise disjoint, we can see that all the arcs of $a_4 \cap Q$ connect a_3 to a_6, all the arcs of $a_7 \cap Q$ connect a_1 to a_6, and all the arcs of $a_2 \cap Q$ connect a_1 to a_3. Similar arguments show that all the arcs of $a_4 \cap P$ connect a_3 to a_5, all the arcs of $a_7 \cap P$ connect a_1 to a_5, and all the arcs of $a_2 \cap P$ connect a_1 to a_3. Then, by looking at the gluing between the arcs in $a_2 \cap Q$ and the arcs in $a_2 \cap P$ to form a_2, we see that $a_2 \cap Q$ has one arc and $a_2 \cap P$ has one arc. Hence, $i(\alpha_1, \alpha_2) = 1$. □

A characterization of geometric intersection 1 property in $\mathcal{C}(R)$ was given by Ivanov [Iv1, Lemma 1]. Our characterization is similar.

LEMMA 2.8. *Suppose $g \geq 2$ and $p \geq 0$. Let $\lambda \colon \mathcal{N}(R) \to \mathcal{N}(R)$ be a superinjective simplicial map, and let α, β be two vertices of $\mathcal{N}(R)$. If $i(\alpha, \beta) = 1$, then $i(\lambda(\alpha), \lambda(\beta)) = 1$.*

Proof. Let α and β be two vertices of $\mathcal{N}(R)$ such that $i(\alpha, \beta) = 1$. Then, by Lemma 2.7, there exist isotopy classes $\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$ in $\mathcal{N}(R)$ satisfying properties (i) and (ii) of that lemma. Then, since λ is superinjective, $i(\lambda(\alpha_i), \lambda(\alpha_j)) = 0$ if and only if the ith and jth circles in Figure 4 are disjoint; using Lemma 2.6 and the properties that λ preserves disjointness and nondisjointness, we can see that there exist pairwise disjoint representatives a'_1, a'_3, a'_5, a'_6 of $\lambda(\alpha_1), \lambda(\alpha_3), \lambda(\alpha_5), \lambda(\alpha_6)$, respectively, such that: $a'_5 \cup a'_6$ divides R into two pieces, one of which is a torus T with two holes and containing some representatives of the isotopy classes $\lambda(\alpha_1)$ and $\lambda(\alpha_2)$; a'_1, a'_3, a'_5 bound a pair of pants in T; and a'_1, a'_3, a'_6 bound a pair of pants in T. Then, by Lemma 2.7, $i(\lambda(\alpha), \lambda(\beta)) = 1$. □

3. Superinjective Simplicial Maps of $\mathcal{C}(R)$ and Injective Homomorphisms of Finite Index Subgroups of Mod_R^*

If $g = 2$ and $p \geq 2$ or if $g \geq 3$ and $p \geq 0$, then—by the results given in [I1] and [I2]—we can make the following statements: (a) a simplicial map $\lambda \colon \mathcal{C}(R) \to \mathcal{C}(R)$ is superinjective if and only if λ is induced by a homeomorphism of R; and (b) if K is a finite index subgroup of Mod_R^* and if $f \colon K \to \mathrm{Mod}_R^*$ is an injective homomorphism, then f is induced by a homeomorphism of R. In this section we prove similar results (i.e., Theorem 1.1 and Theorem 1.2) when $g = 2$ and $p \leq 1$.

If $g = 2$ and $p \leq 1$ and if $\lambda \colon \mathcal{C}(R) \to \mathcal{C}(R)$ is a superinjective simplicial map, then λ is an injective simplicial map that maps a pair of pants decompositions of R to a pair of decompositions of R and also preserves adjacency relations. The proofs of these statements are similar to the proofs given in Lemmas 2.1–2.3. Now, we prove the following lemma.

LEMMA 3.1. *Suppose $g = 2$ and $p \leq 1$. Let $\lambda \colon \mathcal{C}(R) \to \mathcal{C}(R)$ be a superinjective simplicial map, and let α, β, γ be distinct vertices in $\mathcal{C}(R)$ having pairwise disjoint representatives that bound a pair of pants in R. Then $\lambda(\alpha), \lambda(\beta), \lambda(\gamma)$ are* *distinct vertices in $\mathcal{C}(R)$ having pairwise disjoint representatives that bound a pair of pants in R.*

Proof. Let a, b, c be pairwise disjoint representatives of α, β, γ, respectively. If R is a closed surface of genus 2 then a, b, c is a pair of pants decomposition on R and, since a, b, c bound a pair of pants, they must all be nonseparating circles in this case. Let P' be a pair of pants decomposition of R such that $\lambda([P]) = [P']$, and let a', b', c' be (respectively) the representatives of $\lambda([a]), \lambda([b]), \lambda([c])$ in P'. Since a is adjacent to b and c w.r.t. P, we have that a' is adjacent to b' and c' w.r.t. P'. If one of a', b', or c' were a separating curve on R then the other two would not be adjacent to each other w.r.t. P', which would yield a contradiction. So, each of a', b', c' is a nonseparating curve; then clearly they bound a pair of pants on R. Now assume that $g = 2$ and $p = 1$. There are two cases: either each of a, b, c is a nonseparating circle or exactly one of a, b, c is a separating circle.

Case 1. Assume that each of a, b, c is a nonseparating circle, and complete $\{a, b, c\}$ to a pair of pants decomposition $P = \{a, b, c, a_1\}$ consisting of nonseparating circles as shown in Figure 5(i). Let P' be a pair of pants decomposition of R such that $\lambda([P]) = [P']$, and let a', b', c', a_1' be (respectively) the representatives of $\lambda([a]), \lambda([b]), \lambda([c]), \lambda([a_1])$ in P'. Notice that any two curves in P are adjacent w.r.t. P; and, since adjacency is preserved, any two curves in P' must be adjacent w.r.t. P'. If one of a', b', c', a_1' were a separating curve then there would be two circles in P' that are not adjacent. As a result, we conclude that all of a', b', c', a_1' are nonseparating. Then we consider the curve configuration as shown in Figure 5(i), and the proof of the lemma in this case follows as in the proof of Lemma 2.4 for the case $g = 2$ and $p = 1$.

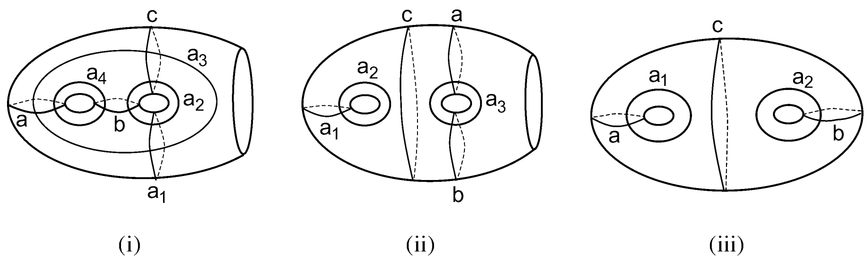


Figure 5 Curve configurations

Case 2. Assume that exactly one of a, b, c is a separating circle. Assume without loss of generality that c is a separating circle. Then c separates R into two subsurfaces R_1, R_2 as shown in Figure 5(ii). Let a_1, a_2, a_3 be as shown in the figure; then a, b, c, a_1 is a pants decomposition P on R. Let P' be a pair of pants decomposition of R such that $\lambda([P]) = [P']$, and let a', b', c', a_1' be the representatives of $\lambda([a]), \lambda([b]), \lambda([c]), \lambda([a_1])$ in P', respectively. Every nonseparating circle x on R could be put inside of a pair of pants decomposition consisting of nonseparating circles, and—using the method given in Case 1—we could see that $\lambda([x])$ has a nonseparating representative. Hence, because a, b, a_1 are nonseparating, also a', b', a_1' are nonseparating. Since a is adjacent to c w.r.t. P, it follows that a' is adjacent to c' w.r.t. P'. Then there is a pair of pants Q_1 having a' and c' on its boundary. Let x be the other boundary component of Q_1. Since a' is not a separating circle, x cannot be c'. If x is the boundary component of R then, since c' is adjacent to a_1' and b', there exists a pair of pants Q_2 in P' having c', b', and a_1' on its boundary. Then, since a_2 intersects a_1 essentially and a_2 is disjoint from b and c, we can choose a representative a_2' of $\lambda([a_2])$ such that there exists an essential arc w of a_2' in Q_2 that starts and ends on a_1' and that does not intersect $b' \cup c'$. Next we consider a_3, which is disjoint from a_1 and a_2. Then there exists a representative a_3' of $\lambda([a_3])$ such that a_3' is disjoint from $a_1' \cup w$. But then a_3' could be isotoped so that it is disjoint from b', since b' is a boundary component of a regular neighborhood of $a_1' \cup w$ in Q_2. This is a contradiction because $i([b], [a_3]) \neq 0$ and so $i(\lambda([b]), \lambda([a_3])) \neq 0$. Hence, x cannot be the boundary component of R.

We are left with three possibilities: x is one of a', b', or a_1'. If $x = a'$ then a' wouldn't be adjacent to b' w.r.t. P', which is a contradiction. Assume $x = a_1'$. Then, since a_2 intersects a_1 essentially and since a_2 is disjoint from a and c, we can choose a representative a_2' of $\lambda([a_2])$ such that there exists an essential arc w of a_2' in Q_1 that starts and ends on a_1' and that does not intersect $a' \cup c'$. Next we consider a_3, which is disjoint from a_1 and a_2. Then there exists a representative a_3' of $\lambda([a_3])$ such that a_3' is disjoint from $a_1' \cup w$. But then a_3' could be isotoped so that it is disjoint from a', since a' is a boundary component of a regular neighborhood of $a_1' \cup w$ in Q_1. This is a contradiction because $i([a], [a_3]) \neq 0$ and so $i(\lambda([a]), \lambda([a_3])) \neq 0$. Therefore, $x \neq a_1'$. Hence $x = b'$, and a', b', c' bound a pair of pants in P'. □

Recall our definition (before Lemma 2.5) of a peripheral pair.

LEMMA 3.2. *Suppose $g = 2$ and $p = 1$. Let $\lambda \colon \mathcal{C}(R) \to \mathcal{C}(R)$ be a superinjective simplicial map, and let (α, β) be a peripheral pair in $\mathcal{C}(R)$. Then $(\lambda(\alpha), \lambda(\beta))$ is a peripheral pair in $\mathcal{C}(R)$.*

Proof. Let x, y be disjoint representatives of α, β (respectively) such that x, y, and a boundary component of R bound a pair of pants in R. There are two cases to consider: either (i) x and y are both nonseparating or (ii) x and y are both separating. The proof in the first case is similar to the proof given in Lemma 2.5. For the second case, we complete x, y to a pair of pants decomposition Q consisting of nonseparating circles a, b on R such that a is in the torus with one hole that comes with the separation by x. Then we will replace y with a nonseparating curve w such that: a, x, b, w is a pants decomposition P on R; x, w, and b bound a pair of pants; and w, b, and the boundary component of R bound a pair of pants on R. Let P' be a pair of pants decomposition of R such that $\lambda([P]) = [P']$, and let x', w', a', b' be (respectively) the representatives of $\lambda([x]), \lambda([w]), \lambda([a]), \lambda([b])$ in P'. Then, using Case 1 of Lemma 3.1, we see that x' is a separating circle of genus 1. Similarly, y' is a separating circle of genus 1 on R. Then it is easy to see that x', y', and the boundary component of R bound a pair of pants. □

LEMMA 3.3. *Suppose $g = 2$ and $p \leq 1$. Let $\lambda : \mathcal{C}(R) \to \mathcal{C}(R)$ be a superinjective simplicial map. Then λ preserves topological equivalence of ordered pairs of pants decompositions on R. (That is: for a given ordered pair of pants decomposition $P = (c_1, c_2, \ldots, c_{3+p})$ of R, where $[c_i] \in \mathcal{C}(R)$, and for a corresponding ordered pair of pants decomposition $P' = (c'_1, c'_2, \ldots, c'_{3+p})$ of R, where $[c'_i] = \lambda([c_i])$ for all $i = 1, 2, \ldots, 3 + p$, there exists a homeomorphism $H : R \to R$ such that $H(c_i) = c'_i$ for all $i = 1, 2, \ldots, 3 + p$.)*

Proof. First we consider the case when $g = 2$ and $p = 0$. Let P be a pair of pants decomposition of R and let A be a nonembedded pair of pants in P. The boundary of A consists of the circles a and c, where c is a 1-separating circle on R and a is a nonseparating circle on R. Let b, a_1, a_2 be as shown in Figure 5(iii); then $P = \{a, b, c\}$ is a pants decomposition on R. Let P' be a pair of pants decomposition of R such that $\lambda([P]) = [P']$, and let a', b', c' be the representatives of $\lambda([a]), \lambda([b]), \lambda([c])$ in P', respectively. Every nonseparating circle x on R could be put inside of a pair of pants decomposition consisting of nonseparating circles and so, using the method given in Case 1 of Lemma 3.1, we can see that $\lambda([x])$ has a nonseparating representative. Hence, because a and b are nonseparating, a' and b' are nonseparating. Assume that c' is also nonseparating. Then there exist two pairs of pants Q_1, Q_2 in P' such that both have a', b', c' on their boundary. Since a_1 intersects a essentially and since a_1 is disjoint from b and c, we can choose a representative a'_1 of $\lambda([a_1])$ such that there exists an essential arc w of a'_1 in Q_1 that starts and ends on a' and that does not intersect $b' \cup c'$. Next we consider a_2, which is disjoint from a and a_1. There exists a representative a'_2 of $\lambda([a_2])$ such that a'_2 is disjoint from $a'_1 \cup a'$. But then a'_2 could be isotoped so that it is disjoint from b', since b' is a boundary component of a regular neighborhood of $a' \cup w$ in Q_1. This is a contradiction because $i([b], [a_2]) \neq 0$ and so $i(\lambda([b]), \lambda([a_2])) \neq 0$. Therefore, c' must be separating. Then clearly a' and c' are the boundary components of a nonembedded pair of pants. Hence, a nonembedded pair of pants in P corresponds to a nonembedded pair of pants in P'. When $g = 2$ and $p = 1$, this result follows from Lemma 3.1 and Lemma 3.2 after considering the circles in Figure 5(ii).

Suppose $g = 2$ and $p \leq 1$. Let $P = (c_1, c_2, \ldots, c_{3+p})$ be an ordered pair of pants decomposition on R. Let $c'_i \in \lambda([c_i])$ be such that the elements of $\{c'_1, c'_2, \ldots, c'_{3+p}\}$ are pairwise disjoint. Then $P' = (c'_1, c'_2, \ldots, c'_{3+p})$ is an ordered pair of pants decomposition of R. Let $(B_1, B_2, \ldots, B_m)$ be an ordered set of all the pairs of pants in P. The foregoing arguments together with Lemmas 3.1 and 3.2 show that there is a corresponding (image) ordered collection of pairs of pants $(B'_1, B'_2, \ldots, B'_m)$. Nonembedded pairs of pants correspond to nonembedded pairs of pants, embedded pairs of pants correspond to embedded pairs of pants, and a pair of pants having an inessential boundary component corresponds to a pair of pants having an inessential boundary component. Then the proof follows as in Lemma 2.6. □

LEMMA 3.4. *Suppose $g = 2$ and $p \leq 1$. Let $\lambda \colon \mathcal{C}(R) \to \mathcal{C}(R)$ be a superinjective simplicial map, and let α, β be two vertices of $\mathcal{C}(R)$. If $i(\alpha, \beta) = 1$, then $i(\lambda(\alpha), \lambda(\beta)) = 1$.*

Proof. The proof follows as in the proof of Lemma 2.8, using Lemma 3.3. □

LEMMA 3.5. *Suppose $g = 2$ and $p \leq 1$. Let $\lambda \colon \mathcal{C}(R) \to \mathcal{C}(R)$ be a superinjective simplicial map. Let α and β be two vertices in $\mathcal{C}(R)$ that have representatives with geometric intersection 2 and algebraic intersection 0 on R. Then $\lambda(\alpha)$ and $\lambda(\beta)$ have representatives with geometric intersection 2 and algebraic intersection 0 on R.*

Proof. Assume that $g = 2$ and $p = 0$. Let h, v be representatives of α, β with geometric intersection 2 and algebraic intersection 0 on R. Let N be a regular neighborhood of $h \cup v$ in R. Then N is a sphere with four boundary components. Let c, x, y, z be boundary components of N such that there exists a homeomorphism $\varphi \colon (N, c, x, y, z, h, v) \to (N_o, c_o, x_o, y_o, z_o, h_o, v_o)$, where N_o is a standard sphere with four holes having c_o, x_o, y_o, z_o on its boundary and h_o, v_o (horizontal, vertical) are two circles as indicated in Figure 6(i). Since h and v have geometric intersection 2 and algebraic intersection 0 on R, none of c, x, y, z bound a disk on R. There are two cases to consider: either exactly one of h or v is separating or both h and v are nonseparating.

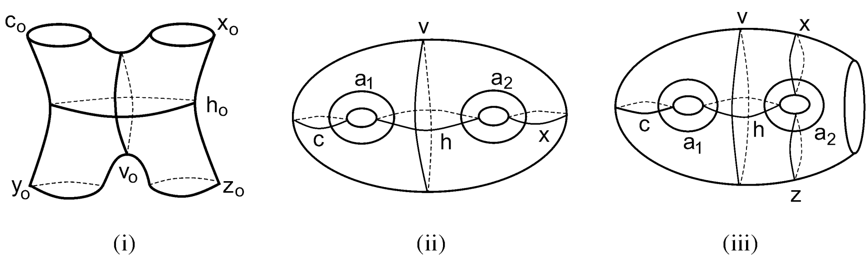
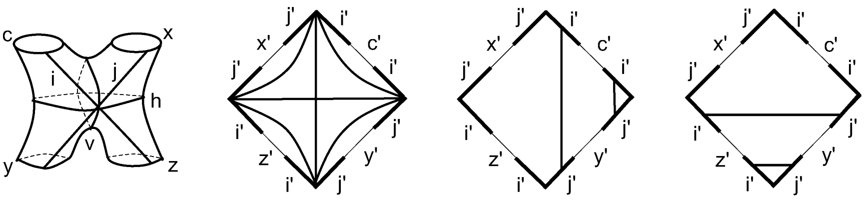


Figure 6 Curves intersecting twice

Case 1. Assume without loss of generality that v is separating and h is nonseparating. Then x is isotopic to z and y is isotopic to c and c, x, h, v are as shown in Figure 6(ii). Let a_1, a_2 be as shown in the figure, and let c', h', x' be pairwise disjoint representatives of $\lambda([c]), \lambda([h]), \lambda([x])$, respectively. Then $\{c', h', x'\}$ is a pair of pants decomposition on R. Since nondisjointness and intersection 1 is preserved by λ, there are disjoint representatives a_1', a_2' of $\lambda([a_1]), \lambda([a_2])$ respectively such that (a) a_1' intersects each of c' and h' exactly once and a_1' is disjoint from x', and similarly (b) a_2' intersects each of x' and h' exactly once and a_2' is disjoint from c'. Then, since v is disjoint from $c \cup a_1 \cup a_2 \cup x$, we can choose a representative v' of $\lambda([v])$ such that it lives inside of the cylinder that we obtain after cutting R along $c' \cup a_1' \cup a_2' \cup x'$. Then it is easy to see that h' and v' intersect twice geometrically and zero times algebraically.



(i) (ii) (iii) (iv)

Figure 7 Curves intersecting twice

Case 2. Assume that both of h, v are nonseparating. Then c and z are the boundary components of an annulus A_1 in $R \setminus N$, and x and y are the boundary components of an annulus A_2 in $R \setminus N$. Let i and j be as shown in Figure 7(i). We connect the endpoints of i on N with an arc in A_1 to get a circle w such that w intersects each of c and z at only one point. Similarly, we connect the endpoints of j on N with an arc in A_2 to get a circle k such that k intersects each of x and y at only one point. Observe that w and k intersect at only one point. Let c', x', h', y', z' be pairwise disjoint representatives of the image curves, and let N' be sphere with four holes bounded by c', x', y', z'. By Lemma 3.3, h' gives a pants decompositions on N'. Now we choose minimally intersecting representatives w', k' of $\lambda([w]), \lambda([k])$ respectively such that each of w' and k' intersects c', x', y', z' minimally. Then, since the intersection 1 is preserved, there exist arcs i', j' of w', k' (respectively) in N' such that i' intersects j' once in N', i' connects c' to z', and j' connects x' to y'. Since v does not intersect any of x, c, y, z and since v intersects w and k exactly once, there is a representative v' of $\lambda([v])$ in N' such that v' intersects each of i' and j' exactly once. Notice that h' also intersects each of i' and j' exactly once. When we cut N' along the arcs of i' and j' we get a disk. The boundary of the disk either is as shown in Figure 7(ii) or has the similar form where only x' and y' are switched. We will consider the first situation (arguments for the second follow similarly). If v' makes its intersection with i' and j' at the intersection point of i' and j' with each other, then v' must be one of the arcs shown in Figure 7(ii), and—by looking at the intersection of v' with the other curves—it is easy to see that v' intersects h' twice geometrically and zero times algebraically on R. Suppose that v' intersects i' and j' at different points. Then there are two arcs of v' connecting i' to j' in the disk. Now, since v' does not intersect any of x', c', y', z', we can see that v' must be one of the curves shown in Figure 7(iii) or (iv). Since v' is not isotopic to h', it follows that v' must be the curve shown in Figure 7(iii) and hence v' intersects h' twice geometrically and zero times algebraically on R.

The proof is similar (using Figure 6(iii) for Case 1) when $g = 2$ and $p = 1$. □

The proof of Theorem 1.1 follows from Lemmas 3.3–3.5 and the techniques given for the proof of Theorem 1.1 in [I1; I2].

A mapping class $g \in \text{Mod}_R^*$ is called *pseudo-Anosov* if $\mathcal{A}$ is nonempty and if $g^n(\alpha) \neq \alpha$ for all α in $\mathcal{A}$ and any $n \neq 0$; g is called *reducible* if there is a nonempty subset $\mathcal{B} \subseteq \mathcal{A}$ such that a set of disjoint representatives can be chosen for $\mathcal{B}$ and $g(\mathcal{B}) = \mathcal{B}$. In this case, $\mathcal{B}$ is called a *reduction system* for g. Each element of $\mathcal{B}$ is called a *reduction class* for g. A reduction class α for g is called an *essential reduction class* for g if, for each $\beta \in \mathcal{A}$ such that $i(\alpha, \beta) \neq 0$ and for each integer $m \neq 0$, we have $g^m(\beta) \neq \beta$. The set $\mathcal{B}_g$ of all essential reduction classes for g is called the *canonical reduction system* for g.

Let $\Gamma' = \ker(\varphi)$, where $\varphi\colon \mathrm{Mod}_R^* \to \mathrm{Aut}(H_1(R, \mathbb{Z}_3))$ is the homomorphism defined by the action of homeomorphisms on the homology. The proofs of Lemma 3.6 and Lemma 3.7 follow by the techniques given in [I1]. Note that we need to use that the maximal rank of an abelian subgroup of Mod_R^* is $3g - 3 + p$ [BLuM].

LEMMA 3.6. *Suppose $g = 2$ and $p \leq 1$. Let K be a finite index subgroup of Mod_R^* and let $f\colon K \to \mathrm{Mod}_R^*$ be an injective homomorphism. Let $\alpha \in \mathcal{A}$. Then there exists an $N \in \mathbb{Z}^*$ such that* $\operatorname{rank} C(C_{\Gamma'}(f(t_\alpha^N))) = 1$.

LEMMA 3.7. *Suppose $g = 2$ and $p \leq 1$. Let K be a finite index subgroup of Mod_R^* and let $f\colon K \to \mathrm{Mod}_R^*$ be an injective homomorphism. Then there exists an $N \in \mathbb{Z}^*$ such that $f(t_\alpha^N)$ is a reducible element of infinite order for all $\alpha \in \mathcal{A}$.*

In the proof of Lemma 3.7, we use that the centralizer of a p-Anosov element in the extended mapping class group is a virtually infinite cyclic group (see [M2]).

LEMMA 3.8. *Suppose $g = 2$ and $p \leq 1$. Let K be a finite index subgroup of Mod_R^* and let $f\colon K \to \mathrm{Mod}_R^*$ be an injective homomorphism. Then, for all $\alpha \in \mathcal{A}$, $f(t_\alpha^N) = t_{\beta(\alpha)}^M$ for some $M, N \in \mathbb{Z}^*$ and $\beta(\alpha) \in \mathcal{A}$.*

Proof. Let $\Gamma = f^{-1}(\Gamma') \cap \Gamma'$. Since Γ is a finite index subgroup, we can choose $N \in Z^*$ such that $t_\alpha^N \in \Gamma$ for all α in $\mathcal{A}$. By Lemma 3.7, $f(t_\alpha^N)$ is a reducible element of infinite order in Mod_R^*. Let C be a realization of the canonical reduction system of $f(t_\alpha^N)$. Let c be the number of components of C and let r be the number of p-Anosov components of $f(t_\alpha^N)$. Since $t_\alpha^N \in \Gamma$, we have $f(t_\alpha^N) \in \Gamma'$. By [IvM, Thm. 5.9], $C(C_{\Gamma'}(f(t_\alpha^N)))$ is a free abelian group of rank $c + r$. By Lemma 3.6, $c + r = 1$. Hence either $c = 1$ and $r = 0$ or $c = 0$ and $r = 1$. Because there is at least one curve in the canonical reduction system, $c = 1$ and $r = 0$. Therefore, since $f(t_\alpha^N) \in \Gamma'$ it follows that $f(t_\alpha^N) = t_{\beta(\alpha)}^M$ for some $M \in \mathbb{Z}^*$ and $\beta(\alpha) \in \mathcal{A}$ [BLuM; IvM]. □

REMARK. Suppose that $f(t_\alpha^M) = t_\beta^P$ for some $\beta \in \mathcal{A}$ and that $M, P \in \mathbb{Z}^*$ and $f(t_\alpha^N) = t_\gamma^Q$ for some $\gamma \in \mathcal{A}$ and $N, Q \in \mathbb{Z}^*$. Since $f(t_\alpha^{MN}) = f(t_\alpha^{NM})$, we have $t_\beta^{PN} = t_\gamma^{QM}$ for $P, Q, M, N \in \mathbb{Z}^*$. Then $\beta = \gamma$. Therefore, by Lemma 3.8, f gives a correspondence between isotopy classes of circles and f induces a map, $f_*\colon \mathcal{A} \to \mathcal{A}$, where $f_*(\alpha) = \beta(\alpha)$.

In the following lemma we use the well-known fact that $f t_\alpha f^{-1} = t_{f(\alpha)}^{\varepsilon(f)}$ for all $\alpha \in \mathcal{A}$ and $f \in \mathrm{Mod}_R^*$, where $\varepsilon(f) = 1$ if f has an orientation-preserving representative and $\varepsilon(f) = -1$ if f has an orientation-reversing representative.

LEMMA 3.9. *Let K be a finite index subgroup of* Mod_R^*. *Let $f: K \to \mathrm{Mod}_R^*$ be an injective homomorphism. Assume that there exists an $N \in \mathbb{Z}^*$ such that, for all $\alpha \in \mathcal{A}$, there is a $Q \in \mathbb{Z}^*$ such that $f(t_\alpha^N) = t_\alpha^Q$. If $g = 2$ and $p = 1$ then f is the identity on K. If $g = 2$ and $p = 0$ then $f(k) = ki^{m(k)}$, where i is the hyperelliptic involution on R and $m(k) \in \{0, 1\}$.*

Proof. We use Ivanov's trick to see that $f(kt_\alpha^N k^{-1}) = f(t_{k(\alpha)}^{\varepsilon(k)\cdot N}) = t_{k(\alpha)}^{Q\cdot\varepsilon(k)}$ and $f(kt_\alpha^N k^{-1}) = f(k)f(t_\alpha^N)f(k)^{-1} = f(k)t_\alpha^Q f(k)^{-1} = t_{f(k)(\alpha)}^{\varepsilon(f(k))\cdot Q}$ for all $\alpha \in \mathcal{A}$ and all $k \in K$. Then we have $t_{k(\alpha)}^{Q\cdot\varepsilon(k)} = t_{f(k)(\alpha)}^{\varepsilon(f(k))\cdot Q}$ $(\forall \alpha \in \mathcal{A}, \forall k \in K)$ and hence $k(\alpha) = f(k)(\alpha)$ $(\forall \alpha \in \mathcal{A}, \forall k \in K)$. Then $k^{-1}f(k)(\alpha) = \alpha$ $(\forall \alpha \in \mathcal{A}, \forall k \in K)$ and therefore $k^{-1}f(k)$ commutes with t_α $(\forall \alpha \in \mathcal{A}, \forall k \in K)$. Then, since Mod_R is generated by Dehn twists when $g = 2$ and $p \le 1$, we have $k^{-1}f(k) \in C(\mathrm{Mod}_R)$ for all $k \in K$. If $g = 2$ and $p = 1$, then $C(\mathrm{Mod}_R)$ is trivial by [IvM, Thm. 5.3]. Hence $k = f(k)$ for all $k \in K$ and so $f = id_K$. If $g = 2$ and $p = 0$ then $C(\mathrm{Mod}_R) = \{id_R, i\} = \mathbb{Z}_2$, where i is the hyperelliptic involution on R. Then, for each $k \in K$, either $k^{-1}f(k) = id_R$ or $k^{-1}f(k) = i$. Thus $f(k) = ki^{m(k)}$, where $m(k) \in \{0, 1\}$. □

COROLLARY 3.10. *Suppose $g = 2$ and $p = 1$. Let $h: \mathrm{Mod}_R^* \to \mathrm{Mod}_R^*$ be an isomorphism and let $f: \mathrm{Mod}_R^* \to \mathrm{Mod}_R^*$ be an injective homomorphism. Assume that there exists an $N \in \mathbb{Z}^*$ such that, for all $\alpha \in \mathcal{A}$, there is a $Q \in \mathbb{Z}^*$ such that $h(t_\alpha^N) = f(t_\alpha^Q)$. Then $h = f$.*

Proof. Apply Lemma 3.9 to $h^{-1}f$ with $K = \mathrm{Mod}_R^*$. Because $h^{-1}f(t_\alpha^N) = t_\alpha^Q$ for all α in $\mathcal{A}$, we have $h^{-1}f = id_K$. Hence, $h = f$. □

By the Remark after Lemma 3.8, we have that $f: K \to \mathrm{Mod}_R^*$ induces a map $f_*: \mathcal{A} \to \mathcal{A}$, where K is a finite index subgroup of Mod_R^*. In the next lemma we prove that f_* is a superinjective simplicial map on $\mathcal{C}(R)$.

LEMMA 3.11. *Suppose $g = 2$ and $p \le 1$. Let $f: K \to \mathrm{Mod}_R^*$ be an injection and let $\alpha, \beta \in \mathcal{A}$. Then $i(\alpha, \beta) = 0$ if and only if $i(f_*(\alpha), f_*(\beta)) = 0$.*

Proof. There exists an $N \in \mathbb{Z}^*$ such that $t_\alpha^N \in K$ and $t_\beta^N \in K$. Then we have the following: $i(\alpha, \beta) = 0$ iff $t_\alpha^N t_\beta^N = t_\beta^N t_\alpha^N$ iff $f(t_\alpha^N)f(t_\beta^N) = f(t_\beta^N)f(t_\alpha^N)$ (since f is injective on K) iff $t_{f_*(\alpha)}^P t_{f_*(\beta)}^Q = t_{f_*(\beta)}^Q t_{f_*(\alpha)}^P$, where $P = M(\alpha, N) \in \mathbb{Z}^*$ and $Q = M(\beta, N) \in \mathbb{Z}^*$ if and only if $i(f_*(\alpha), f_*(\beta)) = 0$. □

Now, we prove the second main theorem of the section.

THEOREM 3.12. *Let K be a finite index subgroup of* Mod_R^* *and let f be an injective homomorphism $f: K \to \mathrm{Mod}_R^*$. If $g = 2$ and $p = 1$, then f has the form $k \to hkh^{-1}$ for some $h \in \mathrm{Mod}_R^*$ and f has a unique extension to an automorphism of Mod_R^*. If R is a closed surface of genus 2 then f has the form $k \to hkh^{-1}i^{m(k)}$ for some $h \in \mathrm{Mod}_R^*$, where m is a homomorphism $K \to \mathbb{Z}_2$ and i is the hyperelliptic involution on R.*

Proof. If $g = 2$ and $p \leq 1$ then, by Lemma 3.11, f_* is a superinjective simplicial map on $\mathcal{C}(R)$. Then, by Theorem 1.1, f_* is induced by a homeomorphism $h\colon R \to R$; that is, $f_*(\alpha) = h_\#(\alpha)$ for all α in $\mathcal{A}$, where $h_\# = [h]$. Let $\chi^{h\#}\colon \mathrm{Mod}_R^* \to \mathrm{Mod}_R^*$ be the isomorphism defined by the rule $\chi^{h_\#}(k) = h_\# k h_\#^{-1}$ for all k in Mod_R^*. Then, for all α in $\mathcal{A}$:

$$\chi^{h_\#^{-1}} \circ f(t_\alpha^N) = \chi^{h_\#^{-1}}(t_{f_*(\alpha)}^M) = \chi^{h_\#^{-1}}(t_{h_\#(\alpha)}^M)$$
$$= h_\#^{-1} t_{h_\#(\alpha)}^M h_\# = t_{h_\#^{-1}(h_\#(\alpha))}^{M\cdot\varepsilon(h_\#^{-1})} = t_\alpha^{M\cdot\varepsilon(h_\#^{-1})}.$$

If $g = 2$ and $p = 1$ then, since $\chi^{h_\#^{-1}} \circ f$ is injective, $\chi^{h_\#^{-1}} \circ f = id_K$ by Lemma 3.9. So, $\chi_\#^h|_K = f$. Hence f is the restriction of an isomorphism that is conjugation by $h_\#$ (i.e., f is induced by h). Suppose there exists an automorphism $\tau\colon \mathrm{Mod}_R^* \to \mathrm{Mod}_R^*$ such that $\tau|_K = f$. Let $N \in Z^*$ be such that $t_\alpha^N \in K$ for all α in $\mathcal{A}$. Since $\chi_\#^h|_K = f = \tau|_K$ and $t_\alpha^N \in K$, it follows that $\tau(t_\alpha^N) = \chi_\#^h(t_\alpha^N)$ for all α in $\mathcal{A}$. Then, by Corollary 3.10, $\tau = \chi^{h_\#}$. Hence the extension of f is unique.

If $g = 2$ and $p = 0$ then, by Lemma 3.9, since $\chi^{h_\#^{-1}} \circ f$ is injective it follows that $\chi^{h_\#^{-1}} \circ f(k) = k i^{m(k)}$, where $m(k) \in \{0, 1\}$. Hence, f has the form $k \to h_\# k h_\#^{-1} i^{m(k)}$. Since $\chi^{h_\#^{-1}} \circ f(k_1 k_2) = k_1 k_2 i^{m(k_1 k_2)}$ and $\chi^{h_\#^{-1}} \circ f(k_1) \chi^{h_\#^{-1}} \circ f(k_2) = k_1 i^{m(k_1)} k_2 i^{m(k_2)} = k_1 k_2 i^{m(k_1)} i^{m(k_2)}$, we have that $m(k_1 k_2) = m(k_1) + m(k_2)$ for all $k_1, k_2 \in K$. Therefore, $m\colon K \to \mathbb{Z}_2$ is a homomorphism. $\square$

REMARK. Note that $k \to hkh^{-1} i^{m(k)}$ defines a homomorphism from $K \to \mathrm{Mod}_R^*$ for every $h \in \mathrm{Mod}_R^*$ and for every homomorphism $m\colon K \to \mathbb{Z}_2$. It is easy to see that $k \to hkh^{-1} i^{m(k)}$ is injective if and only if either $i \notin K$ or $i \in \mathrm{Ker}(m)$. Inner automorphisms of K act on the set of injective homomorphisms from $K \to \mathrm{Mod}_R^*$. Using Theorem 3.12, we can see that the orbit space $\mathrm{Inj\,Hom}(K, \mathrm{Mod}_R^*)/\mathrm{Inn}(K)$ of this action is finite. Then we have the following corollary.

COROLLARY 3.13. *Suppose that $g = 2$ and $p \leq 1$, and let K be a finite index subgroup of* Mod_R^*. *Then* $\mathrm{Out}(K)$ *is finite.*

In the other cases—when R has genus at least 2 and K is a finite index subgroup—we have that $\mathrm{Out}(K)$ is finite as a corollary to the main results in [I1; I2]. See [M1] for an explicit description of automorphisms of Mod_R^* for a closed surface of genus 2.

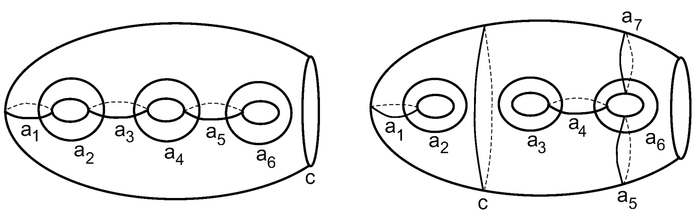
4. Extending Superinjective Simplicial Maps of $\mathcal{N}(R)$ to Superinjective Simplicial Maps of $\mathcal{C}(R)$

In this section we prove Theorem 1.3 and Theorem 1.4.

LEMMA 4.1. *Suppose that $g \geq 2$ and $p \leq 1$. Let $\lambda\colon \mathcal{N}(R) \to \mathcal{N}(R)$ be a superinjective simplicial map. Then λ extends to a superinjective simplicial map $\lambda_*\colon \mathcal{C}(R) \to \mathcal{C}(R)$.*

Proof. For x a nonseparating simple closed curve, we define $\lambda_*([x]) = \lambda([x])$. Let c be a separating simple closed curve on R.

Case 1. Assume that R is closed. Since $g \geq 2$, c separates R into two subsurfaces R_1 and R_2, each of which has genus at least 1. We take a chain on R_1, $\{\alpha_1, \ldots, \alpha_m\}$, with $i(\alpha_i, \alpha_{i+1}) = 1$ and $i(\alpha_i, \alpha_j) = 0$ for $|i - j| > 1$ and where $[a_i] = \alpha_i \in \mathcal{N}(R)$ as shown in Figure 8(i) (for the $g = 4$ case when R_1 has genus 3), such that $R_1 \cup \{c\}$ is a regular neighborhood of $a_1 \cup \cdots \cup a_n$. Since λ preserves disjointness, nondisjointness, and intersection-1 property, we can see that the chain $\{\alpha_1, \ldots, \alpha_n\}$ is mapped by λ into a similar chain, $\{\lambda(\alpha_1), \ldots, \lambda(\alpha_m)\}$, with $i(\lambda(\alpha_i), \lambda(\alpha_{i+1})) = 1$ and $i(\lambda(\alpha_i), \lambda(\alpha_j)) = 0$ for $|i - j| > 1$. Let $a'_i \in \lambda(\alpha_i)$ be such that any two elements in $\{a'_1, \ldots, a'_m\}$ have minimal intersection with each other. Let M be a regular neighborhood of $a'_1 \cup \cdots \cup a'_n$. Then it is easy to see that M is homeomorphic to $R_1 \cup c$. Let a' be the boundary of M. Suppose that we have another chain on R_1, $\{\beta_1, \ldots, \beta_m\}$, with $i(\beta_i, \beta_{i+1}) = 1$ and $i(\beta_i, \beta_j) = 0$ for $|i - j| > 1$ and where $[b_i] = \beta_i \in \mathcal{N}(R)$ such that $R_1 \cup \{c\}$ is a regular neighborhood of $b_1 \cup \cdots \cup b_m$. Again we see that the chain $\{\beta_1, \ldots, \beta_m\}$ is mapped by λ into a similar chain, $\{\lambda(\beta_1), \ldots, \lambda(\beta_m)\}$, with $i(\lambda(\beta_i), \lambda(\beta_{i+1})) = 1$ and $i(\lambda(\beta_i), \lambda(\beta_j)) = 0$ for $|i - j| > 1$. Let $b'_i \in \lambda(\beta_i)$ be such that any two elements in $\{b'_1, \ldots, b'_m\}$ have minimal intersection with each other. Let T be a regular neighborhood of $b'_1 \cup \cdots \cup b'_n$. Then T is homeomorphic to $R_1 \cup c$. Let b' be the boundary of T.



(i) (ii)

Figure 8 Chains

Claim: $[a'] = [b']$.

Proof. We take a similar chain on R_2, $\{\gamma_1, \ldots, \gamma_n\}$, with $i(\gamma_i, \gamma_{i+1}) = 1$ and $i(\gamma_i, \gamma_j) = 0$ for $|i - j| > 1$ and where $[c_i] = \gamma_i \in \mathcal{N}(R)$, such that $R_2 \cup \{c\}$ is a regular neighborhood of $c_1 \cup \cdots \cup c_n$. This chain is mapped into a similar chain, $\{\lambda(\gamma_1), \ldots, \lambda(\gamma_n)\}$. Let $c'_i \in \lambda(\gamma_i)$ be such that any two elements in $\{a'_1, \ldots, a'_n, c'_1, \ldots, c'_m\}$ and $\{b'_1, \ldots, b'_n, c'_1, \ldots, c'_m\}$ have minimal intersection with each other. Then a'_i is disjoint from c'_j for any i, j, and b'_i is disjoint from c'_j for any i, j. Hence we can choose a regular neighborhood N of $c'_1 \cup \cdots \cup c'_n$ in R such that N is disjoint from $M \cup T$. Let c' be the boundary component of N. Then it is easy to see that N is homeomorphic to $R_2 \cup c$ and that the boundary components of M and N are isotopic in R. Similarly, the boundary components of T and N are isotopic in R. Therefore, $[a'] = [c'] = [b']$. We define $\lambda_*([c]) = [a']$.

Claim: $\lambda_* \colon \mathcal{C}(R) \to \mathcal{C}(R)$ *is a simplicial map.*

Proof. Let α, β be two vertices in $\mathcal{C}(R)$ such that $i(\alpha, \beta) = 0$. Let x and y be disjoint representatives of α and β, respectively. If x, y are two nonseparating simple closed curves then $i(\lambda_*(\alpha), \lambda_*(\beta)) = i(\lambda(\alpha), \lambda(\beta)) = 0$. If x is a nonseparating simple closed curve and if y is a separating simple closed curve, then x lives in a subsurface R_1 that comes from separation by y. We could thus choose a chain as described previously that contains x and then see, by the construction of the image of $[y]$, that $i(\lambda_*([x]), \lambda_*([y])) = 0$. If both x and y are separating, then clearly $R \setminus ((R \setminus x) \cap (R \setminus y))$ has two connected components T_1, T_2 such that T_1 is disjoint from T_2 and y (resp. x) is an essential boundary component of T_1 (resp. T_2). As a result, the chains that come from the disjoint subsurfaces T_1 and T_2—and that will be used to define the images of $[x]$ and $[y]$—are disjoint. Because λ preserves disjointness, the "image" chains will be disjoint and hence $i(\lambda_*([x]), \lambda_*([y])) = 0$.

Claim: $\lambda_* \colon \mathcal{C}(R) \to \mathcal{C}(R)$ *is a superinjective simplicial map.*

Proof. Let α, β be two vertices in $\mathcal{C}(R)$ such that $i(\alpha, \beta) \neq 0$. Let x and y be representatives of α and β, respectively.

(i) Assume that x and y are two nonseparating simple closed curves. Then we have $i(\lambda_*(\alpha), \lambda_*(\beta)) = i(\lambda(\alpha), \lambda(\beta)) \neq 0$ (since λ is superinjective).

(ii) Assume y is a nonseparating simple closed curve, and assume x is a separating simple closed curve that separates R into two subsurfaces R_1, R_2. Without loss of generality, assume that y is in R_1. Let $\{\alpha_1, \dots, \alpha_m\}$ be a chain on R_1, with $i(\alpha_i, \alpha_{i+1}) = 1$ and $i(\alpha_i, \alpha_j) = 0$ for $|i - j| > 1$ and where $[a_i] = \alpha_i \in \mathcal{N}(R)$, such that $R_1 \cup \{x\}$ is a regular neighborhood of $a_1 \cup \cdots \cup a_m$. Since $i(\alpha, \beta) \neq 0$, it follows that $i(\beta, \alpha_i) \neq 0$ for some i. Thus $i(\lambda_*(\beta), \lambda_*(\alpha_i)) = i(\lambda(\beta), \lambda(\alpha_i)) \neq 0$ (since λ is superinjective). It is then easy to see that $i(\lambda_*(\alpha), \lambda_*(\beta)) \neq 0$.

(iii) Assume that both x and y are separating and that x separates R into two subsurfaces R_1, R_2. Let $\{\alpha_1, \dots, \alpha_m\}$ be a chain on R_1, with $i(\alpha_i, \alpha_{i+1}) = 1$ and $i(\alpha_i, \alpha_j) = 0$ for $|i - j| > 1$ and where $[a_i] = \alpha_i \in \mathcal{N}(R)$, such that $R_1 \cup \{x\}$ is a regular neighborhood of $a_1 \cup \cdots \cup a_m$. Since $i(\alpha, \beta) \neq 0$, we have $i(\beta, \alpha_i) \neq 0$ for some i. Hence $i(\lambda_*(\beta), \lambda_*(\alpha_i)) \neq 0$ by part (ii), and it is easy to see that $i(\lambda_*(\alpha), \lambda_*(\beta)) \neq 0$. We thus have a superinjective simplicial extension $\lambda_* \colon \mathcal{C}(R) \to \mathcal{C}(R)$ of $\lambda \colon \mathcal{N}(R) \to \mathcal{N}(R)$.

Case 2. Assume that R has one boundary component. Then c separates R into two subsurfaces R_1, R_2. Assume without loss of generality that R_1 is a genus-k subsurface with c as its boundary. We consider chains on R_1 as in the first case and also chains on R_2 such that regular neighborhoods of the curves coming from the chains have c as their essential boundary component and the boundary of R as their inessential boundary component (see Figure 8(ii)). From this point on the proof is similar to that for Case 1. □

REMARK. If $g \geq 2$ and $p \geq 2$, and if C is the set of separating circles on R that separate R into two pieces such that each piece has genus at least 1, then—by using chains on these two pieces and following the techniques of Lemma 4.1—we can extend λ to λ_* on C and obtain a superinjective extension.

Let M be a sphere with k holes, $k \geq 5$. A circle a on M is called an *n-circle* if a bounds a disk with $n \geq 2$ holes on M. A *pentagon* in $\mathcal{C}(M)$ is an ordered 5-tuple $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$, defined up to cyclic permutations, of vertices of $\mathcal{C}(M)$ such that $i(\alpha_j, \alpha_{j+1}) = 0$ for $j = 1, 2, \ldots, 5$ and $i(\alpha_j, \alpha_k) \neq 0$ otherwise, where $\alpha_6 = \alpha_1$. A vertex in $\mathcal{C}(M)$ is called an *n-vertex* if it has a representative that is an n-circle on M.

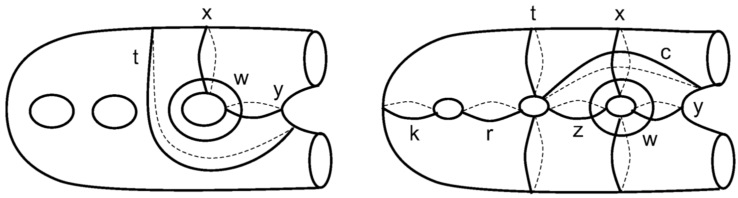
Let x, y be disjoint simple closed curves on R such that $([x], [y])$ is a peripheral pair; that is, x, y and a boundary component of R bound a pair of pants on R. Observe that $x \cup y$ separate R into two subsurfaces. Let $R_{x,y}$ be the positive-genus subsurface of R that comes from this separation. We can identify $\mathcal{N}(R_{x,y})$ with a subcomplex $L_{x,y}$ of $\mathcal{N}(R)$. We use $\lambda_{x,y}$ to denote the restriction of λ on $\mathcal{N}(R_{x,y})$. If $k\colon R \to R$ is a homeomorphism, then $k_\#$ denotes the map induced by k on $\mathcal{N}(R)$ (i.e., $k_\#\colon \mathcal{N}(R) \to \mathcal{N}(R)$ for $k_\# = [k]$).

An arc i on R is *properly embedded* if $\partial i \subseteq \partial R$ and i is transversal to ∂R; arc i is *nontrivial* (or *essential*) if i cannot be deformed into ∂R in such a way that the endpoints of i remain in ∂R during the deformation. If a and b are two disjoint arcs connecting a boundary component of R to itself, then a and b are *linked* if their endpoints alternate on the boundary component; otherwise, a and b are *unlinked*. The *complex of arcs* $\mathcal{B}(R)$ on R is an abstract simplicial complex whose vertices are the isotopy classes of nontrivial properly embedded arcs i in R. A set of vertices forms a simplex if these vertices can be represented by pairwise disjoint arcs. Let i be an essential properly embedded arc on R. Let A be a boundary component of R that has one endpoint of i, and let B be the boundary component of R that has the other endpoint of i. Let N be a regular neighborhood of $i \cup A \cup B$ in R. By Euler characteristic arguments, N is a pair of pants. The boundary components of N are called *encoding circles of i* on R. An essential, properly embedded arc i on R is called *type-1* if it joins one boundary component ∂_k of R to itself; i is *nonseparating* if its complement in R is connected.

The mapping class group Mod_R of R is the group of isotopy classes of orientation-preserving homeomorphisms of R. The pure mapping class group, PMod_R, is the subgroup of Mod_R consisting of isotopy classes of homeomorphisms that preserve each boundary component of R.

LEMMA 4.2. *Suppose $g \geq 3$ and $p = 2$, and let $\lambda\colon \mathcal{N}(R) \to \mathcal{N}(R)$ be a superinjective simplicial map. Assume that, for any peripheral pair $([x], [y])$ on R with x, y disjoint, $\lambda_{x,y}$ agrees with a map $(g_{x,y})_\#\colon \mathcal{N}(R_{x,y}) \to \mathcal{N}(R_{x',y'})$ induced by a homeomorphism $g_{x,y}\colon R_{x,y} \to R_{x',y'}$, where x', y' are disjoint and where $\lambda([x]) = [x'], \lambda([y]) = [y']$ and $g_{x,y}(x) = x', g_{x,y}(y) = y'$. Then λ agrees with a map $h_\#\colon \mathcal{N}(R) \to \mathcal{N}(R)$ that is induced by a homeomorphism $h\colon R \to R$.*

Proof. Let x, y be disjoint simple closed curves such that $([x], [y])$ is a peripheral pair, and let $(g_{x,y})_\#\colon \mathcal{N}(R_{x,y}) \to \mathcal{N}(R_{x',y'})$ be a simplicial map induced by a homeomorphism $g_{x,y}\colon R_{x,y} \to R_{x',y'}$—where x', y' are disjoint, $\lambda([x]) = [x']$,



(i) (ii)

Figure 9 (i) A dual curve; (ii) pants decomposition

$\lambda([y]) = [y']$ and $g_{x,y}(x) = x'$, $g_{x,y}(y) = y'$—such that $\lambda_{x,y}$ agrees with $(g_{x,y})_\#$ on $\mathcal{N}(R_{x,y})$. Let g be a homeomorphism of R that cuts to a homeomorphism $R_{x,y} \to R_{x',y'}$ isotopic to $g_{x,y}$. Then each homeomorphism of R that cuts to a homeomorphism $R_{x,y} \to R_{x',y'}$ that is isotopic to $g_{x,y}$ is itself isotopic to an element in the set $\{gt_x^m t_y^n,\ m,n \in \mathbb{Z}\}$. It is easy to see that $\lambda_{x,y}$ agrees with the restriction of $(gt_x^m t_y^n)_\#$ on $\mathcal{N}(R_{x,y})$ for all $m,n \in \mathbb{Z}$. Let w be a simple closed curve that is dual to both x and y (see Figure 9(i)). Since λ preserves geometric intersection-1 property by Lemma 2.8, $\lambda([w])$ has a representative that is dual to both of x' and y'. Let P be a regular neighborhood of $x \cup y \cup w$ that is a genus-1 surface with two boundary components. Let t be the essential boundary component of P, and let Q be the genus-1 subsurface with two boundary components of R that has $g(t)$ as its boundary. Then, using the properties of λ, it is easy to see that (a) $\lambda([w])$ has a representative that lies in the interior of Q and is dual to both x' and y' and (b) there exist $m_o, n_o \in \mathbb{Z}$ such that $gt_x^{m_o} t_y^{n_o}$ agrees with λ on $[w]$. Let $D_{x,y}$ be the set of isotopy classes of simple closed curves that are dual to each of x and y on R.

Claim 1: $(gt_x^{m_o} t_y^{n_o})_\#$ *agrees with* λ *on* $\{[x]\} \cup \{[y]\} \cup L_{x,y} \cup D_{x,y}$.

Proof. It is clear that $(gt_x^{m_o} t_y^{n_o})_\#([x]) = \lambda([x]) = [x']$ and $(gt_x^{m_o} t_y^{n_o})_\#([y]) = \lambda([y]) = [y']$. Since $\lambda_{x,y}$ agrees with the restriction of $(gt_x^{m_o} t_y^{n_o})_\#$ on $\mathcal{N}(R_{x,y})$, it follows that $(gt_x^{m_o} t_y^{n_o})_\#$ agrees with λ on $L_{x,y}$. We have seen that $(gt_x^{m_o} t_y^{n_o})_\#$ agrees with λ on $[w]$. Let w_1 be a simple closed curve that is disjoint from w and dual to both of x and y. As described previously, there exists $\tilde{m}, \tilde{n} \in \mathbb{Z}$ such that λ agrees with $(gt_x^{\tilde{m}} t_y^{\tilde{n}})_\#$ on $[w_1]$. Since w and w_1 are disjoint nonseparating curves, $i(\lambda([w]), \lambda([w_1])) = 0$. Because λ preserves disjointness and nondisjointness, clearly $m = \tilde{m}$ and $n = \tilde{n}$. This shows that $(gt_x^{m_o} t_y^{n_o})_\#$ also agrees with λ on $[w_1]$. Given any simple closed curve v that is dual to both x and y, we can find a sequence of dual curves to both x and y, connecting w to v, such that each consecutive pair is disjoint; that is, the isotopy classes of these curves define a path between w and v in $\mathcal{N}(R)$. Then, using the argument just given and the sequence, we conclude that $(gt_x^{m_o} t_y^{n_o})_\#$ agrees with λ on $D_{x,y}$. Hence, $(gt_x^{m_o} t_y^{n_o})_\#$ agrees with λ on $\{[x]\} \cup \{[y]\} \cup L_{x,y} \cup D_{x,y}$. This proves Claim 1. Let $h_{x,y} = gt_x^{m_o} t_y^{n_o}$. We have that $(h_{x,y})_\#$ agrees with λ on $\{[x]\} \cup \{[y]\} \cup L_{x,y} \cup D_{x,y}$.

We complete x, y to a pair of pants decomposition P consisting of nonseparating circles as shown in Figure 9(ii) for $g = 3$ and $p = 2$ (similar configurations can be chosen for $g \geq 3$). Let t, z, w, c, k, r be as shown in the figure.

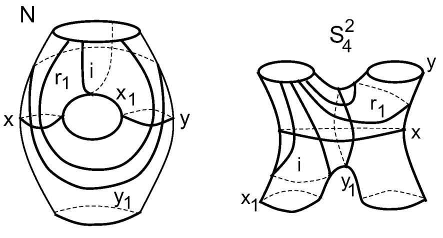
Claim 2. $(h_{x,y})_\#$ *agrees with* λ *on* c.

Proof. Let t', z', x', y', k', r' be (respectively) pairwise disjoint representatives of $\lambda([t]), \lambda([z]), \lambda([x]), \lambda([y]), \lambda([k]), \lambda([r])$, and let w' be a representative of $\lambda([w])$ that has minimal intersection with each of t', z', x', y', k', r'. Since k, r, z, y, ∂_1 bound a sphere T (on R) with five holes and containing x and c, it follows by Lemma 2.6 that k', r', z', y' and a boundary component of R bound a sphere T' with five holes on R. Because x and c have geometric intersection 2 and algebraic intersection 0 in T, there exist vertices $\gamma_1, \gamma_2, \gamma_3$ of $\mathcal{C}(T)$ such that: $(\gamma_1, \gamma_2, [x], \gamma_3, [c])$ is a pentagon in $\mathcal{C}(T)$; γ_1 and γ_3 are 2-vertices; γ_2 is a 3-vertex; $\{[x], \gamma_3\}$, $\{[x], \gamma_2\}$, $\{[c], \gamma_3\}$, and $\{\gamma_1, \gamma_2\}$ are codimension-0 simplices of $\mathcal{C}(T)$; and each γ_i has a representative that is nonseparating on R. Since λ is superinjective, we can see that $(\lambda(\gamma_1), \lambda(\gamma_2), \lambda([x]), \lambda(\gamma_3), \lambda([c]))$ is a pentagon in $\mathcal{C}(T')$. By Lemma 2.6, $\lambda(\gamma_1)$ and $\lambda(\gamma_3)$ are 2-vertices, and $\lambda(\gamma_2)$ is a 3-vertex, in $\mathcal{C}(T')$. Since λ is an injective simplicial map, it follows that $\{\lambda([x]), \lambda(\gamma_3)\}$, $\{\lambda([x]), \lambda(\gamma_2)\}$, $\{\lambda([c]), \lambda(\gamma_3)\}$, and $\{\lambda(\gamma_1), \lambda(\gamma_2)\}$ are all codimension-0 simplices of $\mathcal{C}(T')$. Clearly, $\lambda([c])$ has a representative that is disjoint from t', z', y'. Then (see [K]), $\{\lambda([x]), \lambda([c])\}$ have representatives with geometric intersection 2 and algebraic intersection 0 in the sphere with four holes bounded by t', z', y' and the boundary component of R. Also, $(h_{x,y})_\#([c])$ has a representative c'' that is disjoint from t', z', y', w' and that intersects x' geometrically twice and algebraically zero times. Since $(h_{x,y})_\#$ agrees with λ on $\{[x]\} \cup \{[y]\} \cup L_{x,y} \cup D_{x,y}$, it follows that $[c'] = [c'']$; that is, $(h_{x,y})_\#$ agrees with λ on c.

Claim 3. $(h_{x,y})_\#$ *agrees with* λ *on the class of every nonseparating circle on* R.

Proof. Let z be a nonseparating simple closed curve on R. Let t be another simple closed curve, disjoint from z, such that z, t, and ∂_1 bound a pair of pants in R. Let i and j be nonseparating type-1 arcs connecting ∂_1 to itself such that i has x, y as its encoding circles and j has z, t as its encoding circles. We can assume without loss of generality that i and j have minimal intersection. By [I2, Lemma 3.8] there exists a sequence $i = r_0 \to r_1 \to \cdots \to r_{n+1} = j$ of essential, properly embedded, nonseparating type-1 arcs joining ∂_1 to itself such that each consecutive pair is disjoint—that is, the isotopy classes of these arcs define a path in $\mathcal{B}(R)$ between i and j. Let x_i, y_i be the encoding circles for r_i $(i = 1, \ldots, n)$. For the pair of arcs i and r_1, we will consider two cases, A and B.

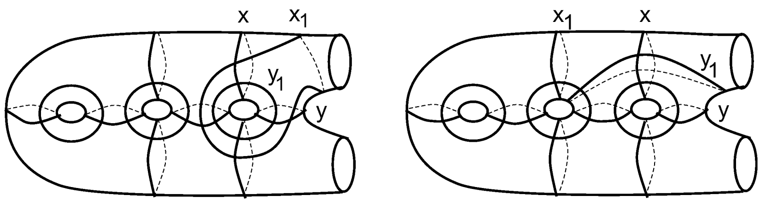
Case A. Assume that i and r_1 are linked (i.e., their endpoints alternate on the boundary component ∂_1). Then a regular neighborhood of $i \cup r_1 \cup \partial_1$ is a genus-1 surface with two boundary components N, and the arcs i, r_1 and their encoding circles x, y, x_1, y_1 on N are as shown in Figure 10(i). In this case, we complete $\{x, y, x_1, y_1\}$ to a curve configuration G consisting of nonseparating circles—which is shown in Figure 11(i) for $g = 3$ and $p = 3$ (see [IvM])—such that the isotopy



(i) (ii)

Figure 10 Arcs and their encoding circles

classes of Dehn twists about the elements of this set generate PMod_R and such that all the curves in this set are (i) either disjoint from x, y or simultaneously dual to x, y and (ii) either disjoint from x_1, y_1 or simultaneously dual to x_1, y_1. Then, since all the curves in G are either disjoint from x, y or simultaneously dual to x, y, by Claim 1 we have $(h_{x,y})_\#([x]) = \lambda([x])$ for every $x \in G$. Similarly, since all the curves in G are either disjoint from x_1, y_1 or simultaneously dual to x_1, y_1, Claim 1 yields $(h_{x_1,y_1})_\#([x]) = \lambda([x])$ for every $x \in G$. Hence $(h_{x,y})_\#([x]) = \lambda([x]) = (h_{x_1,y_1})_\#([x])$ for every $x \in G$, so $(h_{x,y}^{-1} h_{x_1,y_1})_\# \in C_{\mathrm{Mod}_R}(\mathrm{PMod}_R)$. By [IvM, Thm. 5.3], $C_{\mathrm{Mod}_R}(\mathrm{PMod}_R) = \{1\}$; hence $(h_{x,y})_\# = (h_{x_1,y_1})_\#$.



(i) (ii)

Figure 11 Encoding circles in curve configurations

Case B. Assume that i and r_1 are unlinked (i.e., their endpoints do not alternate on the boundary component ∂_1). Then a regular neighborhood of $i \cup r_1 \cup \partial_1$ is a sphere with four boundary components S_4^2, and the arcs i, r_1 and their encoding circles x, y, x_1, y_1 on S_4^2 are as shown in Figure 10(ii). Let w be the boundary component of S_4^2 that is different from x_1, y, ∂_1. If w is a nonseparating curve, then we complete $\{x, y, x_1, y_1\}$ to a curve configuration G consisting of nonseparating circles as shown in Figure 11(ii) and such that the isotopy classes of Dehn twists about the elements of G generate PMod_R. By Claims 1 and 2, respectively, $(h_{x,y})_\#([x]) = \lambda([x])$ for every $x \in G$ and $(h_{x_1,y_1})_\#([x]) = \lambda([x])$ for every $x \in G$. Then $(h_{x,y}^{-1} h_{x_1,y_1})_\#([x]) = [x]$ for every $x \in G$ and so $(h_{x,y}^{-1} h_{x_1,y_1})_\# \in C_{\mathrm{Mod}_R}(\mathrm{PMod}_R)$. Again using [IvM, Thm. 5.3], we have $C_{\mathrm{Mod}_R}(\mathrm{PMod}_R) = \{1\}$. Hence $(h_{x,y})_\# = (h_{x_1,y_1})_\#$.

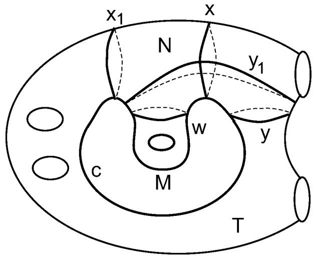


Figure 12 Curves intersecting twice

Suppose that w is a separating curve. By the remark after Lemma 4.1, we can extend λ to a superinjective map λ_* on a subcomplex of $\mathcal{C}(R)$ containing separating circles on R that separate R into two pieces, each of genus ≥ 1. Notice that w is such a circle. For the rest of the proof, we will use λ for this extension. We will demonstrate the case when x_1 and y are not isotopic; the other case is proved similarly. Let M be the subsurface that has w on its boundary and that does not contain N, and let T be the closure of $R \setminus \{M \cup N\}$. The circles x_1 and y are boundary components of T. Since w is an essential separating circle and since $p = 2$, it follows that M has genus at least 1. In Figure 12 we show M, N, T for a special case. By Claim 1, $(h_{x,y})_\#([x]) = \lambda([x])$ for every $x \in \mathcal{N}(T)$, and likewise $(h_{x_1,y_1})_\#([x]) = \lambda([x])$ for every $x \in \mathcal{N}(T)$; hence $(h_{x,y}^{-1} h_{x_1,y_1})_\#([x]) = [x]$ for every $x \in \mathcal{N}(T)$. Then the restriction of $(h_{x,y}^{-1} h_{x_1,y_1})_\#$ on $\mathcal{N}(T)$ is in $C(\mathrm{PMod}_T)$. By Theorem 5.3 in [IvM], $C(\mathrm{PMod}_T) = \{1\}$ and therefore $(h_{x,y})_\# = (h_{x_1,y_1})_\#$ on $\mathcal{N}(T)$. It is easy to see that $(h_{x,y})_\# = (h_{x_1,y_1})_\#$ on the set $\{[x_1], [w], [y]\}$. Following the proof of Claim 2 (considering that we have the extended superinjective simplicial map on "good" separating circles), we see that $(h_{x,y})_\# = (h_{x_1,y_1})_\#$ on $\{[x], [y_1]\}$. Then, since Dehn twists about x and y_1 generate PMod_N, the restriction of $(h_{x,y}^{-1} h_{x_1,y_1})_\#$ on $\mathcal{C}(N)$ is in $C(\mathrm{PMod}_N)$. Again by Theorem 5.3 in [IvM], $C(\mathrm{PMod}_N) = \{1\}$ and so $(h_{x,y})_\# = (h_{x_1,y_1})_\#$ on $\mathcal{C}(N)$. By Claim 1, $(h_{x,y}^{-1} h_{x_1,y_1})_\#([x]) = [x]$ for every $x \in \mathcal{N}(M)$; hence the restriction of $(h_{x,y}^{-1} h_{x_1,y_1})_\#$ on $\mathcal{N}(M)$ is in $C(\mathrm{PMod}_M)$. By considering the action on oriented circles and using [IvM, Thm. 5.3], we see that $(h_{x,y})_\# = (h_{x_1,y_1})_\#$ on $\mathcal{N}(M)$. Thus $(h_{x,y})_\# = (h_{x_1,y_1})_\#$ on $\mathcal{N}(M) \cup \mathcal{C}(N) \cup \mathcal{N}(T)$. Figure 12 shows the curve c, which is dual to each of x, y, x_1, y_1. By Claim 1, $(h_{x,y})_\#([c]) = \lambda([c]) = (h_{x_1,y_1})_\#([c])$. Therefore, since $(h_{x,y})_\#$ and $(h_{x_1,y_1})_\#$ agree on $[c]$, we have $(h_{x,y})_\# = (h_{x_1,y_1})_\#$ on $\mathcal{N}(R)$. Now $g \geq 3$ implies that $(h_{x,y})_\# = (h_{x_1,y_1})_\#$. If w is a boundary component of R, then proving that $(h_{x,y})_\# = (h_{x_1,y_1})_\#$ is similar.

In both cases we have seen that $(h_{x,y})_\# = (h_{x_1,y_1})_\#$. Using our sequence, an inductive argument yields that $(h_{x,y})_\# = (h_{z,t})_\#$ and that $(h_{x,y})_\# = (h_{z,t})_\# = \lambda$ on $\{[x]\} \cup \{[y]\} \cup \{[z]\} \cup \{[t]\} \cup L_{x,y} \cup D_{x,y} \cup L_{z,t} \cup D_{z,t}$. In particular, $(h_{x,y})_\#$ agrees with λ on any nonseparating curve z and so $(h_{x,y})_\#$ agrees with λ on $\mathcal{N}(R)$. This proves the lemma. □

THEOREM 4.3. *Suppose that the genus of R is at least 2 and that R has at most $g-1$ boundary components. Then a simplicial map $\lambda\colon \mathcal{N}(R) \to \mathcal{N}(R)$ is superinjective if and only if λ is induced by a homeomorphism of R.*

Proof. If λ is induced by a homeomorphism of R, then it preserves disjointness and nondisjointness and hence it is superinjective. Assume that λ is superinjective. In the cases when R is a closed surface or when R has exactly one boundary component, λ extends (by Lemma 4.1) to a superinjective simplicial map λ_* on $\mathcal{C}(R)$. Then λ_* is induced by a homeomorphism $h\colon R \to R$; that is, $\lambda_*(\alpha) = h_\#(\alpha)$ for each vertex α in $\mathcal{C}(R)$ by the main results in [I1; I2] and by Theorem 1.1. Hence λ is induced by the homeomorphism h.

Assume that $g \geq 3$ and $p = 2$. Let x and y be disjoint nonseparating circles such that x, y, and a boundary component ∂_1 of R bound a pair of pants on R. Let x' and y' be disjoint representatives of $\lambda([x])$ and $\lambda([y])$, respectively. Using Lemma 2.5 and recalling that λ preserves disjointness and nondisjointness, it is easy to see that λ maps $\mathcal{N}(R_{x\cup y})$ to $\mathcal{N}(R_{x'\cup y'})$. Since every essential separating curve on $R_{x\cup y}$ separates R into two subsurfaces, each with genus ≥ 1, we can use chains on these subsurfaces in order to extend λ to a superinjective simplicial map $\lambda_{x,y}\colon \mathcal{C}(R_{x\cup y}) \to \mathcal{C}(R_{x'\cup y'})$ as in Case 1 of Lemma 4.1. Then, by the main results in [I2], there exists a homeomorphism $h\colon R_{x\cup y} \to R_{x'\cup y'}$ such that $h(x) = x'$, $h(y) = y'$, and $\lambda_{x,y}$ is induced by h. The proof of the theorem now follows from Lemma 4.2.

Next assume that $g \geq 4$ and $3 \leq p \leq g-1$. We will give the proof when $p = g-1$; the proof of the remaining cases is similar. Let $\{a_1, \ldots, a_{2(p-1)}\}$ be a set of pairwise disjoint nonseparating circles such that (a_{2i+1}, a_{2i+2}) is a peripheral pair, as shown in Figure 13 for $i = 0, \ldots, p-2$. Let $R_{a_1\cup a_2\cup\cdots\cup a_{2(p-1)}}$ be the genus-2 surface with $2p-1$ boundary components that comes from the separation by $a_1 \cup a_2 \cup \cdots \cup a_{2(p-1)}$. Exactly one of the boundary components of $R_{a_1\cup a_2\cup\cdots\cup a_{2(p-1)}}$ is a boundary component of R. We identify $\mathcal{N}(R_{a_1\cup a_2\cup\cdots\cup a_{2(p-1)}})$ with a subcomplex $L_{a_1\cup a_2\cup\cdots\cup a_{2(p-1)}}$ of $\mathcal{N}(R)$. Let $\lambda_{a_1\cup a_2\cup\cdots\cup a_{2(p-1)}}$ denote the restriction of λ on $\mathcal{N}(R_{a_1\cup a_2\cup\cdots\cup a_{2(p-1)}})$. Let $a_1', \ldots, a_{2(p-1)}'$ be pairwise disjoint representatives of $\lambda([a_1]), \ldots, \lambda([a_{2(p-1)}])$, respectively. By Lemma 2.5 and the properties of λ, it is clear that λ maps $\mathcal{N}(R_{a_1\cup a_2\cup\cdots\cup a_{2(p-1)}})$ to $\mathcal{N}(R_{a_1'\cup a_2'\cup\cdots\cup a_{2(p-1)}'})$. Since every essential separating curve on $R_{a_1\cup a_2\cup\cdots\cup a_{2(p-1)}}$ separates R into two subsurfaces, each with genus ≥ 1, we can use chains on these subsurfaces and so extend λ to a superinjective simplicial map $\lambda_{a_1\cup a_2\cup\cdots\cup a_{2(p-1)}}\colon \mathcal{C}(R_{a_1\cup a_2\cup\cdots\cup a_{2(p-1)}}) \to \mathcal{C}(R_{a_1'\cup a_2'\cup\cdots\cup a_{2(p-1)}'})$ as in Lemma 4.1. Then, by the main results in [I2], there exists a homeomorphism $h\colon R_{a_1\cup a_2\cup\cdots\cup a_{2(p-1)}} \to R_{a_1'\cup a_2'\cup\cdots\cup a_{2(p-1)}'}$ such that $h(a_i) = a_i'$ and $\lambda_{a_1\cup a_2\cup\cdots\cup a_{2(p-1)}}$ is induced by h. If (a_{2p-3}, a_{2p-2}) is replaced with another peripheral pair (b_{2p-3}, b_{2p-2}), where b_{2p-3} and b_{2p-2} are each disjoint from $a_1 \cup a_2 \cup \cdots \cup a_{2p-4}$, then by similar techniques we obtain a homeomorphism

$$t\colon R_{a_1\cup a_2\cup\cdots\cup a_{2p-4}\cup b_{2p-3}\cup b_{2p-2}} \to R_{a_1'\cup a_2'\cup\cdots\cup a_{2p-4}'\cup b_{2p-3}'\cup b_{2p-2}'}$$

such that $t(a_i) = a_i'$ and $t(b_j) = b_j'$, where b_{2p-3}' and b_{2p-2}' are (respectively) pairwise disjoint representatives of $\lambda([b_{2p-3}])$ and $\lambda([b_{2p-2}])$ and where the map

$\lambda_{a_1 \cup a_2 \cup \cdots \cup a_{2p-4} \cup b_{2p-3} \cup b_{2p-2}}$ is induced by t. Following the techniques in the proof of Lemma 4.2, we see that there exists a homeomorphism $r\colon R_{a_1 \cup a_2 \cup \cdots \cup a_{2p-4}} \to R_{a'_1 \cup a'_2 \cup \cdots \cup a'_{2p-4}}$ such that $r(a_i) = a'_i$ and $\lambda_{a_1 \cup a_2 \cup \cdots \cup a_{2p-4}}$ is induced by r. Then an inductive argument shows that there exists a homeomorphism $q\colon R \to R$ such that λ is induced by q. □

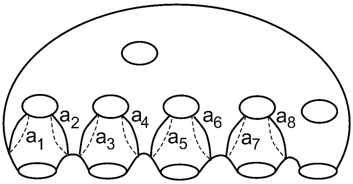


Figure 13 Cutting R along peripheral pairs

Now we consider the graph $\mathcal{G}(R)$ defined by Schaller. The vertex set of $\mathcal{G}(R)$ is the set of isotopy classes of nonseparating simple closed curves on R. Two vertices are connected by an edge if and only if their geometric intersection number is 1. In the proof of the following theorem, we use the result from Schaller [S] that, if $g \geq 2$ and if R is not a closed surface of genus 2, then $\mathrm{Aut}(\mathcal{G}(R)) \cong \mathrm{Mod}_R^*$.

THEOREM 4.4. *Suppose that R has genus ≥ 2. If R is a closed surface of genus 2, then* $\mathrm{Aut}(\mathcal{N}(R)) \cong \mathrm{Mod}_R^*/\mathcal{C}(\mathrm{Mod}_R^*)$. *If R is not a closed surface of genus 2, then* $\mathrm{Aut}(\mathcal{N}(R)) \cong \mathrm{Mod}_R^*$.

Proof. Assume that R is not a closed surface of genus 2. If $[f] \in \mathrm{Mod}_R^*$ then $[f]$ induces an automorphism of $\mathcal{N}(R)$. If $[f]$ fixes the isotopy class of every nontrivial, nonseparating, simple closed curve on R, then f is orientation preserving and it can be shown that f is isotopic to id_R. An automorphism λ of $\mathcal{N}(R)$ is a superinjective simplicial map of $\mathcal{N}(R)$. By Lemma 2.8, λ preserves geometric intersection-1 property and hence induces an automorphism of $\mathcal{G}(R)$. By [S], an automorphism of $\mathcal{G}(R)$ is induced by a homeomorphism of R (note that if R has at most $g-1$ boundary components then Theorem 4.3 also implies that λ is induced by a homeomorphism of R). Then it is easy to see that $\mathrm{Aut}(\mathcal{N}(R)) \cong \mathrm{Mod}_R^*$.

Assume now that R is a closed surface of genus 2. If $[f] \in \mathrm{Mod}_R^*$ then $[f]$ induces an automorphism of $\mathcal{N}(R)$. If $[f]$ fixes the isotopy class of every nonseparating simple closed curve on R, then f is either isotopic to id_R or the hyperelliptic involution. An automorphism of $\mathcal{N}(R)$ is a superinjective simplicial map of $\mathcal{N}(R)$ and, by Theorem 4.3, is induced by a homeomorphism of R. Thus we have $\mathrm{Aut}(\mathcal{N}(R)) \cong \mathrm{Mod}_R^*/\mathcal{C}(\mathrm{Mod}_R^*)$. □

ACKNOWLEDGMENTS. We thank John D. McCarthy for his suggestions and many discussions about this work. We thank Peter Scott for helpful discussions and Mustafa Korkmaz for his comments. We also thank the referee for suggestions about the paper.

References

[BLuM] J. S. Birman, A. Lubotzky, and J. D. McCarthy, *Abelian and solvable subgroups of the mapping class groups,* Duke Math. J. 50 (1983), 1107–1120.

[BrMa] T. E. Brendle and D. Margalit, *Commensurations of the Johnson Kernel,* Geom. Topol. 8 (2004), 1361–1384.

[FIv] B. Farb and N. Ivanov, *The Torelli geometry and its applications: Research announcement,* Math. Res. Lett. 12 (2005), 293–301.

[H] W. J. Harvey, *Geometric structures of surface mapping class groups,* Homological group theory (C. T. Wall, ed.), London Math. Soc. Lecture Note Ser., 36, pp. 255–269, Cambridge Univ. Press, London, 1979.

[I1] E. Irmak, *Superinjective simplicial maps of complexes of curves and injective homomorphisms of subgroups of mapping class groups,* Topology 43 (2004), 513–541.

[I2] ———, *Superinjective simplicial maps of complexes of curves and injective homomorphisms of subgroups of mapping class groups II,* Topology Appl. (to appear), ⟨http://front.math.ucdavis.edu/math.GT/0311407⟩.

[IK] E. Irmak and M. Korkmaz, *Automorphisms of the Hatcher–Thurston complex* (submitted), ⟨http://front.math.ucdavis.edu/math.GT/0409033⟩.

[Iv1] N. V. Ivanov, *Automorphisms of complexes of curves and of Teichmuller spaces,* Internat. Math. Res. Notices 14 (1997), 651–666.

[Iv2] ———, *Subgroups of Teichmuller modular groups,* Transl. Math. Monogr., 115, Amer. Math. Soc., Providence, RI, 1992.

[IvM] N. V. Ivanov and J. D. McCarthy, *On injective homomorphisms between Teichmuller modular groups I,* Invent. Math. 135 (1999), 425–486.

[K] M. Korkmaz, *Automorphisms of complexes of curves on punctured spheres and on punctured tori,* Topology Appl. 95 (1999), 85–111.

[L1] F. Luo, *On non-separating simple closed curves in a compact surface,* Electron. Res. Announc. Amer. Math. Soc. 1 (1995), 18–25.

[L2] ———, *Automorphisms of the complexes of curves,* Topology 39 (2000), 283–298.

[Ma] D. Margalit, *Automorphisms of the pants complex,* Duke Math. J. 121 (2004), 457–479.

[M1] J. D. McCarthy, *Automorphisms of surface mapping class groups. A recent theorem of N. Ivanov,* Invent. Math. 84 (1986), 49–71.

[M2] ———, *Normalizers and centralizers of pseudo-Anosov mapping classes* (manuscript available for informal distribution on request).

[MV] J. D. McCarthy and W. R. Vautaw, *Automorphisms of Torelli groups,* ⟨http://front.math.ucdavis.edu/math.GT/0311250⟩.

[Mo] L. Mosher, *Tiling the projective foliation space of a punctured surface,* Trans. Amer. Math. Soc. 306 (1988), 1–70.

[S] P. S. Schaller, *Mapping class groups of hyperbolic surfaces and automorphism groups of graphs,* Compositio Math. 122 (2000), 243–260.

Department of Mathematics
University of Michigan
Ann Arbor, MI 48109

eirmak@umich.edu

Current address
Department of Mathematics & Statistics
Bowling Green State University
Bowling Green, OH 43403

eirmak@bgsu.edu