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Horocyclically Convex Univalent Functions

D. Mejía & Ch. Pommerenke

1. Introduction: Convexity and Curvature

A domain G in $\mathbb{C}$ is convex if and only if, for every $\omega \in \partial G$, there exists a half-plane H such that $\omega \in \partial H$ and $G \cap H = \emptyset$. The fact that $C = \partial H$ is a line can be expressed in two equivalent ways:

(i) C is a maximal curve of zero curvature;
(ii) C is a noncompact maximal curve of constant curvature.

Now we turn to the hyperbolic metric $ds = (1 - |z|^2)^{-1}|dz|$ in the unit disk $\mathbb{D}$. Let $C : w(s)$, $s \in I$, be a curve of class $\mathcal{C}^2$ in $\mathbb{D}$ that is parameterized by the hyperbolic arc-length s; that is,

$$|w'| = 1 - |w|^2, \quad w'' \text{ is continuous in } I. \tag{1.1}$$

Then the hyperbolic (= geodesic) curvature κ satisfies the differential equation

$$\frac{w''}{w'} + \frac{2\bar{w}w'}{1 - |w|^2} = i\kappa. \tag{1.2}$$

Let $\mathbb{T} = \partial\mathbb{D}$. If $C \subset \mathbb{D}$ is a maximal (= cannot be extended to a larger curve in $\mathbb{D}$) curve of constant hyperbolic curvature κ, then we have three cases as follows.

$|\kappa| \le 2$: C is a circular arc from $\mathbb{T}$ to $\mathbb{T}$;
$|\kappa| = 2$: C is a circle in $\mathbb{D}$ that touches $\mathbb{T}$;
$|\kappa| > 2$: C is a full circle in $\mathbb{D}$.

Thus the noncompact maximal curves of constant curvature are those with $|\kappa| \le 2$, and we see that conditions (i) and (ii) are different in the hyperbolic case.

Ma and Minda [MaM1] considered condition (i). A domain $G \subset \mathbb{D}$ is hyperbolically convex (h-convex) if, for every $\omega \in \mathbb{D} \cap \partial G$, there is a hyperbolic half-plane H with $\omega \in \partial H$ and $G \cap H = \emptyset$. See, for example, [MaM2; MeP1; MeP2; MeP3; MePV] for results on the conformal maps of $\mathbb{D}$ onto h-convex domains.

In this paper we shall consider condition (ii), namely, the extremal case $|\kappa| = 2$. A *horocycle* is, by definition, the inner domain of a circle in $\mathbb{D}$ that touches $\mathbb{T}$. A domain $G \subset \mathbb{D}$ will be called *horocyclically convex* (horo-convex) if, for every $\omega \in \mathbb{D} \cap \partial G$, there exists a horocycle H such that

$$\omega \in \partial H \quad \text{and} \quad G \cap H = \emptyset. \tag{1.3}$$

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A *horocyclically convex function* f is a conformal map of $\mathbb{D}$ onto a horo-convex domain $G \subset \mathbb{D}$. It turns out that these functions are more difficult to study than the h-convex functions.

In Section 2 we show that every horo-convex function f is continuous in $\bar{\mathbb{D}}$, and in Section 5 we establish some sharp estimates for $f(\zeta)$ and $f'(\zeta)$ for $\zeta \in \mathbb{T} = \partial\mathbb{D}$. In Section 6 we study the coefficient growth.

We have not yet been able to obtain any distortion theorems—that is, estimates for $f(z)$ and $f'(z)$ for given $z \in \mathbb{D}$ under the normalization $f(0) = 0$ and $f'(0) = \alpha$. In the final section we list some open problems.

2. Some General Properties

It is clear that horo-convexity is Möbius-invariant: If Möb($\mathbb{D}$) is the group of all Möbius transformations of $\mathbb{D}$ onto $\mathbb{D}$, then

$$f \text{ horo-convex } \ \& \ \ \sigma, \tau \in \text{Möb}(\mathbb{D}) \implies \sigma \circ f \circ \tau \text{ horo-convex.} \tag{2.1}$$

As a consequence we can achieve the normalization $f(0) = 0$ and $f'(0) > 0$.

THEOREM 1. *Every horocyclically convex function has a continuous extension to* $\bar{\mathbb{D}}$.

Proof. Let f be horo-convex with $f(0) = 0$ and let $G = f(\mathbb{D})$. Let (C_n) be a null-chain of G (see e.g. [P2, p. 29]). Let $w_n^{\pm} \in \partial G$ be the two endpoints of the crosscut C_n of G. We have to show that the component V_n of $G \setminus C_n$ with $0 \notin V_n$ satisfies $\operatorname{diam} V_n \to 0$ as $n \to \infty$. Since G is horo-convex there exist horocycles $H_n \subset \mathbb{D} \setminus G$ with $w_n^{\pm} \in \partial H_n^{\pm}$ if $w_n^{\pm} \in \mathbb{D}$. We may assume that $w_n^{\pm} \to w_0 \in \partial G$.

First suppose $w_0 \in \mathbb{D}$, in which case we may assume that the $H_n^{\pm}$ converge to horocycles $H^{\pm}$ as $n \to \infty$. Then $w_0 \in \partial H^{\pm} \subset \mathbb{D} \setminus G$. We construct an arc $A_n \subset \mathbb{D} \setminus G$ from w_n^- to w_n^+ as follows: We take arcs along the inner normal to $\partial H_n^{\pm}$ at $w_n^{\pm}$ until we meet $\partial H^{\pm}$; then we go along $\partial H^{\pm}$ until we meet $\partial H^{\mp}$. It follows from Janiszewski's theorem (as in [P2, p. 2]) that V_n lies in the domain U_n bounded by A_n and C_n. Hence

$$\operatorname{diam} V_n \le \operatorname{diam} U_n \le \operatorname{diam} A_n + \operatorname{diam} C_n \to 0 \ \text{ as } n \to \infty.$$

Next we consider the case $w_0 \in \mathbb{T}$. We begin by assuming that w_n^+ and w_n^- are in $\mathbb{D}$. It is easy to see that $H_n^+ \cap H_n^- = \emptyset$ in this case. Let $\rho_n^{\pm}$ denote the Euclidean radius of $H_n^{\pm}$ and let $\omega_n^{\pm}$ denote the point where $\partial H_n^{\pm}$ touches $\mathbb{T}$. Since $|w_n^{\pm} - \omega_n^{\pm}(1 - \rho_n^{\pm})| = \rho_n^{\pm}$, we have

$$1 \ge \cos(\arg w_n^{\pm} - \arg \omega_n^{\pm}) = \frac{1 + |w_n^{\pm}| - 2\rho_n^{\pm}}{2|w_n^{\pm}|(1 - \rho_n^{\pm})} \to 1 \tag{2.2}$$

as $n \to \infty$, because $|w_n^{\pm}| \to |w_0| = 1$ and $\rho_n < \frac{1}{2}$. It follows that $\omega_n^{\pm} \to w_0$. Since $H_n^{\pm} \subset \mathbb{D} \setminus G$ and $w_n^{\pm} \in \partial H_n^{\pm}$, we see that V_n lies in the component U_n of $\mathbb{D} \setminus (\bar{H}_n^+ \cup \bar{H}_n^- \cup C_n)$ with $0 \notin U_n$, and we conclude that

$$\operatorname{diam} V_n \le \operatorname{diam} U_n \le |\omega_n^+ - \omega_n^-| + \operatorname{diam} C_n \to 0 \ \text{ as } n \to \infty. \tag{2.3}$$

Now we assume that $w_n^+ \in \mathbb{D}$ and $w_n^- \in \mathbb{T}$. Then (2.2) holds for w_n^+ and V_n lies in the component U_n of $\mathbb{D} \setminus (\bar{H}_n^+ \cup C_n)$ with $0 \notin U_n$. We conclude that (2.3) holds. Finally, if $w_n^+ \in \mathbb{T}$ and $w_n^- \in \mathbb{T}$ then $\operatorname{diam} V_n \leq \operatorname{diam} C_n \to 0$. □

Next we show by an example that, in spite of Theorem 1, the family of all horo-convex functions normalized by $f(0) = 0$ is not equicontinuous.

Let $\vartheta_n = \frac{\pi}{6} + \frac{1}{n}$, $n \in \mathbb{N}$. The two horocycles

$$H_n^{\pm} = \left\{ w : |w - \tfrac{2}{3} e^{\pm i\vartheta_n}| < \tfrac{1}{3} \right\}$$

are disjoint and do not meet $\mathbb{R}$. Let f_n map $\mathbb{D}$ onto $G_n = \mathbb{D} \setminus (\bar{H}_n^+ \cup \bar{H}_n^-)$ such that $f_n(0) = 0$ and $f_n(1) = 1$, and determine $z_n^{\pm}$ such that

$$f_n(z_n^{\pm}) = w_n^{\pm} = e^{\pm i/n}/\sqrt{3} \in \partial H_n^{\pm}.$$

Since G_n contains the disk $\left\{|w| < \frac{1}{3}\right\}$ and since $f_n(0) = 0$, we see as in [P1, Cor. 11.5] that

$$|z_n^+ - z_n^-| \leq c|w_n^+ - w_n^-|^{1/2} \to 0 \quad (n \to \infty)$$

for some constant c. We conclude that $z_n^+ \to 1$ whereas $|f_n(z_n^+) - f_n(1)| \geq 1 - 1/\sqrt{3}$.

3. The Canonical Example

The following function plays an important role in the theory of horo-convex functions. Let $0 < \lambda < 1$. We define

$$h_\lambda(z) = \left(\log \frac{1 + e^{i\pi\lambda} z}{1 + e^{-i\pi\lambda} z} \right) \Bigg/ \left(2\pi\lambda - \log \frac{1 + e^{i\pi\lambda} z}{1 + e^{-i\pi\lambda} z} \right). \tag{3.1}$$

This function h_λ is analytic in $\mathbb{D}$ and has a continuous extension to $\bar{\mathbb{D}}$. It is easy to verify that

$$\begin{aligned} h_\lambda &: \{e^{it} : |t| < \pi(1-\lambda)\} \to \mathbb{T} \setminus \{-1\} \quad \text{and} \\ h_\lambda &: \{e^{it} : \pi(1-\lambda) < t < \pi(1+\lambda)\} \to \partial H_\lambda \setminus \{-1\} \end{aligned} \tag{3.2}$$

are bijective maps, where H_λ is the horocycle

$$H_\lambda = \left\{ w : \left| w + \frac{1}{1+\lambda} \right| < \frac{\lambda}{1+\lambda} \right\} \tag{3.3}$$

and we have

$$h_\lambda(1) = 1, \quad h_\lambda(e^{\pm i\pi(1-\lambda)}) = -1, \quad h_\lambda(-1) = -\frac{1-\lambda}{1+\lambda}. \tag{3.4}$$

By the boundary principle [P2, p. 16] it follows from (3.2) and (3.3) that h_λ maps $\mathbb{D}$ conformally onto $\mathbb{D} \setminus \bar{H}_\lambda$. Hence h_λ is horo-convex.

It follows from (3.1) that

$$h_\lambda(z) = \frac{\sin \pi\lambda}{\pi\lambda} z + \left(\left(\frac{\sin \pi\lambda}{\pi\lambda} \right)^2 - \frac{\sin 2\pi\lambda}{2\pi\lambda} \right) z^2 + \cdots . \tag{3.5}$$

The inverse function is

$$h_\lambda^{-1}(w) = \frac{p(w)-1}{e^{i\pi\lambda} - e^{-\pi i\lambda}p(w)} \quad \text{with } \ p(w) = \exp\frac{2\pi i\lambda w}{1+w}. \tag{3.6}$$

Now we shall prove that

$$\min_t |h_\lambda'(e^{it})| = \frac{2\lambda}{\pi(1+\lambda)^2}\cot\frac{\pi\lambda}{2} \tag{3.7}$$

with equality for $t = \pi$.

If $|t| \le \pi(1-\lambda)$ then $h_\lambda(e^{it}) \in \mathbb{T}$, and since $h_\lambda(0) = 0$ it follows from the Julia–Wolff lemma [P2, p. 82] that $|h_\lambda'(e^{it})| \ge 1$, which is greater than the right-hand side of (3.7).

Now let $\pi(1-\lambda) < t < \pi(1+\lambda)$ and write $w = h_\lambda(e^{it})$. It follows from (3.6) that

$$\frac{d}{dw}h_\lambda^{-1}(w) = \frac{2\sin\pi\lambda}{(e^{i\pi\lambda} - e^{-i\pi\lambda}p(w))^2}\,\frac{2\pi\lambda p(w)}{(1+w)^2}. \tag{3.8}$$

By (3.2) we can write $w = -(1+\lambda e^{2i\vartheta})/(1+\lambda)$, so that

$$1+w = -\frac{2i\lambda e^{i\vartheta}\sin\vartheta}{i+\lambda}, \qquad p(w) = -e^{i\pi\lambda}\exp[\pi(1+\lambda)\cot\vartheta].$$

Writing $c = \pi(1+\lambda)^2\sin(\pi\lambda)/(2\lambda)$, we obtain from (3.8) that

$$\begin{aligned} c|h_\lambda'(e^{it})| &= [\cosh(\pi(1+\lambda)\cot\vartheta) + \cos(\pi\lambda)]\sin^2\vartheta \\ &\ge \left(1 + \frac{\pi^2}{2}(1+\lambda)^2\right)\cos^2\vartheta + \cos(\pi\lambda)\sin^2\vartheta \\ &\ge 1 + \cos(\pi\lambda) = 2\cos^2\left(\frac{\pi\lambda}{2}\right), \end{aligned}$$

with equality for $\vartheta = \frac{\pi}{2}$, and thus $t = \pi$. This implies (3.7).

4. The Argument of the Derivative

Let Γ be a Jordan arc or a Jordan curve of the form $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_n$, where the Γ_ν ($\nu = 1, \dots, n$) are smooth arcs from $p_{\nu-1}$ to p_ν that are otherwise disjoint. Let $\triangle(\Gamma_\nu)$ denote the change of the tangent angle along the arc Γ_ν and let $\triangle(p_\nu)$ ($\nu = 1, \dots, n-1$) denote the change of the tangent angle at the corner p_ν. Then

$$\triangle(\Gamma) = \sum_{\nu=1}^{n}\triangle(\Gamma_\nu) + \sum_{\nu=1}^{n-1}\triangle(p_\nu) \tag{4.1}$$

is, by definition, the change of the tangent angle along Γ.

If Γ is a positively oriented Jordan curve, then

$$\triangle(\Gamma) = 2\pi; \tag{4.2}$$

see [H]. We also need the following result [J; K]; see [P2, Prop. 4.22].

PROPOSITION 1. *Let f map $\mathbb{D}$ conformally onto $G \subset \mathbb{C}$ and let S be a hyperbolic segment in $\mathbb{D}$. If D is a disk in G such that ∂D is tangent to $f(S)$, then $D \cap f(S) = \emptyset$.*

First we prove a theorem about general conformal maps.

THEOREM 2. *Let f map $\mathbb{D}$ conformally onto $G \subset \mathbb{C}$ and suppose that, for any pair of points $\omega_1, \omega_2 \in \partial G$, there is a piecewise smooth Jordan arc A from ω_1 to ω_2 such that*

$$A \subset \mathbb{C} \setminus G \quad \text{and} \quad |\triangle(A)| \leq a, \tag{4.3}$$

where a is a constant. If $z \in \mathbb{D}$ then

$$|\arg f'(z) - \arg f'(0)| \leq a + 10\pi, \tag{4.4}$$

and $\log f'$ *belongs to the Hardy spaces H^p* $(0 < p < \infty)$.

Proof. Let $z_1 = 0$, $z_2 = z$, and $S = [0, z]$. Furthermore, let D_j $(j = 1, 2)$ be the largest disk in G such that ∂D_j touches $f(S)$ at $f(z_j)$ from the left. Hence there exists a $\omega_j \in \partial D_j \cap \partial G$. Let C_j be the positively oriented arc of ∂D_j between $f(z_j)$ and ω_j. By assumption, there is a piecewise smooth Jordan arc A satisfying (4.3). Then

$$\Gamma = f(S) \cup C_1 \cup A \cup C_2$$

is a positively oriented Jordan curve by Proposition 1 and (4.3).

Since $C_1 \cup f(S) \cup C_2$ is smooth, it follows from (4.1) and (4.2) that

$$2\pi = \triangle(f(S)) + \triangle(C_1) + \triangle(A) + \triangle(C_2) + \triangle(\omega_1) + \triangle(\omega_2)$$

and thus, by (4.3),

$$|\triangle(f(S))| \leq |\triangle(A)| + 10\pi \leq a + 10\pi.$$

This implies (4.4) because $\arg f'(z) - \arg f'(0) = \triangle(f(S))$.

It then follows from (4.4) that $\log f'(z) - \log f'(0)$ is subordinate to the strip mapping

$$g(z) = \frac{b}{2} \log \frac{1+z}{1-z} \quad (z \in \mathbb{D}), \quad \text{where } b = \frac{4a}{\pi} + 40.$$

Since $g \in H^p$, we have $\log f' \in H^p$. □

5. Some Sharp Estimates

THEOREM 3. *Let f be horocyclically convex and let $f(0) = 0$. Then*

$$|\arg f'(z) - \arg f'(0)| < 2\pi \quad \text{for } z \in \mathbb{D}, \tag{5.1}$$

and 2π cannot be replaced by a smaller constant. Furthermore,

$$f' \in H^p \quad \text{for } 0 < p < \tfrac{1}{4}. \tag{5.2}$$

Proof. Let $G = f(\mathbb{D})$ and let $\zeta \in \mathbb{T}$ be such that the angular derivative $f'(\zeta) \neq 0, \infty$ exists. We assume that $\omega = f(\zeta) \in \mathbb{D}$. Because G is horo-convex, there is a horocycle $H \subset \mathbb{D} \setminus G$ with $\omega \in \partial H$. Since f is isogonal [P2, p. 80] at ζ, the curve $C = \{f(r\zeta) : 0 \leq r \leq 1\}$ is normal to ∂H at ω. Let e^{it} be the point where ∂H touches $\mathbb{T}$.

Let D_0 be the largest disk in G that touches C at $0 = f(0)$ from the right. We assume that $\bar{D}_0 \subset \mathbb{D}$. Then there is a horocycle H_0 that touches D_0 at some point $\omega_0 \in \partial H_0 \cap \partial D_0$. Let e^{it_0} be the point where ∂H_0 touches $\mathbb{T}$. Let Γ_0 be the positively oriented arc of ∂H_0 from e^{it_0} to ω_0 and let Γ_1 be the negatively oriented arc of ∂D_0 from ω_0 to 0; we use T to denote the positively oriented arc of $\mathbb{T}$ from e^{it} to e^{it_0}.

We now assume that $H_0 \neq H$. Let $\Gamma^{\pm}$ be the positively/negatively oriented arcs of ∂H from ω to e^{it}. Then

$$J^{\pm} = C \cup \Gamma^{\pm} \cup T \cup \Gamma_0 \cup \Gamma_1 \tag{5.3}$$

are closed piecewise smooth curves. It follows from Proposition 1 that C is disjoint from the rest of $J^{\pm}$. Hence the only possible multiple points of $J^{\pm}$ lie on $\Gamma^{\pm} \cap \Gamma_0 \subset \partial H \cap \partial H_0$, and it is not difficult to show that either J^+ or J^- is a Jordan curve.

Let $\vartheta_0, \vartheta \in [0, 2\pi)$, denote the geometric tangent angle of C at 0 and ω. Then

$$\triangle(\Gamma_0 \cup \Gamma_1) = \vartheta_0 - t_0 + \frac{3\pi}{2} \quad \text{and} \quad \triangle(T) = t_0 - t.$$

For J^+ we have

$$\triangle(\Gamma^+) = t - \vartheta + \pi, \quad \triangle(\omega) = -\frac{\pi}{2}, \quad \triangle(e^{it}) = 0,$$

whereas for J^- we have

$$\triangle(\Gamma^-) = t - \vartheta - \pi, \quad \triangle(\omega) = \frac{\pi}{2}, \quad \triangle(e^{it}) = \pi.$$

Hence it follows from (5.3) that, in both cases,

$$\triangle(J^{\pm}) = \vartheta_0 - \vartheta + 2\pi + \triangle(C). \tag{5.4}$$

Now $J^{\pm}$ is a positively oriented Jordan curve for a suitable choice of the sign, and thus $\triangle(J^{\pm}) = 2\pi$ by (4.2). Hence we obtain from (5.4) that

$$|\arg f'(\zeta) - \arg f'(0)| = |\triangle(C)| = |\vartheta - \vartheta_0| \leq 2\pi \quad \text{if } f'(\zeta) \neq 0, \infty \text{ exists.} \tag{5.5}$$

We have made the three assumptions that $\omega \in \mathbb{D}$, $\bar{D}_0 \subset \mathbb{D}$, and $H_0 \neq H$. The proof of (5.5) is similar if any one of these assumptions does not hold.

It follows from Theorem 2 that $\log f' \in H^1$. Therefore we have the Poisson integral representation

$$\arg f'(z) = \operatorname{Im}[\log f'(z)] = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} \arg f'(\zeta)\, |d\zeta|.$$

Hence (5.1) follows from (5.5). We conclude from (5.1) that f' is subordinate to the function

$$g(z) = f'(0)\exp\left[4\log\frac{1+z}{1-z}\right] = f'(0)\left(\frac{1+z}{1-z}\right)^4,$$

and since $g \in H^p$ for $p < \frac{1}{4}$ we see that (5.2) holds.

Finally, we show that (5.1) is best possible. Let f be the function h_λ defined in (3.1), and choose $\zeta_\lambda \in \mathbb{T}$ such that

$$\pi(1-\lambda) < \arg\zeta_\lambda < 4(1-\lambda) \quad \text{and} \quad |h_\lambda(\zeta_\lambda)+1| < 1-\lambda;$$

see (3.2). Then

$$\arg\zeta_\lambda \to 0 \quad \text{and} \quad \arg[\zeta_\lambda h'_\lambda(\zeta_\lambda)] \to 2\pi \quad \text{as } \lambda \to 1$$

and thus $\arg h'_\lambda(\zeta_\lambda) \to 2\pi$. □

THEOREM 4. *Let f be horocyclically convex, and let*

$$f(0) = 0 \quad \textit{and} \quad |f'(0)| = \frac{\sin\pi\lambda}{\pi\lambda} \quad (0 < \lambda < 1). \tag{5.6}$$

Then

$$\frac{1-\lambda}{1+\lambda} \le |f(z)| \le 1 \quad \textit{for } z \in \mathbb{T}. \tag{5.7}$$

The angular derivative exists and satisfies

$$\frac{2\lambda}{\pi(1+\lambda)^2}\cot\frac{\pi\lambda}{2} \le |f'(z)| \le +\infty \quad \textit{for } z \in \mathbb{T}. \tag{5.8}$$

If $f = h_\lambda$, then equality holds in all inequalities (5.7) *and* (5.8) *for suitable $z \in \mathbb{T}$.*

Because $f(\mathbb{D}) \subset \mathbb{D}$ we have $0 < |f'(0)| \le 1$, so we can choose λ such that (5.6) holds except in the trivial case $f(z) \equiv e^{i\vartheta}z$. We have $\log f' \in H^1$ by Theorem 3. Hence it follows from (5.8) that

$$|f'(z)| > \frac{2\lambda}{\pi(1+\lambda)^2}\cot\frac{\pi\lambda}{2} \quad \text{for } z \in \mathbb{D}.$$

Proof of Theorem 4. Let $z \in \mathbb{T}$. First we assume that $f(z) \in \mathbb{D}$. Then there exists a horocycle H_μ (see (4.3)) such that

$$e^{-i\vartheta}f(z) \in \partial H_\mu \quad \text{and} \quad (e^{i\vartheta}H_\mu) \cap f(\mathbb{D}) = \emptyset. \tag{5.9}$$

Hence we can write (see (3.5))

$$f = e^{-i\vartheta}h_\mu \circ g \quad \text{with } g(\mathbb{D}) \subset \mathbb{D},\ g(0) = 0. \tag{5.10}$$

It follows from Schwarz's lemma and (5.6) that

$$\frac{\sin\pi\lambda}{\pi\lambda} = |f'(0)| \le |h'_\mu(0)| = \frac{\sin\pi\mu}{\pi\mu}.$$

Hence $\lambda \ge \mu$ and it follows from (5.9) that $|f(z)| \ge (1-\mu)/(1+\mu) \ge (1-\lambda)/(1+\lambda)$. The upper estimate (5.7) is trivial.

We see from (5.9) that the angular derivative exists and satisfies $0 < |f'(z)| \le +\infty$ (see e.g. [P2, p. 83]). Differentiating (5.10) then yields

$$|f'(z)| = |h'_\mu(g(z))||g'(z)|.$$

Since $g(z) \in \mathbb{T}$ by (5.9) and (5.10), it follows from (5.10) and the Julia–Wolff lemma [P2, p. 82] that $|g'(z)| \geq 1$. Hence

$$|f'(z)| \geq |h'_\mu(g(z))| \geq \frac{2\mu\cot(\pi\mu/2)}{\pi(1+\mu)^2} \geq \frac{2\lambda\cot(\pi\lambda/2)}{\pi(1+\lambda)^2} \tag{5.11}$$

by (3.7) and because $\lambda \geq \mu$.

Next we assume that $f(z) \in \mathbb{T}$. Since $f(0) = 0$ it follows from the Julia–Wolff lemma that the angular derivative exists and satisfies

$$+\infty \geq |f'(z)| \geq 1 > \frac{2\lambda\cot(\pi\lambda/2)}{\pi(1+\lambda)^2}.$$

Finally, let $f = h_\lambda$. By (3.2), equality holds in (5.7) for both $z = -1$ and $z = 1$, and by (3.7) and (3.8), respectively, equality holds in (5.8) for both $z = -1$ and $z = -e^{i\pi\lambda}$. □

If f is a conformal map of $\mathbb{D}$ onto any bounded domain G, then by the Koebe "one quarter" theorem we have

$$|f'(z)| \leq \frac{4}{1-|z|^2}\operatorname{dist}(f(z), \partial G) = o\left(\frac{1}{1-|z|}\right) \text{ as } |z| \to 1.$$

Now we show that this trivial estimate is best possible even for the class of horo-convex functions.

THEOREM 5. *For any positive function $\eta(x)$ $(0 \leq x < 1)$ with $\eta(x) \to 0$ $(x \to 1)$, there exists a horocyclically convex function f and a sequence (x_k) with $x_k \to 1$ $(k \to \infty)$ such that*

$$|f'(x_k)| \geq \frac{\eta(x_k)}{1-x_k} \quad \textit{for all } k. \tag{5.12}$$

Proof. We shall recursively construct horo-convex functions f_n $(n = 1, 2, \ldots)$ such that

$$f_n(\mathbb{D} \cap \mathbb{R}) = \mathbb{D} \cap \mathbb{R} \quad \text{and} \quad \{z \in \mathbb{D} : |z-1| < \rho_n\} \subset f_n(\mathbb{D}) \tag{5.13}$$

for some $\rho_n > 0$; we shall also construct numbers $x_n \in \left(1 - \frac{1}{n}, 1\right)$ such that

$$(1-x_k)f'_n(x_k) > \left(1 + \tfrac{1}{n}\right)\eta(x_k) \quad \text{for } k = 1, \ldots, n. \tag{5.14}$$

We start with $f_1(z) \equiv z$ and x_1 sufficiently close to 1. Suppose that f_k and x_k have already been constructed for $k \leq n$ and consider the horocycles

$$H_n^\pm = H_n^\pm(\varepsilon, \delta) = \{|z - (1-\delta)e^{\pm i\varepsilon \pm i\pi\delta/2}| < \delta\},$$

where $0 < \varepsilon < \delta < \rho_n/5$. Then $\mathbb{D} \cap \bar{H}_n^\pm \subset f_n(\mathbb{D})$ by (5.13). Let $\varphi_n(z) = \varphi_n(z, \varepsilon, \delta)$ map $\mathbb{D}$ conformally onto $f_n(\mathbb{D}) \setminus (\bar{H}_n^+ \cup \bar{H}_n^-)$ such that $\varphi_n(0) = 0$ and $\varphi_n(1) = 1$. Then φ_n is horo-convex. If $0 < \varepsilon < \delta \to 0$, then $\varphi_n(z, \delta, \varepsilon) \to f_n(z)$ locally uniformly by the Carathéodory kernel theorem [P2, p. 14]. Hence we can choose δ_n sufficiently small that

$$(1-x_k)\varphi_n'(x_k,\varepsilon,\delta_n) > \left(1+\frac{1}{n+1}\right)\eta(x_k) \quad (k=1,\dots,n) \tag{5.15}$$

for $0<\varepsilon<\delta_n$; see (5.14).

We can find $u_n \in (0,1)$ and $v_n > 0$ independent of ε such that

$$\operatorname{dist}(H_n^{\pm}(\varepsilon,\delta_n),u_n) > v_n \quad \text{and} \quad \operatorname{dist}(\mathbb{T},u_n) > v_n. \tag{5.16}$$

Let $\xi_n(\varepsilon)$ be determined by $\varphi_n(\xi_n(\varepsilon),\varepsilon,\delta_n) = u_n$. Then $\xi_n(\varepsilon)\to 1$ as $\varepsilon\to 0$ by the Carathéodory kernel theorem. Hence there exist $\varepsilon_n \in (0,\delta_n)$ such that

$$\eta(x_{n+1}) < \frac{v_n}{4}, \quad \text{where} \quad x_{n+1} = \xi_n(\varepsilon_n) > 1-\frac{1}{n+1}.$$

We define $f_{n+1}(z) = \varphi_n(z,\varepsilon_n,\delta_n)$ for $z\in\mathbb{D}$. Then

$$\begin{aligned}(1-x_{n+1})f_n'(x_{n+1}) &> \frac{1}{2}\operatorname{dist}(f_n(x_{n+1}),\partial f_{n+1}(\mathbb{D}))\\ &> \frac{1}{2}v_n > 2\eta(x_{n+1}) > \left(1+\frac{1}{n+1}\right)\eta(x_{n+1})\end{aligned}$$

by (5.15). Together with (5.16) this shows that (5.14) is true for $n+1$ instead of n, and (5.13) for $n+1$ holds by our construction.

This completes our construction. Now we let $n\to\infty$. Then $f=\lim_{n\to\infty} f_n$ exists locally uniformly in $\mathbb{D}$, and f is again horo-convex. It follows from (5.14) that (5.12) is satisfied.

6. Coefficient Estimates

Now we consider the Taylor expansion of the conformal map

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

of $\mathbb{D}$ onto G. Let Λ denote the Euclidean length. If $\Lambda(\partial G)<\infty$ then $f'\in H^1$ by the Riesz–Privalov theorem [P2, p. 134] and thus $a_n = o(1/n)$.

It is not difficult to construct a horo-convex domain G of the form

$$G = \mathbb{D}\setminus\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{k_n}\bar{H}_{nk}, \quad \operatorname{diam} H_{nk} = \frac{1}{k_n} \quad (k=1,\dots,k_n), \tag{6.1}$$

where all the horocycles H_{nk} ($k=1,\dots,k_n$; $n=1,2,\dots$) have disjoint closures. Then $\Lambda(\partial G)=\infty$ and thus $f'\notin H^1$. But we do not yet have an example of a horo-convex function with $a_n \neq o(1/n)$.

Theorem 6. *If f is horocyclically convex then*

$$|a_n| < \frac{c\log n}{n} \quad \textit{for } n=2,3,\dots, \tag{6.2}$$

where the constant c depends only on a_0.

The proof is based on the following purely geometrical result, which may well be known. Note that inequality (6.4) is linear whereas the isoperimetric inequality is quadratic.

PROPOSITION 2. *Let H be a bounded open set in $\mathbb{C}$ of the form*

$$H = \bigcup_{n=1}^{m} D_n, \tag{6.3}$$

where the D_n are disks of radius $\geq r > 0$. Then

$$\text{area}\, H \geq \frac{r}{2}\Lambda(\partial H). \tag{6.4}$$

Proof. Let

$$H_n = \bigcup_{\nu=1}^{n} D_\nu \quad (n = 0, \dots, m) \tag{6.5}$$

and

$$b_n = \pi^{-1}\, \text{area}\, H_n, \qquad \ell_n = (2\pi)^{-1}\Lambda(\partial H_n), \tag{6.6}$$

and let r_n be the radius of D_n. Now $H_{n-1} \cap \partial D_n$ consists of finitely many arcs A_{nk} on ∂D_n (possibly none). Let U_{nk} be the component of $D_n \cap H_{n-1}$ with $A_{nk} \subset \partial U_{nk}$, and define λ_{nk} by

$$2\pi\lambda_{nk} = \Lambda(\partial U_{nk}) = \Lambda(A_{nk}) + \Lambda(D_n \cap \partial U_{nk}). \tag{6.7}$$

By (6.5) we have

$$\text{area}\, H_n = \text{area}\, H_{n-1} + \text{area}\, D_n - \text{area}(D_n \cap H_{n-1})$$

and

$$\Lambda(\partial H_n) = \Lambda(\partial H_{n-1}) - \sum_k \Lambda(D_n \cap \partial U_{nk}) + \Lambda(\partial D_n) - \sum_k \Lambda(A_{nk}),$$

and thus by (6.6) and (6.7) we have

$$b_n = b_{n-1} + r_n^2 - \frac{1}{\pi}\sum_k \text{area}\, U_{nk}, \tag{6.8}$$

$$\ell_n = \ell_{n-1} + r_n - \sum_k \lambda_{nk}. \tag{6.9}$$

Now $4\pi\, \text{area}\, U_{nk} \leq \Lambda(\partial U_{nk})^2$ by the isoperimetric inequality [BuZ, p. 69] and therefore, by (6.7) and (6.8),

$$b_n \geq b_{n-1} + r_n^2 - \sum_k \lambda_{n_k}^2. \tag{6.10}$$

With $x_n = \ell_n - \ell_{n-1} \in \mathbb{R}$, we obtain from (6.9) that

$$\sum_k \lambda_{nk} = r_n - x_n \quad \text{and} \quad \max_k \lambda_{nk} \leq r_n - x_n; \tag{6.11}$$

hence, by (6.10),

$$b_n - b_{n-1} \geq r_n^2 - (r_n - x_n)^2 = (2r_n - x_n)x_n.$$

We always have $b_n - b_{n-1} \geq 0$ by (6.5) and (6.6). Since $b_0 = 0$, it follows that

$$b_m \geq \sum_{n=1}^{m}{}' (2r_n - x_n)x_n \geq \sum_{n=1}^{m}{}' r_n x_n,$$

where we sum only over the indices n with $x_n > 0$; note that $x_n \leq r_n$ by (6.11). By assumption we have $r_n \geq r$ and so

$$b_m \geq r \sum_{n=1}^{m}{}' x_n \geq r \sum_{n=1}^{m} x_n = r\ell_m,$$

which by (6.6) yields our assertion. □

Proof of Theorem 6. Let $a = |a_0|$ and $m = [\log(1-a)^{-1}] + 1$. Let the sequence (w_ν) be dense in ∂G. Since G is horo-convex, there is a horocycle $H_\nu \subset \mathbb{D} \setminus G$ with $w_\nu \in \partial H_\nu$. For $N = 1, 2, \ldots$ let

$$f_N(z) = \sum_{n=0}^{\infty} a_{N,n} z^n$$

be the conformal map of $\mathbb{D}$ onto

$$\mathbb{D} \Big\backslash \bigcup_{\nu=1}^{N} H_\nu \tag{6.12}$$

with $f_N(0) = f(0) = a_0$ and $\arg f_N'(0) = \arg f'(0)$. By the Carathéodory kernel theorem, the functions f_N converge to f locally uniformly in $\mathbb{D}$ as $N \to \infty$. It follows that $a_{N,n} \to a_n$ for fixed n. Hence we may assume that G has the form (6.12).

For $k \geq m$, let G_k be the component of $G \cap \{|w| < 1 - e^{-k}\}$ with $a_0 = f(0) \in G$. We define U_j as the union of all the horocycles $H \subset \mathbb{D} \setminus G$ that touch ∂G and satisfy $e^{-j} < \operatorname{diam} H \leq e^{-j+1}$, where $j = 0, 1, \ldots, k$. Then

$$\operatorname{area} U_j \leq \operatorname{area}\{1 - e^{-j+1} < |w| < 1\} \leq 2\pi e^{-j+1}$$

and therefore, by Proposition 2,

$$\Lambda(\partial U_j) \leq \frac{4(\operatorname{area} U_j)}{\inf(\operatorname{diam} H)} \leq 8\pi e \tag{6.13}$$

because $\operatorname{diam} H > e^{-j}$. Furthermore,

$$\partial G_k \subset \{|w| = 1 - e^{-k}\} \cup \bigcup_{j=1}^{k} \partial U_j.$$

Hence we conclude from (6.13) that

$$\Lambda(\partial G_k) \leq 2\pi + 8\pi e k < 80k \quad \text{for } k \geq m. \tag{6.14}$$

Now let $n > e^m$. We determine $k > m$ such that $e^{k-1} \le n < e^k$. Then $C_k = f^{-1}(\partial G_k)$ is a Jordan curve in $\bar{\mathbb{D}}$ around 0 and therefore

$$na_n = \frac{1}{2\pi i}\int_{C_k} \frac{f'(z)}{z^n}\,dz. \tag{6.15}$$

If $z \in \mathbb{D} \cap C_k$, then

$$f(z) \in \partial G_k \subset \{|w| = 1 - e^{-k}\}$$

and thus, with $a = |f(0)|$,

$$1 - e^{-k} \le |f(z)| \le \frac{a + |z|}{1 + a|z|}$$

because $f(\mathbb{D}) \subset \mathbb{D}$. Since $1 - a \ge e^{-m}$, it follows that

$$|z| \ge \frac{1 - e^{-k} - a}{1 - a(1 - e^{-k})} \ge \frac{e^{-m} - e^{-k}}{e^{-m} + e^{-k}} \ge 1 - 2e^{m-k},$$

and this holds trivially if $z \in \mathbb{T}$. Hence we conclude from (6.15) that

$$\begin{aligned} 2\pi n|a_n| &\le (1 - 2e^{m-k})^{-n}\int_{C_k} |f'(z)|\,|dz| \\ &\le (1 - 2e^{m-k})^{-e^k}\Lambda(\partial G_k). \end{aligned}$$

With constants c_1, c_2, c_3 depending only on m and thus on a_0, it follows from (6.13) that

$$n|a_n| \le c_1\Lambda(\partial G_k) \le c_2 k \le c_3 \log n. \qquad \square$$

7. Some Open Problems

1. Internal characterization. Our definition of horo-convexity is in terms of supporting horocycles. It is reasonable to conjecture that a domain $G \subset \mathbb{C}$ is horocyclically convex if and only if every pair of points can be connected by a curve $C \subset G$ of class $\mathcal{C}^2$ with hyperbolic curvature $|\kappa| < 2$. We want to thank Dirk Ferus and Ekkehard Tjaden for our discussions about this problem.

2. Analytic characterization. Now suppose that ∂G is a Jordan curve of class $\mathcal{C}^3$ in $\mathbb{D}$. If f maps $\mathbb{D}$ conformally onto G, then f'' is continuous in $\bar{\mathbb{D}}$ and $f'(z) \neq 0$ for $z \in \bar{\mathbb{D}}$. The hyperbolic curvature of ∂G is given by

$$\kappa(z) = \frac{1 - |f(z)|^2}{|z\,f'(z)|}\,\mathrm{Re}\left(1 + z\frac{f''(z)}{f'(z)} + \frac{2zf'(z)\overline{f(z)}}{1 - |f(z)|^2}\right) \quad (z \in \mathbb{T}). \tag{7.1}$$

The horocycle $H \subset \mathbb{D} \setminus G$ that touches ∂G at $f(z)$ has h-curvature -2, and it follows that $\kappa(z) \ge -2$ for $z \in \mathbb{T}$.

Now let G be any domain in $\mathbb{D}$. We ask if the following statement is true: G is horo-convex if and only if

$$\mathrm{Re}\left(1 + z\frac{f''(z)}{f(z)} + \frac{2zf'(z)\overline{f(z)} + 2|zf'(z)|}{1 - |f(z)|^2}\right) > 0 \quad \text{for } z \in \mathbb{D}. \tag{7.2}$$

A closely related question is whether $\{f(rz) : |z| < 1\}$ is also horo-convex for $0 < r < 1$.

3. Interior estimates. In Theorems 3 and 4, we have several sharp bounds for $z \in \mathbb{T}$ but none for $z \in \mathbb{D}$. We can always achieve the normalization

$$f(z) = \alpha z + a_2 z^2 + a_3 z^3 + \cdots, \quad 0 < \alpha \leq 1. \tag{7.3}$$

What are the sharp bounds for $|f(z)|$ and $|f'(z)|$ for $z \in \mathbb{D}$? These are known for hyperbolically convex functions [MaM1] and related quantities [MeP2]. Another quantity of interest, because of its invariance properties, is the Schwarzian derivative. Barnard and colleagues [BCPW] have recently proved the conjecture of [MeP2] that

$$\sup_{z\in\mathbb{D}}(1 - |z|^2)^2 |S_f(z)| = 2.38\ldots$$

is the sharp bound for the class of h-convex functions.

4. Coefficient estimates. Let f be horo-convex and normalized as in (7.3). In view of the invariance property (2.1), estimates of a_2 and a_3 in terms of α lead, for instance, to estimates of $S_f(z)$ for $z \in \mathbb{D}$. It would be interesting to determine the growth of

$$\max\{|a_n| : f(z) = \alpha z + \cdots \text{ horo-convex}\}$$

as $n \to \infty$, and of related quantities. In particular, is (6.2) a good estimate?

If $f(\mathbb{D})$ is given by (6.1) then $f' \notin H^1$. Is it true that $f' \in H^p$ for all $p < 1$? Compare Theorem 3. The proof of Theorem 6 suggests that

$$\int_0^{2\pi} |f'(re^{it})|\, dt = O\left(\log \frac{1}{1-r}\right) \text{ as } r \to 1$$

might perhaps be true.

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D. Mejía
Universidad Nacional
Departamento de Matemáticas
A.A. 3840 Medellín
Colombia

dmejia@unalmed.edu.co

Ch. Pommerenke
Technische Universität
Institut für Mathematik, MA 8-2
D-10623 Berlin
Germany

pommeren@math.tu-berlin.de