

Graded Cofinite Rings of Differential Operators

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1. Introduction

In this paper we study subalgebras \mathcal{A} of the algebra $\mathcal{D}(X)$ of differential operators on a smooth variety X that are big in the following sense: using the order of a differential operator, the ring $\mathcal{D}(X)$ is equipped with a filtration. Its associated graded algebra $\bar{\mathcal{D}}(X)$ is commutative and can be regarded as the set of regular functions on the cotangent bundle of X . The subalgebra \mathcal{A} inherits a filtration from $\mathcal{D}(X)$, and its associated graded algebra $\bar{\mathcal{A}}$ is a subalgebra of $\bar{\mathcal{D}}(X)$. We call \mathcal{A} *graded cofinite* in $\mathcal{D}(X)$ if $\bar{\mathcal{D}}(X)$ is a finitely generated $\bar{\mathcal{A}}$ -module.

Our guiding example of a graded cofinite subalgebra is the algebra of invariants $\mathcal{D}(X)^W$, where W is a finite group acting on X . Other examples can be constructed as follows. Let $\varphi: X \rightarrow Y$ be a finite dominant morphism onto a normal variety Y . Then we put

$$\mathcal{D}(X, Y) = \{D \in \mathcal{D}(X) \mid D(\mathcal{O}(Y)) \subseteq \mathcal{O}(Y)\}. \quad (1.1)$$

We show (Corollary 3.6) that this subalgebra is graded cofinite if and only if the ramification of φ is uniform—that is, if the ramification degree of φ along a divisor $D \subset X$ depends only on the image $\varphi(D)$.

It should be noted that these two constructions are in fact more or less equivalent. In Theorem 3.1 we show that $\mathcal{D}(X)^W = \mathcal{D}(X, X/W)$. Conversely, we show in Proposition 3.3 that $\mathcal{D}(X, Y) = \mathcal{D}(\tilde{X})^W$, where $\tilde{X} \rightarrow X$ is a suitable finite cover of X and W is a finite group acting on \tilde{X} .

Our main result is that, up to automorphisms, every graded cofinite subalgebra is of the form just described.

1.1. THEOREM. *Let X be a smooth variety and \mathcal{A} a graded cofinite subalgebra of $\mathcal{D}(X)$. Then there is an automorphism Φ of $\mathcal{D}(X)$, inducing the identity on $\bar{\mathcal{D}}(X)$, such that $\mathcal{A} = \Phi\mathcal{D}(X, Y)$ for some uniformly ramified morphism $\varphi: X \rightarrow Y$.*

The main motivation for this notion came from the following result of Levasseur and Stafford [LSt]. Let W be a finite group acting linearly on a vector space V ; then $\mathcal{D}(V)^W$ is generated by the W -invariant functions $\mathcal{O}(V)^W$ and the W -invariant constant coefficient differential operators $S^*(V)^W$. For general varieties X , there

is no notion of constant coefficient differential operators. Since the algebra generated by $\mathcal{O}(V)^W$ and $S^*(V)^W$ is clearly graded cofinite, our main theorem can be seen as a nonlinear generalization of the theorem of Levasseur–Stafford.

Our main theorem has several applications concerning generating elements of rings of W -invariant differential operators that go beyond the theorem of Levasseur–Stafford. For example, we prove that $\mathcal{D}(X)^W$ can be generated by at most $2n + 1$ elements when V is an n -dimensional representation of W . Moreover, we establish a kind of Galois correspondence for graded cofinite subalgebras. Finally, we determine all graded cofinite subalgebras of $\mathcal{D}(\mathbf{A}^1)$, the Weyl algebra in two generators.

The proof consists essentially of five steps.

1. We show the aforementioned claim that $\mathcal{D}(X, Y)$ is graded cofinite if and only if φ is uniformly ramified.
2. Then we show that, under these conditions, $\mathcal{D}(X, Y)$ is a simple ring; here we follow an argument of Wallach [W].
3. We show that the theorem holds over the generic point of X .
4. Then we construct the automorphism Φ . This is the most tedious part of the paper and rests on explicit computations in codimension 1.
5. Finally, we paste all this information together by showing that two graded cofinite subalgebras $\mathcal{A} \subseteq \mathcal{A}'$ which coincide generically and for which \mathcal{A}' is a simple ring are actually equal; here we follow the argument in [LSt].

Finally, it should be mentioned that the actual main Theorem 7.1 is more general in that it allows for certain singularities of X .

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2. Graded Cofinite Subalgebras: Definition and Base Change

All varieties and algebras will be defined over \mathbb{C} . Moreover, varieties are irreducible by definition.

Recall that a \mathbb{C} -linear endomorphism D of a commutative algebra B is a *differential operator of order $\leq d$* if

$$[b_0, [b_1, \dots, [b_d, D], \dots]] = 0 \quad \text{for all } b_0, b_1, \dots, b_d \in B. \quad (2.1)$$

Let $\mathcal{D}(B)_{\leq d}$ be the set of differential operators of order $\leq d$ and let $\mathcal{D}(B) = \bigcup_d \mathcal{D}(B)_{\leq d}$. Then $\mathcal{D}(B)$ is a filtered algebra; that is, $\mathcal{D}(B)_{\leq d} \mathcal{D}(B)_{\leq e} \subseteq \mathcal{D}(B)_{\leq d+e}$ for all integers d and e . Let $\tilde{\mathcal{D}}(B)$ be its *associated graded algebra*: $\tilde{\mathcal{D}}(X) := \bigoplus_d \tilde{\mathcal{D}}(X)_d$ with $\tilde{\mathcal{D}}(X)_d = \mathcal{D}(X)_{\leq d} / \mathcal{D}(X)_{\leq d-1}$. This is a graded *commutative* algebra. If X is a variety with ring of functions $\mathcal{O}(X)$, then we define $\mathcal{D}(X) = \mathcal{D}(\mathcal{O}(X))$.

Every subalgebra $\mathcal{A} \subseteq \mathcal{D}(X)$ inherits the filtration by $\mathcal{A}_{\leq d} = \mathcal{A} \cap \mathcal{D}(X)_{\leq d}$. This way, the associated graded algebra $\bar{\mathcal{A}}$ is a subalgebra of $\bar{\mathcal{D}}(X)$ and we have the following definition.

DEFINITION. A subalgebra \mathcal{A} of $\mathcal{D}(X)$ is called *graded cofinite* if $\bar{\mathcal{D}}(X)$ is a finitely generated $\bar{\mathcal{A}}$ -module.

EXAMPLE. Let W be a finite group acting on X and assume $\bar{\mathcal{D}}(X)$ to be finitely generated (e.g., X is smooth). We claim that $\mathcal{A} = \mathcal{D}(X)^W$ is graded cofinite in $\mathcal{D}(X)$. In fact, since W is linearly reductive it follows that $\bar{\mathcal{A}} = \bar{\mathcal{D}}(X)^W$, which is well known to be cofinite in $\bar{\mathcal{D}}(X)$.

The ring $A := \mathcal{A}_{\leq 0} = \mathcal{A} \cap \mathcal{O}(X)$ is called the *base* of \mathcal{A} .

2.1. PROPOSITION. *Let $\mathcal{A} \subseteq \mathcal{D}(X)$ be graded cofinite. Then the base A of \mathcal{A} is a finitely generated algebra that is cofinite in $\mathcal{O}(X)$. In other words, if $Y = \text{Spec } A$ then $X \rightarrow Y$ is a finite surjective morphism of affine varieties.*

Proof. Since $\bar{\mathcal{A}}$ is cofinite in $\bar{\mathcal{D}}(X)$, its 0-component A is cofinite in the 0-component $\mathcal{O}(X)$ of $\bar{\mathcal{D}}(X)$. Now the assertion follows from our next lemma. \square

2.2. LEMMA. *Let $A \subseteq B$ be an integral extension of commutative \mathbb{C} -algebras. Assume B is a finitely generated algebra. Then A is finitely generated as well, and A is cofinite in B .*

Proof. This is the Artin–Tate lemma. For a proof see [E, p. 143]. \square

In the sequel we shall need some auxiliary results concerning the behavior of \mathcal{A} with respect to extension of scalars. Let X be an affine variety, $B := \mathcal{O}(X)$, and $J \subseteq B$ an ideal. Let \hat{B} be the J -adic completion of B and let $\hat{X} := \text{Spec } \hat{B}$. Let $\mathcal{D}_c(\hat{X}) \subseteq \text{End}_{\mathbb{C}}^{\text{cont}}(\hat{B})$ be the algebra of *continuous* differential operators on \hat{X} . We show that this is also the algebra of differential operators on X with coefficients in \hat{B} . More precisely, we have the following lemma.

2.3. LEMMA. *Fix $d \geq 0$. Then the left J -adic topology and the right J -adic topology of $\mathcal{D}(X)_{\leq d}$ coincide. Its completion with respect to this topology equals $\mathcal{D}_c(\hat{X})_{\leq d}$. In particular, the two natural maps*

$$\hat{B} \otimes_B \mathcal{D}(X) \rightarrow \mathcal{D}_c(\hat{X}) \quad \text{and} \quad \mathcal{D}(X) \otimes_B \hat{B} \rightarrow \mathcal{D}_c(\hat{X}) \quad (2.2)$$

are isomorphisms of filtered vector spaces.

Proof. We recall Grothendieck’s description of $\mathcal{D}(X)$. Let δ be the kernel of the multiplication map $B \otimes_{\mathbb{C}} B \rightarrow B$; it is the ideal of $C := B \otimes B$ generated by all elements of the form $b \otimes 1 - 1 \otimes b$ for $b \in B$. Let $\mathcal{P}^d := C/\delta^{d+1}$. This is a C -module; that is, it carries a left and a right B -module structure. Moreover, it is a finitely generated module with respect to both structures. Now we have $\mathcal{D}(X)_{\leq d} = \text{Hom}_B(\mathcal{P}^d, B)$, where we use the left B -module structure of \mathcal{P}^d .

Now consider the completed ring \hat{B} . Then $\text{End}_{\mathbb{C}}(\hat{B}) = \text{Hom}_{\hat{B}}(\hat{B} \otimes_{\mathbb{C}} \hat{B}, \hat{B})$. It is easy to see that the continuous endomorphisms correspond exactly to those homomorphisms $\hat{B} \otimes_{\mathbb{C}} \hat{B} \rightarrow \hat{B}$ that extend to the completed tensor product $\hat{C} := \hat{B} \hat{\otimes}_{\mathbb{C}} \hat{B}$. Thus, $\text{End}_{\mathbb{C}}^{\text{cont}}(\hat{B}) = \text{Hom}_{\hat{B}}(\hat{C}, \hat{B})$. Let $\hat{\delta}$ be the kernel of $\hat{C} = \hat{B} \hat{\otimes}_{\mathbb{C}} \hat{B} \rightarrow \hat{B}$ and let $\hat{\mathcal{P}}^d = \hat{C}/\hat{\delta}^{d+1}$. Then $\mathcal{D}_c(\hat{X})_{\leq d} = \text{Hom}_{\hat{B}}(\hat{\mathcal{P}}^d, \hat{B})$.

Let $K := J \otimes B + B \otimes J \subseteq C$. Then \hat{C} is the K -adic completion of C . Moreover, $\hat{\delta}$ is the K -adic completion of δ . Thus, everything boils down to the following statement: The left J -adic, the right J -adic, and the K -adic topologies of \mathcal{P}^d all coincide.

For $b \in B$ we have $1 \otimes b = b \otimes 1 + c$ with $c = b \otimes 1 - 1 \otimes b \in \delta$. As a result, $B \otimes J \subseteq J \otimes B + \delta$ and, for any $n \geq d$,

$$J^n \otimes B \subseteq K^n \subseteq (J \otimes B + \delta)^n \subseteq J^{n-d} \otimes B + \delta^{d+1}. \quad (2.3)$$

This shows that the left J -adic and the K -adic topologies of \mathcal{P}^d coincide. The argument for the right J -adic topology is the same. \square

Now let $\mathcal{A} \subseteq \mathcal{D}(X)$ be graded cofinite with base A . Let $I \subseteq A$ be an ideal and let \hat{A} be the I -adic completion of A . Set $J := IB \subseteq B$. Since $J^n = I^n B$, the I -adic completion \hat{B} of B is the same as its J -adic completion.

2.4. COROLLARY. *Let $\hat{\mathcal{A}} \subseteq \mathcal{D}_c(\hat{X})$ be the subalgebra generated by \mathcal{A} and \hat{A} . Then $\hat{\mathcal{A}}$ is a graded cofinite subalgebra of $\mathcal{D}_c(\hat{X})$ with base \hat{A} . Moreover, the maps*

$$\hat{A} \otimes_{\hat{A}} \mathcal{A} \rightarrow \hat{\mathcal{A}} \quad \text{and} \quad \mathcal{A} \otimes_{\hat{A}} \hat{A} \rightarrow \hat{\mathcal{A}} \quad (2.4)$$

are isomorphisms of filtered vector spaces.

Proof. Redefine $\hat{\mathcal{A}}$ to be the closure of \mathcal{A} in $\mathcal{D}_c(\hat{X})$ with respect to either left or right J -adic topology. Then Lemma 2.3 implies that the maps (2.4) are isomorphisms. In particular, $\hat{\mathcal{A}}$ is an algebra and thus is the algebra generated by \hat{A} and \mathcal{A} . \square

Now we deduce the same thing for étale morphisms. Again, let $\mathcal{A} \subseteq \mathcal{D}(X)$ be graded cofinite with base A . Let $\tilde{Y} \rightarrow Y := \text{Spec } A$ be an étale morphism, where \tilde{Y} is another affine variety. Then also $\tilde{X} := \tilde{Y} \times_Y X \rightarrow X$ is étale. Now put $B := \mathcal{O}(X)$, $\tilde{A} := \mathcal{O}(\tilde{Y})$, and $\tilde{B} := \mathcal{O}(\tilde{X}) = \tilde{A} \otimes_A B$. Then

$$\tilde{B} \otimes_B \mathcal{D}(X) \xrightarrow{\sim} \mathcal{D}(\tilde{X}) \quad \text{and} \quad \mathcal{D}(X) \otimes_B \tilde{B} \xrightarrow{\sim} \mathcal{D}(\tilde{X}) \quad (2.5)$$

are isomorphisms. For a proof see [M, Thm. 2.2.10, Prop. 2.2.12] or [S2, Thm. 4.2]. Both references state that the first isomorphism is an isomorphism of filtered rings (i.e., that there is an isomorphism on the associated graded level). This, in turn, implies the second isomorphism.

2.5. LEMMA. *Let \mathcal{A} , A , \tilde{A} , and \tilde{X} be as before. Let $\tilde{\mathcal{A}} \subseteq \mathcal{D}(\tilde{X})$ be the subalgebra generated by \mathcal{A} and \tilde{A} . Then $\tilde{\mathcal{A}}$ is a graded cofinite subalgebra of $\mathcal{D}(\tilde{X})$ with base \tilde{A} . Moreover, the maps*

$$\tilde{A} \otimes_A \mathcal{A} \rightarrow \tilde{\mathcal{A}} \quad \text{and} \quad \mathcal{A} \otimes_A \tilde{A} \rightarrow \tilde{\mathcal{A}} \quad (2.6)$$

are isomorphisms of filtered vector spaces.

Proof. We start with a general remark. Let Z be an affine variety, and let $\mathfrak{m}_z \subset \mathcal{O}(Z)$ be the maximal ideal corresponding to a point $z \in Z$. It is known that \mathfrak{m}_z -adic completion is exact on finitely generated $\mathcal{O}(Z)$ -modules. Moreover, a finitely generated $\mathcal{O}(Z)$ -module M is 0 if and only if it is so after \mathfrak{m}_z -adic completion for every $z \in Z$. Now let $N \subseteq M$ be a submodule, N' another $\mathcal{O}(Z)$ -module, and $N' \rightarrow M$ an $\mathcal{O}(Z)$ -homomorphism. Then one sees from the foregoing remarks that φ induces an isomorphism of N' onto N if and only if this is so after \mathfrak{m}_z -adic completion for every $z \in Z$.

We apply this to $Z = \tilde{X}$ and $M = \mathcal{D}(\tilde{X})_{\leq d}$. Let $\tilde{x} \in \tilde{X}$ with image $x \in X$. Since $\tilde{X} \rightarrow X$ is étale, the $\mathfrak{m}_{\tilde{x}}$ -adic completion of $\mathcal{D}(\tilde{X})_{\leq d}$ is the same as the \mathfrak{m}_x -adic completion of $\mathcal{D}(X)_{\leq d}$. Thus, Corollary 2.4 implies that the two homomorphisms

$$\tilde{A} \otimes_A \mathcal{A}_{\leq d} \rightarrow \mathcal{D}(\tilde{X})_{\leq d} \quad \text{and} \quad \mathcal{A}_{\leq d} \otimes_A \tilde{A} \rightarrow \mathcal{D}(\tilde{X})_{\leq d} \quad (2.7)$$

are injective with the same image after $\mathfrak{m}_{\tilde{x}}$ -adic completion for every $\tilde{x} \in \tilde{X}$. \square

2.6. COROLLARY. *Let $\mathcal{A} \subseteq \mathcal{D}(X)$ be graded cofinite with base A and let $S \subseteq A$ be a multiplicatively closed subset defining localizations $A_S \subseteq B_S$ (with $B := \mathcal{O}(X)$). Let $\mathcal{A}_S \subseteq \mathcal{D}(B_S)$ be the subalgebra generated by \mathcal{A} and A_S . Then \mathcal{A}_S is a graded cofinite subalgebra of $\mathcal{D}(B_S)$ with base A_S . Moreover, the maps $\mathcal{A}_S \otimes_A \mathcal{A} \rightarrow \mathcal{A}_S$ and $\mathcal{A} \otimes_A \mathcal{A}_S \rightarrow \mathcal{A}_S$ are filtered isomorphisms.*

Proof. If S is finite then $A \rightarrow A_S$ is an open embedding and, in particular, is étale. It follows from Lemma 2.5 that \mathcal{A}_S has base A_S . For the general case, use that S is the union of its finite subsets and that all objects behave well under inductive limits. \square

An important consequence is that we can “normalize” graded cofinite subalgebras.

2.7. COROLLARY. *Let X be normal and let $\mathcal{A} \subseteq \mathcal{D}(X)$ be a graded cofinite subalgebra. Let A' be the normalization of the base A , regarded as a subalgebra of $\mathcal{O}(X)$. Let $\mathcal{A}' \subseteq \mathcal{D}(X)$ be the subalgebra generated by \mathcal{A} and A' . Then \mathcal{A}' is a graded cofinite subalgebra of $\mathcal{D}(X)$ with base A' .*

Proof. Let $B := \mathcal{O}(X)$. Both algebras A and A' have the same quotient field $K = A_S$ with $S = A \setminus \{0\}$. Thus we have $A' \subseteq \mathcal{A}' \cap B \subseteq \mathcal{A}_K \cap B = K \cap B = A'$. \square

REMARK. It is possible to combine étale base change, localization, and completion. More precisely, we will use this twice in the following situation: let $\mathcal{A} \subseteq \mathcal{D}(X)$ be graded cofinite with base A and assume that X and $Y = \text{Spec } A$ are normal. Let $\tilde{Y} \rightarrow Y$ be étale and $\tilde{D} \subseteq \tilde{Y}$ a prime divisor. Take \hat{A} to be the completion of the local ring $\mathcal{O}_{\tilde{Y}, \tilde{D}}$. Then $\hat{A} \cong E[[t]]$ is a discrete valuation ring with $E = \mathbb{C}(\tilde{D})$, and $\hat{B} = \hat{A} \otimes_A \mathcal{O}(X)$ is a finite normal extension. It follows that $\hat{B} = \hat{B}_1 \times \cdots \times \hat{B}_s$, where each $\hat{B}_i \cong E_i[[t^{1/n_i}]]$ with $n_i \in \mathbb{Z}_{>0}$ and $[E_i : E] < \infty$. In

that case, we have that $\hat{\mathcal{A}} = \hat{A} \otimes_A \mathcal{A} = \mathcal{A} \otimes_A \hat{A}$ is a graded cofinite subalgebra of $\mathcal{D}_c(\hat{B}) = \mathcal{D}_c(\hat{B}_1) \times \cdots \times \mathcal{D}_c(\hat{B}_s)$ with base \hat{A} . Finally, we may choose $\tilde{Y} \rightarrow Y$ in such a way that $E_i = E$ for all i : let D be the image of \tilde{D} in Y . Assume the preimage of D in X has irreducible components D_1, \dots, D_r . Then it suffices to require that $E = \mathbb{C}(\tilde{D})$ is a splitting field for all the finite extensions $\mathbb{C}(D_j)|\mathbb{C}(D)$.

3. Certain Rings of Differential Operators

In this section we construct a certain class of graded cofinite subalgebras (see Corollary 3.6). Later we show that, under mild conditions, all examples are basically of this kind (Theorem 7.1).

For a dominant morphism $\varphi: X \rightarrow Y$, we have $\mathcal{O}(Y) \hookrightarrow \mathcal{O}(X)$ and can define the subalgebra

$$\mathcal{D}(X, Y) := \{D \in \mathcal{D}(X) \mid D(\mathcal{O}(Y)) \subseteq \mathcal{O}(Y)\}. \quad (3.1)$$

Its associated graded algebra is denoted by $\bar{\mathcal{D}}(X, Y)$.

Now assume that the field extension $\mathbb{C}(X)|\mathbb{C}(Y)$ is finite. This means that there is a nonempty open subset $X_0 \subseteq X$ such that $\varphi: X_0 \rightarrow Y$ is étale. Then every differential operator D on Y can be uniquely lifted to a differential operator D_0 on X_0 (see (2.5)). Thus, $\mathcal{D}(X, Y)$ can be also interpreted as the set of $D \in \mathcal{D}(Y)$ such that D_0 extends to a (regular) differential operator on X . In other words, the diagram

$$\begin{array}{ccc} \mathcal{D}(X, Y) & \hookrightarrow & \mathcal{D}(X) \\ \downarrow & & \downarrow \\ \mathcal{D}(Y) & \hookrightarrow & \mathcal{D}(X_0) \end{array} \quad (3.2)$$

is Cartesian. Note that the filtrations of $\mathcal{D}(X, Y)$ induced by those on $\mathcal{D}(X)$, $\mathcal{D}(Y)$, and $\mathcal{D}(X_0)$ are the same. Thus we get an analogous diagram of inclusions for the associated graded rings:

$$\begin{array}{ccc} \bar{\mathcal{D}}(X, Y) & \hookrightarrow & \bar{\mathcal{D}}(X) \\ \downarrow & & \downarrow \\ \bar{\mathcal{D}}(Y) & \hookrightarrow & \bar{\mathcal{D}}(X_0), \end{array} \quad (3.3)$$

which, however, may not be Cartesian.

First we show that this class of algebras includes rings of invariant differential operators.

3.1. THEOREM. *Let W be a finite group acting on X . Then*

$$\mathcal{D}(X, X/W) = \mathcal{D}(X)^W \quad \text{and} \quad \bar{\mathcal{D}}(X, X/W) = \bar{\mathcal{D}}(X)^W. \quad (3.4)$$

Proof. Clearly $\mathcal{D}(X, X/W) \supseteq \mathcal{D}(X)^W$. Conversely, for $D \in \mathcal{D}(X, X/W)$, put $D' = \frac{1}{|W|} \sum_{w \in W} wD$. If $f \in \mathcal{O}(X)^W$ then $(wD)(f) = w(D(w^{-1}f)) = f$. This

implies that $D - D'$ is a differential operator which is zero on $\mathcal{O}(X)^W$ and hence on all of $\mathcal{O}(X)$. Thus, $D = D' \in \mathcal{D}(X)^W$. The second equality follows because forming the associated graded algebra commutes with taking W -invariants. \square

EXAMPLE. Let $X = \mathbf{A}^1$ be the affine line with coordinate ring $\mathcal{O}(X) = \mathbb{C}[x]$ and let $W = \mu_n \cong \mathbb{Z}/n\mathbb{Z}$ acting by multiplication. Define $Y \cong \mathbf{A}^1$ by $\mathcal{O}(Y) = \mathbb{C}[t]$, where $t = x^n$. The chain rule yields $\partial_x = nx^{n-1}\partial_t = nt^{1-1/n}\partial_t$. Let ξ and τ be the symbols of ∂_x and ∂_t , respectively. Then $\zeta \in W$ acts on (x, ξ) by $(\zeta^{-1}x, \zeta\xi)$. Moreover, $\xi = nx^{n-1}\tau$. Thus, we have

$$\begin{array}{ccccccc} \mathcal{D}(X, Y) = \mathbb{C}\langle t, t\partial_t, (t^{1-1/n}\partial_t)^n \rangle & = & \mathbb{C}\langle x^n, x\partial_x, \partial_x^n \rangle & \hookrightarrow & \mathbb{C}\langle x, \partial_x \rangle & = & \mathcal{D}(X) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{D}(Y) & = & \mathbb{C}\langle t, \partial_t \rangle & = & \mathbb{C}\langle x^n, x^{1-n}\partial_x \rangle & \hookrightarrow & \mathbb{C}\langle x, x^{-1}, \partial_x \rangle = \mathcal{D}(X_0), \end{array} \quad (3.5)$$

while the corresponding diagram for the associated graded rings is

$$\begin{array}{ccccccc} \bar{\mathcal{D}}(X, Y) = \mathbb{C}\langle t, t\tau, t^{n-1}\tau^n \rangle & = & \mathbb{C}\langle x^n, x\xi, \xi^n \rangle & \hookrightarrow & \mathbb{C}\langle x, \xi \rangle & = & \bar{\mathcal{D}}(X) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bar{\mathcal{D}}(Y) & = & \mathbb{C}\langle t, \tau \rangle & = & \mathbb{C}\langle x^n, x^{1-n}\xi \rangle & \hookrightarrow & \mathbb{C}\langle x, x^{-1}, \xi \rangle = \bar{\mathcal{D}}(X_0). \end{array} \quad (3.6)$$

In general, not all subalgebras of the form $\mathcal{D}(X, Y)$ are graded cofinite. In order to formulate a criterion, we introduce the following notions.

DEFINITION. Let X and Y be normal varieties and $\varphi: X \rightarrow Y$ a finite surjective morphism. Let $D \subseteq Y$ be a prime divisor and consider the divisor $\varphi^{-1}(D) = r_1E_1 + \cdots + r_sE_s$, where the E_i are pairwise distinct prime divisors and $r_i > 0$. We say that φ is *uniformly ramified over D* if $r_1 = \cdots = r_s$. Moreover, φ is *uniformly ramified* if it is uniformly ramified over every $D \subseteq Y$. If all the ramification numbers r_i are 1 for all D then we call φ *unramified in codimension 1*. Equivalently, there is an open subset $X_0 \subseteq X$ with $\text{codim}_X(X \setminus X_0) \geq 2$ on which φ is étale.

3.2. PROPOSITION. *Let $X \rightarrow Y$ be a finite dominant morphism between normal affine varieties that is unramified in codimension 1. Then $\mathcal{D}(X, Y) = \mathcal{D}(Y)$.*

Proof. Let $D \in \mathcal{D}(Y)$. Then D can be uniquely lifted to a differential operator D_0 on the set $X_0 \subseteq X$ on which φ is étale. Since $\text{codim}_X(X \setminus X_0) \geq 2$ and since X is normal, it follows that $\mathcal{O}(X_0) = \mathcal{O}(X)$. Hence one can extend D_0 uniquely to all of X , which proves that $\mathcal{D}(Y) \subseteq \mathcal{D}(X, Y)$. \square

Now we show that uniformly ramified morphisms are just a slight generalization of quotients by finite groups. For this we introduce the following notation. Let W be a finite group acting on a normal variety X . For a prime divisor $Z \subseteq X$, let

$W_Z \subseteq W$ be the pointwise stabilizer of Z in W (the *inertia* group). This group is always a cyclic group and its order is the ramification number of $X \rightarrow X/W$.

Now assume that $\varphi: X \rightarrow Y$ is a finite surjective morphism between normal varieties. Then the field extension $\mathbb{C}(X)|\mathbb{C}(Y)$ is finite and hence has a Galois cover L with Galois group W . Let $H \subseteq W$ be the Galois group of $L|\mathbb{C}(X)$ and let \tilde{X} be the normal affine variety such that $\mathcal{O}(\tilde{X})$ is the integral closure of $\mathcal{O}(Y)$ in L . Then \tilde{X} carries a W -action with $\tilde{X}/W = Y$ and $\tilde{X}/H = X$, and we have the diagram

$$\begin{array}{ccccc}
 \tilde{X} & \longrightarrow & \tilde{X}/H & \equiv & X \\
 & \searrow & \downarrow & & \downarrow \varphi \\
 & & \tilde{X}/W & \equiv & Y.
 \end{array} \tag{3.7}$$

3.3. PROPOSITION. *With notation as before, the following statements are equivalent.*

- (i) *The morphism $\varphi: X \rightarrow Y$ is uniformly ramified.*
- (ii) *The morphism $\tilde{X} \rightarrow \tilde{X}/H$ is unramified in codimension 1.*
- (iii) *For all prime divisors Z of \tilde{X} , the condition $W_Z \cap H = 1$ holds.*

Moreover, under these conditions we have $\mathcal{D}(X) = \mathcal{D}(\tilde{X})^H$ and $\mathcal{D}(X, Y) = \mathcal{D}(\tilde{X})^W$.

Proof. The inertia group of $\tilde{X} \rightarrow \tilde{X}/H$ at Z is $H_Z = W_Z \cap H$, which shows the equivalence (ii) \Leftrightarrow (iii).

Let D be the image of Z in Y . Then the divisors of \tilde{X} lying over D are precisely the translates wZ , $w \in W$. For fixed $w \in W$ let E be the image of wZ in \tilde{X}/H . Then E is a prime divisor of X lying over D and every such divisor is of this kind.

The inertia groups of $\tilde{X} \rightarrow \tilde{X}/H$ and $\tilde{X} \rightarrow X/W$ at wZ are $H \cap W_{wZ} = H \cap wW_Zw^{-1}$ and $W_{wZ} = wW_Zw^{-1}$, respectively. Therefore, the ramification number of $X \rightarrow Y$ at E is $[wW_Zw^{-1} : H \cap wW_Zw^{-1}]$. Thus, condition (i) means that the order of $w^{-1}Hw \cap W_Z \cong H \cap wW_Zw^{-1}$ is independent of $w \in W$. This means, in turn, that all isotropy groups of W_Z acting on W/H have the same order. Now W_Z (being cyclic) has at most one subgroup of any given order. Therefore, (i) means that all isotropy groups in W_Z on W/H are the same. We assumed that $L|\mathbb{C}(Y)$ is the Galois cover of $\mathbb{C}(X)|\mathbb{C}(Y)$ —that is, the smallest Galois extension of $\mathbb{C}(Y)$ containing $\mathbb{C}(X)$. This means that H does not contain any nontrivial normal subgroup of W (i.e., that the action of W on W/H is effective). We conclude that (i) is equivalent to the statement that, for all Z , the isotropy groups of W_Z on W/H are trivial. This is precisely the content of (ii).

Finally, $\mathcal{D}(X) = \mathcal{D}(\tilde{X}, X) = \mathcal{D}(\tilde{X})^H$ follows from Proposition 3.2 and Theorem 3.1. Moreover, $D \in \mathcal{D}(X, Y)$ implies $D \in \mathcal{D}(X) = \mathcal{D}(\tilde{X})^H \subseteq \mathcal{D}(\tilde{X})$. Hence $D \in \mathcal{D}(\tilde{X}, Y) = \mathcal{D}(\tilde{X})^W$. This shows that $\mathcal{D}(X, Y) \subseteq \mathcal{D}(\tilde{X})^W$. The opposite inclusion is obvious. \square

This result makes it easy to construct uniformly ramified morphisms. Take, for example, $W = S^n$ with its standard action on the affine space \mathbf{A}^n . Let $H \subseteq S_n$ be a subgroup of odd order (or any other subgroup not containing a transposition). Then the morphism $\mathbf{A}^n/H \rightarrow \mathbf{A}^n/S_n$ is uniformly ramified. For any fixed X it appears to be quite difficult to construct uniformly unramified morphisms. For $X = \mathbf{A}^1$, the affine line, see Theorem 7.11 and its proof.

The following technical consequence will be crucial in the proof of Theorem 4.2.

3.4. COROLLARY. *For any uniformly ramified morphism $\varphi: X \rightarrow Y$, the inclusion $\mathcal{D}(X, Y) \hookrightarrow \mathcal{D}(X)$ has a left inverse $\rho: \mathcal{D}(X) \rightarrow \mathcal{D}(X, Y)$ that is a homomorphism of $\mathcal{D}(X, Y)$ -bimodules.*

Proof. The map ρ is just the averaging operator $\frac{1}{|W|} \sum_w w$ restricted to H -invariants $\mathcal{D}(\tilde{X})^H \rightarrow \mathcal{D}(\tilde{X})^W$. \square

Now we show that uniform ramification is also necessary for $\mathcal{D}(X, Y)$ to be graded cofinite. In fact, we prove something stronger as follows.

3.5. THEOREM. *Let $\varphi: X \rightarrow Y$ be a dominant morphism between normal affine varieties. Assume $\mathcal{D}(X)$ contains a graded cofinite subalgebra \mathcal{A} with base $\mathcal{O}(Y)$. Then φ is uniformly ramified.*

Proof. That φ is finite follows from Proposition 2.1. Suppose that φ is not uniformly ramified and let $D \subseteq Y$ be a prime divisor over which φ has nonuniform ramification. Choose $\tilde{Y} \rightarrow Y$ étale and a prime divisor $\tilde{D} \subset \tilde{Y}$ that maps to D as in the Remark at the end of Section 2. Then $\hat{\mathcal{A}} = \hat{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{A}$ is graded cofinite in $\mathcal{D}_c(\hat{B}) = \mathcal{D}_c(\hat{B}_1) \times \cdots \times \mathcal{D}_c(\hat{B}_s)$. This means that $\hat{\mathcal{A}}$ is actually graded cofinite in each of the algebras $\mathcal{D}_c(\hat{B}_i)$.

We have $\hat{\mathcal{A}} = E[[t]]$ and $\hat{B}_i = E[[t^{1/n_i}]]$. Let u_2, \dots, u_m be a transcendence basis of E and put $u_1 := t$. Let ∂_i be the associated partial derivatives of $\hat{\mathcal{A}}$; their symbols are denoted by η_i , with the special notation $\tau := \eta_1$. Then $\bar{\mathcal{D}}_c(\hat{\mathcal{A}}) = E[[t]][\tau, S]$ with $S = \{\eta_2, \dots, \eta_m\}$. Much as in the Example following Theorem 3.1, we have $\bar{\mathcal{D}}_c(\hat{B}_i, \hat{\mathcal{A}}) = E[[t]][t\tau, t^{n_i-1}\tau^{n_i}, S]$ (see diagram (3.6)).

Because φ is nonuniformly ramified over D , the n_i are not all equal. Hence, after relabeling we may assume $n_1 < n_2$. Then we see that $\bar{\mathcal{D}}_c(\hat{B}_2, \hat{\mathcal{A}}) \subseteq \bar{\mathcal{D}}_c(\hat{B}_1, \hat{\mathcal{A}})$ (considered as subrings of $\bar{\mathcal{D}}_c(\hat{\mathcal{A}})$). Since $\tilde{\mathcal{A}} \subseteq \bar{\mathcal{D}}_c(\hat{B}_1, \hat{\mathcal{A}}) \cap \bar{\mathcal{D}}_c(\hat{B}_2, \hat{\mathcal{A}})$ is cofinite in $\bar{\mathcal{D}}_c(\hat{B})$, we conclude that $\bar{\mathcal{D}}_c(\hat{B}_2, \hat{\mathcal{A}})$ is cofinite in $\bar{\mathcal{D}}_c(\hat{B}_1)$. Now put $x := t^{1/n_1}$ and let ξ be the symbol of ∂_x . Then $\xi = n_1 x^{n_1-1} \tau$ implies

$$n_1^{n_2} t^{n_2-1} \tau^{n_2} = x^{(n_2-1)n_1+n_2(1-n_1)} \xi^{n_2} = x^{n_2-n_1} \xi^{n_2} \quad (3.8)$$

and therefore

$$\begin{aligned} \bar{\mathcal{D}}_c(\hat{B}_2, \hat{\mathcal{A}}) &= E[[t]][t\tau, t^{n_2-1}\tau^{n_2}, S] \\ &= E[[x^{n_2}][x\xi, x^{n_2-n_1}\xi^{n_2}, S] \subseteq E[[x]][\xi, S] = \bar{\mathcal{D}}_c(\hat{B}_1). \end{aligned} \quad (3.9)$$

Now put $x = 0$. Then $\bar{\mathcal{D}}_c(\hat{B}_2, \hat{\mathcal{A}})$ becomes $E[S]$, which is clearly not cofinite in $E[\xi, S]$. \square

DEFINITION. The affine variety X is called \mathcal{D} -finite if $\bar{\mathcal{D}}(X)$ is a finitely generated \mathbb{C} -algebra.

All smooth varieties are \mathcal{D} -finite. The cubic $x^3 + y^3 + z^3 = 0$ is the standard example of a variety that is not \mathcal{D} -finite (see [BGG]).

3.6. COROLLARY. *Let $\varphi: X \rightarrow Y$ be a dominant morphism between normal affine varieties, and assume X to be \mathcal{D} -finite. Then the following are equivalent:*

- (i) φ is uniformly ramified;
- (ii) $\mathcal{D}(X, Y)$ is graded cofinite in $\mathcal{D}(X)$.

Proof. If φ is uniformly ramified then $\bar{\mathcal{D}}(X) \subseteq \bar{\mathcal{D}}(\tilde{X})$ is integral over $\bar{\mathcal{D}}(X, Y) = \bar{\mathcal{D}}(\tilde{X})^W$ (notation as in Proposition 3.3). Since $\bar{\mathcal{D}}(X)$ is finitely generated, it is even finite over $\bar{\mathcal{D}}(X, Y)$. The converse follows from Theorem 3.5. \square

4. Simplicity

In this section we derive a simplicity criterion for the ring $\mathcal{D}(X, Y)$.

DEFINITION. A \mathcal{D} -finite affine variety X is called \mathcal{D} -simple if $\mathcal{D}(X)$ is a simple ring.

It is well known that smooth varieties are \mathcal{D} -simple. A curve is \mathcal{D} -simple if and only if its normalization map is bijective (see [SmSt]). Further examples include quotients X/W of smooth varieties by finite groups. More generally, Schwarz [S3] conjectured that any categorical quotient $X//G := \text{Spec } \mathcal{O}(X)^G$ is \mathcal{D} -finite, where G is a reductive group and X is a smooth G -variety. This has been confirmed in many cases [S1; S2; V2]. It should be added that \mathcal{D} -simple varieties are automatically Cohen–Macaulay (see [V1]). In particular, for a \mathcal{D} -simple variety, normality is equivalent to smoothness in codimension 1.

4.1. LEMMA. *For an affine variety Y , let $I \subseteq \mathcal{D}(Y)$ be a nonzero subspace with $[\mathcal{O}(Y), I] \subseteq I$. Then $I \cap \mathcal{O}(Y) \neq 0$.*

Proof. Let $0 \neq D \in I$ be of minimal order. From minimality and $[\mathcal{O}(Y), D] \in I$ we obtain $[\mathcal{O}(Y), D] = 0$; that is, $D \in \text{End}_{\mathcal{O}(Y)} \mathcal{O}(Y) = \mathcal{O}(Y)$. \square

4.2. THEOREM. *Let $X \rightarrow Y$ be a uniformly ramified morphism between normal affine varieties.*

- (i) X is \mathcal{D} -finite if and only if $\bar{\mathcal{D}}(X, Y)$ is finitely generated.
- (ii) If X is \mathcal{D} -simple then $\mathcal{D}(X, Y)$ is simple.
- (iii) If $\mathcal{D}(X, Y)$ is simple then $\mathcal{D}(X)$ and $\mathcal{D}(Y)$ are simple.

Proof. (i) The algebra $S := \bar{\mathcal{D}}(X)$ is integral over $R := \bar{\mathcal{D}}(X, Y)$. Thus, if S is finitely generated then R is so as well, by Lemma 2.2. Let L be the field of fractions of S . Then $L|\mathbb{C}$ is a finitely generated field extension. Thus, if R is finitely

generated then its integral closure in L is a finite R -module. This implies that S is finitely generated.

(ii) Let $I \subseteq \mathcal{D}(X, Y)$ be a nonzero two-sided ideal. By Lemma 4.1 we may choose a nonzero function $f \in I \cap \mathcal{O}(Y)$. Corollary 3.6 states that $\bar{\mathcal{D}}(X)$ is a finitely generated $\bar{\mathcal{D}}(X, Y)$ -module, which implies that $\mathcal{D}(X)$ is a finitely generated right $\mathcal{D}(X, Y)$ -module. In other words, there are operators $D_1, \dots, D_s \in \mathcal{D}(X)$ such that $\mathcal{D}(X) = \sum_i D_i \mathcal{D}(X, Y)$. Let k be an integer that is strictly larger than the order of every D_i . Then $(\text{ad } f)^k(D_i) = 0$. On the other side we have $(\text{ad } f)^k(D_i) = \sum_{v=0}^k (-1)^v \binom{k}{v} f^{k-v} D_i f^v$, hence $f^k D_i \in \mathcal{D}(X) f \subseteq \mathcal{D}(X) I$. This means that f^k annihilates the $\mathcal{D}(X)$ -module $\mathcal{D}(X)/\mathcal{D}(X)I$. The annihilator is a two-sided ideal and $\mathcal{D}(X)$ is a simple ring, so $\mathcal{D}(X) = \mathcal{D}(X)I$. Applying the retraction $\mathcal{D}(X) \rightarrow \mathcal{D}(X, Y)$ from Corollary 3.4 shows that $\mathcal{D}(X, Y) = \mathcal{D}(X, Y)I = I$.

(iii) Let Z denote either X or Y and assume that $I \neq 0$ is a two-sided ideal of $\mathcal{D}(Z)$. Lemma 4.1 implies that there is an $f \neq 0$ such that $f \in I \cap \mathcal{O}(Z)$. Since $Z \rightarrow Y$ is finite we have $L := \mathcal{O}(Z)f \cap \mathcal{O}(Y) \neq 0$. Since $L \subseteq I \cap \mathcal{D}(X, Y)$, this shows that $I \cap \mathcal{D}(X, Y) \neq 0$. Hence $1 \in I \cap \mathcal{D}(X, Y) \subseteq I$. \square

4.3. COROLLARY. *Let $X \rightarrow Y$ be a finite morphism between normal affine varieties that is unramified in codimension 1. Then X is \mathcal{D} -simple if and only if Y is.*

Proof. In this case, $\mathcal{D}(X, Y) = \mathcal{D}(Y)$. \square

Thus, one may assume in the sequel that we are always in the situation where $X \rightarrow Y$ is a quotient by a finite group.

5. The Associated Graded Algebra

The next result is the beginning of the classification of all graded cofinite subalgebras.

5.1. LEMMA. *For a field extension $L|\mathbb{C}$, let $\bar{\mathcal{A}} \subseteq L[\xi_1, \dots, \xi_n]$ be a cofinite homogeneous subalgebra. Then its base $K = L \cap \bar{\mathcal{A}}$ is a field.*

- (i) *Assume $\frac{\partial}{\partial \xi_i} \bar{\mathcal{A}} \subseteq \bar{\mathcal{A}}$ for all $i = 1, \dots, n$. Then $\bar{\mathcal{A}} = K[\xi_1, \dots, \xi_n]$.*
- (ii) *Assume $\frac{\partial}{\partial \xi_i} \bar{\mathcal{A}} \subseteq \bar{\mathcal{A}}$ just for $i = 1, \dots, n-1$. Then there exist a positive integer k and $a_1, \dots, a_n \in L$ with $a_n \neq 0$ such that $K[\xi_1 + a_1 \xi_n, \dots, \xi_{n-1} + a_{n-1} \xi_n, a_n \xi_n^k] \subseteq \bar{\mathcal{A}}$.*

Proof. Clearly, K is cofinite in L . This implies that $IL \subseteq L$ is a proper ideal whenever $I \subset K$ is a proper ideal. This forces $I = 0$ and implies that K is a field.

(i) For a multi-index $\alpha \in \mathbb{N}^n$, define $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$ and analogously ∂^α . Let $f = \sum_\alpha c_\alpha \xi^\alpha \in \bar{\mathcal{A}}$ be homogeneous. Then $\partial^\alpha(f) = \alpha! c_\alpha \in L \cap \bar{\mathcal{A}} = K$, which implies $\bar{\mathcal{A}} \subseteq K[\xi, \dots, \xi_n]$.

For the reverse inclusion it suffices to show that $\xi_i \in \bar{\mathcal{A}}$ for all i . Let S be the intersection of $\bar{\mathcal{A}}$ with $\langle \xi_1, \dots, \xi_n \rangle_K$. Then, after a linear change of coordinates, we may assume $S = \langle \xi_1, \dots, \xi_m \rangle_K$ for some $m \leq n$. Since $\bar{\mathcal{A}}$ is cofinite, there is a

homogeneous $f \in \bar{\mathcal{A}}$ such that the variable ξ_n occurs in f . Assume that the monomial ξ^α occurs in f with $\alpha_n > 0$. Put $\beta = \alpha - e_n$, where $e_n = (0, \dots, 0, 1)$. Then $\partial^\beta(f)$ is an element of S that contains ξ_n . This implies $m \geq n$ and we are done.

(ii) Let $\pi : L[\xi_1, \dots, \xi_n] \rightarrow L[\xi_1, \dots, \xi_{n-1}]$ be the projection obtained by setting $\xi_n = 0$. Then part (i) implies that $\pi(\bar{\mathcal{A}})$ contains ξ_1, \dots, ξ_{n-1} . Thus $\bar{\mathcal{A}}$ contains elements of the form $\xi_i + a_i \xi_n$ for $i = 1, \dots, n-1$.

Now perform the coordinate change $\xi_i \mapsto \xi_i - a_i \xi_n$ for $i = 1, \dots, n-1$ and $\xi_n \mapsto \xi_n$. This is allowed because the partial derivatives $\partial/\partial \xi_i$ ($i = 1, \dots, n-1$) stay unchanged. So we may assume that $a_1 = \dots = a_{n-1} = 0$.

Since $\bar{\mathcal{A}}$ is cofinite, there is an element $f = \sum_\alpha c_\alpha \xi^\alpha$ that contains the variable ξ_n ; that is, $c_\alpha \neq 0$ and $\alpha_n > 0$ for some multi-index α . Assume that $k := \alpha_n > 0$ is as small as possible and put $\beta := (\alpha_1, \dots, \alpha_{n-1}, 0)$. Then $g = \partial^\beta(f) \in \bar{\mathcal{A}}$ is of the form $g = a_n \xi^k + h(\xi_1, \dots, \xi_{n-1})$. Moreover, each coefficient of h appears as a derivative $\partial^\gamma(g)$ for a convenient multi-index γ with $\gamma_n = 0$. This implies $h \in K[\xi_1, \dots, \xi_{n-1}] \subseteq \bar{\mathcal{A}}$ and we are done. \square

6. Automorphisms

For a normal affine variety X , let $\Omega(X)$ be the module of Kähler differentials. Then

$$\mathcal{T}(X) = \text{Hom}_{\mathcal{O}(X)}(\Omega(X), \mathcal{O}(X)) \quad (6.1)$$

is the module of vector fields, and we have a canonical isomorphism

$$\mathcal{D}(X)_{\leq 1} = \mathcal{O}(X) \oplus \mathcal{T}(X). \quad (6.2)$$

Let $\bar{\Omega}(X) := \text{Hom}_{\mathcal{O}(X)}(\mathcal{T}(X), \mathcal{O}(X))$, the double dual of $\Omega(X)$. Since X is normal, elements of $\bar{\Omega}(X)$ can be characterized as those rational 1-forms on X that are regular in codimension 1 (or, equivalently, on the smooth part X^s of X). Let $Z(X)$ be the set of $\omega \in \bar{\Omega}(X)$ with $d\omega|_{X^s} = 0$. Our interest in $Z(X)$ comes from the following well-known lemma.

6.1. LEMMA. *For every $\omega \in Z(X)$ there exists a unique automorphism Φ_ω of $\mathcal{D}(X)$, with $\Phi_\omega(f) = f$ for all $f \in \mathcal{O}(X)$ and $\Phi_\omega(\xi) = \xi + \omega(\xi)$ for all $\xi \in \mathcal{T}(X)$. This automorphism induces the identity on $\bar{\mathcal{D}}(X)$.*

Proof. First assume X to be smooth. Then $\mathcal{D}(X)$ is generated by $\mathcal{O}(X) \cup \mathcal{T}(X)$ subject to the relations

$$\xi f - f \xi = \xi(f) \quad \text{and} \quad \xi \eta - \eta \xi = [\xi, \eta]. \quad (6.3)$$

The first relation is clearly satisfied by Φ_ω . The second relation is preserved because of Cartan's formula,

$$0 = d\omega(\xi, \eta) = \omega([\xi, \eta]) - \xi(\omega(\eta)) + \eta(\omega(\xi)). \quad (6.4)$$

This shows that Φ_ω exists. Clearly, it is the identity on $\bar{\mathcal{D}}(X)$.

In general, we have shown that $\Phi_\omega(D)$ is a differential operator on the smooth part of X . By normality, it is regular on all of X and still induces the identity on $\bar{\mathcal{D}}(X)$. \square

Let $\varphi : X \rightarrow Y$ be a finite morphism of normal varieties. Subsequently, we want to study the twists $\mathcal{A} = \Phi_\omega \mathcal{D}(X, Y)$ with $\omega \in Z(X)$. It is obvious that \mathcal{A} does not determine ω because $\Phi_\omega \mathcal{D}(X, Y) = \mathcal{D}(X, Y)$ if $\omega \in Z(Y)$. To pin down a unique ω we consider the trace map $\text{tr}_{L|K} : L \rightarrow K$, where K and L are the function fields $\mathbb{C}(Y)$ and $\mathbb{C}(X)$. This map induces a trace map $\Omega(L) \rightarrow \Omega(K)$ that is characterized by the property

$$\text{tr}_{L|K}(f\varphi^*\omega) = \text{tr}_{L|K}(f)\omega, \quad f \in L, \quad \omega \in \Omega(K). \quad (6.5)$$

It commutes with the derivative d and splits, up to the factor $[L : K]$, the inclusion $\Omega(K) \hookrightarrow \Omega(L)$. We define $Z_K(L)$ as the set of $\omega \in \Omega(L)$ with $d\omega = 0$ and $\text{tr}_{L|K} \omega = 0$.

Recall the following property of the trace: let $\partial_K : K \rightarrow K$ be a derivation and $\partial_L : L \rightarrow L$ its unique extension to L . Then

$$\text{tr}_{L|K}(\partial_L f) = \partial_K \text{tr}_{L|K}(f), \quad f \in L. \quad (6.6)$$

Indeed, we may assume that $L|K$ is Galois with group Γ . Then $\text{tr}_{L|K} f = \sum_{\gamma \in \Gamma} \gamma(f)$. Since the extension ∂_L is unique, it commutes with Γ and the claim follows.

All notions have global counterparts: there are induced trace maps $\mathcal{O}(X) \rightarrow \mathcal{O}(Y)$ and $\bar{\Omega}(X) \rightarrow \bar{\Omega}(Y)$ (see [Z]). We put $Z_Y(X) = Z_K(L) \cap \bar{\Omega}(X)$.

In the next result, we classify graded cofinite subalgebras of $\mathcal{D}(X)$ generically.

6.2. PROPOSITION. *Let X be an affine variety with quotient field $\mathbb{C}(X) = L$, and let $\mathcal{A} \subseteq \mathcal{D}(L)$ be a graded cofinite algebra with base $K = \mathcal{A} \cap L$. Then K is a field with $[L : K] < \infty$. Furthermore, there is a unique $\omega \in Z_K(L)$ with $\mathcal{A} = \Phi_\omega \mathcal{D}(K)$.*

Proof. That $K \subseteq L$ is a cofinite subfield is proved in the same way as in Lemma 5.1. Let $u_1, \dots, u_n \in K$ be a transcendence basis. Then there are unique derivations $\partial_1, \dots, \partial_n$ of L (or K) with $\partial_i(u_j) = \delta_{ij}$. Moreover, these derivations together with L generate the ring $\mathcal{D}(L)$. Let ξ_i be the symbol of ∂_i . Then we have

$$K \subseteq \bar{\mathcal{A}} \subseteq \bar{\mathcal{D}}(L) = L[\xi_1, \dots, \xi_n]. \quad (6.7)$$

Observe that $\bar{\mathcal{D}}(L)$ is a Poisson algebra and $\bar{\mathcal{A}}$ is a sub-Poisson algebra. We have $\{f, u_i\} = \partial f / \partial \xi_i$, which means that $\bar{\mathcal{A}}$ is stable under the operators $\partial / \partial \xi_i$. Lemma 5.1 implies $\bar{\mathcal{A}} = K[\xi_1, \dots, \xi_n]$. This means in particular that \mathcal{A} contains elements of the form $\delta_i := \partial_i + b_i$ with $b_i \in L$. We may replace b_i by the unique element of $b_i + K$ with trace zero. If $\omega := b_1 du_1 + \dots + b_n du_n$, then $\text{tr}_{L|K} \omega = 0$.

Observe that $a_{ij} := [\delta_i, \delta_j] = \partial_i(b_j) - \partial_j(b_i) \in \mathcal{A} \cap L = K$. From $\text{tr}_{L|K} a_{ij} = 0$ (see (6.6)) we deduce $a_{ij} = 0$. This means $d\omega = 0$ and therefore $\omega \in Z_K(L)$. Since $\delta_i = \Phi_\omega(\partial_i)$ it follows that $\Phi_\omega \mathcal{D}(K) \subseteq \mathcal{A}$, and from $\bar{\mathcal{A}} = \bar{\mathcal{D}}(K)$ we obtain $\mathcal{A} = \Phi_\omega \mathcal{D}(K)$. \square

The 1-form ω from Proposition 6.2 may have poles. Our goal is to show that this won't happen if ω comes from a graded cofinite subalgebra \mathcal{A} of $\mathcal{D}(X)$. First we present a very local version of this result.

6.3. LEMMA. *Let E be a finitely generated field extension of \mathbb{C} and put $B = E[[x]]$ and $A = E[[t]] \subseteq B$, with $t = x^p$ for some integer $p \geq 1$. Let $K = E((t))$ and $L = E((x))$ be the fields of fractions of A and B . For $\omega \in Z(L)$ assume that $\text{tr}_{L|K} \omega$ is regular at $t = 0$ and that $\mathcal{A} = \mathcal{D}_c(B) \cap \Phi_\omega \mathcal{D}_c(K)$ is graded cofinite in $\mathcal{D}_c(B)$. Then ω is regular at $x = 0$.*

Proof. If $p = 1$ then $\omega = \text{tr}_{L|K} \omega$ is regular. Assume $p \geq 2$ from now on. Let u_1, \dots, u_{n-1} be a transcendence basis of E and put $u_n = x$. Let $\partial_1, \dots, \partial_n$ be the corresponding differentials of B . Put $b_i := \omega(\partial_i) \in L$. Then we need only show that $b_i \in B$ for all i . We have

$$\omega = \sum_{i=1}^{n-1} b_i du_i + b_n dt^{1/p} = \sum_{i=1}^{n-1} b_i du_i + \frac{1}{p} b_n x t^{-1} dt. \quad (6.8)$$

Hence the condition that $\text{tr}_{L|K} \omega$ is regular means

$$\text{tr}_{L|K} b_1, \dots, \text{tr}_{L|K} b_{n-1}, t^{-1} \text{tr}_{L|K} x b_n \in A. \quad (6.9)$$

Note also the explicit formula

$$\text{tr}_{L|K} x^d = \begin{cases} px^d = pt^{d/p} & \text{if } p|d, \\ 0 & \text{otherwise.} \end{cases} \quad (6.10)$$

Let $\xi_i \in \bar{\mathcal{D}}_c(B)$ be the symbol of ∂_i . Then $\bar{\mathcal{A}} \subseteq \bar{\mathcal{D}}_c(B) = B[\xi_1, \dots, \xi_n]$ is cofinite. Since $A \subseteq \bar{\mathcal{A}}$ we have $u_1, \dots, u_{n-1} \in \bar{\mathcal{A}}$. As in the proof of Proposition 6.2, this implies that $\bar{\mathcal{A}}$ is stable under partial differentiation by ξ_i , $i = 1, \dots, n-1$. Let $\bar{\mathcal{A}}'$ be the image of $\bar{\mathcal{A}}$ in $B/xB = E[\xi_1, \dots, \xi_n]$. Then Lemma 5.1(ii) applied to $\bar{\mathcal{A}}'$ yields elements $a_{ij} \in \delta_{ij} + xB$ ($i, j = 1, \dots, n-1$) and $c_i \in B$ ($i = 1, \dots, n-1$) such that

$$\sum_{j=1}^{n-1} a_{ij} \xi_j + c_i \xi_n \in \bar{\mathcal{A}} \quad \text{for } i = 1, \dots, n-1. \quad (6.11)$$

From $\mathcal{A} \subseteq \Phi_\omega \mathcal{D}_c(K)$ we infer (since $\partial_n = px^{p-1} \partial_t$)

$$\bar{\mathcal{A}} \subseteq \bar{\mathcal{D}}_c(K) = K[\xi_1, \dots, \xi_{n-1}, x^{1-p} \xi_n]. \quad (6.12)$$

This implies in particular that $a_{ij} \in K \cap B = A$. Since $A \subseteq \bar{\mathcal{A}}$ and since the matrix $(a_{ij}) \in M_{n-1}(A)$ is invertible, we may assume $a_{ij} = \delta_{ij}$; that is,

$$\xi_1 + c_1 \xi_n, \dots, \xi_{n-1} + c_{n-1} \xi_n \in \bar{\mathcal{A}} \quad \text{with } c_1, \dots, c_{n-1} \in x^{1-p} K \cap B = xA. \quad (6.13)$$

In the last equation we used $p \geq 2$. Lifting to \mathcal{A} , we obtain operators

$$\delta_i := \partial_i + c_i \partial_n + d_i \in \mathcal{A} \quad \text{with } c_i \in xA, d_i \in B \quad (i = 1, \dots, n-1). \quad (6.14)$$

Now we use $\Phi_{-\omega}(\mathcal{A}) \subseteq \mathcal{D}_c(K)$. More precisely, from

$$\Phi_{-\omega}(\delta_i) = (\partial_i - b_i) + c_i(\partial_n - b_n) + d_i \quad (6.15)$$

we get $d_i - b_i - c_i b_n \in K$. Therefore, $b_i \in K + B + A(xb_n)$. From (6.9) we obtain

$$b_i \in B + Ax b_n, \quad i = 1, \dots, n-1. \quad (6.16)$$

Now we use that Lemma 5.1(ii) gives us also an element of $\bar{\mathcal{A}}$ of the form $a\xi_n^k \bmod x$ with $a \in B^\times$. From (6.12) it then follows that

$$a \in (x^{1-p})^k K \cap B. \quad (6.17)$$

Since a has a nonzero constant term, this is possible only if p divides k . Then $a \in K \cap B^\times = A^\times$, so we can make $a = 1$. Summarizing, we have found an operator D in \mathcal{A} of the form

$$D = \partial_n^k + (u_1 \partial_1 + \cdots + u_{n-1} \partial_{n-1} + u) \partial_n^{k-1} + \cdots \\ \text{with } u_i \in xB, u \in B \text{ and } p|k. \quad (6.18)$$

As before, we want to use that $\Phi_{-\omega}(D) \in \mathcal{D}_c(K)$; more precisely, we want to look at the coefficient of ∂_t^{k-1} . Write $\partial_n = f \partial_t$ with $f = px^{p-1} = pt^{1-1/p}$. Using the easily verified formulas

$$\partial_n^k = (f \partial_t)^k = f^k \partial_t^k + \alpha_{k,p} f^{k-1} x^{-1} \partial_t^{k-1} + \cdots \\ \text{with } \alpha_{k,p} = (p-1) \binom{k}{2}, \quad (6.19)$$

$$\partial_x^{k-1} = f^{k-1} \partial_t^{k-1} + \cdots, \quad (6.20)$$

$$(\partial_n - b_n)^k = f^k \partial_t^k + f^{k-1} (\alpha_{k,p} x^{-1} - kb_n) \partial_t^{k-1} + \cdots, \quad (6.21)$$

the coefficient of ∂_t^{k-1} in $\Phi_{-\omega}(D)$ can be computed:

$$f^{k-1} (\alpha_{k,p} x^{-1} - kb_n - u_1 b_1 - \cdots - u_{n-1} b_{n-1} + u) \in K. \quad (6.22)$$

From (6.16) we get elements $v \in B$ with $w \in B^\times$ such that

$$f^{k-1} (\alpha_{k,p} x^{-1} - wb_n + v) \in K. \quad (6.23)$$

Since $(p-1)(k-1) \equiv 1 \pmod p$, we have $x/f^{k-1} \in K$. This implies

$$wxb_n \in K + B = K + xB = E((x^p)) + xE[[x]]. \quad (6.24)$$

Let $d \in \mathbb{Z}$ be the order of zero of $x b_n$ or, equivalently, wxb_n . If $d \leq 0$ then (6.24) implies $p|d$. On the other hand, $\text{tr}(x b_n) \in tE[[t]]$ (see (6.9)) means that $x b_n$ contains no monomials x^d with $p|d$ and $d \leq 0$. Therefore $d > 0$; that is, $b_n \in B$. Finally, (6.16) implies that the other b_i are in B and we are done. \square

The next statement is similar but much easier to prove.

6.4. LEMMA. *Let $B = E[[x]]$ with quotient field $L = E((x))$ and let $\omega \in Z(L)$. Assume that $\mathcal{A} = \mathcal{D}_c(B) \cap \Phi_\omega \mathcal{D}_c(B)$ is cofinite in $\mathcal{D}_c(B)$. Then $\omega \in Z(B)$.*

Proof. The base of \mathcal{A} is $E[[x]]$. Thus we have $u_1, \dots, u_{n-1}, u_n = x \in \mathcal{A}$ and we can immediately apply Lemma 5.1(i). Hence $\bar{\mathcal{A}} \bmod x = E[\xi_1, \dots, \xi_n]$. The Nakayama lemma implies $\bar{\mathcal{A}} = \bar{\mathcal{D}}_c(E[[x]])$, so $\mathcal{A} = \mathcal{D}_c(E[[x]])$. In particular, $\Phi_\omega(\partial_i) = \partial_i + \omega(\partial_i) \in \mathcal{D}(E[[x]])$ means that ω is regular. \square

Now we globalize these local computations.

6.5. THEOREM. *Let $\varphi: X \rightarrow Y$ be a finite dominant morphism between normal varieties, and let L and K be the fields of rational functions of X and Y , respectively. For $\omega \in Z_K(L)$, assume that $\mathcal{A} = \mathcal{D}(X) \cap \Phi_\omega \mathcal{D}(K)$ is graded cofinite in $\mathcal{D}(X)$. Then $\omega \in Z_Y(X)$; that is, ω is regular on all of X .*

Proof. Because X is normal, it suffices to prove the regularity of ω in codimension 1. Let $D \subseteq Y$ be a prime divisor and choose $\tilde{D} \subseteq \tilde{Y} \rightarrow Y$ as in the Remark that concludes Section 2. Theorem 3.5 implies that φ is uniformly ramified. Therefore, the rings \hat{B}_i are all the same—say, equal to $E[[x]]$ with $x^p = t$. The form ω gives rise to forms ω_i over $\hat{B}_i[x^{-1}] \cong E((x))$. From $\mathcal{A} \subseteq \mathcal{D}(X) \cap \Phi_\omega \mathcal{D}(K)$ (with $K = \mathbb{C}(Y)$) we deduce that

$$\hat{\mathcal{A}} \subseteq \mathcal{D}_c(E[[x]])^s \cap (\Phi_{\omega_1} \times \cdots \times \Phi_{\omega_s}) \Delta \mathcal{D}_c(E((t))), \quad (6.25)$$

where Δ is the diagonal embedding. As a result, $\hat{\mathcal{A}}$ is contained in the set of all $(D_1, \dots, D_s) \in \mathcal{D}_c(E[[x]])^s$ with

$$\Phi_{-\omega_1}(D_1) = \cdots = \Phi_{-\omega_s}(D_s) \in \mathcal{D}_c(E((t))). \quad (6.26)$$

Solving for D_2, \dots, D_s , we see that $\hat{\mathcal{A}}$ is contained in

$$\mathcal{D}_c(E[[x]]) \cap \Phi_{\omega_2 - \omega_1} \mathcal{D}_c(E[[x]]) \cap \cdots \cap \Phi_{\omega_s - \omega_1} \mathcal{D}_c(E[[x]]). \quad (6.27)$$

In particular, the latter algebra is graded cofinite in $\mathcal{D}(X)$, which by Lemma 6.4 implies that $\delta_i := \omega_i - \omega_1$ is regular for all i . Then

$$0 = \mathrm{tr}_{L|K} \omega = s \mathrm{tr}_{E((x))|E((t))} \omega_1 + \sum_i \mathrm{tr}_{E((x))|E((t))} \delta_i \quad (6.28)$$

implies that $\mathrm{tr}_{E((x))|E((t))} \omega_1$ is regular. Since clearly $\hat{\mathcal{A}} \subseteq \mathcal{D}_c(E[[x]]) \cap \Phi_{\omega_1} \mathcal{D}_c(E((t)))$, it follows from Lemma 6.3 that ω_1 itself is regular. \square

7. The Main Theorem and Its Applications

The main result of this paper is as follows.

7.1. THEOREM. *Let X be a normal \mathcal{D} -simple variety, and let $\mathcal{A} \subseteq \mathcal{D}(X)$ be a graded cofinite subalgebra. Then $\mathcal{A} = \Phi_\omega \mathcal{D}(X, Y)$, where $Y = \mathrm{Spec} \mathcal{A} \cap \mathcal{O}(X)$ and where $\omega \in Z_Y(X)$ is unique. The variety Y is normal and the morphism $X \rightarrow Y$ is uniformly ramified.*

Proof. Let \mathcal{A}' be the normalization of \mathcal{A} with base A' (Corollary 2.7). Put $Y' := \mathrm{Spec} A'$, and let L and K be the quotient fields of X and Y' , respectively. Theorem 3.5 implies that $X \rightarrow Y'$ is uniformly ramified. We conclude that $\mathcal{D}(X, Y')$ is simple (Theorem 4.2).

From Proposition 6.2 we get a unique $\omega \in Z_L(K)$ such that $\mathcal{A}'_K = \Phi_\omega \mathcal{D}(K)$. By Theorem 6.5, this ω is regular on all of X and so we may replace \mathcal{A} by $\Phi_{-\omega} \mathcal{A}$. By these means we obtain

$$\mathcal{A} \subseteq \mathcal{D}(X) \cap \mathcal{D}(K) = \mathcal{D}(X, Y') \subseteq \mathcal{D}(X). \quad (7.1)$$

We know that $K \otimes_A \mathcal{A} = \mathcal{D}(K) = K \otimes_A \mathcal{D}(X, Y')$. Hence, for every $D \in \mathcal{D}(X, Y')$ we have $0 \neq f \in A$ such that $fD \in \mathcal{A}$. Now (7.1) implies that $\mathcal{D}(X, Y')$ is a finitely generated \mathcal{A} -module, both left and right. Thus there is a single $f \in A$ with $f\mathcal{D}(X, Y') \subseteq \mathcal{A}$ (and where $f \neq 0$). Likewise, we have $0 \neq g \in A$ with $\mathcal{D}(X, Y')g \subseteq \mathcal{A}$. This implies that $\mathcal{D}(X, Y')gf\mathcal{D}(X, Y')$ is a nonzero two-sided ideal of $\mathcal{D}(X, Y')$ contained in \mathcal{A} . We conclude that $\mathcal{A} = \mathcal{D}(X, Y')$. From this we get $\mathcal{O}(Y) = \mathcal{O}(X) \cap \mathcal{A} = \mathcal{O}(Y')$, hence $Y = Y'$ is normal. \square

For the applications we start with a well-known cofiniteness criterion.

7.2. LEMMA. *Let $R = \bigoplus_{d=0}^{\infty} R_d$ be a finitely generated graded \mathbb{C} -algebra. Let $F \subseteq R_0$ be a subset such that R_0 is finite over $\mathbb{C}[F]$. Let $G \subseteq R_{>0}$ be a set of homogeneous elements that has the same zero set in $\text{Spec } R$ as $R_{>0}$. Then the subalgebra generated by $F \cup G$ is cofinite in R .*

Proof. Hilbert's Nullstellensatz implies that there is an $N > 0$ with $(R_{>0})^N \subseteq RG$. Since R is finitely generated, there also is an $M \geq N$ with $R_{>M} \subseteq (R_{>0})^N$. Put $S := R_{\leq M}$; this is a finitely generated R_0 -module with $R = S + RG$. Thus we have

$$R = S + RG = S + SG + RG^2 = \cdots = S + SG + \cdots + SG^{d-1} + RG^d \quad (7.2)$$

for all $d \geq 1$. Since the minimal degree of an element of G^d goes to ∞ as d goes to ∞ , we see that $R = S[G]$ and hence is a finitely generated $R_0[G]$ -module. Thus it is also a finitely generated $\mathbb{C}[F \cup G]$ -module. \square

7.3. THEOREM. *Let W be a finite group acting on the normal \mathcal{D} -simple affine variety X . Let $F \subseteq \mathcal{O}(X)^W$ and $G \subseteq \mathcal{D}(X)^W$, where*

- (i) *the normalization of $\mathbb{C}[F]$ is $\mathcal{O}(X)^W$, and*
- (ii) *the set of symbols \bar{G} of G vanishes simultaneously only on the zero section of the cotangent bundle $\text{Spec } \bar{\mathcal{D}}(X)$ of X .*

Then $\mathcal{D}(X)^W$ is, as an algebra, generated by $F \cup G$.

Proof. Let $\mathcal{A} \subseteq \mathcal{D}(X)$ be the subalgebra generated by F and G . Then F and \bar{G} meet the assumptions of Lemma 7.2, and we conclude that \mathcal{A} is graded cofinite in $\mathcal{D}(X)$.

Let A be the base of \mathcal{A} ; by Theorem 7.1, A is integrally closed. We have $\mathcal{A} \subseteq \mathcal{D}(X)^W$ and hence $\mathbb{C}[F] \subseteq A \subseteq \mathcal{O}(X)^W$, which implies $A = \mathcal{O}(X)^W$. Finally, Theorem 7.1 implies that $\bar{\mathcal{A}} = \bar{\mathcal{D}}(X, X/W) = \bar{\mathcal{D}}(X)^W$. Since $\mathcal{A} \subseteq \mathcal{D}(X)^W$ we have $\mathcal{A} = \mathcal{D}(X)^W$. \square

As mentioned in the Introduction, we obtain the following result of Levasseur and Stafford [LSt] as an application.

7.4. COROLLARY. *Let V be a finite-dimensional representation of W . Then $\mathcal{D}(V)^W$ is generated by the invariant polynomials along with the invariant constant coefficient differential operators.*

Observe that—even for vector spaces—Theorem 7.3 is more general than the Levasseur–Stafford theorem: it suffices to take invariant functions that generate the ring of invariants only up to normalization and to take invariant constant coefficient operators that generate all invariant constant coefficient operators up to integral closure. In practice, this leads to much smaller generating sets, as our next result shows.

7.5. COROLLARY. *Let V be an n -dimensional representation of W . Then $\mathcal{D}(V)^W$ can be generated by $2n + 1$ elements.*

Proof. First, choose homogeneous systems of parameters f_1, \dots, f_n and d_1, \dots, d_n of $\mathcal{O}(V)^W$ and $\mathcal{O}(V^*)^W$, respectively. Then choose a generator $f_0 \in \mathcal{O}(V)^W$ of the finite field extension $\mathbb{C}(V)^W | \mathbb{C}(f_1, \dots, f_n)$. Then $F = \{f_0, \dots, f_n\}$ and $G = \{d_1, \dots, d_n\}$ satisfy the assumptions of Theorem 7.3. \square

We need the following.

DEFINITION. Let $\mathcal{A} \subseteq \mathcal{D}(X)$ be a graded cofinite subalgebra with base A and $Y = \text{Spec } A$. Then \mathcal{A} is called *untwisted* if $\mathcal{A} = \mathcal{D}(X, Y)$.

7.6. PROPOSITION. *Let X be a normal \mathcal{D} -simple affine variety, and let $\mathcal{A} \subseteq \mathcal{A}' \subseteq \mathcal{D}(X)$ be graded cofinite subalgebras. If \mathcal{A} is untwisted, then so is \mathcal{A}' .*

Proof. Let K, K' , and L be the field of fractions of $\mathcal{A} \cap \mathcal{O}(X)$, $\mathcal{A}' \cap \mathcal{O}(X)$, and $\mathcal{O}(X)$, respectively. Moreover, let \mathcal{A}_K (resp. $\mathcal{A}'_{K'}$) be the algebra generated by \mathcal{A} and K (resp. \mathcal{A}' and K'). Choose a transcendence basis $u_1, \dots, u_n \in K$ and let $\partial_1, \dots, \partial_n$ be the derivations of K, K', L with $\partial_i(u_j) = \delta_{ij}$. If \mathcal{A} is untwisted then $\partial_i \in \mathcal{A}_K$. Thus $\partial_i \in \mathcal{A}'_{K'}$, which means that \mathcal{A}' is untwisted as well. \square

Now we derive a Galois correspondence for graded cofinite subalgebras.

7.7. THEOREM. *Let X be a normal \mathcal{D} -simple affine variety and W a finite group acting on X . Then the map $H \mapsto \mathcal{D}(X)^H$ establishes a bijective correspondence between subgroups of W and subalgebras of $\mathcal{D}(X)$ containing $\mathcal{D}(X)^W$.*

Proof. The only nontrivial thing to show is that every subalgebra \mathcal{A} containing $\mathcal{D}(X)^W$ is of the form $\mathcal{D}(X)^H$. Let A be the base of \mathcal{A} and let $Y = \text{Spec } A$. By Theorem 7.1 and Proposition 7.6, $\mathcal{A} = \mathcal{D}(X, Y)$. Since $\mathcal{O}(X)^W \subseteq A \subseteq \mathcal{O}(X)$ and since A is integrally closed, there exists an $H \subseteq W$ with $A = \mathcal{O}(X)^H$. Thus $\mathcal{A} = \mathcal{D}(X, X/H) = \mathcal{D}(X)^H$. \square

REMARK. The preceding result could have been as well derived from a noncommutative version of Galois theory due to Kharchenko. Recall that a subalgebra \mathcal{A} of $\mathcal{D}(X)$ is called an *anti-ideal* if, for any $a \in \mathcal{D}(X)$ and $b, c \in \mathcal{A} \setminus \{0\}$, $ab, ca \in \mathcal{A}$ implies $a \in \mathcal{A}$ (see e.g. [C, Sec. 6.6, p. 334]). This is a noncommutative version of integral closedness. Now Theorem 7.7 follows from Kharchenko’s Galois correspondence [C, Thm. 11.7] with the aid of our next proposition.

7.8. PROPOSITION. *Let X be a normal \mathcal{D} -simple affine variety. Then every graded cofinite subalgebra of $\mathcal{D}(X)$ is an anti-ideal.*

Proof. Let $\mathcal{A} \subseteq \mathcal{D}(X)$ be a graded cofinite subalgebra. By Theorem 7.1 we may assume that $\mathcal{A} = \mathcal{D}(X, Y)$ for some uniformly ramified morphism $X \rightarrow Y$. Now, using Proposition 3.3, we have

$$\mathcal{A} = \mathcal{D}(\tilde{X})^W \subseteq \mathcal{D}(X) \subseteq \mathcal{D}(\tilde{X}). \quad (7.3)$$

It is clear (see the proof of [C, Thm. 11.7]) that $\mathcal{D}(\tilde{X})^W$ is an anti-ideal of $\mathcal{D}(\tilde{X})$. A fortiori, it is an anti-ideal of $\mathcal{D}(X)$. \square

Here is another example of how one can play with Theorem 7.1.

7.9. THEOREM. *Let V be a finite-dimensional representation of W . Then the ring $\mathcal{D}(V \oplus V^*)^W$ is generated by*

$$\mathcal{D}(V \oplus 0)^W \cup \mathcal{D}(0 \oplus V^*)^W \cup \{\omega\}, \quad (7.4)$$

where $\omega: V \times V^* \rightarrow \mathbb{C}$ is the evaluation map.

Proof. Let \mathcal{A} be the subalgebra generated by this set. The first two pieces generate the subalgebra $\mathcal{D}(V \oplus V^*)^{W \times W}$. Theorem 7.7 implies that $\mathcal{A} = \mathcal{D}(V \oplus V^*)^H$ for some subgroup H of $W \times W$. But the isotropy group of ω inside $W \times W$ is just W embedded diagonally, which implies $H = W$. \square

One remarkable feature of the subalgebras of noncommutative rings is that they are much scarcer. An argument similar to Theorem 7.7 shows the following.

7.10. COROLLARY. *Let X be normal and \mathcal{D} -simple and let $\mathcal{A} \subseteq \mathcal{D}(X)$ be graded cofinite. Then there are only finitely many intermediate subalgebras.*

Proof. By applying an automorphism to $\mathcal{D}(X)$, we may assume \mathcal{A} to be untwisted: $\mathcal{A} = \mathcal{D}(X, Y)$. Then every intermediate algebra is also untwisted and hence of the form $\mathcal{D}(X, Y')$, with $\mathcal{O}(Y) \subseteq \mathcal{O}(Y') \subseteq \mathcal{O}(X)$ and $\mathcal{O}(Y')$ integrally closed. Galois theory tells us that there are only finitely many of these. \square

For $X = \mathbf{A}^1$, the next theorem makes things quite explicit.

7.11. THEOREM. *Let $\mathcal{A} \subseteq \mathcal{D}(\mathbf{A}^1) = \mathbb{C}\langle x, \partial_x \rangle$ be graded cofinite. Then there exist $a \in \mathbb{C}$, $p \in \mathbb{C}[x]$, and $m \in \mathbb{Z}_{>0}$ such that $\mathcal{A} = \mathbb{C}\langle u^m, \eta^m \rangle$, where $u = x - a$ and $\eta = \partial_x + p(x)$. Moreover, p may be chosen in such a way that xp does not contain monomials whose exponent is divisible by m . In that case, the triple (a, m, p) is uniquely determined by \mathcal{A} .*

Proof. Clearly we may assume that \mathcal{A} is untwisted. Then we have to determine all uniformly ramified morphism $\varphi: \mathbf{A}^1 \rightarrow Y$.

First, Y is a smooth rational curve with $\mathcal{O}(Y)^\times = \mathbb{C}^*$, which implies $Y \cong \mathbf{A}^1$. Thus, φ extends to a morphism $\varphi: \mathbf{P}^1 \rightarrow \mathbf{P}^1$ such that $\varphi^{-1}(\infty) = \infty$. Let d be the degree of φ . Assume φ is ramified over the points $y_1, \dots, y_s \in Y$ with ramification

numbers $r_1, \dots, r_s \geq 2$; then $\varphi^{-1}(y_i)$ will consist of d/r_i points. The ramification number at ∞ is d , so Hurwitz's formula implies that

$$-2 = -2d + \sum_{i=1}^s \frac{d}{r_i}(r_i - 1) + (d - 1) = (s - 1)d - 1 - \sum_{i=1}^s \frac{d}{r_i}. \quad (7.5)$$

From $r_i \geq 2$ we obtain

$$-1 = (s - 1)d - \sum_{i=1}^s \frac{d}{r_i} \geq (s - 1)d - s \frac{d}{2} = \left(\frac{s}{2} - 1\right)d. \quad (7.6)$$

This implies that $s = 0$ and $d = 1$ (i.e., φ is an isomorphism) or that $s = 1$ and $d = r_1$. In the latter case, φ is (up to a translation) just the quotient $\mathbf{A}^1 \rightarrow \mathbf{A}^1/\mu_d$. \square

FINAL REMARK. The stipulation that our subalgebras are *graded* cofinite in $\mathcal{D}(X)$ is essential. It would be interesting to classify all subalgebras \mathcal{A} for which $\mathcal{D}(X)$ itself is a finitely generated left and right \mathcal{A} -module. Take, for example, the affine space $X = \mathbf{A}^n$. Here $\mathcal{D}(X)$ is the Weyl algebra on which the symplectic group $\mathrm{Sp}_{2n}(\mathbb{C})$ acts by automorphisms. Now take any irreducible $2n$ -dimensional representation of a finite group W that preserves a symplectic form. Then $\mathcal{A} = \mathcal{D}(\mathbf{A}^n)^W$ will have the required property even though it is not graded cofinite. The point, of course, is that the W -action does not preserve the standard filtration. Nevertheless, it preserves the so-called Bernstein filtration, for which linear functions have degree 1. One might therefore want to start with the problem: What are the subalgebras of a Weyl algebra that are graded cofinite with respect to the Bernstein filtration?

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