Michigan Math. J. 54 (2006)

On Some Lacunary Power Series

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1. Introduction

Consider a lacunary power series given by

$$f(z) = \sum_{n=0}^{\infty} a_n z^{k_n}, \tag{1}$$

where $k_{n+1}/k_n \ge b > 1$ for every $n \ge 0$ and where $a_n \in \mathbb{C}$ such that $\sum_{n=0}^{\infty} |a_n| < \infty$. Then f is holomorphic in the unit disc $\mathbb{D}$ and continuous in $\bar{\mathbb{D}}$.

In 1945, Salem and Zygmund showed in [SZ] that if $b > b_0$ for a constant $b_0 \approx 45$ and if the a_n satisfy some conditions (so that the convergence of $\sum_{n=0}^{\infty} |a_n|$ is slow enough), then the image of the unit circle under f is a Peano curve—that is, it contains an open set in the plane. In 1963, Kahane, M. Weiss, and G. Weiss in [KWW] extended the result, showing that for every $b > 1$ there exists a constant $\gamma > 0$ depending only on b and such that, if

$$|a_n| \le \gamma \sum_{m=n+1}^{\infty} |a_m| \tag{2}$$

for every n, then the image of the unit circle under f is a Peano curve. In fact, they proved that there exist constants K, ξ, ν (depending only on b) such that, if

- inequality (2) is fulfilled and
- E is any Cantor set in the unit circle obtained by taking an arc I of length at least ξ/k_0, removing the middle subarc of I of length K times the length of I and repeating the procedure inductively, always removing the middle subarc of length K times the length of the larger one,

then $f(E)$ contains the disc centered at 0 of radius $\nu \sum_{n=0}^{\infty} |a_n|$.

In [CGP] it was noticed by Cantón, Granados, and Pommerenke that the Kahane–Weiss–Weiss result implies the following.

CGP Theorem. *If f is a map of the form* (1) *satisfying* (2) *and if k_0 is sufficiently large, then f does not preserve Borel sets on the unit circle.*

Received July 7, 2004. Revision received September 2, 2005.
Research supported by Polish KBN Grant no. 2 P03A 034 25

We say that a map f defined on a set A *preserves Borel sets* on A if $f(B)$ is Borel for every Borel set B. (Recall that the family of Borel sets in a topological space X is, by definition, the smallest σ-algebra containing open sets in X; here we consider $X = \hat{\mathbb{C}}$.) The idea for proving the CGP Theorem is to show that f assumes every value from an open disc uncountable many times on the unit circle and then use the following Lusin–Purves Theorem (see e.g. [Ku]), which gives a characterization of maps preserving Borel sets.

LUSIN–PURVES THEOREM. *Let $A \subset \hat{\mathbb{C}}$ be a Borel set and let $f\colon A \to \hat{\mathbb{C}}$. Suppose that, for every Borel set $B \subset \hat{\mathbb{C}}$, the set $f^{-1}(B)$ is Borel. Then f preserves Borel sets on A if and only if the set $\{w \in \hat{\mathbb{C}} : w = f(z)$ for uncountable many $z \in A\}$ is countable.*

In particular, the Lusin–Purves Theorem can be applied provided f is continuous. Note that, by holomorphicity, a function of the form (1) preserves Borel sets on $A = \mathbb{D}$. The interesting case is when A is the unit circle. In [CGP], some criteria were given to characterize holomorphic maps on the unit disc preserving Borel sets on the unit circle at points, where the radial limits exist.

In 1990, the following result was proved by Belov in [Be].

BELOV THEOREM. *Let f be a map of the form* (1). *Suppose that $b > 2$, $a_m \neq 0$ for some $m > 0$, and there exist $\alpha, \beta > 0$ such that $\alpha(1+\beta) < 1$ and the following conditions are satisfied for sufficiently large n:*

(a) $|a_n| \le \beta \sum_{m=n+1}^{\infty} |a_m|$;
(b) $2\pi \frac{b-1}{b-2} \sum_{m=1}^{n} |a_m| b^m \le \alpha |a_{n+1}| b^{n+1}$.

Then f assumes every value from a certain disc uncountable many times on the unit circle, so it does not preserve Borel sets.

In this paper we consider the well-known case

$$k_n = b^n \tag{3}$$

for integers $b \ge 2$. We prove the following.

THEOREM A. *There exists a $q \in (0,1)$ with the following property. Let $b = 2, 3, \ldots$ and let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^{b^n}, \tag{4}$$

where $a_n \in \mathbb{C}$, such that $\sum_{n=0}^{\infty} |a_n| < \infty$ and

$$|a_{n+1}| > q|a_n| \tag{5}$$

for sufficiently large n. Then there exists a disc $D \subset \mathbb{C}$ such that f assumes every value from D uncountable many times on the unit circle. Hence, f does not preserve Borel sets on the unit circle.

In comparison with the Belov Theorem, observe that Theorem A has no condition (b) and that Belov's (a) is replaced by our simpler condition (5).

Theorem A immediately implies the next two results.

COROLLARY 1. *There exists an* $a_0 \in (0,1)$ *such that, for every* $a \in \mathbb{C}$, $|a| \in (a_0, 1)$, *and* $b = 2, 3, \ldots$, *the Weierstrass function*

$$f(z) = \sum_{n=0}^{\infty} a^n z^{b^n}$$

does not preserve Borel sets on the unit circle.

COROLLARY 2. *If* $b = 2, 3, \ldots, a_n \in \mathbb{C}$ *such that* $\sum_{n=0}^{\infty} |a_n| < \infty$ *and* $|a_{n+1}/a_n| \to 1$, *then the map*

$$f(z) = \sum_{n=0}^{\infty} a_n z^{b^n}$$

does not preserve Borel sets on the unit circle. In particular, this holds for

$$f(z) = \sum_{n=1}^{\infty} \frac{z^{b^n}}{n^\alpha},$$

where $\alpha > 1$.

Note that the CGP Theorem implies that the map

$$f(z) = \sum_{n=n_0}^{\infty} a^n z^{b^n}$$

does not preserve Borel sets on the unit circle provided $b = 2, 3, \ldots$, $a \in \mathbb{C}$, and $|a| \in (a_0, 1)$ for a constant $0 < a_0 < 1$ and sufficiently large n_0. The same is true for

$$f(z) = \sum_{n=n_0}^{\infty} a_n z^{b^n}$$

provided that $b = 2, 3, \ldots$, that $a_n \in \mathbb{C}$ with $\sum_{n=0}^{\infty} |a_n| < \infty$ and $|a_{n+1}/a_n| \to 1$, and that n_0 is sufficiently large. Similarly, it is easy to check that the Belov Theorem shows the result for the maps from Corollaries 1 and 2 when $b \geq 9$. Here we were able to extend it to the case $b \geq 2$ because of equation (3). The proof of our theorem, which is contained in Section 3, is based on two technical lemmas proved in Section 2; these lemmas use ideas from [Ba] and [CGP].

NOTATION. The open disc in $\mathbb{C}$ of radius r centered at z is denoted by $\mathbb{D}_r(z)$. For $z \in \mathbb{C}$ and $A \subset \mathbb{C}$, set

$$\operatorname{dist}(z, A) = \inf\{|z - x| : x \in A\}.$$

For $z \in \mathbb{C} \setminus \{0\}$ we assume $\operatorname{Arg}(z) \in [0, 2\pi)$. By an arithmetic constant we mean a constant that is independent of all variables appearing in the paper.

ACKNOWLEDGMENT. The author is grateful to Prof. Ch. Pommerenke for introducing him to the subject and for helpful remarks.

2. Technical Lemmas

Consider a map f of the form (4) and fulfilling condition (5) for sufficiently large n and some fixed $q \in (3/4, 1)$. Let

$$\zeta_{n,k}(t) = f(\exp[2\pi i(t + k/b^{n+1})])$$

for $n \geq 0$, $k \in \mathbb{Z}$, and $t \in \mathbb{R}$. By definition, for any $j \in \mathbb{Z}$ we have

$$\begin{aligned}\zeta_{n,k+j}(t) - \zeta_{n,k}(t) &= 2i \sum_{l=0}^{n} a_l \sin \frac{\pi j}{b^{n+1-l}} \exp\left[\pi i\left(2tb^l + \frac{2k+j}{b^{n+1-l}}\right)\right] \\ &= 2i \sum_{m=1}^{n+1} a_{n+1-m} \sin \frac{\pi j}{b^m} \exp\left[\pi i\left(2tb^{n+1-m} + \frac{2k+j}{b^m}\right)\right].\end{aligned}$$

Fix $N > 0$ such that $|a_{n+1}| > q|a_n|$ for every $n \geq N-1$. In particular, this implies $a_n \neq 0$ for every $n \geq N$. Then for every $n > N+1$ and $N < \tilde{n} < n - N$,

$$\begin{aligned}&\zeta_{n,kb^{n-\tilde{n}}+j}(t) - \zeta_{\tilde{n},k}(t) \\ &= \zeta_{n,kb^{n-\tilde{n}}+j} - \zeta_{n,kb^{n-\tilde{n}}} \\ &= 2ia_n \exp[2\pi itb^n] \sum_{m=1}^{n+1} \frac{a_{n+1-m}}{a_n} \sin \frac{\pi j}{b^m} \exp\left[\pi i\left(2t(b^{n+1-m} - b^n) + \frac{2kb^{n-\tilde{n}} + j}{b^m}\right)\right] \\ &= 2ia_n \exp[2\pi itb^n]\left(\sum_{m=1}^{n-\tilde{n}} \frac{a_{n+1-m}}{a_n} \sin \frac{\pi j}{b^m} \exp\left[\pi i\left(2tb^{n+1-m}(1 - b^{m-1}) + \frac{j}{b^m}\right)\right]\right. \\ &\qquad \left. + \sum_{m=n+1-\tilde{n}}^{n+1} \frac{a_{n+1-m}}{a_n} \sin \frac{\pi j}{b^m} \exp\left[\pi i\left(2tb^{n+1-m}(1 - b^{m-1}) + \frac{2kb^{n-\tilde{n}} + j}{b^m}\right)\right]\right).\end{aligned}$$

Hence, denoting

$$u_{n,t}(j) = \sum_{m=1}^{n+1-N} \frac{a_{n+1-m}}{a_n} \sin \frac{\pi j}{b^m} \exp\left[\pi i\left(2tb^{n+1-m}(1 - b^{m-1}) + \frac{j}{b^m}\right)\right] \tag{6}$$

and

$$\begin{aligned}\Delta_{n,\tilde{n},k,t}(j) = &\sum_{m=n+1-\tilde{n}}^{n+1} \frac{a_{n+1-m}}{a_n} \sin \frac{\pi j}{b^m} \exp\left[\pi i\left(2tb^{n+1-m}(1 - b^{m-1}) + \frac{2kb^{n-\tilde{n}} + j}{b^m}\right)\right] \\ &- \sum_{m=n+1-\tilde{n}}^{n+1-N} \frac{a_{n+1-m}}{a_n} \sin \frac{\pi j}{b^m} \exp\left[\pi i\left(2tb^{n+1-m}(1 - b^{m-1}) + \frac{j}{b^m}\right)\right],\end{aligned}$$

we obtain

$$\zeta_{n,kb^{n-\tilde{n}}+j}(t)-\zeta_{\tilde{n},k}(t)=2ia_n\exp[2\pi itb^n](u_{n,t}(j)+\Delta_{n,\tilde{n},k,t}(j)). \tag{7}$$

Observe that, by assumption, for every $m=2,\dots,n+1-N$ we have

$$\left|\frac{a_{n+1-m}}{a_n}\right|<\frac{1}{q^{m-1}},$$

so

$$\begin{aligned}|\Delta_{n,\tilde{n},k,t}(j)|&<2\pi|j|q\sum_{m=n+1-\tilde{n}}^{n+1-N}\frac{1}{(qb)^m}+\frac{\pi|j|}{q^{n-N}|a_N|}\sum_{m=n+2-N}^{n+1}\frac{|a_{n+1-m}|}{b^m}\\&<\pi|j|\left(\frac{2}{(qb-1)(qb)^{n-\tilde{n}}}+\frac{\max(|a_0|,\dots,|a_{N-1}|)}{|a_N|b(b-1)(qb)^{n-N}}\right).\end{aligned}$$

Since $q>3/4$, it follows that $1/qb<2/3$ and so

$$|\Delta_{n,\tilde{n},k,t}(j)|<\frac{\pi|j|}{b}\left(8+\frac{\max(|a_0|,\dots,|a_{N-1}|)}{|a_N|}\left(\frac{2}{3}\right)^{\tilde{n}-N}\right)\left(\frac{2}{3}\right)^{n-\tilde{n}}. \tag{8}$$

The proof of Theorem A is based on the two following lemmas.

Lemma 1. *There exist constants $q_0\in(0,1)$ and $M_0,c_1,c_2>0$ such that—for every $q\in(q_0,1)$, $M\in\mathbb{Z}$, and $M>M_0$ and for every map f of the form* (4)*—if $|a_{n+1}|>q|a_n|$ for sufficiently large n then there exist a subsequence n_l ($l=0,1,\dots$), numbers $\sigma_l,\tilde{\sigma}_l\in[0,2b^3]$, and integers $j_0,j_1,j_2\in[-b,b]$ (depending on l) such that, for every $l\ge 1$, the following statements hold.*

(a) $n_l-n_{l-1}\ge M$.
(b) $|a_{n_l}|>q^M|a_{n_{l-1}}|$.
(c) $|\sigma_l-\tilde{\sigma}_l|>3b$.
(d) *For every $t\in\mathcal{T}$, where*

$$\mathcal{T}=\left\{\sum_{l=1}^{\infty}\frac{r_l}{b^{n_l+1}}:r_l\in\{\sigma_l,\tilde{\sigma}_l\}\text{ for }l\ge 1\right\},$$

the points $u_{n_l,t}(j_0),u_{n_l,t}(j_1),u_{n_l,t}(j_2)$ as defined in (6) *are noncollinear and form a triangle with sides of length L_0,L_1,L_2 and angles $\alpha_0,\alpha_1,\alpha_2$ such that*

$$\frac{1}{c_1}<L_0,L_1,L_2<c_1,\qquad c_2<\alpha_0,\alpha_1,\alpha_2<\pi-c_2.$$

Proof. Assume q_0 is closer to 1 than some small arithmetic constant: say $q_0>0.9999$ and $M_0>10000$. First we define inductively a subsequence n_l fulfilling conditions (a) and (b) as well as the additional condition

(e) $|a_{n_l}|<|a_{n_l-2}|$.

Fix some n_0 larger than N. Suppose we have defined $n_0,\dots,n_l$ such that conditions (a), (b), and (e) hold up through l. Let

$$n_{l+1}=\min\{n\ge n_l+M:|a_n|<|a_{n-2}|\}.$$

Since the series $\sum|a_n|$ is convergent and since $a_n \neq 0$ for large n, we must have $|a_n| < |a_{n-2}|$ for infinitely many n. Hence the minimum is over a nonempty set, so it is well-defined. The conditions (a) and (e) for $l+1$ are obvious. Moreover, if $n_{l+1} - n_l - M$ is even then

$$|a_{n_{l+1}}| > q|a_{n_{l+1}-1}| \geq q|a_{n_l+M-1}| > q^M|a_{n_l}|,$$

and if $n_{l+1} - n_l - M$ is odd then

$$|a_{n_{l+1}}| > q|a_{n_{l+1}-1}| \geq q|a_{n_l+M-2}| > q^{M-1}|a_{n_l}| > q^M|a_{n_l}|.$$

Hence, condition (b) holds for $l+1$, which ends the inductive step.

Using $\lfloor x \rfloor$ to denote the largest integer not greater than x, we now define

$$\sigma_l = \frac{\lfloor b^{M-1}\varphi_l/(b-1)\rfloor}{b^{M-3}} + \varrho_l b^2, \qquad \tilde{\sigma}_l = \frac{\lfloor b^{M-1}\varphi_l/(b-1)\rfloor}{b^{M-3}} + \varrho_l b^2 + b^3,$$

where

$$\varphi_l = \frac{1}{2\pi}\operatorname{Arg}\left(\frac{a_{n_l-1}}{a_{n_l}}\right), \qquad \psi_l = \frac{1}{2\pi}\operatorname{Arg}\left(\frac{a_{n_l-2}}{a_{n_l}}\right),$$

and

$$\varrho_l = \begin{cases} 0 & \text{if } b > 2, \\ 0 & \text{if } b = 2 \text{ and } \operatorname{dist}(\psi_l - 3\lfloor 2^{M-1}\varphi_l\rfloor/2^M, \mathbb{Z}) \leq 1/4, \\ 1 & \text{if } b = 2 \text{ and } \operatorname{dist}(\psi_l - 3\lfloor 2^{M-1}\varphi_l\rfloor/2^M, \mathbb{Z}) > 1/4. \end{cases}$$

It is clear that $\sigma_l, \tilde{\sigma}_l \in [0, 2b^3]$ and, moreover, condition (c) holds. Now we define the numbers j_0, j_1, j_2 and show condition (d). Take $t \in \mathcal{T}$. Note that for all integers $j \in [-b, b]$ we have

$$|u_{n_l,t}(j)| < \frac{\pi|j|}{b}\sum_{m=1}^{\infty}\frac{1}{(qb)^{m-1}} \leq \frac{\pi}{1-1/2q_0} < 3\pi. \tag{9}$$

We will consider several cases depending on b. In all cases we set

$$j_0 = 0,$$

which implies that

$$u_{n_l,t}(j_0) = 0. \tag{10}$$

Define

$$j_1 = \begin{cases} \lceil b/4 \rceil & \text{for } b \geq 6, \\ 1 & \text{for } b = 2,3,4,5, \end{cases} \qquad \text{and} \qquad j_2 = -j_1,$$

where $\lceil x \rceil$ denotes the smallest integer not smaller than x.

First, let $b \geq 4$. By (9) and (10), to satisfy condition (d) it is enough to show

$$\operatorname{Re}(u_{n_l,t}(j_1)), \operatorname{Im}(u_{n_l,t}(j_1)), \operatorname{Im}(u_{n_l,t}(j_2)) > c_0, \qquad \operatorname{Re}(u_{n_l,t}(j_2)) < -c_0 \tag{11}$$

for some arithmetic constant $c_0 > 0$. Let

$$u_{n_l,t}(j_s) = v_s + \xi_s$$

for $s = 1, 2$, where

$$v_s = \sin\frac{\pi j_s}{b}\exp\left[\frac{\pi i j_s}{b}\right] \quad \text{and}$$

$$\xi_s = \sum_{m=2}^{n+1-N} \frac{a_{n_l+1-m}}{a_{n_l}} \sin\frac{\pi j_s}{b^m} \exp\left[\pi i\left(2tb^{n_l+1-m}(1-b^{m-1}) + \frac{j_s}{b^m}\right)\right].$$

Observe that

$$|\xi_s| < \frac{\pi |j_s|}{b} \sum_{m=2}^{\infty} \frac{1}{(qb)^{m-1}} < \frac{\pi |j_s|}{b(q_0 b - 1)}. \tag{12}$$

Since $j_2 = -j_1$ we have

$$\mathrm{Re}(v_2) = -\mathrm{Re}(v_1) \quad \text{and} \quad \mathrm{Im}(v_2) = \mathrm{Im}(v_1);$$

hence, for (11) to hold it is sufficient that

$$\min(\mathrm{Re}(v_1), \mathrm{Im}(v_1)) - \max(|\xi_1|, |\xi_2|) > c_0. \tag{13}$$

For $b \geq 6$,

$$\frac{1}{4} \leq \frac{j_1}{b} \leq \frac{1}{3};$$

therefore,

$$\mathrm{Im}(v_1) \geq \mathrm{Re}(v_1) = \frac{1}{2}\sin\frac{2\pi j_1}{b} \geq \frac{1}{2}\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{4}$$

and, by (12),

$$|\xi_s| < \frac{\pi}{3(6q_0 - 1)} < \frac{\pi}{14} < \frac{\sqrt{3}}{4},$$

so we have (13).

For $b = 5$,

$$\mathrm{Re}(v_1) > \mathrm{Im}(v_1) = \sin^2\frac{\pi}{5}$$

and, by (12),

$$|\xi_s| < \frac{\pi}{5(5q_0 - 1)} < \frac{\pi}{19} < \sin^2\frac{\pi}{5},$$

which implies (13).

Finally, for $b = 4$,

$$\mathrm{Re}(v_1) = \mathrm{Im}(v_1) = \frac{1}{2}$$

and then (12) yields

$$|\xi_s| < \frac{\pi}{4(4q_0 - 1)} < \frac{\pi}{11} < \frac{1}{2},$$

so (13) holds. This ends the proof for $b \geq 4$.

We are left with the case $b = 2, 3$. Write $u_{n_l}(j_s)$ for $s = 1, 2$ in the form

$$u_{n_l}(j_s) = v_s + w_s + z_s + \eta_s,$$

where

$$
\begin{aligned}
v_s &= \sin\frac{\pi j_s}{b}\exp\left[\frac{\pi i j_s}{b}\right],\\
w_s &= \frac{a_{n_l-1}}{a_{n_l}}\sin\frac{\pi j_s}{b^2}\exp\left[\pi i\left(2tb^{n_l-1}(1-b)+\frac{j_s}{b^2}\right)\right],\\
z_s &= \frac{a_{n_l-2}}{a_{n_l}}\sin\frac{\pi j_s}{b^3}\exp\left[\pi i\left(2tb^{n_l-2}(1-b^2)+\frac{j_s}{b^3}\right)\right],\\
\eta_s &= \sum_{m=4}^{n+1-N}\frac{a_{n_l+1-m}}{a_{n_l}}\sin\frac{\pi j_s}{b^m}\exp\left[\pi i\left(2tb^{n_l+1-m}(1-b^{m-1})+\frac{j_s}{b^m}\right)\right].
\end{aligned}
$$

By condition (e) we have

$$q<\left|\frac{a_{n_l-1}}{a_{n_l}}\right|<\frac{1}{q},$$

so

$$q_0\sin\frac{\pi}{b^2}<|w_s|<\frac{1}{q_0}\sin\frac{\pi}{b^2}. \tag{14}$$

Moreover, by definition,

$$t=\sum_{l=1}^{\infty}\left(\frac{\lfloor b^{M-1}\varphi_l/(b-1)\rfloor}{b^{n_l+M-2}}+\frac{\varrho_l}{b^{n_l-1}}+\frac{i_l}{b^{n_l-2}}\right) \tag{15}$$

for some $i_l\in\{0,1\}$; hence, by condition (a),

$$
\begin{aligned}
&\exp[2\pi i(\varphi_l+tb^{n_l-1}(1-b))]\\
&\quad=\exp\Bigg[2\pi i\Bigg(\varphi_l-\frac{\lfloor b^{M-1}\varphi_l/(b-1)\rfloor}{b^{M-1}/(b-1)}\\
&\qquad\qquad+(1-b)\sum_{p=1}^{\infty}\left(\frac{\lfloor b^{M-1}\varphi_l/(b-1)\rfloor}{b^{n_{l+p}-n_l+M-1}}+\frac{\varrho_{l+p}}{b^{n_{l+p}-n_l}}+\frac{i_{l+p}}{b^{n_{l+p}-n_l-1}}\right)\Bigg)\Bigg]\\
&\quad=\exp[i\tau],
\end{aligned}
$$

where

$$-\frac{2\pi b^2}{b^M-1}=2\pi(1-b)\sum_{p=1}^{\infty}\left(\frac{b^{M-1}/(b-1)}{b^{M(p+1)-1}}+\frac{1}{b^{Mp}}+\frac{1}{b^{Mp-1}}\right)\le\tau<\frac{2\pi(b-1)}{b^{M-1}},$$

and so

$$|\tau|\le\frac{2\pi b^2}{b^{M_0}-1}<0.001. \tag{16}$$

This implies

$$w_s=\left|\frac{a_{n_l-1}}{a_{n_l}}\right|\sin\frac{\pi j_s}{b^2}\exp\left[\frac{\pi i j_s}{b^2}+i\tau\right],$$

so

$$\operatorname{Arg}(w_1)=\frac{\pi}{b^2}+\tau\quad\text{and}\quad\operatorname{Arg}(w_2)=\pi\left(1-\frac{1}{b^2}\right)+\tau. \tag{17}$$

Moreover,

$$|\eta_s| < \frac{\pi|j_s|}{b}\sum_{m=4}^{\infty}\frac{1}{(qb)^{m-1}} < \frac{\pi|j_s|}{q_0^2 b^3(q_0 b-1)}. \tag{18}$$

Consider the case $b = 3$. Then

$$v_1 = \frac{\sqrt{3}}{4} + \frac{3}{4}i \quad \text{and} \quad v_2 = -\frac{\sqrt{3}}{4} + \frac{3}{4}i.$$

By (14), (17), and (16),

$$\mathrm{Re}(w_1), \mathrm{Im}(w_1), \mathrm{Im}(w_2) > 0 \quad \text{and} \quad \mathrm{Re}(w_2) < 0,$$

so

$$\mathrm{Re}(v_1 + w_1) > \frac{\sqrt{3}}{4}, \qquad \mathrm{Re}(v_2 + w_2) < -\frac{\sqrt{3}}{4},$$
$$\mathrm{Im}(v_1 + w_1), \mathrm{Im}(v_2 + w_2) > \frac{3}{4}.$$

On the other hand, by (18),

$$|z_s + \eta_s| < \frac{1}{q_0^2}\sin\frac{\pi}{27} + \frac{\pi}{27q_0^2(3q_0-1)} < 0.2.$$

This gives

$$\mathrm{Re}(u_{n_l}(j_1)) > \frac{\sqrt{3}}{4} - 0.2, \qquad \mathrm{Re}(u_{n_l}(j_2)) < -\frac{\sqrt{3}}{4} + 0.2,$$
$$\mathrm{Im}(u_{n_l}(j_1)), \mathrm{Im}(u_{n_l}(j_2)) > 0.55,$$

which together with (9) and (10) show condition (d).

We are left with the case $b = 2$. In this case

$$v_1 = v_2 = i, \tag{19}$$

and then (14), (17), and (16) yield

$$\begin{gathered}\mathrm{Re}(w_1), \mathrm{Im}(w_1) > 0.49, \qquad \mathrm{Re}(w_2) < -0.49,\\ \mathrm{Im}(w_2) \in (0.49, 0.51).\end{gathered} \tag{20}$$

By condition (e),

$$1 < \left|\frac{a_{n_l-2}}{a_{n_l}}\right| < \frac{1}{q^2}$$

and so

$$\sin\frac{\pi}{8} < |z_s| < \frac{1}{q_0^2}\sin\frac{\pi}{8}. \tag{21}$$

It is easy to see that the definition of ϱ_l implies

$$\mathrm{dist}\left(\psi_l - \frac{3\lfloor 2^{M-1}\varphi_l\rfloor}{2^M} - \frac{3\varrho_l}{2}, \mathbb{Z}\right) \le \frac{1}{4},$$

which together with (15) and condition (a) yields

$$\begin{aligned}
&\exp[2\pi i(\psi_l + tb^{n_l-2}(1-b^2))] \\
&\quad= \exp[2\pi i(\psi_l - 3t2^{n_l-2})] \\
&\quad= \exp\biggl[2\pi i\biggl(\psi_l - \frac{3\lfloor 2^{M-1}\varphi_l\rfloor}{2^M} - \frac{3\varrho_l}{2} \\
&\qquad\qquad - 3\sum_{p=1}^{\infty}\biggl(\frac{\lfloor 2^{M-1}\varphi_l\rfloor}{2^{n_{l+p}-n_l+M}} + \frac{\varrho_{l+p}}{2^{n_{l+p}-n_l+1}} + \frac{i_{l+p}}{2^{n_{l+p}-n_l}}\biggr)\biggr)\biggr] \\
&\quad= \exp[i\kappa],
\end{aligned}$$

where

$$-\frac{\pi}{2} - \frac{12\pi}{2^M-1} = -2\pi\biggl(\frac{1}{4} + 3\sum_{p=1}^{\infty}\biggl(\frac{2^{M-1}}{2^{M(p+1)}} + \frac{1}{2^{Mp+1}} + \frac{1}{2^{Mp}}\biggr)\biggr) \le \kappa \le \frac{\pi}{2};$$

therefore,

$$|\kappa| \le \frac{\pi}{2} + \frac{12\pi}{2^{M_0}-1} < \frac{\pi}{2} + 0.001. \tag{22}$$

This gives

$$z_s = \left|\frac{a_{n_l-2}}{a_{n_l}}\right| \sin\frac{\pi j_s}{8} \exp\biggl[\frac{\pi i j_s}{8} + i\kappa\biggr],$$

so

$$\operatorname{Arg}(z_1) = \frac{\pi}{8} + \kappa \quad \text{and} \quad \operatorname{Arg}(z_2) = \frac{7\pi}{8} + \kappa. \tag{23}$$

Suppose first that $\kappa \ge 0$. Then, by (19)–(23),

$$\begin{aligned}
\operatorname{Re}(v_1 + w_1 + z_1) &> 0.49 - \frac{1}{q_0^2}\sin\frac{\pi}{8}\sin\biggl(\frac{\pi}{8} + 0.001\biggr) \\
&> 0.34, \\
\operatorname{Im}(v_1 + w_1 + z_1) &> 1.49 + \sin^2\frac{\pi}{8} > 1.63, \\
\operatorname{Re}(v_2 + w_2 + z_2) &< -0.49 - \sin\frac{\pi}{8}\sin\biggl(\frac{\pi}{8} - 0.001\biggr) \\
&< -0.63,
\end{aligned}$$

$$1 < 1.49 - \frac{1}{q_0^2}\sin\frac{\pi}{8} < \operatorname{Im}(v_2 + w_2 + z_2) < 1.51 + \frac{1}{q_0^2}\sin^2\frac{\pi}{8} < 1.66.$$

Moreover, by (18) we have

$$|\eta_s| < \frac{\pi}{8q_0^2(2q_0-1)} < 0.4,$$

so

$$\begin{aligned}
\operatorname{Re}(u_{n_l,t}(j_1)) &> -0.06, & \operatorname{Im}(u_{n_l,t}(j_1)) &> 1.23, \\
\operatorname{Re}(u_{n_l,t}(j_2)) &< -0.23, & 0.6 < \operatorname{Im}(u_{n_l,t}(j_2)) &< 2.06.
\end{aligned}$$

This implies

$$\mathrm{Arg}(u_{n_l,t}(j_2)) - \mathrm{Arg}(u_{n_l,t}(j_1)) > \arctan\frac{23}{206} - \arctan\frac{2}{41} > 0,$$
$$\mathrm{Im}(u_{n_l,t}(j_1)), \mathrm{Im}(u_{n_l,t}(j_2)) > 0.6,$$

which together with (9) and (10) give condition (d).

The proof for the remaining case $\kappa < 0$ is the same as for $\kappa \geq 0$, because the points z_1 and z_2 for given κ are symmetric (with respect to the imaginary axis) to the points z_2 and z_1 for $-\kappa$. □

The following elementary geometric lemma is a more precise version of Lemma 3.3 in [Ba].

LEMMA 2. *Let $\delta_1, \delta_2 > 0$ and let $\mathcal{A}$ be a family of all triangles Δ for which*

$$\frac{1}{\delta_1} < L_0, L_1, L_2 < \delta_1 \quad \textit{and} \quad \delta_2 < \alpha_0, \alpha_1, \alpha_2 < \pi - \delta_2,$$

where L_0, L_1, L_2 are the side lengths and $\alpha_0, \alpha_1, \alpha_2$ the angles of Δ. Then there exist $Q \in (0,1)$ and $r, \varepsilon > 0$ such that, for every triangle $\Delta \in \mathcal{A}$ with vertices A, B, C, there exists a point $P \in \Delta$ such that

$$\mathbb{D}_r(P) \subset \mathbb{D}_{Qr}(\tilde{A}) \cup \mathbb{D}_{Qr}(\tilde{B}) \cup \mathbb{D}_{Qr}(\tilde{C})$$

for every $\tilde{A} \in \mathbb{D}_\varepsilon(A)$, $\tilde{B} \in \mathbb{D}_\varepsilon(B)$, and $\tilde{C} \in \mathbb{D}_\varepsilon(C)$.

Proof. It is easy to check (see the proof of Lemma 3.3 in [Ba]) that, if we take P as the unique point in the interior of Δ such that $\measuredangle APB = \measuredangle BPC = \measuredangle CPA = 2\pi/3$, then for Q, r, ε it is sufficient to satisfy the three inequalities

$$(1 - Q^2)r^2 - (X - 2\varepsilon Q)r + X^2 - \varepsilon^2 < 0 \tag{24}$$

for $X = |\overline{AP}|, |\overline{BP}|, |\overline{CP}|$. Take

$$Q = 1 - \frac{\inf_{\Delta\in\mathcal{A}} \min(|\overline{AP}|, |\overline{BP}|, |\overline{CP}|)}{100 \sup_{\Delta\in\mathcal{A}} \max(|\overline{AP}|, |\overline{BP}|, |\overline{CP}|)},$$
$$\varepsilon = \frac{\inf_{\Delta\in\mathcal{A}} \min(|\overline{AP}|, |\overline{BP}|, |\overline{CP}|)}{4}.$$

By the definition of $\mathcal{A}$ it follows that $0.99 \leq Q < 1$ and $\varepsilon > 0$. Moreover, for every $\Delta \in \mathcal{A}$ and for $X = |\overline{AP}|, |\overline{BP}|, |\overline{CP}|$ we have

$$\varepsilon \leq \frac{X}{4},$$

so

$$(X - 2\varepsilon Q)^2 - 4(1 - Q^2)(X^2 - \varepsilon^2) > \frac{X^2}{10} > 0$$

and the quadratic function of r in (24) has two roots, $0 < r_{\min} < r_{\max}$, such that

$$r_{\min} = \frac{2(X^2 - \varepsilon^2)}{X - 2\varepsilon Q + \sqrt{(X - 2\varepsilon Q)^2 - 4(1 - Q^2)(X^2 - \varepsilon^2)}}$$
$$< \frac{2X^2}{X - 2\varepsilon Q} < 4X \leq 4 \sup_{\Delta\in\mathcal{A}} \max(|\overline{AP}|, |\overline{BP}|, |\overline{CP}|) = r'$$

and

$$r_{\max} = \frac{X - 2\varepsilon Q + \sqrt{(X-2\varepsilon Q)^2 - 4(1-Q^2)(X^2-\varepsilon^2)}}{2(1-Q^2)}$$
$$> \frac{X - 2\varepsilon Q}{2(1-Q^2)} > \frac{X}{8(1-Q)} \geq \frac{\inf_{\Delta\in\mathcal{A}} \min(|\overline{AP}|, |\overline{BP}|, |\overline{CP}|)}{8(1-Q)} = r''.$$

By the definition of Q we have $0 < r' < r''$, so the three inequalities (24) hold for any $r \in (r', r'')$. □

3. Proof of Theorem A

Consider the constants q_0, M_0, c_1, c_2 from Lemma 1 and let Q, r, ε be the constants from Lemma 2 applied for $\delta_1 = c_1$ and $\delta_2 = c_2$. Take $M > M_0$ such that $M > N$ and

$$9\pi \left(\frac{2}{3}\right)^M < \varepsilon. \tag{25}$$

Fix $q \in (0,1)$ such that $q > q_0$ and

$$q > Q^{1/M}, \tag{26}$$

and now consider the set $\mathcal{T}$, the subsequence n_l, and j_0, j_1, j_2 from Lemma 1 applied for M, q and the function f. Let $t \in \mathcal{T}$ and

$$A_{l,t} = u_{n_l,t}(j_0) = 0, \quad B_{l,t} = u_{n_l,t}(j_1), \quad C_{l,t} = u_{n_l,t}(j_2)$$

for $l \geq 1$. Then, by Lemmas 1 and 2, there exists a point $P_{l,t}$ in the triangle of vertices $A_{l,t}, B_{l,t}, C_{l,t}$ such that

$$\mathbb{D}_r(P_{l,t}) \subset \mathbb{D}_{Qr}(\tilde{A}) \cup \mathbb{D}_{Qr}(\tilde{B}) \cup \mathbb{D}_{Qr}(\tilde{C}) \tag{27}$$

for every $\tilde{A} \in \mathbb{D}_\varepsilon(A_{l,t})$, $\tilde{B} \in \mathbb{D}_\varepsilon(B_{l,t})$, and $\tilde{C} \in \mathbb{D}_\varepsilon(C_{l,t})$. Take l_0 such that

$$\frac{\max(|a_0|, \ldots, |a_{N-1}|)}{|a_N|} \left(\frac{2}{3}\right)^{n_{l_0} - N} < 1.$$

Then—using (8), (25), and Lemma 1—for $n = n_l$ and $\tilde{n} = n_{l-1}$ we have

$$|\Delta_{n_l, n_{l-1}, k, t}(j_s)| < \varepsilon$$

for $s = 0, 1, 2$ and for every $l \geq l_0$, $k \in \mathbb{Z}$, and $t \in \mathcal{T}$. This together with (27) yields

$$\mathbb{D}_r(P_{l,t}) \subset \bigcup_{s=0}^{2} \mathbb{D}_{Qr}(u_{n_l,t}(j_s) + \Delta_{n_l,n_{l-1},k,t}(j_s))$$

for every $l \geq l_0$ and so, by (7),

$$\mathbb{D}_{2|a_{n_l}|r}(\zeta_{n_{l-1},k}(t) + 2ia_{n_l}\exp[2\pi i t b^{n_l}] P_{l,t}) \subset \bigcup_{s=0}^{2} \mathbb{D}_{2Q|a_{n_l}|r}(\zeta_{n_l, kb^{n_l - n_{l-1}} + j_s}(t)).$$

By (26) and Lemma 1 we have

$$2Q|a_{n_{l-1}}|r < 2|a_{n_l}|r,$$

so

$$\mathbb{D}_{2Q|a_{n_{l-1}}|r}(\zeta_{n_{l-1},k}(t) + 2ia_{n_l}\exp[2\pi itb^{n_l}]P_{l,t}) \subset \bigcup_{s=0}^{2}\mathbb{D}_{2Q|a_{n_l}|r}(\zeta_{n_l,kb^{n_l-n_{l-1}}+j_s}(t)) \quad (28)$$

for every $l \geq l_0$, $k \in \mathbb{Z}$, and $t \in \mathcal{T}$. Starting from $l = l_0$ and $k = 0$ and then applying (28) inductively, we obtain

$$\mathbb{D}_{2Q|a_{n_{l_0-1}}|r}\left(\zeta_{n_{l_0-1},0}(t) + 2i\sum_{l=l_0}^{m} a_{n_l}\exp[2\pi itb^{n_l}]P_{l,t}\right) \subset \bigcup_{s_1,\ldots,s_m\in\{0,1,2\}} \mathbb{D}_{2Q|a_{n_m}|r}(\zeta_{n_m,b^{n_m}\sum_{l=l_0}^{m} j_{s_l}/b^{n_l}}(t)) \quad (29)$$

for every $m \geq l_0$. Observe that $\zeta_{n_{l_0-1},0}(t) = \zeta_{0,0}(t) = f(e^{2\pi it})$. Let

$$\mathcal{C}_t = \left\{\exp\left[2\pi i\left(t + \sum_{l=1}^{\infty}\frac{j_{s_l}}{b^{n_l+1}}\right)\right] : s_l \in \{0,1,2\} \text{ for } l \geq 1\right\}.$$

Then $\mathcal{C}_t$ are Cantor sets in the unit circle and

$$\zeta_{n_m,b^{n_m}\sum_{l=l_0}^{m} j_{s_l}/b^{n_l}}(t) \in f(\mathcal{C}_t)$$

for every $m \geq l_0$, so (29) gives

$$\mathbb{D}_{2Q|a_{n_{l_0-1}}|r}\left(f(e^{2\pi it}) + 2i\sum_{l=l_0}^{m} a_{n_l}\exp[2\pi itb^{n_l}]P_{l,t}\right) \subset \{w \in \mathbb{C} : \operatorname{dist}(w, f(\mathcal{C}_t)) < 2Q|a_{n_m}|r\}. \quad (30)$$

Note that by (9) we have

$$\sup\{|P_{l,t}| : l \geq 1,\ t \in \mathcal{T}\} < 3\pi,$$

so the series $\sum_{l=l_0}^{\infty} a_{n_l}\exp[2\pi itb^{n_l}]P_{l,t}$ is convergent to some $P_t \in \mathbb{C}$ such that

$$\sup\{|P_t| : t \in \mathcal{T}\} < \infty. \quad (31)$$

Moreover, $a_{n_m} \to 0$ as $m \to \infty$, so (30) and the compactness of $f(\mathcal{C}_t)$ yield

$$\mathbb{D}_{2Q|a_{n_{l_0-1}}|r}(f(e^{2\pi it}) + 2iP_t) \subset f(\mathcal{C}_t). \quad (32)$$

It is clear that the set $\mathcal{T}$ is uncountable. Moreover, by (31),

$$\sup\{|f(e^{2\pi it}) + 2iP_t| : t \in \mathcal{T}\} < \infty.$$

Hence, there exist $\tilde{z} \in \mathbb{C}$ and an uncountable set $\mathcal{S} \subset \mathcal{T}$ such that

$$|f(e^{2\pi it}) + 2iP_t - \tilde{z}| < |a_{n_{l_0-1}}|r/2$$

for every $t \in \mathcal{S}$. This together with (32) gives

$$D \subset \bigcap_{t\in\mathcal{S}} f(\mathcal{C}_t) \quad \text{for} \quad D = \mathbb{D}_{|a_{n_{l_0-1}}|r/2}(\tilde{z}).$$

Therefore, to use the Lusin–Purves Theorem and finish the proof of Theorem A it is enough to show that the sets $\mathcal{C}_t$ are pairwise disjoint for $t \in \mathcal{T}$. In order to do this, suppose that $\mathcal{C}_t \cap \mathcal{C}_{t'} \neq \emptyset$ for some $t, t' \in \mathcal{T}$ $(t \neq t')$. Then, for every $l \geq 1$, there exist $r_l, r_l' \in \{\sigma_l, \tilde{\sigma}_l\}$ for $\sigma_l, \tilde{\sigma}_l$ from Lemma 1 and $s_l, s_l' \in \{0, 1, 2\}$ such that

$$\exp\left[2\pi i \sum_{l=1}^{\infty} \frac{r_l + j_{s_l}}{b^{n_l+1}}\right] = \exp\left[2\pi i \sum_{l=1}^{\infty} \frac{r_l' + j_{s_l'}}{b^{n_l+1}}\right]. \tag{33}$$

Recall that by Lemma 1 we have $\sigma_l, \tilde{\sigma}_l \in [0, 2b^3]$, $|\sigma_l - \tilde{\sigma}_l| > 3b$, and $j_{s_l}, j_{s_l'} \in [-b, b] \cap \mathbb{Z}$. Therefore,

$$\begin{aligned} \left|2\pi \sum_{l=1}^{\infty} \frac{r_l + j_{s_l}}{b^{n_l+1}} - 2\pi \sum_{l=1}^{\infty} \frac{r_l' + j_{s_l'}}{b^{n_l+1}}\right| &\leq 2\pi \sum_{l=1}^{\infty} \frac{|r_l - r_l'| + |j_{s_l} - j_{s_l'}|}{b^{n_l+1}} \\ &\leq \frac{4\pi(b^2+1)}{b^{n_0}} \sum_{l=1}^{\infty} \frac{1}{b^{Ml}} \\ &= \frac{4\pi(b^2+1)}{b^{n_0}(b^{M_0}-1)} < 2\pi, \end{aligned}$$

so (33) yields

$$\sum_{l=1}^{\infty} \frac{r_l + j_{s_l}}{b^{n_l}} = \sum_{l=1}^{\infty} \frac{r_l' + j_{s_l'}}{b^{n_l}}.$$

Since $t \neq t'$, there exists an l_1 such that

$$r_{l_1} \neq r_{l_1}'.$$

Then

$$|r_{l_1} + j_{s_{l_1}} - r_{l_1}' - j_{s_{l_1}'}| \geq |\sigma_{l_1} - \tilde{\sigma}_{l_1}| - |j_{s_{l_1}} - j_{s_{l_1}'}| > b > 0,$$

so

$$r_{l_1} + j_{s_{l_1}} \neq r_{l_1}' + j_{s_{l_1}'}.$$

Hence, there exists an

$$l_2 = \min\{l \geq 1 : r_l + j_{s_l} \neq r_l' + j_{s_l'}\}.$$

If $r_{l_2} \neq r_{l_2}'$ then

$$|r_{l_2} + j_{s_{l_2}} - r_{l_2}' - j_{s_{l_2}'}| \geq |\sigma_{l_2} - \tilde{\sigma}_{l_2}| - |j_{s_{l_2}} - j_{s_{l_2}'}| > b,$$

and if $r_{l_2} = r_{l_2}'$ then $j_{s_{l_2}} \neq j_{s_{l_2}'}$ and we have

$$|r_{l_2} + j_{s_{l_2}} - r_{l_2}' - j_{s_{l_2}'}| = |j_{s_{l_2}} - j_{s_{l_2}'}| \geq 1.$$

This implies

$$\begin{aligned}\left|\sum_{l=1}^{\infty}\frac{r_l+j_{s_l}}{b^{n_l}}-\sum_{l=1}^{\infty}\frac{r'_l+j_{s'_l}}{b^{n_l}}\right| &\geq \frac{|r_{l_2}+j_{s_{l_2}}-r'_{l_2}-j_{s'_{l_2}}|}{b^{n_{l_2}}}\\ &\quad-\sum_{l=l_2+1}^{\infty}\frac{|r_l-r'_l|+|j_{s_l}-j'_{s_l}|}{b^{n_l}}\\ &> \frac{1}{b^{n_{l_2}}}-\frac{2b(b^2+1)}{b^{n_{l_2}}}\sum_{l=l_2+1}^{\infty}\frac{1}{b^{Ml}}\\ &= \left(1-\frac{2b(b^2+1)}{b^{M_0}-1}\right)\frac{1}{b^{n_{l_2}}}>0,\end{aligned}$$

which is a contradiction. Hence, the $\mathcal{C}_t$ are pairwise disjoint for $t \in \mathcal{T}$, which completes the proof.

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