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# Normality and Shared Functions of Holomorphic Functions and Their Derivatives

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## 1. Introduction

Let $D$ be a domain in $\mathbb{C}$ and let $\mathcal{F}$ be a family of meromorphic functions defined in $D$. The family $\mathcal{F}$ is said to be *normal* in $D$, in the sense of Montel, if each sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence $\{f_{n_j}\}$ that converges, spherically locally uniformly in $D$, to a meromorphic function or to $\infty$ (see [7; 12; 14]).

Let $f$ and $g$ be meromorphic functions in a domain $D$ in $\mathbb{C}$, and let $a$ and $b$ be complex numbers. If $g(z) = b$ whenever $f(z) = a$, we write $f(z) = a \Rightarrow g(z) = b$. If $f(z) = a \Rightarrow g(z) = b$ and $g(z) = b \Rightarrow f(z) = a$, we write $f(z) = a \Leftrightarrow g(z) = b$. If $f(z) = a \Leftrightarrow g(z) = a$ then we say that $f$ and $g$ *share* $a$ in $D$.

Schwick [13] was the first to draw a connection between values shared by functions in $\mathcal{F}$ (and their derivatives) and the normality of the family $\mathcal{F}$. Specifically, he showed that if there exist three distinct complex numbers $a_1, a_2, a_3$ such that $f$ and $f'$ share $a_j$ $(j = 1,2,3)$ in $D$ for each $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in $D$. Pang and Zalcman [10] extended this result as follows.

Theorem A. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D$, and let $a, b, c, d$ be complex numbers such that $c \neq a$ and $d \neq b$. If for each $f \in \mathcal{F}$ we have $f(z) = a \Leftrightarrow f'(z) = b$ and $f(z) = c \Leftrightarrow f'(z) = d$, then $\mathcal{F}$ is normal in $D$.*

Chen and Hua proved the following.

Theorem B ([4], cf. [5; 9]). *Let $\mathcal{F}$ be a family of holomorphic functions in a domain $D$, and let $a$ $(\neq 0)$ be a finite complex value. If, $f$, $f'$, and $f''$ share $a$ in $D$ for each $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in $D$.*

In this paper, we extend Theorem B as follows.

Theorem 1. *Let $\mathcal{F}$ be a family of holomorphic functions in a domain $D$, and let $a(z)$ be an analytic function in $D$ such that $a' \not\equiv a$. If, for each $f \in \mathcal{F}$, $f(z) = a(z) \Leftrightarrow f'(z) = a(z) \Leftrightarrow f''(z) = a(z)$ and $f(z) - a(z) = 0 \to f'(z) - a(z) = 0$ in $D$, then $\mathcal{F}$ is normal in $D$.*

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Here $f(z)-a(z)=0 \to f'(z)-a(z)=0$ means: if $z_0$ is a zero of $f(z)-a(z)$ with multiplicity $n$, then $z_0$ is a zero of $f'(z)-a(z)$ with multiplicity at least $n$.

Theorem B is an instant corollary of Theorem 1, which yields also our next result.

COROLLARY 1. *Let $\mathcal{F}$ be a family of holomorphic functions in a domain $D$. If, for each $f \in \mathcal{F}$, $f, f', f''$ have the same fixed points in $D$, then $\mathcal{F}$ is normal in $D$.*

The following two examples show that the conditions $a' \not\equiv a$ and $f(z)-a(z) = 0 \to f'(z)-a(z)=0$ in Theorem 1 are necessary.

EXAMPLE 1. Let $D=\{z : |z|<1\}$ and $a(z)=ce^z$ for $c$ a finite value. Let $\mathcal{F} = \{f_n\}$, where

$$f_n(z) = e^{nz} + ce^z.$$

Then, for any $f \in \mathcal{F}$, it is easy to see that $f(z)-a(z) \neq 0$, $f'(z)-a(z) \neq 0$, and $f''(z)-a(z) \neq 0$. But $\mathcal{F}$ is not normal in $D$.

EXAMPLE 2. Let $D=\{z : |z|<1\}$, $a(z)=z^2+2z+2$, and $\mathcal{F}=\{f_n : n = 2,3,\dots\}$, where

$$f_n(z) = nz^3+z^2+2z+2.$$

Then, for any $f_n(z) = nz^3+z^2+2z+2 \in \mathcal{F}$,

$$f_n(z)-a(z) = nz^3,$$
$$f_n'(z)-a(z) = (3n-1)z^2,$$
$$f_n''(z)-a(z) = (6n-2-z)z.$$

Thus $f_n(z)-a(z)$, $f_n'(z)-a(z)$, and $f_n''(z)-a(z)$ have the same zeros in $D$. But $\mathcal{F}$ is not normal in $D$.

The following example shows that there are normal families that do not satisfy the conditions of Theorem B yet do satisfy the conditions of our results.

EXAMPLE 3. Let $D=\{z \in \mathbb{C} : \mathrm{Re}(z) > -3/2\}$ and $\mathcal{F}=\{f_n : n=1,2,3,\dots\}$, where

$$f_n(z) = \frac{i}{2}nz^2 + (n^2+ni)z + n^2 - \frac{i}{2}(n^3-2n).$$

(Here, as usual, $i=\sqrt{-1}$.) Then $\mathcal{F}$ is normal in $D$. In fact, $f_n \to \infty$ locally uniformly in $D$ as $n \to \infty$. We may compute

$$f_n(z)-z = \frac{i}{2}n(z-ni)\left[z - \left(-2+ni-\frac{2i}{n}\right)\right],$$
$$f_n'(z)-z = (-1+ni)(z-ni),$$
$$f_n''(z)-z = -(z-ni).$$

It follows that $f_n(z)$, $f_n'(z)$, $f_n''(z)$ have the same fixed points in $D$, so the functions $f_n$ satisfy the conditions of Corollary 1.

However, there does not exist a number $a \in \mathbb{C}$ such that $f_n, f_n', f_n''$ share $a$ in $D$. Let $a = x_0 + y_0 i$. Then, for sufficiently large $n$, $f_n''(z) = ni \neq a$, but $z_n = -1 + y_0/n + (n - x_0/n)i \in D$ and so $f_n'(z_n) = a$. Thus the functions $f_n$ do not satisfy the conditions of Theorem B.

In order to prove Theorem 1, we need the following results.

PROPOSITION 1. *Let $\mathcal{F}$ be a family of holomorphic functions in a domain $D$, and let $A(z) \neq 0$ be a zero-free analytic function in $D$. If for each $f \in \mathcal{F}$ we have $f(z) = 0 \Rightarrow f'(z) = A(z)$ and $f'(z) = A(z) \Rightarrow f''(z) = A(z) + A'(z)$ in $D$, then $\mathcal{F}$ is normal in $D$.*

PROPOSITION 2. *Let $\mathcal{F}$ be a family of holomorphic functions in a domain $D$, and let $A(z) \not\equiv 0$ be an analytic function in $D$ that is not equal to zero identically. If, for each $f \in \mathcal{F}$, we have $A(z) = 0 \Rightarrow f(z) = 0$, $f(z) = 0 \Leftrightarrow f'(z) = A(z)$, and $f'(z) = A(z) \Leftrightarrow f''(z) = A(z) + A'(z)$ and also $f(z) = 0 \to f'(z) = A(z)$, then $\mathcal{F}$ is normal in $D$.*

PROPOSITION 3. *Let $\mathcal{F}$ be a family of holomorphic functions in a domain $D$, and let $A(z) \not\equiv 0$ be an analytic function in $D$ that is not equal to zero identically. If, for each $f \in \mathcal{F}$, we have $A(z) = 0 \Rightarrow f(z) \neq 0$ and $f(z) = 0 \Leftrightarrow f'(z) = A(z)$ and $f'(z) = A(z) \Rightarrow f''(z) = A(z) + A'(z)$, then $\mathcal{F}$ is normal in $D$.*

## 2. Some Lemmas

Let $f$ be a nonconstant meromorphic function in $D_R = \{z : |z| < R\}$ $(R \leq \infty)$. Throughout this paper we use the basic results and notation of Nevanlinna theory, such as $T(r, f)$, $m(r, f)$, $N(r, f)$, ... (cf. [6; 7; 12; 14]). In particular, $S(r, f)$ denotes any function satisfying

$$S(r, f) = o\{T(r, f)\}$$

as $r \to +\infty$ and possibly outside of a set of finite linear measure, where $T(r, f)$ is Nevanlinna's characteristic function. As usual, the order $\rho(f)$ of $f$ is defined as

$$\rho(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}.$$

In order to prove our theorems, we require the following results.

LEMMA 1 ([11, Lemma 2]; cf. [15, p. 217]). *Let $\mathcal{F}$ be a family of meromorphic functions in the domain $D \subset \mathbb{C}$, all of whose zeros have multiplicity at least $k$, and suppose there exists an $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. Then, if $\mathcal{F}$ is not normal at some point $z_0 \in D$, for each $0 \leq \alpha \leq k$ there exist*

(a) *points $z_n \in D$, $z_n \to z_0$,*
(b) *functions $f_n \in \mathcal{F}$, and*
(c) *positive numbers $\rho_n \to 0$*

*such that* $\rho_n^{-\alpha} f_n(z_n + \rho_n \zeta) = g_n(\zeta) \to g(\zeta)$ *locally uniformly, where* $g$ *is a nonconstant meromorphic function in* $\mathbb{C}$, *all of whose zeros have multiplicity at least* $k$, *such that* $g^{\#}(\zeta) \le g^{\#}(0) = kA + 1$. *In particular, if* $\mathcal{F}$ *is a family of holomorphic functions, then* $g$ *is of exponential type.*

Here, as usual, $g^{\#}(\zeta) = |g'(\zeta)|/(1 + |g(\zeta)|^2)$ is the spherical derivative.

LEMMA 2 ([9]; cf. [3]). *Let* $g$ *be a nonconstant entire function of exponential type. If* $g(z) = 0 \Rightarrow g'(z) = 1$ *and* $g'(z) = 1 \Rightarrow g''(z) = 0$, *then* $g'(z) \equiv 1$.

LEMMA 3 [7; 14]. *Let* $f$ *be a nonconstant meromorphic function and let* $k$ *be a positive integer. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f);$$

*in particular, if* $f$ *is of finite order then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r).$$

LEMMA 4 [6, Lemma 7.1]. *Let* $\phi_1(z), \phi_2(z), \dots, \phi_n(z)$ *be* $n$ *entire functions such that* $\phi_i - \phi_j$ *is nonconstant for* $i \ne j$. *Let* $g_1(z), g_2(z), \dots, g_n(z)$ *be* $n$ *meromorphic functions of finite order such that*

$$\rho(g_i) < \min_{1 \le s < t \le n} \{\rho(e^{\phi_t - \phi_s})\}, \quad i = 1, 2, \dots, n.$$

*If*

$$\sum_{i=1}^{n} g_i(z) e^{\phi_i(z)} = 0,$$

*then*

$$g_1 = g_2 = \cdots = g_n = 0.$$

Here and in the sequel, $\rho(g)$ denotes the order of $g$.

LEMMA 5. *Let* $g$ *be an entire function whose order is at most* 1, *and let* $k$ *be a positive integer. If* $g(z) = 0 \Leftrightarrow g'(z) = z^k$ *and* $g'(z) = z^k \Leftrightarrow g''(z) = kz^{k-1}$, *then* $g(z) = cz^{k+1}$, *where* $c$ *is a nonzero constant.*

*Proof.* Set

$$\phi(z) = \frac{zg''(z) - kg'(z)}{g(z)}. \tag{2.1}$$

Now we consider two cases.

*Case 1:* $\phi \equiv 0$. Then $zg''(z) - kg'(z) = 0$ for any $z \in \mathbb{C}$. It follows that

$$g(z) = cz^{k+1} + d, \tag{2.2}$$

where $c$ and $d$ are constants. Thus by $g(z) = 0 \Leftrightarrow g'(z) = z^k$, we know that $d = 0$. Hence $g(z) = cz^{k+1}$, where $c$ is a nonzero constant.

*Case 2:* $\phi \not\equiv 0$. Then, by the conditions of the lemma, $\phi(z)$ has only one possible simple pole $z = 0$ (if $g(0) = 0$). Since $\rho(g) \le 1$, by Lemma 3 we have

$$\begin{aligned} T(r,\phi) &= m(r,\phi) + N(r,\phi) \\ &\le m(r,z) + m\left(r, \frac{g'}{g}\right) + m\left(r, \frac{g''}{g}\right) + \log r + O(1) \\ &= O(\log r). \end{aligned} \tag{2.3}$$

It follows that

$$\phi(z) = \alpha_n z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_1 z + \alpha_0 + \frac{\gamma}{z}, \tag{2.4}$$

where $\alpha_n, \ldots, \alpha_0$ and $\gamma$ are constants and where $\gamma = 0$ if $g(0) \neq 0$.

Set

$$L(z) = \frac{g(z)[g''(z) - kz^{k-1}]}{g'(z) - z^k}. \tag{2.5}$$

If $L \equiv 0$ then $g(z)[g''(z) - kz^{k-1}] \equiv 0$. Thus, by $g(z) = 0 \Leftrightarrow g''(z) - kz^{k-1} = 0$ we deduce that $g(z) \equiv 0$, which is impossible.

Hence $L \not\equiv 0$. Since $g(z) = 0 \Leftrightarrow g'(z) - z^k = 0 \Leftrightarrow g''(z) - kz^{k-1} = 0$, it follows that $L(z)$ is an entire function that has only one possible zero $z = 0$ with multiplicity $s$ if $z = 0$ is a zero of $g$ with multiplicity $s + 1$. Since $\rho(g) \le 1$, we deduce from Lemma 3 that $\rho(L) \le 1$. Thus we have

$$L(z) = az^s e^{\lambda z}, \tag{2.6}$$

where $\lambda$ and $a \neq 0$ are constants and where $s$ is a nonnegative integer.

Thus, by (2.5) and (2.6),

$$g(z)[g''(z) - kz^{k-1}] = az^s e^{\lambda z}[g'(z) - z^k]. \tag{2.7}$$

This together with (2.1) yields

$$g(z)[kg'(z) + \phi(z)g(z) - kz^k] = az^{s+1} e^{\lambda z}[g'(z) - z^k],$$

so that

$$[g'(z) - z^k][kg(z) - az^{s+1}e^{\lambda z}] = -\phi(z)[g(z)]^2. \tag{2.8}$$

It follows that $kg(z) - az^{s+1}e^{\lambda z}$ has only finitely many zeros.

Since $g$ is an entire function and since $\rho(g) \le 1$, we may assume that

$$g(z) = \frac{a}{k} z^{s+1} e^{\lambda z} + P(z)e^{\mu z}, \tag{2.9}$$

where $P(z)$ is a polynomial and $\mu$ is a constant. It is obvious that $P(z) \not\equiv 0$.

Using (2.9), we obtain

$$g'(z) = \frac{a}{k}[\lambda z^{s+1} + (s+1)z^s]e^{\lambda z} + [P'(z) + \mu P(z)]e^{\mu z}. \tag{2.10}$$

Thus by (2.8)–(2.10) and some calculation, we have

$$A_1(z)e^{2\lambda z} + A_2(z)e^{(\lambda+\mu)z} + A_3(z)e^{2\mu z} + A_4(z)e^{\mu z} = 0,$$

where

$$A_1(z) = \frac{a^2}{k^2} z^{2(s+1)} \phi(z) \not\equiv 0,$$

$$A_2(z) = \frac{2a}{k} z^{s+1} P(z)\phi(z) + a[\lambda z^{s+1} + (s+1)z^s]P(z),$$

$$A_3(z) = [P(z)]^2 \phi(z) + kP(z)[P'(z) + \mu P(z)],$$

$$A_4(z) = -kz^k P(z) \not\equiv 0.$$

Next we show that $\lambda = \mu = 0$. First, by Lemma 4, we know that at least one of $\mu$, $\lambda$, $\lambda - \mu$, and $2\lambda - \mu$ is zero. And again by Lemma 4 with $A_1 \not\equiv 0$ and $A_4 \not\equiv 0$, we see that either (a) $\lambda \neq 0$, $\mu \neq 0$, and $\lambda - \mu \neq 0$ or (b) $\lambda = \mu = 0$.

In case (a) we have $\mu = 2\lambda$. Thus, by Lemma 4 again, we know that $A_2 \equiv 0$ and $A_3 \equiv 0$.

Hence by (2.4) and $A_2 \equiv 0$ we have

$$2\alpha_n z^{s+n+1} + \cdots + 2\alpha_1 z^{s+2} + [k\lambda + 2\alpha_0]z^{s+1} + [k(s+1) + 2\gamma]z^s \equiv 0,$$

so that

$$\alpha_n = \cdots = \alpha_1 = 0, \quad \alpha_0 = -\frac{1}{2}k\lambda, \qquad \gamma = -\frac{1}{2}k(s+1) \neq 0.$$

Therefore, (2.4) allows us to obtain

$$\phi(z) = -\frac{1}{2}k\lambda - \frac{k(s+1)}{2z};$$

together with $A_3 \equiv 0$, this yields

$$\left(-\frac{1}{2}k\lambda z - \frac{k(s+1)}{2}\right)P(z) + kz[P'(z) + \mu P(z)] \equiv 0.$$

It follows that $\mu = -\lambda/2$, which together with $\mu = 2\lambda$ gives that $\lambda = \mu = 0$, a contradiction. Hence we have proved that $\lambda = \mu = 0$. Thus, by (2.9), $g(z)$ is a polynomial.

By (2.7) and $\lambda = 0$, we have

$$g(z)[g''(z) - kz^{k-1}] = az^s[g'(z) - z^k]. \tag{2.11}$$

By (2.1) and (2.4),

$$z^2 g''(z) - kzg'(z) - (\alpha_n z^{n+1} + \cdots + \alpha_0 z + \gamma)g(z) \equiv 0.$$

It follows that $\alpha_n = \cdots = \alpha_1 = \alpha_0 = 0$ and $\gamma \neq 0$, so that $g(0) = 0$. Thus, by (2.11), $z = 0$ is a zero of $g$ with multiplicity $s + 1$.

If

$$g(z) = a_0 z^l + a_1 z^m + \cdots, \tag{2.12}$$

where $a_0, a_1$ are nonzero constants and $l > m$ are nonnegative integers, then $m \geq s + 1$ and so $l \geq s + 2$. On the other hand, by (2.11) it follows that $l = s + 1$, a contradiction.

Hence $g(z) = cz^l$, where $c \neq 0$ is a constant and $l$ is a positive integer. Thus, by $g(z) = 0 \Leftrightarrow g'(z) = z^k$, it follows that $l = k + 1$. This completes the proof of Lemma 5. □

LEMMA 6. *Let $f(z)$ be analytic in the disc $\Delta = \{z : |z| < r_0\}$; let $A(z) = z^k\phi(z)$, where $k \in \mathbb{N}$ and $\phi \neq 0$ is analytic on $\bar{\Delta}$; and let $a$ be a complex number such that $|a| < r_0$. If $f(0) \neq 0$, $f(-a) \neq 0$, $(f'/\phi_a)^{(2k)}_{z=0} \neq 0$, and $L_a(0) \neq 0$ and if $f(z) = 0 \Leftrightarrow f'(z) = A_a(z)$ and $f'(z) = A_a(z) \Rightarrow f''(z) = A_a(z) + A'_a(z)$, where $A_a(z) = A(z+a)$, $\phi_a(z) = \phi(z+a)$, and*

$$L_a(z) = [A_a(z) - A'_a(z)]\frac{f'(z)}{f(z)} + A_a(z)\frac{f''(z)}{f(z)} - 2A_a(z)\frac{f''(z) - A'_a(z)}{f'(z) - A_a(z)} + A'_a(z), \tag{2.13}$$

*then for $0 < r < r_0 - |a|$ we have*

$$T(r, f) \le LD[r, f] + \log\frac{|[f'(0) - A_a(0)]f(0)|}{|[L_a(0)]^2(f'/\phi_a)^{(2k)}_{z=0}|}, \tag{2.14}$$

*where*

$$\begin{aligned} LD[r, f] = {} & 3m\left(r, \frac{f'}{f}\right) + 2m\left(r, \frac{f''}{f}\right) + 2m\left(r, \frac{f'' - A'_a}{f' - A_a}\right) \\ & + m\left(r, \frac{(f'/\phi_a)^{(k)}}{f'/\phi_a}\right) + m\left(r, \frac{(f'/\phi_a)^{(2k)}}{(f'/\phi_a)^{(k)}}\right) + m\left(r, \frac{(f'/\phi_a)^{(2k)}}{(f'/\phi_a)^{(k)} - k!}\right) \\ & + m\left(r, \frac{[(f'/\phi_a) - (z+a)^k]^{(k)}}{(f'/\phi_a) - (z+a)^k}\right) + m\left(r, \frac{(f'/\phi_a)^{(2k)}}{(f'/\phi_a) - (z+a)^k}\right) \\ & + 3m\left(r, \frac{1}{\phi_a}\right) + 6m(r, A_a) + 4m(r, A'_a) + 12\log 2. \end{aligned} \tag{2.15}$$

*Proof.* Using a standard argument in Nevanlinna's theory, we have

$$\begin{aligned} & m\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{f' - A_a}\right) \\ & \quad = m\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{f' - (z+a)^k\phi_a}\right) \\ & \quad \le m\left(r, \frac{1}{f'/\phi_a}\right) + m\left(r, \frac{1}{(f'/\phi_a) - (z+a)^k}\right) + 2m\left(r, \frac{1}{\phi_a}\right) + m\left(r, \frac{f'}{f}\right) \\ & \quad \le m\left(r, \frac{1}{(f'/\phi_a)^{(k)}}\right) + m\left(r, \frac{1}{(f'/\phi_a)^{(k)} - k!}\right) + m\left(r, \frac{f'}{f}\right) \\ & \qquad + m\left(r, \frac{(f'/\phi_a)^{(k)}}{f'/\phi_a}\right) + m\left(r, \frac{[(f'/\phi_a) - (z+a)^k]^{(k)}}{(f'/\phi_a) - (z+a)^k}\right) + 2m\left(r, \frac{1}{\phi_a}\right) \\ & \quad \le m\left(r, \frac{1}{(f'/\phi_a)^{(k)}} + \frac{1}{(f'/\phi_a)^{(k)} - k!}\right) + m\left(r, \frac{f'}{f}\right) + \log 2 + \log^+\frac{4}{k!} \\ & \qquad + m\left(r, \frac{(f'/\phi_a)^{(k)}}{f'/\phi_a}\right) + m\left(r, \frac{[(f'/\phi_a) - (z+a)^k]^{(k)}}{(f'/\phi_a) - (z+a)^k}\right) + 2m\left(r, \frac{1}{\phi_a}\right) \le \end{aligned}$$

$$\leq m\left(r, \frac{1}{(f'/\phi_a)^{(2k)}}\right) + m\left(r, \frac{(f'/\phi_a)^{(2k)}}{(f'/\phi_a)^{(k)}}\right) + m\left(r, \frac{(f'/\phi_a)^{(2k)}}{(f'/\phi_a)^{(k)} - k!}\right)$$
$$+ m\left(r, \frac{f'}{f}\right) + m\left(r, \frac{(f'/\phi_a)^{(k)}}{f'/\phi_a}\right) + m\left(r, \frac{[(f'/\phi_a) - (z+a)^k]^{(k)}}{(f'/\phi_a) - (z+a)^k}\right)$$
$$+ 2m\left(r, \frac{1}{\phi_a}\right) + 4\log 2. \tag{2.16}$$

Since $f'/\phi_a$ is holomorphic in $\Delta$, by Nevanlinna's first fundamental theorem it follows that

$$\begin{aligned} m\left(r, \frac{1}{(f'/\phi_a)^{(2k)}}\right) &\leq T\left(r, \frac{1}{(f'/\phi_a)^{(2k)}}\right) \\ &= T(r, (f'/\phi_a)^{(2k)}) + \log \frac{1}{|(f'/\phi_a)^{(2k)}_{z=0}|} \\ &= m\left(r, \frac{(f'/\phi_a)^{(2k)}}{(f'/\phi_a) - (z+a)^k} \cdot \frac{f' - A_a}{\phi_a}\right) + \log \frac{1}{|(f'/\phi_a)^{(2k)}_{z=0}|} \\ &\leq m(r, f' - A_a) + m\left(r, \frac{(f'/\phi_a)^{(2k)}}{(f'/\phi_a) - (z+a)^k}\right) \\ &\quad + m\left(r, \frac{1}{\phi_a}\right) + \log \frac{1}{|(f'/\phi_a)^{(2k)}_{z=0}|}. \end{aligned} \tag{2.17}$$

Thus, by (2.16) and (2.17) we have

$$\begin{aligned} T(r, f) &= m(r, f) + m(r, f' - A_a) - m(r, f' - A_a) \\ &= T(r, f) + T(r, f' - A_a) - m(r, f' - A_a) \\ &= T\left(r, \frac{1}{f}\right) + T\left(r, \frac{1}{f' - A_a}\right) - m(r, f' - A_a) \\ &\quad + \log|f(0)[f'(0) - A_a(0)]| \\ &\leq N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f' - A_a}\right) + LD_1[r, f] \\ &\quad + \log\left|\frac{f(0)[f'(0) - A_a(0)]}{(f'/\phi_a)^{(2k)}_{z=0}}\right|, \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} LD_1[r, f] = m\left(r, \frac{f'}{f}\right) &+ m\left(r, \frac{(f'/\phi_a)^{(2k)}}{(f'/\phi_a)^{(k)}}\right) + m\left(r, \frac{(f'/\phi_a)^{(2k)}}{(f'/\phi_a)^{(k)} - k!}\right) \\ &+ m\left(r, \frac{[(f'/\phi_a) - (z+a)^k]^{(k)}}{(f'/\phi_a) - (z+a)^k}\right) + m\left(r, \frac{(f'/\phi_a)^{(2k)}}{(f'/\phi_a) - (z+a)^k}\right) \\ &+ m\left(r, \frac{(f'/\phi_a)^{(k)}}{f'/\phi_a}\right) + 3m\left(r, \frac{1}{\phi_a}\right) + 4\log 2. \end{aligned} \tag{2.19}$$

Because $f(-a) \neq 0$, $A_a(-a) = 0$, and $f(z) = 0 \Leftrightarrow f'(z) = A_a(z)$, we can see that $f'(-a) \neq 0$. Since $f(z) = 0 \Leftrightarrow f'(z) = A_a(z)$ and $f'(z) = A_a(z) \Rightarrow f''(z) = A_a(z) + A'_a(z)$, it follows that all zeros of $f(z)$ and $f'(z) - A_a(z)$ are simple. Hence we have

$$N\left(r, \frac{1}{f' - A_a}\right) = N\left(r, \frac{1}{f}\right). \tag{2.20}$$

This with (2.18) yields

$$T(r, f) \leq 2N\left(r, \frac{1}{f}\right) + LD_1[r, f] + \log\left|\frac{f(0)[f'(0) - A_a(0)]}{(f'/\phi_a)^{(2k)}_{z=0}}\right|. \tag{2.21}$$

Now let $f(z_0) = 0$. Then, by $f(-a) \neq 0$, we have $z_0 + a \neq 0$ and so $A_a(z_0) = A(z_0 + a) \neq 0$. By assumption, $f'(z_0) = A_a(z_0)$ and $f''(z_0) = A_a(z_0) + A'_a(z_0)$. Thus, near $z_0$:

$$\begin{aligned}
&\frac{f'(z)}{f(z)} \\
&\quad = \frac{A_a(z_0) + [A_a(z_0) + A'_a(z_0)](z - z_0) + O[(z - z_0)^2]}{A_a(z_0)(z - z_0) + \frac{1}{2}[A_a(z_0) + A'_a(z_0)](z - z_0)^2 + O[(z - z_0)^3]} \\
&\quad = \frac{1}{z - z_0} + \frac{A_a(z_0) + A'_a(z_0)}{2A_a(z_0)} + O(z - z_0);
\end{aligned}$$

$$\begin{aligned}
&\frac{f''(z)}{f(z)} \\
&\quad = \frac{A_a(z_0) + A'_a(z_0) + f'''(z_0)(z - z_0) + O[(z - z_0)^2]}{A_a(z_0)(z - z_0) + \frac{1}{2}[A_a(z_0) + A'_a(z_0)](z - z_0)^2 + O[(z - z_0)^3]} \\
&\quad = \frac{A_a(z_0) + A'_a(z_0)}{A_a(z_0)} \cdot \frac{1}{z - z_0} + \frac{f'''(z_0)}{A_a(z_0)} - \frac{[A_a(z_0) + A'_a(z_0)]^2}{2[A_a(z_0)]^2} \\
&\qquad + O(z - z_0);
\end{aligned}$$

$$\begin{aligned}
&\frac{f''(z) - A'_a(z)}{f'(z) - A_a(z)} \\
&\quad = \frac{A_a(z_0) + [f'''(z_0) - A''_a(z_0)](z - z_0) + O[(z - z_0)^2]}{A_a(z_0)(z - z_0) + \frac{1}{2}[f'''(z_0) - A''_a(z_0)](z - z_0)^2 + O[(z - z_0)^3]} \\
&\quad = \frac{1}{z - z_0} + \frac{f'''(z_0) - A''_a(z_0)}{2A_a(z_0)} + O(z - z_0).
\end{aligned}$$

Hence, by definition of the function $L_a(z)$, near $z_0$ we have $L_a(z) = O(z - z_0)$, and it follows that $L_a(z_0) = 0$. Combining this with the fact that all zeros of $f(z)$ are simple, we get

$$N\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{1}{L_a}\right) \quad \text{and} \quad N(r, L_a) = 0. \tag{2.22}$$

This, together with (2.21) and Nevanlinna's first fundamental theorem, yields

$$
\begin{aligned}
T(r,f) &\le 2N\left(r,\frac{1}{L_a}\right) + LD_1[r,f] + \log\left|\frac{f(0)[f'(0)-A_a(0)]}{(f'/\phi_a)^{(2k)}_{z=0}}\right| \\
&\le 2m\left(r,L_a\right) + LD_1[r,f] + \log\left|\frac{f(0)[f'(0)-A_a(0)]}{[L_a(0)]^2(f'/\phi_a)^{(2k)}_{z=0}}\right|. \qquad (2.23)
\end{aligned}
$$

By (2.13), we have

$$
\begin{aligned}
m(r,L_a) \le m\left(r,\frac{f'}{f}\right) + m\left(r,\frac{f''}{f}\right) + m\left(r,\frac{f''-A'_a}{f'-A_a}\right) \\
+ 3m(r,A_a) + 2m(r,A'_a) + 4\log 2. \qquad (2.24)
\end{aligned}
$$

Thus, by (2.23), (2.24), and (2.29) we obtain (2.14) and (2.15). This completes the proof of Lemma 6. □

LEMMA 7 [1]. *Let $U(r)$ be a nonnegative, increasing function on an interval $[R_1,R_2]$ $(0<R_1<R_2<+\infty)$; let $a,b$ be two positive constants satisfying $b>(a+2)^2$; and let*

$$
U(r) < a\left\{\log^+ U(\rho) + \log\frac{\rho}{\rho-r}\right\} + b
$$

*whenever $R_1<r<\rho<R_2$. Then, for $R_1<r<R_2$,*

$$
U(r) < 2a\log\frac{R_2}{R_2-r} + 2b.
$$

LEMMA 8 [8]. *Let $f(z)$ be meromorphic in $|z|<R$. If $f(0)\ne 0,\infty$ then, for every positive integer $k$,*

$$
\begin{aligned}
m\left(r,\frac{f^{(k)}}{f}\right) \le C_k\Bigg\{1 + \log^+\log^+\frac{1}{|f(0)|} + \log^+\frac{1}{r} \\
+ \log^+\frac{1}{\rho-r} + \log^+\rho + \log^+ T(\rho,f)\Bigg\},
\end{aligned}
$$

*where $0<r<\rho<R$ and $C_k$ is a constant depending only on $k$.*

In the sequel, $C_k$ may vary with each occurrence.

## 3. Proofs

### *3.1. Proof of Proposition 1*

Suppose that $\mathcal{F}$ is not normal at some point $z_0\in D$. Since $D$ is open, there exists a positive number $\delta$ such that $\{z:|z-z_0|<\delta\}\subset D$. Hence, by Lemma 1 there exist $z_n\to z_0$, $\rho_n\to 0$, and $f_n\in\mathcal{F}$ such that

$$
g_n(\zeta) = \rho_n^{-1}f_n(z_n+\rho_n\zeta) \to g(\zeta)
$$

locally uniformly on $\mathbb{C}$, where $g$ is a nonconstant entire function such that $g^{\#}(\zeta) \le g^{\#}(0) = \max_{|z-z_0|\le\delta/2}|A(z)| + 1$. In particular, $g$ is of exponential type.

We claim that:

(i) $g(\zeta) = 0 \Rightarrow g'(\zeta) = A(z_0)$;
(ii) $g'(\zeta) = A(z_0) \Rightarrow g''(\zeta) = 0$.

Suppose now that $g(\zeta_0) = 0$. Then, by Hurwitz's theorem, there exist points $\zeta_n \to \zeta_0$ such that $g_n(\zeta_n) = \rho_n^{-1} f_n(z_n + \rho_n\zeta_n) = 0$. Thus $f_n(z_n + \rho_n\zeta_n) = 0$ and so $f_n'(z_n + \rho_n\zeta_n) = A(z_n + \rho_n\zeta_n)$. Hence $g_n'(\zeta_n) = A(z_n + \rho_n\zeta_n)$ and $g'(\zeta_0) = \lim_{n\to\infty} g_n'(\zeta_n) = A(z_0)$. This proves (i).

Next we prove (ii). Suppose that $g'(\zeta_0) = A(z_0)$. Obviously $g'(\zeta) \not\equiv A(z_0)$, for otherwise $g^{\#}(0) \le |g'(0)| = |A(z_0)|$, which contradicts

$$g^{\#}(0) = \max_{|z-z_0|\le\delta/2}|A(z)| + 1.$$

Hence, by Hurwitz's theorem there exist points $\zeta_n \to \zeta_0$ such that $g_n'(\zeta_n) = A(z_n+\rho_n\zeta_n)$, so $f_n'(z_n+\rho_n\zeta_n) = A(z_n+\rho_n\zeta_n)$ and $g_n''(\zeta_n) = \rho_n f_n''(z_n+\rho_n\zeta_n) = \rho_n[A(z_n + \rho_n\zeta_n) + A'(z_n + \rho_n\zeta_n)]$. It follows that $g''(\zeta_0) = \lim_{n\to\infty} g_n''(\zeta_n) = 0$, which proves (ii).

Therefore, Lemma 2 implies that $g'(\zeta) \equiv A(z_0)$—a contradiction. Thus, the proof of Proposition 1 is complete.

### *3.2. Proof of Proposition 2*

Let $z_0 \in D$. If $A(z_0) \ne 0$ then, by Proposition 1, $\mathcal{F}$ is normal at $z_0$. Now suppose that $A(z_0) = 0$. Then there exists a positive number $\delta$ such that $A(z) \ne 0$ for $z \in \{z : 0 < |z - z_0| \le \delta\} \subset D$. Hence, again by Proposition 1, $\mathcal{F}$ is normal in $\{z : 0 < |z - z_0| < \delta\}$. Without loss of generality, we assume that $z_0 = 0$. Let $\Delta = \{z : |z| < \delta\}$. Then $\mathcal{F}$ is normal in $\Delta \setminus \{0\}$. Let $A(z) = z^k\phi(z)$, where $k$ is a positive integer and $\phi$ is a zero-free analytic function on $\bar{\Delta}$. We shall prove that $\mathcal{F}$ is normal at $z = 0$, but first we prove three claims as follows.

Claim 1. *Let $f \in \mathcal{F}$. Then $z = 0$ is a zero of $f$ with multiplicity $k + 1$ and $f^{(k+1)}(0) = k!\,\phi(0)$.*

*Proof.* Indeed, by $A(z) = 0 \Rightarrow f(z) = 0$ and $A(0) = 0$ it follows that $f(0) = 0$. Thus $f(z) = z^l f_1(z)$, where $l$ is a positive integer and where $f_1(z)$ is analytic at $z = 0$ and satisfies $f_1(0) \ne 0$. Hence we have

$$f'(z) - A(z) = z^{l-1}[l f_1(z) + z f_1'(z)] - z^k\phi(z). \tag{3.2.1}$$

If $l - 1 \ne k$, then by $f(z) = 0 \to f'(z) = A(z)$ we see that $\min(l - 1, k) \ge l$, which is impossible.

Thus $l = k + 1$, so

$$f(z) = z^{k+1} f_1(z), \tag{3.2.2}$$

$$f'(z) - A(z) = z^k[(k + 1) f_1(z) + z f_1'(z) - \phi(z)]. \tag{3.2.3}$$

Since $f(z) = 0 \to f'(z) = A(z)$, we know that $(k+1)f_1(0) - \phi(0) = 0$. Hence, by (3.2.2), $f^{(k+1)}(0) = k!\,\phi(0)$. This proves Claim 1. □

CLAIM 2. *Let $f \in \mathcal{F}$ and let $F(z) = z^{-k}f(z)$. Then $F(z)$ is analytic in $\Delta$ and $|F'(z)| \le M$ whenever $F(z) = 0$ in $\Delta$, where $M = \max_{z\in\bar{\Delta}}|\phi(z)|$.*

*Proof.* In fact, by Claim 1 we know that $F(z)$ is analytic in $\Delta$. Now suppose $F(z_0) = 0$, so that $f(z_0) = 0$. If $z_0 \neq 0$ then, by $f(z_0) = 0$, it follows that $f'(z_0) = A(z_0) = z_0^k\phi(z_0)$. Thus

$$|F'(z_0)| = |z_0^{-k}f'(z_0) - kz_0^{-k-1}f(z_0)| = |\phi(z_0)| \le M.$$

If $z_0 = 0$ then by Claim 1 we know that, near $z_0 = 0$,

$$f(z) = \frac{\phi(0)}{k+1}z^{k+1} + O(z^{k+2})$$

and so

$$F(z) = \frac{\phi(0)}{k+1}z + O(z^2).$$

Thus we have

$$|F'(z_0)| = |F'(0)| = \frac{|\phi(0)|}{k+1} \le M,$$

and Claim 2 is proved. □

CLAIM 3. *If $\{F(z) = z^{-k}f(z) : f \in \mathcal{F}\}$ is normal at $z = 0$, then $\mathcal{F}$ is also normal at $z = 0$.*

*Proof.* Let $\{f_n\}$ be a sequence in $\mathcal{F}$. Then $\{F_n(z) = z^{-k}f_n(z)\}$ is normal at $z = 0$. By Claim 1, $F_n(0) = 0$. It follows that there exists a subsequence $\{F_{n_j}\}$ of $\{F_n\}$ such that, in a neighborhood $U \subset \Delta$ of $z = 0$, $\{F_{n_j}\}$ converges uniformly to an analytic function $h(z)$. Thus $f_{n_j}(z) = z^kF_{n_j}(z)$ converges uniformly to $z^kh(z)$ in $U$. Hence $\mathcal{F}$ is normal at $z = 0$, which proves Claim 3. □

Now we prove that $\mathcal{F}$ is normal at $z = 0$. Suppose on the contrary that $\mathcal{F}$ is not normal at $z = 0$. Then, by Claim 3, the family $\{F(z) = z^{-k}f(z) : f \in \mathcal{F}\}$ is not normal at $z = 0$. Thus, by Claim 2 and Lemma 1, we can find $z_n \to 0$, $\rho_n \to 0^+$, and $f_n \in \mathcal{F}$ such that

$$g_n(\zeta) = \rho_n^{-1}(z_n + \rho_n\zeta)^{-k}f_n(z_n + \rho_n\zeta) \to g(\zeta) \tag{3.2.4}$$

locally uniformly on $\mathbb{C}$, where $g$ is a nonconstant entire function such that $g^\#(\zeta) \le g^\#(0) = M + 1$ for $M = \max_{z\in\bar{\Delta}}|\phi(z)|$. In particular, $\rho(g) \le 1$.

Without loss of generality, we assume that

$$\lim_{n\to\infty}\frac{z_n}{\rho_n} = c \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}. \tag{3.2.5}$$

Next we consider two cases.

*Case 1:* $c \neq \infty$. Let

$$h_n(\zeta) = \rho_n^{-k-1}f_n(z_n + \rho_n\zeta). \tag{3.2.6}$$

Then (3.2.4) and (3.2.5) yield

$$h_n(\zeta) = \left(\zeta + \frac{z_n}{\rho_n}\right)^k g_n(\zeta) \to (\zeta + c)^k g(\zeta) = h(\zeta) \tag{3.2.7}$$

locally uniformly on $\mathbb{C}$. We claim that:

(i) $h(\zeta) = 0 \Leftrightarrow h'(\zeta) = \phi(0)(\zeta + c)^k$;
(ii) $h'(\zeta) = \phi(0)(\zeta + c)^k \Leftrightarrow h''(\zeta) = k\phi(0)(\zeta + c)^{k-1}$;
(iii) $h^{(k+1)}(-c) = k!\,\phi(0)$.

Suppose $h(\zeta_0) = 0$. Obviously, $h(\zeta) \not\equiv 0$. Thus, by Hurwitz's theorem, there exist points $\zeta_n \to \zeta_0$ such that $h_n(\zeta_n) = 0$, so that $f_n(z_n + \rho_n\zeta_n) = 0$. Then, since $f(z) = 0 \Rightarrow f'(z) = A(z)$, it follows that $f_n'(z_n + \rho_n\zeta_n) = A(z_n + \rho_n\zeta_n)$ and so

$$h_n'(\zeta_n) = \rho_n^{-k} f_n'(z_n + \rho_n\zeta_n) = \left(\zeta_n + \frac{z_n}{\rho_n}\right)^k \phi(z_n + \rho_n\zeta_n).$$

Hence

$$h'(\zeta_0) = \lim_{n\to\infty} h_n'(\zeta_n) = (\zeta_0 + c)^k\phi(0).$$

Thus we have proved that $h(\zeta) = 0 \Rightarrow h'(\zeta) = \phi(0)(\zeta + c)^k$. On the other hand, suppose $h'(\zeta_0) = \phi(0)(\zeta_0 + c)^k$. Then $h'(\zeta) \not\equiv \phi(0)(\zeta + c)^k$. For otherwise, if $h'(\zeta) \equiv \phi(0)(\zeta + c)^k$, then

$$h(\zeta) = \frac{\phi(0)}{k+1}(\zeta + c)^{k+1} + d,$$

where $d$ is a constant. Since $h(-c) = 0$, we get $d = 0$. Thus we obtain

$$g(\zeta) = \frac{h(\zeta)}{(\zeta + c)^k} = \frac{\phi(0)}{k+1}(\zeta + c).$$

It follows that $g^{\#}(0) \le |g'(0)| = |\phi(0)|/(k+1) < M + 1$, a contradiction.

Therefore, $h'(\zeta) \not\equiv \phi(0)(\zeta + c)^k$. Hence, by Hurwitz's theorem there exist points $\zeta_n \to \zeta_0$ such that

$$h_n'(\zeta_n) = \phi(z_n + \rho_n\zeta_n)\left(\zeta_n + \frac{z_n}{\rho_n}\right)^k = \rho_n^{-k} A(z_n + \rho_n\zeta_n),$$

so that $f_n'(z_n + \rho_n\zeta_n) = \rho_n^k h_n'(\zeta_n) = A(z_n + \rho_n\zeta_n)$. Then, since $f'(z) = A(z) \Rightarrow f(z) = 0$, it follows that $f_n(z_n + \rho_n\zeta_n) = 0$ and so $h_n(\zeta_n) = 0$. Because $h(\zeta_0) = \lim_{n\to\infty} h_n(\zeta_n) = 0$, we have proved that $h'(\zeta) = \phi(0)(\zeta + c)^k \Rightarrow h(\zeta) = 0$. Thus claim (i) is proved.

Next we prove (ii). Suppose $h'(\zeta_0) = \phi(0)(\zeta_0 + c)^k$; then the foregoing argument shows that there exist points $\zeta_n \to \zeta_0$ such that $f_n'(z_n + \rho_n\zeta_n) = A(z_n + \rho_n\zeta_n)$. As a result, by $f'(z) = A(z) \Rightarrow f''(z) = A(z) + A'(z)$ we have

$$\begin{aligned} f_n''(z_n + \rho_n\zeta_n) &= A(z_n + \rho_n\zeta_n) + A'(z_n + \rho_n\zeta_n) \\ &= k(z_n + \rho_n\zeta_n)^{k-1}\phi(z_n + \rho_n\zeta_n) \\ &\quad + (z_n + \rho_n\zeta_n)^k[\phi(z_n + \rho_n\zeta_n) + \phi'(z_n + \rho_n\zeta_n)]. \end{aligned}$$

Hence, by (3.2.6),

$$\begin{aligned} h_n''(\zeta_n) &= \rho_n^{-k+1} f_n''(z_n + \rho_n\zeta_n) \\ &= k\left(\zeta_n + \frac{z_n}{\rho_n}\right)^{k-1} \phi(z_n + \rho_n\zeta_n) \\ &\quad + \rho_n\left(\zeta_n + \frac{z_n}{\rho_n}\right)^{k} [\phi(z_n + \rho_n\zeta_n) + \phi'(z_n + \rho_n\zeta_n)] \end{aligned}$$

and so

$$h''(\zeta_0) = \lim_{n\to\infty} h_n''(\zeta_n) = k(\zeta_0 + c)^{k-1}\phi(0).$$

This proves that $h'(\zeta) = \phi(0)(\zeta + c)^k \Rightarrow h''(\zeta) = k\phi(0)(\zeta + c)^{k-1}$.

Suppose now $h''(\zeta_0) = k\phi(0)(\zeta_0 + c)^{k-1}$. If $h''(\zeta) \equiv k\phi(0)(\zeta + c)^{k-1}$, then

$$h(\zeta) = \frac{\phi(0)}{k+1}(\zeta + c)^{k+1} + d_1\zeta + d_2,$$

where $d_1$ and $d_2$ are constants. Since $h(-c) = 0$, we have $d_2 = cd_1$. Therefore,

$$g(\zeta) = \frac{h(\zeta)}{(\zeta + c)^k} = \frac{\phi(0)}{k+1}(\zeta + c) + \frac{d_1}{(\zeta + c)^{k-1}}.$$

Because $g$ is an entire function, it follows that either $d_1 = 0$ or $k = 1$, so that $g'(\zeta) \equiv \phi(0)/(k+1)$. Thus $g^{\#}(0) \le |g'(0)| = |\phi(0)|/(k+1) < M + 1$, a contradiction.

Hence $h''(\zeta) \not\equiv k\phi(0)(\zeta + c)^{k-1}$. Thus, by $h''(\zeta_0) = k\phi(0)(\zeta_0 + c)^{k-1}$ and Hurwitz's theorem, there exist points $\zeta_n \to \zeta_0$ such that

$$\begin{aligned} h_n''(\zeta_n) &= k\left(\zeta_n + \frac{z_n}{\rho_n}\right)^{k-1} \phi(z_n + \rho_n\zeta_n) \\ &\quad + \rho_n\left(\zeta_n + \frac{z_n}{\rho_n}\right)^{k} [\phi(z_n + \rho_n\zeta_n) + \phi'(z_n + \rho_n\zeta_n)] \\ &= \rho_n^{-k+1}[A(z_n + \rho_n\zeta_n) + A'(z_n + \rho_n\zeta_n)]. \end{aligned}$$

It follows from $h_n''(\zeta) = \rho_n^{-k+1} f_n''(z_n + \rho_n\zeta)$ that

$$f_n''(z_n + \rho_n\zeta_n) = A(z_n + \rho_n\zeta_n) + A'(z_n + \rho_n\zeta_n).$$

As a result, we may use $f''(z) = A(z) + A'(z) \Rightarrow f'(z) = A(z)$ to deduce $f_n'(z_n + \rho_n\zeta_n) = A(z_n + \rho_n\zeta_n)$, so that by $h_n'(\zeta) = \rho_n^{-k} f_n'(z_n + \rho_n\zeta)$ we have

$$h_n'(\zeta_n) = \rho_n^{-k} A(z_n + \rho_n\zeta_n) = \left(\zeta_n + \frac{z_n}{\rho_n}\right)^{k} \phi(z_n + \rho_n\zeta_n).$$

Consequently,

$$h'(\zeta_0) = \lim_{n\to\infty} h_n'(\zeta_n) = \phi(0)(\zeta_0 + c)^k.$$

Thus we have proved that $h''(\zeta) = k\phi(0)(\zeta + c)^{k-1} \Rightarrow h'(\zeta) = \phi(0)(\zeta + c)^k$. This, together with $h'(\zeta) = \phi(0)(\zeta + c)^k \Rightarrow h''(\zeta) = k\phi(0)(\zeta + c)^{k-1}$, proves (ii).

Finally, we prove (iii). By Claim 1, $f_n^{(k+1)}(0) = k!\,\phi(0)$. Thus, using (3.2.6) yields

$$h^{(k+1)}(-c) = \lim_{n\to\infty} h_n^{(k+1)}\left(\frac{-z_n}{\rho_n}\right)$$
$$= \lim_{n\to\infty} f_n^{(k+1)}\left(z_n + \rho_n\left(\frac{-z_n}{\rho_n}\right)\right) = \lim_{n\to\infty} f_n^{(k+1)}(0) = k!\,\phi(0).$$

This proves (iii).

Now let

$$H(\zeta) = \frac{h(\zeta - c)}{\phi(0)}.$$

Then claims (i)–(iii) yield:

(i′) $H(\zeta) = 0 \Leftrightarrow H'(\zeta) = \zeta^k$;
(ii′) $H'(\zeta) = \zeta^k \Leftrightarrow H''(\zeta) = k\zeta^{k-1}$;
(iii′) $H^{(k+1)}(0) = k!$.

Thus, by Lemma 5, it follows that $H(\zeta) = \zeta^{k+1}/(k+1)$. Hence we have $h(\zeta) = \frac{\phi(0)}{k+1}(\zeta + c)^{k+1}$, so that $g(\zeta) = (\zeta + c)^{-k}h(\zeta) = \frac{\phi(0)}{k+1}(\zeta + c)$. Therefore $g^{\#}(0) \le |g'(0)| = |\phi(0)|/(k+1) < M + 1$, a contradiction.

*Case 2:* $c = \infty$. Then $z_n \ne 0$ and $\rho_n/z_n \to 0$ as $n \to \infty$. Set

$$h_n(\zeta) = \rho_n^{-1} z_n^{-k} f_n(z_n + \rho_n\zeta).$$

Then, by (3.2.4),

$$h_n(\zeta) = \left(1 + \frac{\rho_n}{z_n}\zeta\right)^k g_n(\zeta) \to g(\zeta)$$

locally uniformly on $\mathbb{C}$. Next, using the same argument as in the proof of Case 1, we have

(iv) $g(\zeta) = 0 \Rightarrow g'(\zeta) = \phi(0)$;
(v) $g'(\zeta) = \phi(0) \Rightarrow g''(\zeta) = 0$.

Thus, by Lemma 2, $g'(\zeta) \equiv \phi(0)$. It follows that $g^{\#}(0) \le |g'(0)| = |\phi(0)| < M + 1$, a contradiction. Hence $\mathcal{F}$ is normal in $D$ and Proposition 2 is proved.

### 3.3. *Proof of Proposition 3*

Let $z_0 \in D$. If $A(z_0) \ne 0$ then, by Proposition 1, $\mathcal{F}$ is normal at $z_0$. Now suppose $A(z_0) = 0$. Then there exists a positive number $\delta$ such that $A(z) \ne 0$ for $z \in \{z : 0 < |z - z_0| \le \delta\} \subset D$. Hence, by Proposition 1, $\mathcal{F}$ is normal in $\{z : 0 < |z - z_0| < \delta\}$. Without loss of generality, we assume that $z_0 = 0$. Let $\Delta = \{z : |z| < \delta\}$. Then $\mathcal{F}$ is normal in $\Delta \setminus \{0\}$. Let $A(z) = z^k\phi(z)$, where $k$ is a positive integer and $\phi$ is a zero-free analytic function on $\bar{\Delta}$. We shall prove that $\mathcal{F}$ is normal at $z = 0$.

Suppose on the contrary that $\mathcal{F}$ is not normal at $z = 0$. Then, by Lemma 1 there exist points $z_n \to 0$, positive numbers $\rho_n \to 0$, and functions $f_n \in \mathcal{F}$ such that

$$g_n(\zeta) = f_n(z_n + \rho_n\zeta) \to g(\zeta) \tag{3.3.1}$$

locally uniformly on $\mathbb{C}$, where $g$ is a nonconstant entire function. Without loss of generality, we assume that

$$\lim_{n\to\infty} \frac{z_n}{\rho_n} = c \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}. \tag{3.3.2}$$

First, we prove that $g(\zeta)$ is a transcendental entire function. Suppose now that $g$ is a polynomial. The argument given in the proof of Proposition 1 shows that

$$g(\zeta) = 0 \Longleftrightarrow g'(\zeta) = 0. \tag{3.3.3}$$

It follows that $g(\zeta) = C(\zeta - \zeta_0)^d$, where $C$ $(\neq 0)$ is a constant and $d \geq 2$ is an integer.

Let $\varepsilon < 1$ be a positive number. Then by (3.3.1) and Hurwitz's theorem, for sufficiently large $n$ in $D_\varepsilon = \{\zeta \in \mathbb{C} : |\zeta - \zeta_0| < \varepsilon\}$, $g_n(\zeta)$ has $d$ zero points $\zeta_{n,j}$ $(j = 1, 2, \ldots, d)$ counting multiplicity. Thus we have $f_n(z_n + \rho_n\zeta_{n,j}) = 0$. It follows from $A(z) = 0 \Rightarrow f(z) \neq 0$ and $f(z) = 0 \Rightarrow f'(z) = A(z)$ that $f_n'(z_n + \rho_n\zeta_{n,j}) = A(z_n + \rho_n\zeta_{n,j}) \neq 0$. Therefore, by $g_n'(\zeta) = \rho_n f_n'(z_n + \rho_n\zeta)$ we see that $g_n'(\zeta_{n,j}) = \rho_n A(z_n + \rho_n\zeta_{n,j}) \neq 0$. Hence each $\zeta_{n,j}$ is a simple zero of $g_n(\zeta)$, so that $\zeta_{n,j} \neq \zeta_{n,l}$ for $1 \leq j < l \leq d$. Thus, for sufficiently large $n$, the function $h_n(\zeta) = g_n'(\zeta) - \rho_n A(z_n + \rho_n\zeta)$ has at least $d$ distinct zero points in $D_\varepsilon$. Obviously, we have

$$h_n(\zeta) = g_n'(\zeta) - \rho_n A(z_n + \rho_n\zeta) \to g'(\zeta)$$

uniformly on $D_\varepsilon$. By Hurwitz's theorem we then know that $g'(\zeta)$ has at least $d$ zero points in $D_\varepsilon$ counting multiplicity. Since we can choose $\varepsilon$ to be as small as we like, $\zeta_0$ is a zero of $g'$ with multiplicity at least $d$, which contradicts $g'(\zeta) = dC(\zeta - \zeta_0)^{d-1}$. Hence $g$ is a transcendental entire function.

Now we consider four cases.

*Case 1.* There exist infinitely many $\{n_j\}$ such that

$$f_{n_j}'(z_{n_j} + \rho_{n_j}\zeta) \equiv A(z_{n_j} + \rho_{n_j}\zeta).$$

It follows that $g_{n_j}'(\zeta) \equiv \rho_{n_j} A(z_{n_j} + \rho_{n_j}\zeta)$. Letting $j \to \infty$, we deduce that $g'(\zeta) \equiv 0$, which contradicts that $g$ is transcendental.

*Case 2.* There exist infinitely many $\{n_j\}$ such that

$$\left(\frac{f_{n_j}'(z)}{\phi(z)}\right)^{(2k)}_{z=z_{n_j}+\rho_{n_j}\zeta} \equiv 0.$$

Thus, we have

$$\sum_{i=0}^{2k} \binom{2k}{i} f_{n_j}^{(i+1)}(z_{n_j} + \rho_{n_j}\zeta)\left(\frac{1}{\phi(z)}\right)^{(2k-i)}_{z=z_{n_j}+\rho_{n_j}\zeta} \equiv 0,$$

so that

$$\sum_{i=0}^{2k} \binom{2k}{i} \rho_{n_j}^{2k-i} g_{n_j}^{(i+1)}(\zeta)\left(\frac{1}{\phi(z)}\right)^{(2k-i)}_{z=z_{n_j}+\rho_{n_j}\zeta} \equiv 0.$$

Letting $j \to \infty$, we deduce that $g^{(2k+1)}(\zeta)/\phi(0) \equiv 0$, which also contradicts that $g$ is transcendental.

*Case 3.* There exist infinitely many $\{n_j\}$ such that

$$L_{n_j}(z_{n_j} + \rho_{n_j}\zeta) \equiv 0,$$

where

$$L_n(z) = [A(z) - A'(z)]\frac{f_n'(z)}{f_n(z)} + A(z)\frac{f_n''(z)}{f_n(z)} - 2A(z)\frac{f_n''(z) - A'(z)}{f_n'(z) - A(z)} + A'(z).$$

Thus we have

$$\begin{aligned}\rho_{n_j}&\left(1 - \frac{k}{z_{n_j} + \rho_{n_j}\zeta} - \frac{\phi'(z_{n_j} + \rho_{n_j}\zeta)}{\phi(z_{n_j} + \rho_{n_j}\zeta)}\right) \cdot \frac{g_{n_j}'(\zeta)}{g_{n_j}(\zeta)} + \frac{g_{n_j}''(\zeta)}{g_{n_j}(\zeta)} \\ &+ 2\rho_{n_j}\frac{g_{n_j}''(\zeta) - \rho_{n_j}^2 A'(z_{n_j} + \rho_{n_j}\zeta)}{g_{n_j}'(\zeta) - \rho_{n_j}A(z_{n_j} + \rho_{n_j}\zeta)} + \rho_{n_j}^2\left(\frac{k}{z_{n_j} + \rho_{n_j}\zeta} + \frac{\phi'(z_{n_j} + \rho_{n_j}\zeta)}{\phi(z_{n_j} + \rho_{n_j}\zeta)}\right) \equiv 0.\end{aligned} \tag{3.3.4}$$

If $c \neq \infty$, then letting $j \to \infty$ in (3.3.4) yields

$$-\frac{k}{\zeta + c} \cdot \frac{g'(\zeta)}{g(\zeta)} + \frac{g''(\zeta)}{g(\zeta)} \equiv 0.$$

It follows that $g$ is a polynomial, a contradiction. If instead $c = \infty$, then letting $j \to \infty$ in (3.3.4) yields $g''(\zeta)/g(\zeta) \equiv 0$. Hence $g$ again is a polynomial, a contradiction.

*Case 4.* There exist finitely many $\{n_j\}$ such that

$$f_{n_j}'(z_{n_j} + \rho_{n_j}\zeta) \equiv A(z_{n_j} + \rho_{n_j}\zeta) \quad \text{or}$$

$$\left(\frac{f_{n_j}'(z)}{\phi(z)}\right)^{(2k)}_{z=z_{n_j}+\rho_{n_j}\zeta} \equiv 0 \quad \text{or}$$

$$L_{n_j}(z_{n_j} + \rho_{n_j}\zeta) \equiv 0,$$

where $L_n$ is defined as in Case 3. For all $n$ we may suppose that $f_{n_j}'(z_{n_j} + \rho_{n_j}\zeta) \not\equiv A(z_{n_j} + \rho_{n_j}\zeta)$, $(f_{n_j}'(z)/\phi(z))^{(2k)}_{z=z_{n_j}+\rho_{n_j}\zeta} \not\equiv 0$, and $L_{n_j}(z_{n_j} + \rho_{n_j}\zeta) \not\equiv 0$.

Take $\zeta_0 \in \mathbb{C}$ such that $g^{(j)}(\zeta_0) \neq 0$ for $j = 0, 1, 2, \ldots, 2k+1$. In case $c \neq \infty$, choose $\zeta_0$ to satisfy the additional conditions that $\zeta_0 \neq -c$ and

$$g''(\zeta_0) - \frac{k}{\zeta_0 + c} \cdot g'(\zeta_0) \neq 0.$$

The argument given previously now shows that, as $n \to \infty$:

$$\rho_n[f_n'(z_n + \rho_n\zeta_0) - A(z_n + \rho_n\zeta_0)] \to g'(\zeta_0) \neq 0, \infty;$$

$$\rho_n^{2k+1}\left(\frac{f_n'(z)}{\phi(z)}\right)^{(2k)}_{z=z_n+\rho_n\zeta_0} \to \frac{g^{(2k+1)}(\zeta_0)}{\phi(0)} \neq 0, \infty;$$

$$\rho_n^{-k+2}L_n(z_n+\rho_n\zeta_0) \to \phi(0)(\zeta_0+c)^k\left[\frac{g''(\zeta_0)}{g(\zeta_0)} - \frac{k}{\zeta_0+c}\cdot\frac{g'(\zeta_0)}{g(\zeta_0)}\right] \neq 0,\infty, \quad c\neq\infty;$$

$$\rho_n^2 z_n^{-k}L_n(z_n+\rho_n\zeta_0) \to \phi(0)\frac{g''(\zeta_0)}{g(\zeta_0)} \neq 0,\infty, \quad c=\infty.$$

These facts imply that

$$K_n = \frac{[f_n'(z_n+\rho_n\zeta_0) - A(z_n+\rho_n\zeta_0)]f_n(z_n+\rho_n\zeta_0)}{[L_n(z_n+\rho_n\zeta_0)]^2(f_n'(z)/\phi(z))^{(2k)}_{z=z_n+\rho_n\zeta_0}} \to 0$$

as $n\to\infty$, so that

$$\log|K_n| \to -\infty \text{ as } n\to\infty. \tag{3.3.5}$$

For $n=1,2,3,\ldots$, put

$$h_n(z) = f_n(z_n+\rho_n\zeta_0+z).$$

Since $z_n+\rho_n\zeta_0 \to 0$ as $n\to\infty$, it follows that (for sufficiently large $n$) $h_n$ is defined and holomorphic on $|z|<1/2$. Denote

$$a_n = z_n+\rho_n\zeta_0.$$

Then, for sufficiently large $n$, $h_n(0)\neq 0$, $h_n(-a_n)\neq 0$, $[h_n'(z)/\phi_{a_n}(z)]^{(2k)}_{z=0}\neq 0$, $L_{a_n}(0)=L_n(a_n)\neq 0$, and

$$\frac{[h_n'(0)-A_{a_n}(0)]h_n(0)}{[L_{a_n}(0)]^2[h_n'(z)/\phi_{a_n}(z)]^{(2k)}_{z=0}} = K_n,$$

as well as $h_n(z)=0 \Leftrightarrow h_n'(z)=A_{a_n}(z)$ and $h_n'(z)=A_{a_n}(z) \Rightarrow h_n''(z) = A_{a_n}(z)+A_{a_n}'(z)$.

Now applying Lemma 6 to $h_n(z)$ with $r_0=1/2$ and $a=a_n$, using (3.3.5), and noting that the last four terms in (2.15) are bounded for $0<r<1/3$, we obtain that, for sufficiently large $n$ and $0<r<1/3$,

$$\begin{aligned} T(r,h_n) \le\ & 3m\left(r,\frac{h_n'}{h_n}\right) + 2m\left(r,\frac{h_n''}{h_n}\right) + 2m\left(r,\frac{h_n''-A_{a_n}'}{h_n'-A_{a_n}}\right) \\ & + m\left(r,\frac{(h_n'/\phi_{a_n})^{(k)}}{h_n'/\phi_{a_n}}\right) + m\left(r,\frac{(h_n'/\phi_{a_n})^{(2k)}}{(h_n'/\phi_{a_n})^{(k)}}\right) \\ & + m\left(r,\frac{(h_n'/\phi_{a_n})^{(2k)}}{(h_n'/\phi_{a_n})^{(k)}-k!}\right) + m\left(r,\frac{(h_n'/\phi_{a_n}-(z+a_n)^k)^{(k)}}{h_n'/\phi_{a_n}-(z+a_n)^k}\right) \\ & + m\left(r,\frac{(h_n'/\phi_{a_n})^{(2k)}}{h_n'/\phi_{a_n}-(z+a_n)^k}\right). \end{aligned}$$

We know that

$$h_n(0) = g_n(\zeta_0) \to g(\zeta_0),$$

$$h_n'(0) - A_{a_n}(0) = \rho_n^{-1} g_n'(\zeta_0) - A(z_n + \rho_n\zeta_0) \to \infty,$$

$$\frac{h_n'(0)}{\phi_{a_n}(0)} = \frac{g_n'(\zeta_0)}{\rho_n \phi(z_n + \rho_n\zeta_0)} \to \infty,$$

$$\begin{aligned}\left(\frac{h_n'(z)}{\phi_{a_n}(z)}\right)^{(k)}_{z=0} &= \sum_{j=0}^{k} \binom{k}{j} [h_n'(z)]^{(j)}_{z=0} \left(\frac{1}{\phi_{a_n}(z)}\right)^{(k-j)}_{z=0} \\ &= \frac{1}{\rho_n^{k+1}} \left( g_n^{(k+1)}(\zeta_0) + \sum_{j=0}^{k-1} \binom{k}{j} \rho_n^{k-j} g_n^{(j+1)}(\zeta_0) \left(\frac{1}{\phi(z)}\right)^{(k-j)}_{z=z_n+\rho_n\zeta_0} \right) \\ &\to \infty,\end{aligned}$$

so by Lemma 8 we obtain, for $0 < r < \tau < 1/3$,

$$\begin{aligned}T(r, h_n) \le C_k \Bigg\{ & 1 + \log^+ \frac{1}{r} + \log^+ \frac{1}{\tau - r} \\ & + \log^+ T(\tau, h_n) + \log^+ T(\tau, h_n' - A_{a_n}) + \log^+ T\left(\tau, \frac{h_n'}{\phi_{a_n}}\right) \\ & + \log^+ T\left(\tau, \left(\frac{h_n'}{\phi_{a_n}}\right)^{(k)}\right) + \log^+ T\left(\tau, \left(\frac{h_n'}{\phi_{a_n}}\right)^{(k)} - k!\right) \\ & + \log^+ T\left(\tau, \frac{h_n'}{\phi_{a_n}} - (z + a_n)^k\right) \Bigg\} \\ \le C_k \Bigg\{ & 1 + \log^+ \frac{1}{r} + \log^+ \frac{1}{\tau - r} + \log^+ T(\tau, h_n) \\ & + \log^+ T(\tau, h_n') + \log^+ T\left(\tau, \left(\frac{h_n'}{\phi_{a_n}}\right)^{(k)}\right) \Bigg\}. \end{aligned} \tag{3.3.6}$$

Observe that $T(\tau, h_n') = m(\tau, h_n') \le m(\tau, h_n) + m(\tau, h_n'/h_n)$ and

$$\begin{aligned}T\left(\tau, \left(\frac{h_n'}{\phi_{a_n}}\right)^{(k)}\right) &= m\left(\tau, \left(\frac{h_n'}{\phi_{a_n}}\right)^{(k)}\right) \\ &\le m\left(\tau, \frac{h_n'}{\phi_{a_n}}\right) + m\left(\tau, \frac{(h_n'/\phi_{a_n})^{(k)}}{h_n'/\phi_{a_n}}\right) \\ &\le m(\tau, h_n) + m(\tau, \phi_{a_n}) + m\left(\tau, \frac{h_n'}{h_n}\right) + m\left(\tau, \frac{(h_n'/\phi_{a_n})^{(k)}}{h_n'/\phi_{a_n}}\right).\end{aligned} \tag{3.3.7}$$

Hence, for $1/4 < r < \rho < 1/3$ with $\tau = (r + \rho)/2$, we can use (3.3.6), (3.3.7), and Lemma 8 to obtain

$$T(r, h_n) \le C_k \left(1 + \log^+ \frac{1}{\rho - r} + \log^+ T(\rho, h_n)\right).$$

By Lemma 7 it then follows that

$$T\left(\frac{1}{4}, h_n\right) \le A,$$

where $A$ is a constant independent of $n$. Thus $\{f_n(z)\}$ is uniformly bounded for sufficiently large $n$ and $|z| < 1/8$. However, from $\rho_n^2 f_n'(z_n + \rho_n\zeta_0) = g_n''(\zeta_0) \to g''(\zeta_0) \neq 0$ we see that $f(z)$ cannot bounded in $|z| < 1/8$. This is a contradiction, so the proof is complete.

### 3.4. *Proof of Theorem 1*

Let $\mathcal{G} = \{g = f - a : f \in \mathcal{F}\}$ and $A(z) = a(z) - a'(z) \not\equiv 0$. Obviously, $\mathcal{G}$ is normal in $D$ if and only if $\mathcal{F}$ is normal in $D$. It follows from our assumptions that, for any $g \in \mathcal{G}$, we have $g(z) = 0 \Leftrightarrow g'(z) = A(z)$ and $g'(z) = A(z) \Leftrightarrow g''(z) = A(z) + A'(z)$ and $g(z) = 0 \to g'(z) = A(z)$.

Let $z_0 \in D$. Now we prove that $\mathcal{G}$ is normal at $z_0$. Let $\{g_n\} \subset \mathcal{G}$ be a sequence.

If $A(z_0) \neq 0$, then there exists a positive number $\delta$ such that $\Delta_\delta(z_0) = \{z \in D : |z - z_0| < \delta\} \subset D$ and $A(z) \neq 0$ in $\Delta_\delta(z_0)$. Thus, by Proposition 1, $\{g_n\}$ is normal in $\Delta_\delta(z_0)$.

If $A(z_0) = 0$, then there exists a positive number $\delta$ such that $\Delta_\delta(z_0) = \{z \in D : |z - z_0| < \delta\} \subset D$ and $A(z) \neq 0$ in $\Delta_\delta(z_0) \setminus \{z_0\}$. If $\{g_n\}$ has a subsequence—say, without loss of generality, itself—such that $g_n(z_0) = 0$, then $\{g_n\}$ is normal in $\Delta_\delta(z_0)$ by Proposition 2. If $g_n(z_0) \neq 0$ for all but finitely many of $\{g_n\}$, then $\{g_n\}$ is normal in $\Delta_\delta(z_0)$ by Proposition 3.

Thus $\mathcal{F}$ is normal in $D$ and so Theorem 1 is proved.

### 3.5. *Proof of Corollary 1*

By Theorem 1, we need only show that $f(z) - z = 0 \to f'(z) - z = 0$ in $D$. Let $z_0$ be a zero of $f(z) - z$ in $D$. Then, since $f(z) = z \Leftrightarrow f'(z) = z$ and $f'(z) = z \Leftrightarrow f''(z) = z$, it follows that $f(z_0) = f'(z_0) = f''(z_0) = z_0$. Thus we obtain that

$$[f(z) - z]'_{z=z_0} = z_0 - 1, \quad [f(z) - z]''_{z=z_0} = z_0, \quad [f'(z) - z]'_{z=z_0} = z_0 - 1.$$

If $z_0 \neq 1$, then $z_0$ is a simple zero of $f(z) - z$; if $z_0 = 1$, then $z_0$ is a double zero of $f(z) - z$ and $z_0$ is a multiple zero of $f'(z) - z$. Consequently, $f(z) - z = 0 \to f'(z) - z = 0$ in $D$. Thus $\mathcal{F}$ is normal in $D$ by Theorem 1, completing the proof of Corollary 1.

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