A Linear Bound for Frobenius Powers and an Inclusion Bound for Tight Closure

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Introduction

Let R denote a Noetherian ring, let $\mathfrak{m}$ denote a maximal ideal in R, and let I denote an $\mathfrak{m}$-primary ideal. This means by definition that $\mathfrak{m}$ is the radical of I. Then there exists a (minimal) number k such that $\mathfrak{m}^k \subseteq I \subseteq \mathfrak{m}$ holds. If R contains a field of positive characteristic p then the Frobenius powers of the ideal I, that is,

$$I^{[q]} = \{f^q : f \in I\}, \quad q = p^e,$$

are also $\mathfrak{m}$-primary and hence there exists a minimal number $k(q)$ such that $\mathfrak{m}^{k(q)} \subseteq I^{[q]}$ holds. In this paper we deal with the question of how $k(q)$ behaves as a function of q; in particular, we look for linear bounds for $k(q)$ from above. If $\mathfrak{m}^k \subseteq I$ and if l denotes the number of generators for $\mathfrak{m}^k$, then we obtain the trivial linear inclusion $(\mathfrak{m}^k)^{lq} \subseteq (\mathfrak{m}^k)^{[q]} \subseteq I^{[q]}$.

The main motivation for this question comes from the theory of tight closure. Recall that the tight closure of an ideal I in a domain R containing a field of positive characteristic p is the ideal

$$I^* = \{f \in R : \exists\, 0 \neq c \in R \text{ such that } cf^q \in I^{[q]} \text{ for all } q = p^e\}.$$

A linear inclusion relation $\mathfrak{m}^{\lambda q + \gamma} \subseteq I^{[q]}$ for all $q = p^e$ implies the inclusion $\mathfrak{m}^\lambda \subseteq I^*$, since then we can take any element $0 \neq c \in \mathfrak{m}^\gamma$ to show for $f \in \mathfrak{m}^\lambda$ that $cf^q \in \mathfrak{m}^{\lambda q + \gamma} \subseteq I^{[q]}$ and hence $f \in I^*$. The trivial bound mentioned previously yields $\mathfrak{m}^{kl} \subseteq I^*$, but this does not yield anything of interest because, in fact, we have already $\mathfrak{m}^{kl} \subseteq \mathfrak{m}^k \subseteq I$.

We restrict our attention in this paper to the case of a normal standard-graded domain R over an algebraically closed field $K = R_0$ of positive characteristic p and to a homogeneous R_+-primary ideal I. The question is then to find the minimal degree $k(q)$ such that $R_{\geq k(q)} \subseteq I^{[q]}$ or at least to find a good linear bound $k(q) \leq \lambda q + \gamma$. In this setting we work mainly over the normal projective variety $Y = \operatorname{Proj} R$ endowed with the very ample invertible sheaf $\mathcal{O}_Y(1)$. If $I = (f_1, \ldots, f_n)$ is given by homogeneous ideal generators f_i of degree $d_i = \deg(f_i)$, then on Y we have the following short exact sequences of locally free sheaves:

$$0 \longrightarrow \operatorname{Syz}(f_1^q, \ldots, f_n^q)(m) \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_Y(m - qd_i) \xrightarrow{\ f_1^q, \ldots, f_n^q\ } \mathcal{O}_Y(m) \longrightarrow 0.$$

Another homogeneous element $h \in R$ of degree m yields a cohomology class $\delta(h) \in H^1(Y, \text{Syz}(f_1^q, \ldots, f_n^q)(m))$, and therefore the question of whether $h \in (f_1^q, \ldots, f_n^q) = I^{[q]}$ is equivalent to the question of whether $\delta(h) = 0$. Since $\text{Syz}(f_1^q, \ldots, f_n^q)(0) = F^{*e}(\text{Syz}(f_1, \ldots, f_n)(0))$ is the pull-back under the eth absolute Frobenius morphism $F^e : Y \to Y$, our question is an instance of the following more general one: Given a locally free sheaf $\mathcal{S}$ on a normal projective variety $(Y, \mathcal{O}_Y(1))$, find an (affine-linear) bound $\ell(q)$ such that for $m \geq \ell(q)$ we have $H^1(Y, \mathcal{S}^q(m)) = 0$, where we set $\mathcal{S}^q = F^{e*}(\mathcal{S})$. Using a resolution $\mathcal{G}. \to \mathcal{S} \to 0$ with $\mathcal{G}_j = \bigoplus_{(k, j)} \mathcal{O}_Y(-\alpha_{k, j})$, we can shift the problem (at least if $Y = \text{Proj } R$ with R Cohen–Macaulay, so that $H^i(Y, \mathcal{O}_Y(m)) = 0$ for $0 < i < \dim(Y)$) to the problem of finding a bound such that $H^t(Y, \mathcal{S}_t^q(m)) = 0$, where $\mathcal{S}_t = \text{kern}(\mathcal{G}_t \to \mathcal{G}_{t-1})$ and $t = \dim(Y)$. By Serre duality this translates to $\text{Hom}(\mathcal{S}_t^q(m), \omega_Y) = 0$. Now the existence of such mappings is controlled by the minimal slope of $\mathcal{S}_t^q(m)$. Let $\bar{\mu}_{\min}(\mathcal{S}_t) = \liminf_{q=p^e} \mu_{\min}(\mathcal{S}_t^q)/q$ and set $\nu = -\bar{\mu}_{\min}(\mathcal{S}_t)/\deg(Y)$. With this notation applied to $\mathcal{S} = \text{Syz}(f_1, \ldots, f_n)(0)$, our main results are the following theorems (Theorems 2.2 and 2.4).

Theorem 1. *Let R denote a standard-graded normal Cohen–Macaulay domain over an algebraically closed field K of characteristic $p > 0$. Suppose that the dualizing sheaf ω_Y of $Y = \text{Proj } R$ is invertible, and let I denote a homogeneous R_+-primary ideal. Then $R_{>q\nu+\deg(\omega_Y)/\deg(Y)} \subseteq I^{[q]}$.*

From this linear bound for the Frobenius powers we derive the following inclusion bound for tight closure.

Theorem 2. *Under the assumptions of Theorem 1 we have the inclusion $R_{\geq \nu} \subseteq I^*$, where I^* denotes the tight closure of I.*

This theorem generalizes [2, Thm. 6.4] from dimension 2 to higher dimensions. We also obtain an inclusion bound for the Frobenius closure (Corollary 2.12) and a linear bound for the Castelnuovo–Mumford regularity of the Frobenius powers $I^{[q]}$ (Theorem 3.1), which improves a result of Chardin [6].

I thank M. Blickle for useful remarks.

1. Some Projective Preliminaries

Let K denote an algebraically closed field, and let Y denote a normal projective variety over K of dimension t together with a fixed ample Cartier divisor H with corresponding ample invertible sheaf $\mathcal{O}_Y(1)$. The degree of a coherent torsion-free sheaf $\mathcal{S}$ (with respect to H) is defined by the intersection number $\deg(\mathcal{S}) = \deg(c_1(\mathcal{S})) = c_1(\mathcal{S}).H^{t-1}$; see [17, Preliminaries] for background on this notion. The degree is additive on short exact sequences [17, Lemma 1.5(2)].

The slope of $\mathcal{S}$ (with respect to H), written $\mu(\mathcal{S})$, is defined by dividing the degree through the rank. The slope fulfills the property that $\mu(\mathcal{S}_1 \otimes \mathcal{S}_2) = \mu(\mathcal{S}_1) + \mu(\mathcal{S}_2)$ [17, Lemma 1.5(4)]. The minimal slope of $\mathcal{S}$, $\mu_{\min}(\mathcal{S})$, is given by

$$\mu_{\min}(\mathcal{S}) = \inf\{\mu(\mathcal{Q}) \colon \mathcal{S} \to \mathcal{Q} \to 0 \text{ is a torsion-free quotient sheaf}\}.$$

If $\mathcal{S}_1 \subset \cdots \subset \mathcal{S}_k = \mathcal{S}$ is the Harder–Narasimhan filtration of $\mathcal{S}$ [17, Prop. 1.13], then $\mu_{\min}(\mathcal{S}) = \mu(\mathcal{S}/\mathcal{S}_{k-1})$. If $\mathcal{L}$ is an invertible sheaf and $\mu_{\min}(\mathcal{S}) > \deg(\mathcal{L})$, then there does not exist any nontrivial sheaf homomorphism $\mathcal{S} \to \mathcal{L}$. The sheaf $\mathcal{S}$ is called *semistable* if $\mu(\mathcal{S}) = \mu_{\min}(\mathcal{S})$.

Suppose now that the characteristic of K is positive and let $F^e \colon Y \to Y$ denote the eth absolute Frobenius morphism. We denote the pull-back of $\mathcal{S}$ under this morphism by $\mathcal{S}^q = F^{e*}(\mathcal{S})$, $q = p^e$. The slope behaves like $\mu(\mathcal{S}^q) = q\mu(\mathcal{S})$ (this follows from [17, Lemma 1.6], for which it is enough to assume that the finite mapping is flat in codimension 1; note that we compute the slope always with respect to $\mathcal{O}_Y(1)$, not with respect to $F^{*e}(\mathcal{O}_Y(1)) = \mathcal{O}_Y(q))$. However, it may happen that $\mu_{\min}(\mathcal{S}^q) < q\mu_{\min}(\mathcal{S})$. It is therefore useful to consider the number

$$\bar{\mu}_{\min}(\mathcal{S}) = \liminf_{q=p^e} \mu_{\min}(\mathcal{S}^q)/q$$

(cf. [16]). This limit exists, since for some number k there exists a surjection $\bigoplus_j \mathcal{O}(\beta_j) \to \mathcal{S}(k)$ such that all β_j are positive. Then $\mathcal{S}(k)$ is a quotient of an ample bundle and so all its quotients have positive degree. This holds also for all its Frobenius pull-backs, so $\mu_{\min}((\mathcal{S}(k))^q) \geq 0$ and the limit is ≥ 0. Thus $\mu_{\min}(\mathcal{S}^q) \geq -qk\deg(\mathcal{O}_Y(1))$ for all q. Moreover, a theorem of Langer [16] implies that this limit is even a rational number. The sheaf $\mathcal{S}$ is called *strongly semistable* if $\mu(\mathcal{S}) = \bar{\mu}_{\min}(\mathcal{S})$ or (equivalently) if all Frobenius pull-backs $\mathcal{S}^q$ are semistable.

The degree of the variety Y (with respect to H) is by definition the top self-intersection number $\deg(Y) = \deg(\mathcal{O}_Y(1)) = H^t$. In the following we will impose on a polarized variety $(Y, \mathcal{O}_Y(1))$ of dimension t the condition that $H^i(Y, \mathcal{O}_Y(m)) = 0$ for $i = 1, \ldots, t-1$ and all m. If $Y = \operatorname{Proj} R$, where R is a standard-graded Cohen–Macaulay ring, then this property holds as a result of [5, Thm. 3.5.7].

PROPOSITION 1.1.   *Let Y denote a normal projective variety of dimension $t \geq 1$ over an algebraically closed field K of positive characteristic p. Let $\mathcal{O}_Y(1)$ denote a very ample invertible sheaf on Y such that $H^i(Y, \mathcal{O}(m)) = 0$ for $i = 1, \ldots, t-1$. Suppose that the dualizing sheaf ω_Y on Y is invertible. Let $\mathcal{S}$ denote a torsion-free coherent sheaf on Y, and suppose that the stalk $\mathcal{S}_y$ is free for every singular point $y \in Y$. Let*

$$\cdots \longrightarrow \mathcal{G}_3 \longrightarrow \mathcal{G}_2 \longrightarrow \mathcal{S} \longrightarrow 0$$

denote an exact complex of sheaves, where $\mathcal{G}_j$ has type $\mathcal{G}_j = \bigoplus_{(k,j)} \mathcal{O}_Y(-\alpha_{k,j})$. Set $\mathcal{S}_1 = \mathcal{S}$ and set $\mathcal{S}_j = \operatorname{im}(\mathcal{G}_{j+1} \to \mathcal{G}_j) = \operatorname{kern}(\mathcal{G}_j \to \mathcal{G}_{j-1})$ for $j \geq 2$. Fix $i = 1, \ldots, t$. Then, for

$$m > -q\frac{\bar{\mu}_{\min}(\mathcal{S}_{t-i+1})}{\deg(Y)} + \frac{\deg(\omega_Y)}{\deg(Y)},$$

we have $H^i(Y, \mathcal{S}^q(m)) = 0$.

Proof. Note first that the Frobenius acts flat on the exact complex and on the corresponding short exact sequences $0 \to \mathcal{S}_{j+1} \to \mathcal{G}_{j+1} \to \mathcal{S}_j \to 0$. This can be

checked locally and is true for the smooth points of Y. Over a singular point $y \in Y$ the sheaf S is free, so these short exact sequences split locally in a neighborhood of such a point and hence all the S_j are also free in y. Hence in these points also the Frobenius preserves the exactness of the complex.

By our assumption on $\mathcal{O}_Y(1)$ we have $H^i(Y, \mathcal{G}_j(m)) = 0$ for $i = 1, \dots, t-1$, all m, and all $j \geq 2$. As a result, from the short exact sequences $0 \to S_{j+1}(m) \to \mathcal{G}_{j+1}(m) \to S_j(m) \to 0$ we can infer that

$$H^i(Y, S_j(m)) \cong H^{i+1}(Y, S_{j+1}(m)) \quad \text{(isomorphisms for } i = 1, \dots, t-2\text{)},$$

$$H^{t-1}(Y, S_j(m)) \subseteq H^t(Y, S_{j+1}(m)) \quad \text{(injection for } t \geq 2\text{)},$$

$$H^t(Y, \mathcal{G}_{j+1}(m)) \to H^t(Y, S_j(m)) \quad \text{(surjection)}.$$

The same is true if we replace S_j and G_j by their respective Frobenius pull-backs S_j^q and G_j^q. For $i = 1, \dots, t$, we find that

$$H^i(Y, S_1^q(m)) \cong H^{i+1}(Y, S_2^q(m)) \cong \cdots$$
$$\cong H^{t-1}(Y, S_{t-i}^q(m)) \subseteq H^t(Y, S_{t-i+1}^q(m)).$$

Hence we need only look at $H^t(Y, S_{t-i+1}^q(m))$, which by Serre duality is dual to $\mathrm{Hom}(S_{t-i+1}^q(m), \omega_Y)$; see [10, Thm. III.7.6]. Suppose now that m fulfills the numerical condition. Then

$$\mu_{\min}(S_{t-i+1}^q(m)) = \mu_{\min}(S_{t-i+1}^q) + m \deg(Y)$$
$$\geq q\bar{\mu}_{\min}(S_{t-i+1}) + m \deg(Y)$$
$$> q\bar{\mu}_{\min}(S_{t-i+1}) + \left(-q\frac{\bar{\mu}_{\min}(S_{t-i+1})}{\deg(Y)} + \frac{\deg(\omega_Y)}{\deg(Y)}\right) \deg(Y)$$
$$= \deg(\omega_Y).$$

So for these m there are no nontrivial mappings from $S_{t-i+1}^q(m)$ to ω_Y, and therefore $H^t(Y, S_{t-i+1}^q(m)) = 0$. $\qquad\square$

REMARK 1.2.   The dualizing sheaf ω_Y on the projective variety $Y \subseteq \mathbb{P}^N$ is invertible under the condition that Y is locally a complete intersection in $\mathbb{P}^N$ and, in particular, if Y is smooth (see [10, Thm. III.7.11 & Cor. III.7.12]). If ω_Y is not invertible but is torsion free, then we may replace $\deg(\omega_Y)$ by $\mu_{\max}(\omega_Y)$ to obtain the same statement as in Proposition 1.1.

2. An Inclusion Bound for Tight Closure

We first fix the following situation, with which we shall deal in this section.

SITUATION 2.1.   Let K denote an algebraically closed field of characteristic $p > 0$. Let R denote a standard-graded normal Cohen–Macaulay domain of dimension $t + 1 \geq 2$ over K with corresponding projective normal variety $Y = \mathrm{Proj}\, R$. Suppose that the dualizing sheaf ω_Y of Y is invertible. Let $I \subseteq R$ denote a homogeneous R_+-primary ideal, and let

$$\cdots \longrightarrow F_2 = \bigoplus_{(k,2)} R(-\alpha_{k,2}) \longrightarrow F_1 = \bigoplus_{(k,1)} R(-\alpha_{k,1}) \longrightarrow I \longrightarrow 0$$

denote a homogeneous complex of graded R-modules that is exact on $D(R_+)$. Let

$$\cdots \longrightarrow \mathcal{G}_2 = \bigoplus_{(k,2)} \mathcal{O}(-\alpha_{(k,2)}) \longrightarrow \mathcal{G}_1 = \bigoplus_{(k,1)} \mathcal{O}(-\alpha_{k,1}) \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

denote the corresponding exact complex of sheaves on Y. Denote by $\mathrm{Syz}_j = \mathrm{kern}(\mathcal{G}_j \to \mathcal{G}_{j-1})$ the locally free kernel sheaves on Y, and set $\mathrm{Syz}_j(m) = \mathrm{Syz}_j \otimes \mathcal{O}_Y(m)$. Let $\nu = -\bar{\mu}_{\min}(\mathrm{Syz}_t)/\deg(Y)$, where t is the dimension of Y.

THEOREM 2.2. *Suppose Situation 2.1. Then, for all prime powers $q = p^e$, we have the inclusion $R_{>q\nu+\deg(\omega_Y)/\deg(Y)} \subseteq I^{[q]}$.*

Proof. Because I is primary, all the syzygy sheaves occurring in the resolution on Y are locally free and hence we may apply Proposition 1.1. Fix a prime power $q = p^e$. Let $h \in R$ denote a homogeneous element of degree $m > q\nu + \deg(\omega_Y)/\deg(Y)$. This gives rise, via the short exact sequence

$$0 \longrightarrow \mathrm{Syz}(f_1^q, \ldots, f_n^q)(m) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_Y(m - qd_i) \xrightarrow{f_1^q, \ldots, f_n^q} \mathcal{O}_Y(m) \longrightarrow 0$$

on Y, to a cohomology class $\delta(h) \in H^1(Y, \mathrm{Syz}(f_1^q, \ldots, f_n^q)(m))$, where

$$\mathrm{Syz}(f_1^q, \ldots, f_n^q)(m) = (F^{e*}(\mathrm{Syz}(f_1, \ldots, f_n)))(m) = \mathcal{S}^q(m)$$

and $\mathcal{S} = \mathcal{S}_1 = \mathrm{Syz}(f_1, \ldots, f_n)$. It is enough to show that $\delta(h) = 0$, for then $h \in I^{[q]}\Gamma(D(R_+), \mathcal{O}) = I^{[q]}$ since R is normal. But this follows from Proposition 1.1 applied to $\mathcal{S} = \mathrm{Syz}(f_1, \ldots, f_n)$ and $i = 1$. □

REMARK 2.3. We do not insist that the "resolution" of the ideal be exact on the whole Spec R nor that it be minimal, but it is likely that a minimal resolution will give us in general a better bound ν. For example, we can always use the Koszul complex given by ideal generators of the R_+-primary ideal I.

The next theorem gives an inclusion bound for tight closure. Recall that the tight closure of an ideal $I \subseteq R$ in a Noetherian domain containing a field of positive characteristic p is by definition the ideal

$$I^* = \{f \in R : \exists 0 \neq c \in R \text{ such that } cf^q \in I^{[q]} \text{ for all } q = p^e\}.$$

See [11] for basic properties of this closure operation.

THEOREM 2.4. *Suppose Situation 2.1. Then we have the inclusion $R_{\geq \nu} \subseteq I^*$.*

Proof. Let $f \in R$ be a homogeneous element of degree $\deg(f) = m \geq \nu = -\bar{\mu}_{\min}(\mathrm{Syz}_t)/\deg(Y)$. By the definition of tight closure we must show that $cf^q \in I^{[q]}$ holds for some $c \neq 0$ and all prime powers q. Let $c \neq 0$ be any homogeneous element of degree $> \deg(\omega_Y)/\deg(Y)$. Then $\deg(cf^q) = qm + \deg(c) > q\nu + \deg(\omega_Y)/\deg(Y)$ and therefore $cf^q \in I^{[q]}$ by Theorem 2.2. □

REMARK 2.5.   Suppose R fulfills the conditions of Situation 2.1, and let $I = (f_1, \ldots, f_n)$ denote an ideal generated by a full regular system of homogeneous parameters of degree $\deg(f_i) = d_i$ (so $n = t + 1$). Then the Koszul resolution of these elements gives a resolution on $Y = \operatorname{Proj} R$ such that the top-dimensional syzygy bundle is invertible; namely,

$$\operatorname{Syz}_t(m) = \mathcal{G}_{t+1}(m) = \mathcal{O}_Y(m - d_1 - \cdots - d_{t+1}).$$

Then Theorem 2.4 gives the known (even without the Cohen–Macaulay condition) inclusion bound $R_{\geq d_1 + \cdots + d_n} \subseteq (f_1, \ldots, f_n)^*$ (see [12, Thm. 2.9]).

The next easiest case is when the R_+-primary homogeneous ideal I has finite projective dimension (it is again enough to impose the exactness only on $D(R_+)$). In this case the resolution on Y looks like

$$0 \longrightarrow \mathcal{G}_{t+1} \longrightarrow \mathcal{G}_t \longrightarrow \cdots \longrightarrow \mathcal{G}_1 \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

and the top-dimensional syzygy bundle is $\operatorname{Syz}_t = \mathcal{G}_{t+1} = \bigoplus_k \mathcal{O}_Y(-\alpha_{k,t+1})$, so

$$\mu_{\min}(\operatorname{Syz}_t) = \deg(Y) \min_k \{-\alpha_{k,t+1}\} = -\deg(Y) \max_k \{\alpha_{k,t+1}\}.$$

The corresponding inclusion bound was proved in [13, Thm. 5.11]. Such a situation arises, for example, if I is generated by a set of monomials in a system of homogeneous parameters.

The following easy corollary unifies two known inclusion bounds for tight closure given by Smith (see [19, Props. 3.1 & 3.3])—namely, that $R_{\geq \sum_{i=1}^{n} \deg(f_i)} \subseteq I^*$ and that $R_{\geq \dim(R) \max_i \{\deg(f_i)\}} \subseteq I^*$.

COROLLARY 2.6.   *Suppose Situation 2.1 and suppose that the homogeneous R_+-primary ideal $I = (f_1, \ldots, f_n)$ is generated by homogeneous elements of degree $d_i = \deg(f_i)$. Set $d = \max_{1 \leq i_1 < \cdots < i_{\dim(R)} \leq n} (d_{i_1} + \cdots + d_{i_{\dim(R)}})$. Then $R_{\geq d} \subseteq I^*$.*

Proof. We consider the Koszul resolution of $I = (f_1, \ldots, f_n)$, which is exact outside the origin. This gives the surjection

$$\bigoplus_{1 \leq i_1 < \cdots < i_{\dim(R)} \leq n} \mathcal{O}(-d_{i_1} - \cdots - d_{i_{\dim(R)}}) \longrightarrow \operatorname{Syz}_{\dim(R)-1} \longrightarrow 0,$$

which shows that

$$\bar{\mu}_{\min}(\operatorname{Syz}_{\dim(R)-1}) \geq \bar{\mu}_{\min}\left(\bigoplus_{1 \leq i_1 < \cdots < i_{\dim(R)} \leq n} \mathcal{O}(-d_{i_1} - \cdots - d_{i_{\dim(R)}}) \right)$$

$$= - \max_{1 \leq i_1 < \cdots < i_{\dim(R)} \leq n} \{d_{i_1} + \cdots + d_{i_{\dim(R)}}\} \deg(Y).$$

Hence $\nu = -\bar{\mu}_{\min}(\operatorname{Syz}_{\dim(R)-1})/\deg(Y) \leq \max\{d_{i_1} + \cdots + d_{i_{\dim(R)}}\}$ and Theorem 2.4 applies. $\square$

REMARK 2.7.   Theorem 2.4 was proved for $\dim(R) = 2$ in [2, Thm. 6.4] using somewhat more geometric methods. In this case $Y = \operatorname{Proj} R$ is a smooth projective curve and the top syzygy bundle is just the first syzygy bundle, and the result

also holds in characteristic 0 for solid closure; see [2] and [3] for concrete computations of the number ν in this case. In general it is difficult to compute the number ν of the theorem, just as it is difficult to compute the minimal slope of a locally free sheaf.

The following corollary gives an inclusion bound for tight closure under the condition that the top-dimensional syzygy bundle is strongly semistable. In the 2-dimensional situation this bound is exact in the sense that, below this bound, an element belongs to the tight closure only if it belongs to the ideal itself; see [2, Thm. 8.4].

COROLLARY 2.8. *Suppose Situation 2.1 and let $I = (f_1, \ldots, f_n)$ be generated by homogeneous elements of degree $d_i = \deg(f_i)$. Let $F_\bullet \to I$ denote the Koszul complex and suppose that the top-dimensional syzygy bundle Syz_t is strongly semistable. Set $d = (\dim(R) - 1)(d_1 + \cdots + d_n)/(n - 1)$. Then $R_{\geq d} \subseteq I^*$.*

Proof. The condition "strongly semistable" means that $\mu(\mathrm{Syz}_t) = \bar{\mu}_{\min}(\mathrm{Syz}_t)$, so we only have to compute the degree and the rank of Syz_t. It is easy to compute that $\det(\mathrm{Syz}_t) = \mathcal{O}_Y\big(\binom{n-2}{t-1}\big(-\sum_{i=1}^n d_i\big)\big)$; hence

$$\deg(\mathrm{Syz}_t) = \binom{n-2}{t-1}\bigg(-\sum_{i=1}^n d_i\bigg)\deg(Y)$$

and $\mathrm{rk}(\mathrm{Syz}_t) = \binom{n-1}{t}$. Therefore,

$$\mu(\mathrm{Syz}_t) = \binom{n-2}{t-1}\bigg(-\sum_{i=1}^n d_i\bigg)\deg(Y)\bigg/\binom{n-1}{t} = \frac{t}{n-1}\bigg(-\sum_{i=1}^n d_i\bigg)\deg(Y)$$

and $\nu = \frac{t}{n-1}\big(\sum_{i=1}^n d_i\big)$. $\square$

REMARK 2.9. As the proofs of Theorem 2.4 and Proposition 1.1 show, Corollary 2.8 is also true under the weaker condition that there does not exist any nontrivial mapping $\mathrm{Syz}_t^q \to \mathcal{L}$ to any invertible sheaf $\mathcal{L}$ that contradicts the semistability of Syz_t^q for all $q = p^e$.

EXAMPLE 2.10. Theorem 2.4 applies in particular when R is a normal complete intersection domain. Let $R = K[X_1, \ldots, X_N]/(H_1, \ldots, H_r)$, where the H_j are homogeneous forms of degree δ_j. Then $\omega_Y = \mathcal{O}\big(\sum_j \delta_j - N\big)$. Therefore, the number $\deg(\omega_Y)/\deg(Y) = \sum_j \delta_j - N$ is just the a-invariant of R.

EXAMPLE 2.11. We want to apply Corollary 2.8 to the computation of the tight closure $(x^a, y^a, z^a, w^a)^*$ in $R = K[x, y, z, w]/(H)$, where H is supposed to be a polynomial of degree 4 defining a smooth projective (hyper-)surface

$$Y = V_+(H) = \mathrm{Proj}\, R \subset \mathbb{P}^3 = \mathrm{Proj}\, K[x, y, z, w]$$

of degree 4; hence Y is a $K3$ surface. Our result will hold true only for generic choice of H. We look at the Koszul complex on $\mathbb{P}^3$ defined by x^a, y^a, z^a, w^a and break it up to get

$$0 \longrightarrow \mathrm{Syz}_2 \cong \bigwedge^2 \mathrm{Syz} \longrightarrow \bigoplus_6 \mathcal{O}_{\mathbb{P}^3}(-2a)$$
$$\longrightarrow \bigoplus_4 \mathcal{O}_{\mathbb{P}^3}(-a) \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow 0.$$

Suppose first that K is an algebraically closed field of characteristic 0. It is easy to see that the syzygy bundle $\mathrm{Syz} = \mathrm{Syz}(x^a, y^a, z^a, w^a)$ is semistable on $\mathbb{P}^3$ [1, Cor. 3.6 or Cor. 6.4]. Hence also the exterior power $\mathrm{Syz}_2 \cong \bigwedge^2 \mathrm{Syz}$ is semistable on $\mathbb{P}^3$. By the restriction theorem of Flenner [7, Thm. 1.2] it follows that the restriction $\mathrm{Syz}_2|_Y$ is also semistable on the generic hypersurface $Y = V_+(H)$.

On the other hand, by Noether's theorem (see [9, Sec. IV.4]), every curve on the generic surface of degree 4 in $\mathbb{P}^3$ is a complete intersection and $R = K[x, y, z, w]/(H)$ is a factorial domain for generic H of degree 4. It follows that the cotangent bundle Ω_Y on $Y = V_+(H)$ is semistable. For the semistability of a rank-2 bundle we need only look at mappings $\mathcal{L} \to \Omega_Y$, where $\mathcal{L}$ is invertible. But since $\mathcal{L} = \mathcal{O}_Y(k)$, the semistability follows because Y is a $K3$ surface and so Ω_Y has degree 0 but does not have any global nontrivial section (see [8, Sec. IV.5]).

So for H generic the relevant second syzygy bundle $\mathrm{Syz}_2|_Y$ and the cotangent bundle Ω_Y are both semistable in characteristic 0. Since the $\mathbb{Q}$-rational points are dense in $\mathbb{A}_K^N$, there exist also such polynomials H with rational coefficients and then also with integer coefficients. We consider such a polynomial H with integer coefficients as defining a family of quartics over $\mathrm{Spec}\,\mathbb{Z}$. Since semistability is an open property, we infer that the second syzygy bundle and the cotangent bundle are also semistable on $Y_p = V_+(H_p)$ for $p \gg 0$.

By the semistability of Ω_{Y_p} ($p \gg 0$), the maximal slope of Ω_{Y_p} is ≤ 0. A theorem of Langer [16, Cors. 2.4 & 6.3] shows then that every semistable bundle on Y_p is already strongly semistable. Hence the second syzygy bundle is also strongly semistable. We are thus in the situation of Corollary 2.8 and so may compute $d = 8a/3$. Therefore,
$$R_{8a/3} \subseteq (x^a, y^a, z^a, w^a)^*$$
holds in $R = K[x, y, z, w]/(H)$ for H generic of degree 4 and for $p \gg 0$. The first nontrivial instance is for $a = 3$. In fact, for the (nongeneric) Fermat quartic $x^4 + y^4 + z^4 + w^4 = 0$ it was proved directly by Singh [18, Thm. 4.1] that $x^2 y^2 z^2 w^2 \in (x^3, y^3, z^3, w^3)^*$.

For the next corollary we recall the definition of the Frobenius closure. Suppose that R is a Noetherian ring containing a field of positive characteristic $p > 0$, and let I denote an ideal. Then the Frobenius closure of I is defined by
$$I^F = \{f \in R : \exists q = p^e \text{ such that } f^q \in I^{[q]}\}.$$

It is easy to see that the Frobenius closure of an ideal is contained in its tight closure.

COROLLARY 2.12.   *Suppose Situation 2.1. Then $R_{>\nu} \subseteq I^F$, the Frobenius closure of I.*

Proof. Let f denote a homogeneous element of degree $m = \deg(f) > v = -\bar{\mu}_{\min}(\mathrm{Syz}_t)/\deg(Y)$. Then we need only take a prime power $q = p^e$ such that $\deg(f^q) = qm > qv + \deg(\omega_Y)/\deg(Y)$ holds. Now $f^q \in I^{[q]}$ holds on account of Theorem 2.2. $\square$

EXAMPLE 2.13. Corollary 2.12 is not true for $R_{\geq v}$ instead of $R_{>v}$. This is already clear for parameter ideals in dimension 2—say, for (x, y) in $R = K[x, y, z]/(H)$, where H defines a smooth projective curve $Y = \mathrm{Proj}\, R = V_+(H) \subset \mathbb{P}^2$. Here we have the resolution

$$0 \longrightarrow \mathcal{O}_Y(-2) \cong \mathrm{Syz}(x, y)(0) \longrightarrow \mathcal{O}_Y(-1) \oplus \mathcal{O}_Y(-1) \xrightarrow{x,y} \mathcal{O}_Y \longrightarrow 0.$$

Hence we get $v = 2$, but an element of degree 2 (say, z^2) does not in general belong to the Frobenius closure of (x, y).

REMARK 2.14. A problem of Katzman and Sharp [15] asks, in its strongest form: Does there exist a number b such that, if $f \in I^F$ holds, then already $f^{p^b} \in I^{[p^b]}$ holds? A positive answer (together with knowledge of a bound for the number b) to this question would give a finite test for checking whether or not a given element f belongs to the Frobenius closure I^F. For those elements that belong to I^F by virtue of Corollary 2.12 (owing to, say, degree reasons) the answer is Yes—at least in the sense that for f fulfilling $\deg(f) \geq v + \varepsilon$ ($\varepsilon > 0$) we have $\deg(f^q) = q \deg(f) \geq qv + q\varepsilon$, so the condition $q\varepsilon > \deg(\omega_Y)/\deg(Y)$ is sufficient to ensure that $f^q \in I^{[q]}$. It is possible, however, that elements of degree $\deg(f) \leq v$ belong to the Frobenius closure.

3. The Castelnuovo–Mumford Regularity of Frobenius Powers

We recall briefly the notion of Castelnuovo–Mumford regularity, following [4, Def. 15.2.9]. Let R denote a standard-graded ring and let M denote a finitely generated graded R-module. Then the Castelnuovo–Mumford regularity of M (or regularity of M, for short) is

$$\mathrm{reg}(M) = \sup\{\mathrm{end}(H^i_{R_+}(M)) + i : 0 \leq i \leq \dim M\},$$

where $\mathrm{end}(N)$ of a graded R-module N denotes the maximal degree e such that $N_e \neq 0$. For a number l we define the regularity $\mathrm{reg}^l(M)$ at and above level l by

$$\mathrm{reg}^l(M) = \sup\{\mathrm{end}(H^i_{R_+}(M)) + i : l \leq i \leq \dim M\}.$$

A question raised by Katzman in the Introduction of [14] asks how the regularity of the Frobenius powers $I^{[q]}$ behaves, in particular whether there exists a linear bound $\mathrm{reg}(I^{[q]}) \leq C_1 q + C_0$. Such a linear bound for the regularity of the Frobenius powers of an ideal was given by Chardin in [6, Thm. 2.3]. The following theorem gives a better linear bound for the regularity of Frobenius powers of I in terms of the slope of the syzygy bundles.

THEOREM 3.1.   *Let K denote an algebraically closed field of positive characteristic p. Let R denote a standard-graded normal Cohen–Macaulay K-domain of dimension $t + 1 \geq 2$. Let $I = (f_1, \ldots, f_n) \subseteq R$ denote a homogeneous ideal generated by homogeneous elements of degree $d_i = \deg(f_i)$. Suppose that the dualizing sheaf ω_Y on $Y = \operatorname{Proj} R$ is invertible, and suppose that the points $y \in \sup(\mathcal{O}_Y/\mathcal{I})$ are smooth points of Y. Let $F_\bullet \to I$ denote a graded free resolution with corresponding exact complex of sheaves on Y, $\mathcal{G}_\bullet \to \mathcal{I} \subseteq \mathcal{O}_Y$. Set $\operatorname{Syz}_j = \operatorname{kern}(\mathcal{G}_j \to \mathcal{G}_{j-1})$. Then for the Castelnuovo–Mumford regularity of the Frobenius powers $I^{[q]}$ we have the linear bound $\operatorname{reg}(I^{[q]}) \leq C_1 q + C_0$, where*

$$C_1 = \max\left\{ d_i \ (i = 1, \ldots, n), \ -\frac{\bar{\mu}_{\min}(\operatorname{Syz}_j)}{\deg(Y)} \ (j = 1, \ldots, t = \dim(Y)) \right\},$$

$$C_0 = \max\left\{ \operatorname{reg}(R), \frac{\deg(\omega_Y)}{\deg(Y)} \right\}.$$

Proof. For $q = p^e$, the ideal generators define the homogeneous short exact sequences

$$0 \longrightarrow \operatorname{Syz}(f_1^q, \ldots, f_n^q) \longrightarrow \bigoplus_{i=1}^n R(-qd_i) \xrightarrow{f_1^q, \ldots, f_n^q} I^{[q]} \longrightarrow 0$$

of graded R-modules. It is an easy exercise [4, Ex. 15.2.15] to show that $\operatorname{reg}(N) \leq \max\{\operatorname{reg}^1(L) - 1, \operatorname{reg}(M)\}$ for a short exact sequence $0 \to L \to M \to N \to 0$. We have $\operatorname{reg}(R(-qd)) = \operatorname{reg}(R) + qd$ and

$$\operatorname{reg}\left(\bigoplus_{i=1}^n R(-qd_i) \right) = \max_i\{\operatorname{reg}(R(-qd_i))\} = \operatorname{reg}(R) + q \max_i\{d_i\},$$

which gives the first terms in the definition of C_1 and C_0, respectively. Hence it is enough to give a linear bound for $\operatorname{reg}^1(\operatorname{Syz}(f_1^q, \ldots, f_n^q))$. Moreover, the long exact local cohomology sequence associated to the preceding short exact sequence gives

$$\cdots \longrightarrow H^0_{R_+}(I^{[q]}) \longrightarrow H^1_{R_+}(\operatorname{Syz}(f_1^q, \ldots, f_n^q))$$

$$\longrightarrow \bigoplus_{i=1}^n H^1_{R_+}(R(-qd_i)) \longrightarrow \cdots.$$

The term on the right is 0 because R is Cohen–Macaulay, and the term on the left is 0 because R is a domain. Hence $H^1_{R_+}(\operatorname{Syz}(f_1^q, \ldots, f_n^q)) = 0$ and we must find a linear bound for $\operatorname{reg}^2(\operatorname{Syz}(f_1^q, \ldots, f_n^q)) = \operatorname{reg}^1(\operatorname{Syz}(f_1^q, \ldots, f_n^q))$. We have $H^i_{R_+}(\operatorname{Syz}(f_1^q, \ldots, f_n^q)) = H^{i-1}(D(R_+), \operatorname{Syz}(f_1^q, \ldots, f_n^q)\tilde{\ })$ for $i \geq 2$ owing to the long exact sequence relating local cohomology with sheaf cohomology. Denote now by $\operatorname{Syz}(f_1^q, \ldots, f_n^q)$ the corresponding torsion-free sheaf on $Y = \operatorname{Proj} R$. On Y we have the short exact sequences of sheaves

$$0 \longrightarrow \operatorname{Syz}(f_1^q, \ldots, f_n^q) \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_Y(-qd_i) \longrightarrow \mathcal{I}^{[q]} \longrightarrow 0.$$

We may compute the cohomology as

$$H^i(D_+(R), \mathrm{Syz}(f_1^q, \ldots, f_n^q)^\sim)_m = H^i(Y, \mathrm{Syz}(f_1^q, \ldots, f_n^q)(m)).$$

Note that, by assumption, the syzygy bundle $\mathrm{Syz}(f_1, \ldots, f_n)$ is free in the singular points of Y. Hence we are in the situation of Proposition 1.1 with $\mathcal{S} = \mathrm{Syz}(f_1, \ldots, f_n)$; therefore, $H^i(Y, \mathrm{Syz}(f_1^q, \ldots, f_n^q)(m)) = 0$ $(i = 1, \ldots, t)$ holds for $m > \max_{j=1,\ldots,t}\{-q(\bar\mu_{\min}(\mathrm{Syz}_j)/\deg(Y))\} + \deg(\omega_Y)/\deg(Y)$, which proves the theorem. $\qquad\square$

REMARK 3.2. The Castelnuovo–Mumford regularity of a standard-graded Cohen–Macaulay domain R is just $\mathrm{reg}(R) = \mathrm{end}(H_{R_+}^{\dim(R)}(R)) + \dim(R)$. The end of the top-dimensional local cohomology module of a graded ring is also called its a-invariant (see [4, 13.4.7]), hence $\mathrm{reg}(R) = a + \dim(R)$. If R is Gorenstein, then $R(a)$ is the canonical module of R and $\omega_Y = \mathcal{O}_Y(a)$ is the dualizing sheaf on $Y = \mathrm{Proj}\, R$. So in this case the quotient $\deg(\omega_Y)/\deg(Y) = a\deg(Y)/\deg(Y) = a$ equals also the a-invariant.

REMARK 3.3. The surjection $\bigoplus_{(k,j+1)} \mathcal{O}_Y(-\alpha_{k,j+1}) \to \mathrm{Syz}_j \to 0$ gives at once the bound $\bar\mu_{\min}(\mathrm{Syz}_j) \geq \bar\mu_{\min}(\bigoplus_{(k,j+1)} \mathcal{O}_Y(-\alpha_{k,j+1})) = -\max\{\alpha_{k,j+1}\}\deg(Y)$. Hence for the constant C_1 from Theorem 3.1 we obtain the estimate $C_1 \leq \max\{\alpha_{k,j} : j = 1, \ldots, t+1 = \dim(R)\} = C_1'$. This number C_1' is the coefficient for the linear bound that Chardin obtained in [6, Thm. 2.3]. This bound corresponds to the inclusion bounds for tight closure of K. Smith that we obtained in Corollary 2.6. The following standard example of tight closure theory shows already the difference between the Chardin–Smith bound and the slope bound.

EXAMPLE 3.4. Consider the ideal $I = (x^2, y^2, z^2)$ in

$$R = \frac{K[z, y, z]}{x^3 + y^3 + z^3}, \quad \mathrm{char}(K) \neq 3.$$

We compute the bound from Theorem 3.1 for the regularity of the Frobenius powers $I^{[q]} = (x^{2q}, y^{2q}, z^{2q})$. First observe that we may consider the curve equation $0 = x^3 + y^3 + z^3 = xx^2 + yy^2 + zz^2$ as a global section of the syzygy bundle of degree 3. Since this section has no zero on $Y = \mathrm{Proj}\, R$, we get the short exact sequence

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathrm{Syz}(x^2, y^2, z^2)(3) \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

This shows that the syzygy bundle is strongly semistable, and therefore

$$\bar\mu_{\min}(\mathrm{Syz}(x^2, y^2, z^2)(0)) = -6\deg(Y)/2 = -9.$$

So $C_1 = 3$ and we thus obtain the bound $\mathrm{reg}(I^{[q]}) \leq 3q + 2$.

Note that $\mathrm{Syz}(x^2, y^2, z^2)(3)$ is not generated by its global sections, because the section just mentioned is the only section. Consequently, a surjection $\bigoplus_k \mathcal{O}(-\alpha_k) \to \mathrm{Syz}(x^2, y^2, z^2)(0)$ is possible only for $\max_k\{\alpha_k\} \geq 4$. Hence the slope bound for regularity is better than the linear bound obtained by considering only the degrees in a resolution.

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