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On Representation of Integers by Sums of a Cube and Three Cubes of Primes

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1. Introduction

We consider the expression of positive integers n as the sum of a cube and three cubes of primes; that is,

$$n = m^3 + p_2^3 + p_3^3 + p_4^3, \tag{1.1}$$

where m is a positive integer and p_j are primes. In 1949, Roth [8] proved that almost all positive integers n can be written as (1.1). More precisely, let $E(N)$ denote the number of positive integers $n \le N$ that cannot be written in the form (1.1); then Roth's theorem actually stated that $E(N) \ll N \log^{-A} N$ for arbitrary $A > 0$. In 1995, Brüdern [1] proved that the same exceptional set estimate holds for the number of positive integers $n \equiv 4 \pmod{18}$, not exceeding N, that cannot be written in the form (1.1) with m restricted to a P_4-number. Later Kawada [2] further strengthened this by replacing m by a P_3-number. All these results can be viewed as approximations to the conjecture that all sufficiently large integers satisfying some necessary congruence conditions are the sum of four cubes of primes. As is well known, the quality of the approximation is indicated in the upper bound of $E(N)$. Roth's theorem has been improved by Ren [5] to $E(N) \ll N^{169/170}$ and by Ren and Tsang [7] to $E(N) \ll N^{1271/1296+\varepsilon}$. These improvements were obtained via new approaches for enlarging major arcs in the circle method used (see e.g. [4; 5; 7]). In this paper, based on the major arcs estimate in [7], we use some new ideas to handle the minor arcs and prove the following.

Theorem 1. *For $E(N)$ as just defined, we have*

$$E(N) \ll N^{17/18+\varepsilon}.$$

Notation. As usual, $\Lambda(n)$ stands for the von Mangoldt function. In our statement, N is a large positive integer and $L = \log N$. The notation $r \sim R$ means $R < r \le 2R$. The letters ε and A denote positive constants that are (respectively) arbitrarily small and arbitrarily large; they may assume different values at each occurrence.

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2. Proof of Theorem 1

Following [7], we introduce the notation

$$U = (N/9)^{1/3} \quad \text{and} \quad V = U^{5/6}. \tag{2.1}$$

In order to apply the circle method, for large positive integer N and positive real number θ we let

$$P(\theta) = U^{\theta} \quad \text{and} \quad Q(\theta) = NP^{-1} = 9U^{3-\theta}. \tag{2.2}$$

As usual, we shall define the major arcs $\mathfrak{M}(\theta)$ to be the union of all intervals $[a/q - 1/(qQ(\theta)), a/q + 1/(qQ(\theta))]$, where a and q are coprime integers and $1 \le a \le q \le P(\theta)$. Let the minor arcs $\mathfrak{m}(\theta)$ be the complement of $\mathfrak{M}(\theta)$ in the unit interval $I(\theta) = [1/Q(\theta), 1 + 1/Q(\theta)]$.

We define

$$T(\alpha) = \sum_{m \sim U} e(m^3 \alpha)$$

and, for $W > 0$,

$$S(\alpha, W) = \sum_{m \sim W} \Lambda(m) e(m^3 \alpha).$$

Let

$$R(n) = \sum_{\substack{n = m_1^3 + \cdots + m_4^3 \\ m_1, m_4 \sim U,\, m_2, m_3 \sim V}} \Lambda(m_1)\Lambda(m_2)\Lambda(m_3).$$

Then

$$R(n) = \int_{I(\theta)} S(\alpha, U) S^2(\alpha, V) T(\alpha) e(-n\alpha)\, d\alpha = \int_{\mathfrak{M}(\theta)} + \int_{\mathfrak{m}(\theta)}. \tag{2.3}$$

For the major arcs estimate, we quote [7, Thm. 2] and record it in the following lemma.

LEMMA 2.1. *Let $\theta < 25/72$. For all integers n with $N/2 \le n \le N$,*

$$\int_{\mathfrak{M}(\theta)} S(\alpha, U) S^2(\alpha, V) T(\alpha) e(-n\alpha)\, d\alpha = \mathfrak{S}(n) J(n) + O(V^2 U^{-1} L^{-A}),$$

where $\mathfrak{S}(n)$ is the singular series in this problem that satisfies

$$(\log\log n)^{-c_0} \ll \mathfrak{S}(n) \ll \log n$$

for a certain positive constant c_0 and where $J(n)$ is a multiple integral that satisfies

$$V^2 U^{-1} \ll J(n) \ll V^2 U^{-1}.$$

In this paper we will concentrate on the minor arcs estimates. Our main result is the following.

LEMMA 2.2. *We have*

$$\int_{\mathfrak{m}(25/72 - \varepsilon)} |S(\alpha, U)|^2 |S(\alpha, V)|^4 |T(\alpha)|^2\, d\alpha \ll U^{5/2+\varepsilon} V^2.$$

This lemma is proved in Section 3.

Proof of Theorem 1. We start from (2.3), where the major arcs estimate is taken care of by Lemma 2.1. As regards the minor arcs, by Bessel's inequality and Lemma 2.2 we have

$$\sum_{N/2<n\le N}\left|\int_{\mathfrak{m}(25/72-\varepsilon)} S(\alpha,U)S^2(\alpha,V)T(\alpha)e(-n\alpha)\,d\alpha\right|^2$$
$$\ll \int_{\mathfrak{m}(25/72-\varepsilon)} |S(\alpha,U)|^2|S(\alpha,V)|^4|T(\alpha)|^2\,d\alpha \ll U^{5/2+\varepsilon}V^2. \quad (2.4)$$

By a standard argument we derive that, for all $N/2 < n \le N$ and with at most $O(U^{9/2+3\varepsilon}V^{-2})$ exceptions,

$$\int_{\mathfrak{m}(25/72-\varepsilon)} S(\alpha,U)S^2(\alpha,V)T(\alpha)e(-n\alpha)\,d\alpha \ll V^2U^{-1-\varepsilon}.$$

This together with Lemma 2.1 proves that, for these n,

$$R(n) = \mathfrak{S}(n)J(n) + O(V^2U^{-1}L^{-A})$$

and hence n can be written as (1.1). Let $F(N)$ be the number of the foregoing exceptional n; then we have

$$F(N) \ll U^{9/2+3\varepsilon}V^{-2} = (N/9)^{17/18+\varepsilon}.$$

The assertion of Theorem 1 now follows because $E(N) = \sum_{j\ge 0} F(N/2^j)$. $\square$

3. Proof of Lemma 2.2

In order to prove Lemma 2.2, we need the following lemmas. Lemma 3.1 is obtained by letting $k = 3$ in Theorem 1 of [6]; Lemma 3.2 is due to Vaughan [9]; and Lemma 3.3 is Lemma 2.4 in [3].

LEMMA 3.1. *Suppose $\alpha = a/q + \lambda$, where a, q are integers with $q \ge 1$ and $(a,q) = 1$ and where $\lambda \in \mathbb{R}$. Then we have*

$$S(\alpha,W) \ll q^{\varepsilon}(\log^c W)\left\{W^{1/2}q^{1/2}\sqrt{1+|\lambda|W^3} + W^{4/5} + \frac{Wq^{-1/2}}{\sqrt{1+|\lambda|W^3}}\right\},$$

where c is an absolute positive constant.

LEMMA 3.2. *Let Z_0 denote the number of solutions of the equation $m_1^3+n_1^3+n_2^3 = m_2^3+n_3^3+n_4^3$, subject to $m_j \sim U$ and $n_j \sim V$. Then $Z_0 \ll U^{1+\varepsilon}V^2$.*

LEMMA 3.3. *For $k \ge 3$, let $\omega_k(q)$ be the multiplicative function defined by*

$$\omega_k(p^{ku+v}) = \begin{cases} kp^{-u-1/2} & \text{if } u \ge 0 \text{ and } v = 1, \\ p^{-u-1} & \text{if } u \ge 0 \text{ and } 2 \le v \le k. \end{cases}$$

Suppose that η and ξ are real numbers satisfying $\eta > 0$, $\xi \ge 2\eta + 2$, and $\xi \ge k\eta + 1$. Then, whenever $X \ge 2$,

$$\sum_{1\le q\le X} q^{\eta}\omega_k^{\xi}(q) \ll \begin{cases} 1 & \text{if } \xi > k\eta+1, \\ \log X & \text{if } \xi = k\eta+1. \end{cases}$$

The implied constant depends at most on k, η, *and* ξ.

Proof of Lemma 2.2. Let $\alpha \in \mathfrak{m}(25/72-\varepsilon)$. Then, by Dirichlet's lemma on rational approximations, there exist coprime integers a, q and a real number λ satisfying

$$1 \le q \le 24U^2 \quad \text{and} \quad |\lambda| \le 1/(24qU^2) \tag{3.1}$$

such that $\alpha = a/q + \lambda$. If $U < q \le 24U^2$, we apply Weyl's inequality to get

$$|T(\alpha)| \ll U^{3/4+\varepsilon}. \tag{3.2}$$

If $1 \le q \le U$, we combine the conclusions of Lemmas 6.1 and 6.2 in [10] and (2.1)–(2.3) in [3] to obtain

$$|T(\alpha)| \ll \frac{\omega(q)U}{1+|\lambda|U^3} + q^{1/2+\varepsilon}, \tag{3.3}$$

where $\omega(q) = \omega_3(q)$ is as defined in Lemma 3.3 and also satisfies

$$q^{-1/2} \ll \omega(q) \ll q^{-1/3}. \tag{3.4}$$

Let $0 < b < 1$. Then, for q, λ satisfying either $U^b \le q \le U$ or

$$1 \le q \le U^b \quad \text{and} \quad \omega(q)U^{b/3-3} < |\lambda| \le 1/(24qU^2),$$

it follows that

$$|T(\alpha)| \ll U^{1-b/3}. \tag{3.5}$$

Now let $\mathfrak{D}(b)$ be the set of all $\alpha = a/q + \lambda \in \mathfrak{m}(25/72-\varepsilon)$ with q, λ satisfying

$$1 \le q \le U^b \quad \text{and} \quad |\lambda| \le \omega(q)U^{b/3-3}.$$

Then one concludes from (3.2)–(3.5) that

$$\max_{\mathfrak{m}(25/72-\varepsilon)\setminus\mathfrak{D}(b)} |T(\alpha)| \ll \max\{U^{1-b/3}, U^{3/4+\varepsilon}\}.$$

Therefore, on choosing $b = 3/4$ we obtain

$$\begin{aligned} &\int_{\mathfrak{m}(25/72-\varepsilon)} |S(\alpha,U)|^2 |S(\alpha,V)|^4 |T(\alpha)|^2 \, d\alpha \\ &\ll \int_{\mathfrak{D}(3/4)} |S(\alpha,U)|^2 |S(\alpha,V)|^4 |T(\alpha)|^2 \, d\alpha + U^{3/2+\varepsilon} \int_0^1 |S(\alpha,U)|^2 |S(\alpha,V)|^4 \, d\alpha. \end{aligned}$$

By Lemma 3.2, the last term is $\le U^{3/2+\varepsilon} Z_0 \ll U^{5/2+\varepsilon} V^2$. So it remains to prove

$$\int_{\mathfrak{D}(3/4)} |S(\alpha,U)|^2 |S(\alpha,V)|^4 |T(\alpha)|^2 \, d\alpha \ll U^{5/2+\varepsilon} V^2. \tag{3.6}$$

For $\alpha = a/q + \lambda \in \mathfrak{D}(3/4)$, we have either

$$U^{25/72-\varepsilon} < q \le U^{3/4} \quad \text{and} \quad |\lambda| \le \omega(q)U^{1/4-3}$$

or

$$1 \le q \le U^{25/72-\varepsilon} \quad \text{and} \quad 1/(qU^{191/72+\varepsilon}) < |\lambda| \le \omega(q)U^{1/4-3}.$$

By (3.3), where the right-hand side is dominated by $\omega(q)U(1+|\lambda|U^3)^{-1}$, it follows that

$$\begin{aligned}
&\int_{\mathfrak{D}(3/4)} |S(\alpha,U)|^2|S(\alpha,V)|^4|T(\alpha)|^2\,d\alpha \\
&\quad \ll U^2 \sum_{1\le q\le U^{25/72-\varepsilon}} \sum_{\substack{a=1\\(a,q)=1}}^{q} \omega^2(q) \\
&\qquad \times \int_{1/(qU^{191/72+\varepsilon})<|\lambda|\le\omega(q)U^{1/4-3}} \frac{|S(a/q+\lambda,U)|^2|S(a/q+\lambda,V)|^4}{(1+|\lambda|U^3)^2}\,d\lambda \\
&\qquad + U^2 \sum_{U^{25/72-\varepsilon}<q\le U^{3/4}} \sum_{\substack{a=1\\(a,q)=1}}^{q} \omega^2(q) \\
&\qquad \times \int_{|\lambda|\le\omega(q)U^{1/4-3}} \frac{|S(a/q+\lambda,U)|^2|S(a/q+\lambda,V)|^4}{(1+|\lambda|U^3)^2}\,d\lambda \\
&\quad := M_1 + M_2 \quad \text{(say)}.
\end{aligned} \tag{3.7}$$

To estimate M_1 and M_2, we observe that $|\lambda|V^3 \le 1$ for $|\lambda| \le \omega(q)U^{1/4-3}$. Hence, by Lemma 3.1 we have

$$S(a/q+\lambda,V) \ll V^{\varepsilon}\{V^{1/2}q^{1/2} + V^{4/5} + Vq^{-1/2}\}. \tag{3.8}$$

For $q \le U^{25/72-\varepsilon}$, this gives

$$|S(a/q+\lambda,V)|^4 \ll V^{\varepsilon}\{V^{16/5} + V^4q^{-2}\}.$$

By Lemma 3.1, we also have

$$\frac{|S(a/q+\lambda,U)|^2}{1+|\lambda|U^3} \ll U^{\varepsilon}\left\{Uq + \frac{U^{8/5}}{1+|\lambda|U^3} + \frac{U^2q^{-1}}{(1+|\lambda|U^3)^2}\right\}. \tag{3.9}$$

For $|\lambda| > 1/(qU^{191/72-\varepsilon})$, this gives

$$\frac{|S(a/q+\lambda,U)|^2}{1+|\lambda|U^3} \ll qU^{47/36+\varepsilon}.$$

Therefore,

$$\begin{aligned}
M_1 &\ll U^{2+47/36+\varepsilon} \sum_{1\le q\le U^{25/72-\varepsilon}} q^2\omega^2(q)\{V^{16/5}+V^4q^{-2}\} \int_0^1 \frac{d\lambda}{1+|\lambda|U^3} \\
&\ll U^{11/36+\varepsilon} \sum_{1\le q\le U^{25/72-\varepsilon}} \omega^2(q)\{q^2V^{16/5}+V^4\}.
\end{aligned}$$

By Lemma 3.3, for any $X > Y \ge 1$ we have

$$\sum_{Y\le q\le X} \omega^2(q) \ll \sum_{Y\le q\le X} q^{1/2}\omega^3(q) \ll 1. \tag{3.10}$$

This yields
$$M_1 \ll U^{11/36+\varepsilon}\{V^{16/5}U^{25/36} + V^4\} \ll U^{2+\varepsilon}V^2. \tag{3.11}$$

We now turn to M_2. Again by (3.8) and (3.9), we have
$$|S(a/q+\lambda, V)|^2 \ll V^\varepsilon\{Vq + V^{8/5} + V^2q^{-1}\}$$
and
$$\frac{|S(a/q+\lambda, U)|^2}{1+|\lambda|U^3} \ll U^\varepsilon\{Uq + U^{8/5} + U^2q^{-1}\}.$$
Thus,
$$M_2 \ll U^{2+\varepsilon} \sum_{U^{25/72-\varepsilon}<q\le U^{3/4}} \omega^2(q)\{Vq + V^{8/5} + V^2q^{-1}\}\{Uq + U^{8/5} + U^2q^{-1}\}$$
$$\times \int_{|\lambda|\le\omega(q)U^{1/4-3}} \sum_{a=1}^{q} |S(a/q+\lambda, V)|^2\, d\lambda. \tag{3.12}$$
It follows that
$$\sum_{a=1}^{q} |S(a/q+\lambda, V)|^2 = \sum_{m_i\sim V} \Lambda(m_1)\Lambda(m_2) \sum_{a=1}^{q} e((a/q+\lambda)(m_1^3 - m_2^3))$$
$$\ll qL^2 \sharp\{m_i \sim V : m_1^3 \equiv m_2^3 \ (\mathrm{mod}\, q),\ \Lambda(m_i) \ne 0\}.$$

Consider the congruence $m_1^3 \equiv m_2^3$ (mod q), where $m_i \sim V$ are prime powers. If $(m_1m_2, q) > 1$ then m_1 must be equal to m_2. On the other hand, if $(m_1m_2, q) = 1$ then $m_1 \equiv \eta m_2$ (mod q) for a certain root η of the cubic congruence $x^3 \equiv 1$ (mod q). Clearly there are $\ll q^\varepsilon$ such cubic roots of unity. Hence,
$$\sharp\{m_i \sim V : m_1^3 \equiv m_2^3 \ (\mathrm{mod}\, q),\ \Lambda(m_i) \ne 0\} \ll q^\varepsilon(V^2/q) + V$$
and
$$\sum_{a=1}^{q} |S(a/q+\lambda, V)|^2 \ll q^\varepsilon V^2 L^2.$$

Putting this into (3.12) and then applying the second inequality in (3.10), we obtain
$$M_2 \ll U^{-3/4+\varepsilon}V^2$$
$$\times \max_{U^{25/72-\varepsilon}<q\le U^{3/4}} q^{-1/2}\{Vq + V^{8/5} + V^2q^{-1}\}\{Uq + U^{8/5} + U^2q^{-1}\}.$$
Since $V = U^{5/6}$, for $q > U^{25/72-\varepsilon}$ we have
$$\{Vq + V^{8/5} + V^2q^{-1}\}\{Uq + U^{8/5} + U^2q^{-1}\}$$
$$\ll UVq^2 + U^{8/5}Vq + U^{8/5}V^{8/5} + U^2V^{8/5}q^{-1}.$$
Consequently,
$$M_2 \ll U^{-3/4+\varepsilon}V^2\{U^{1+9/8}V + U^{8/5+3/8}V + U^{8/5-25/144}V^{8/5} + U^{2-75/144}V^{8/5}\}$$
$$\ll U^{53/24+\varepsilon}V^2.$$

This, together with (3.7) and (3.11), proves (3.6). The proof of Lemma 2.2 is thus complete. □

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