Michigan Math. J. 53 (2005)

Distortion of Quasiconformal and Quasiregular Mappings at Extremal Points

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a domain, $n \geq 2$. We call a mapping $f : \Omega \to \mathbb{R}^n$ *quasiregular* if $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and if there exists $1 \leq K_O < \infty$ such that

$$|Df(x)|^n \leq K_O J_f(x)$$

for almost all $x \in \Omega$. Here $|Df(x)|$ is the operator norm of the differential matrix of f at x, and $J_f(x)$ is the Jacobian determinant of $Df(x)$. Quasiregular homeomorphisms are called quasiconformal maps. See [4; 5; 7; 8] for the basic theory of quasiconformal and quasiregular mappings.

One of the properties of quasiregular mappings is that they are locally Hölder continuous with exponent $\alpha(m, K_I) = (m/K_I)^{1/(n-1)}$. Here K_I is the inner distortion of f, and m is the local index of f at the point at which the local modulus of continuity is measured; see Section 2 for the definitions. Also, for nonconstant quasiregular mappings it is true that the local modulus of continuity at a single point with local index m cannot be better than Hölder continuity with exponent $\beta(m, K_O) = (mK_O)^{1/(n-1)}$. See [5, III, Thm. 4.7] for the proofs of these properties.

The purpose of this paper is to study the asymptotic behavior of quasiconformal and quasiregular mappings at points at which the mappings behave in an extremal way—in the sense of the local modulus of continuity. That is, we look at the local distortion of a mapping f at points at which the local modulus of continuity is either no better than $\alpha(m, K_I)$ or no worse than $\beta(m, K_O)$. This kind of study was initiated by Kovalev in [2], where the planar case was considered. For quasiconformal mappings with distortion K_O close to unity, good estimates for the linear dilatation have been proved in [6; 9].

The basic principle of the results in [2] is that, at the points that are extremal in the sense just described, the mapping distorts spheres somewhat like analytic functions do. Here we show that a similar phenomenom also occurs in higher dimensions. The results in [2] were proved by using estimates for solutions of the Beltrami equation, a method not available in higher dimensions. Here we will use methods similar to the ones established in [3]. Namely, we will use inequalities and estimates concerning the conformal modulus of families of $(n-1)$-dimensional spheres and their images and preimages under the mappings in question.

Received May 12, 2004. Revision received August 25, 2004.

Our first result is stated for quasiconformal maps. Since the result is local, it also holds true for quasiregular mappings at points at which the mapping is a local homeomorphism. The concepts used to state Theorems 1.1 and 1.4 are defined in Section 2. In particular, we will use the notion of infinitesimal space, introduced in [1], in order to describe the asymptotic behavior.

Fix $K_I \geq 1$, and set

$$\omega_f(x_0, \delta) := \delta^{-K_I^{-1/(n-1)}} \max_{|x-x_0|\leq\delta} |f(x) - f(x_0)|$$

and

$$\omega_f(x_0) := \limsup_{\delta\to 0} \omega_f(x_0, \delta).$$

We now have the following result, which generalizes Theorem 2.1 of [2] to all dimensions $n \geq 2$.

THEOREM 1.1. *Let $f\colon \Omega \to \mathbb{R}^n$ be a quasiconformal map, so that $K_I(x) \leq K_I$ for almost every $x \in \Omega$. Let $x_0 \in \Omega$, and suppose that $\omega_f(x_0) > 0$. Then $H(x_0, f) = 1$. Furthermore, for all $g\colon \mathbb{R}^n \to \mathbb{R}^n$ in the infinitesimal space of f at x_0,*

$$g(x) = |x|^{K_I^{-1/(n-1)}} h\left(\frac{x}{|x|}\right) \quad \text{for all } x \in \mathbb{R}^n, \tag{1.1}$$

where $h\colon S^{n-1}(0,1) \to S^{n-1}(0,1)$ is a homeomorphism that preserves surface measure. In particular, if $n = 2$ then h is a rotation.

REMARK 1.2. Although the mappings in the infinitesimal space satisfy the strong property (1.1), a mapping satisfying the assumptions of Theorem 1.1 need not be "differentiable" at x_0, meaning that the infinitesimal space of f at x_0 may include more than one mapping. See the proof of [2, Thm. 2.2] for an example where such differentiability fails.

REMARK 1.3. For planar mappings f, the mapping h in (1.1) is an isometry. In higher dimensions this may not be true; for any $n \geq 3$ and any $K_I > 1$ there exists a $g\colon \mathbb{R}^n \to \mathbb{R}^n$ satisfying the assumptions of Theorem 1.1 for $x_0 = 0$ and K_I, so that the mapping h in (1.1) is not an isometry. To see this, fix $K_I > 1$ and take a quasiconformal mapping g of the form (1.1). Furthermore, choose h to be a sense-preserving $K_I^{1/(n-1)}$-bilipschitz homeomorphism that preserves surface measure but is not an isometry. Then g satisfies the assumptions of Theorem 1.1. In particular,

$$K_I(x, g) \leq K_I$$

almost everywhere: If we denote $v(t) = t^{K_I^{-1/(n-1)}}$ then, at a point of differentiability x,

$$l(Dg(x)) := \inf_{\{y\in\mathbb{R}^n : |y|=1\}} |Dg(x)y| = K_I^{-1/(n-1)} |x|^{K_I^{-1/(n-1)}-1} = v'(|x|)$$

while

$$\begin{aligned} J_g(x) &= l(Dg(x)) \cdot |x|^{(n-1)(K_I^{-1/(n-1)}-1)} \\ &= K_I^{-1/(n-1)} |x|^{n(K_I^{-1/(n-1)}-1)} = K_I l(Dg(x))^n. \end{aligned}$$

Also, the infinitesimal space of g at the origin consists of one mapping, namely the mapping g itself. Mappings h as the one needed here do indeed exist. For instance, for the two-dimensional sphere $S^2(0,1) \subset \mathbb{R}^3$ such mappings can be constructed as follows. Take the caps $C_1 = S^2(0,1) \cap \{x_3 > 1/2\}$ and $C_2 = S^2(0,1) \cap \{x_3 < -1/2\}$. Now define h so that h restricted to $C_1 \cup C_2$ is the identity. For $|x_3| \leq 1/2$, set

$$h(r, \phi, x_3) = (r, \phi + \varphi(|x_3|), x_3)$$

in cylindrical coordinates, where $\varphi\colon [0,1] \to [0,1]$ is a nontrivial differentiable function such that $\varphi(1/2) = 0$ and $|\varphi'(t)| \leq K_I^{1/2} - 1$ for all $x \in [0, 1/2]$. Then h has the desired properties.

We do not know how to prove a result like Theorem 1.1 for quasiregular mappings at branch points, at which the local modulus of continuity is roughly $\alpha(m, K_I)$. The reason for this is that our methods (seem to) require that boundaries of certain balls be mapped into the boundaries of the images. In dimension 2 this problem does not exist because, by the Stoilow factorization theorem, the result for quasiconformal maps implies that the corresponding result holds true also for quasiregular mappings.

However, for general points we have the following result concerning the inverse distortion. Fix $K_O \geq 1$, and set

$$\sigma_f^m(x_0, \delta) := \delta^{-(mK_O)^{-1/(n-1)}} \max_{U(x_0, f, \delta)} |x - x_0|$$

(see Section 2 for the definition of $U(x, f, r)$) and

$$\sigma_f^m(x_0) := \limsup_{\delta \to 0} \sigma_f^m(x_0, \delta).$$

THEOREM 1.4. *Let $f : \Omega \to \mathbb{R}^n$ be a quasiregular mapping, so that $K_O(x) \leq K_O$ for almost every $x \in \Omega$. Let $x_0 \in \Omega$, $i(x_0, f) = m$, and suppose that $\sigma_f^m(x_0) > 0$. Then $H^*(x_0, f) = 1$ and*

$$\sigma_f^m(x_0) = \lim_{\delta \to 0} \sigma_f^m(x_0, \delta).$$

In Section 2 we recall some definitions and the main tools that are needed to prove the foregoing results; the material is mainly from [3] and [5]. Theorems 1.1 and 1.4 are proved in Section 3.

ACKNOWLEDGMENTS. We thank Pekka Koskela and Eero Saksman for useful discussions.

2. Preliminaries

We will denote open Euclidean balls with center x and radius r by $B(x, r)$, while the corresponding $(n-1)$-dimensional spheres are denoted by $S(x, r)$; when $x = 0$ the notations B_r and S_r are used. The boundary of a general set E is denoted by

∂E. Let f be a nonconstant quasiregular mapping and let $Df(x)$ be the differential matrix of f at a point of differentiability x. Set

$$|Df(x)| = \sup_{\{y\in\mathbb{R}^n : |y|=1\}} |Df(x)y| \quad \text{and} \quad l(Df(x)) = \inf_{\{y\in\mathbb{R}^n : |y|=1\}} |Df(x)y|.$$

The inner and outer distortion functions of f are defined by

$$K_I(x) = K_I(x,f) = \frac{J_f(x)}{l(Df(x))^n}, \qquad K_O(x) = K_O(x,f) = \frac{|Df(x)|^n}{J_f(x)},$$

respectively, whenever $J_f(x) \neq 0$; otherwise, set $K_I(x) = K_O(x) = 0$. Recall that, for nonconstant quasiregular mappings f, $J_f > 0$ almost everywhere (see [5, II, Thm. 7.4]). If f is quasiconformal and $K_I(x,f) \leq K_I$ almost everywhere, then f^{-1} is also quasiconformal and

$$K_O(x, f^{-1}) \leq K_I \tag{2.1}$$

almost everywhere (see [7, Cor. 13.3 & Thm. 34.4]). The Lebesgue n-measure of a measurable set A is denoted by $|A|$, and the Lebesgue measure of the unit n-ball is denoted by α_n. The $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ is normalized so that $\mathcal{H}^{n-1}(B_1) = \omega_{n-1}$, where ω_{n-1} denotes the surface measure of the unit sphere.

Let $f : \Omega \to \mathbb{R}^n$ be a nonconstant quasiregular mapping. A domain $D \subset \Omega$ is called a *normal* domain (of f) if $f(\partial D) = \partial f(D)$. Furthermore, if D is a normal domain and $x \in D$ so that $f^{-1}(f(x)) \cap D = \{x\}$, then D is called a normal *neighborhood* of x. The x-component of the preimage of the ball $B(f(x), r)$ under f is denoted by $U(x,f,r)$. By [5, II, Lemma 4.1], for each $x \in \Omega$ there exists an $s_x > 0$ such that, for each $s < s_x$, the following properties hold:

(1) $U(x,f,s)$ is a normal neighborhood of x;
(2) $U(x,f,s) = U(x,f,s_x) \cap f^{-1}(B(f(x),s))$;
(3) $\partial U(x,f,s) = U(x,f,s_x) \cap f^{-1}(S(f(x),s))$;
(4) $\mathbb{R}^n \setminus U(x,f,s)$ and $\mathbb{R}^n \setminus \bar{U}(x,f,s)$ are connected.

The local index $i(x,f)$ of a quasiregular mapping f at a point $x \in \Omega$ can be defined by

$$i(x,f) = \lim_{r\to 0} \sup_{y\in B(f(x),r)} \#\{f^{-1}(y) \cap U(x,f,r)\}.$$

We will use the following dilatation functions:

$$L(x,f,r) = \sup_{|x-y|=r} |f(y) - f(x)|, \qquad l(x,f,r) = \inf_{|x-y|=r} |f(y) - f(x)|;$$

$$L^*(x,f,r) = \sup_{z\in\partial U(x,f,r)} |x - z|, \qquad l^*(x,f,r) = \inf_{z\in\partial U(x,f,r)} |x - z|;$$

$$H(x,f,r) = \frac{L(x,f,r)}{l(x,f,r)}, \qquad H^*(x,f,r) = \frac{L^*(x,f,r)}{l^*(x,f,r)};$$

$$H(x,f) = \limsup_{r\to 0} H(x,f,r), \qquad H^*(x,f) = \limsup_{r\to 0} H^*(x,f,r).$$

If U is a normal domain of f and if $B_r \subset f(U)$, then we denote $B_r' := f^{-1}(B_r) \cap U$. Also, a similar notation for components of preimages of spheres will be used. We define the surface modulus M_S of a family Λ of Borel-measurable subsets of $\mathbb{R}^n$ by setting

$$M_S(\Lambda) = \inf\left\{ \int_{\mathbb{R}^n} \rho(x)^{n/(n-1)}\, dx : \rho\colon \mathbb{R}^n \to [0,\infty] \text{ is Borel measurable}, \int_S \rho(x)\, d\mathcal{H}^{n-1}(x) \geq 1\ \forall S \in \Lambda \right\}.$$

The surface modulus is a conformal invariant, and we have an inequality for quasiregular mappings as follows (see [3, Thm. 3.3]). Suppose that $f\colon \Omega \to \mathbb{R}^n$ is a quasiregular mapping. Furthermore, assume that $f(0) = 0$, $i(0, f) = m$, and that $U(0, f, 1)$ is a normal neighborhood of 0. Let $I \subset (0,1)$ be a Borel measurable set. If $\Lambda := \{S_t : t \in I\}$ and $\Lambda' = \{S'_t : t \in I\}$, then

$$M_S\Lambda \leq (mK_O)^{1/(n-1)} M_S\Lambda'. \tag{2.2}$$

Also, by [5, III, Lemma 4.1] there exists a constant d such that, for all $r < d$, we have

$$H^*(0, f, r) \leq C, \tag{2.3}$$

where C is a constant depending only on n and K_O.

The isoperimetric defect $\delta(\Omega)$ of a bounded domain Ω is defined as

$$\delta(\Omega) = 1 - \frac{|\Omega|}{C_I \mathcal{H}^{n-1}(\partial\Omega)^{n/(n-1)}},$$

where C_I is the sharp constant in the isoperimetric inequality, so that $\delta(B) = 0$ for all balls $B \subset \mathbb{R}^n$. Hence $\delta(\Omega) \in [0,1)$ for every Ω. In addition to the isoperimetric defect, we will use the metric distortion. Let f be as in (2.2). For $t \in (0,1)$, set

$$\alpha(B'_t) = \inf\left\{ \frac{R}{r} : S'_t \subset B(x,R) \setminus \bar{B}(x,r),\ x \in \mathbb{R}^n \right\}.$$

We then have the following connection between the isoperimetric defect and the metric distortion (see [3, Prop. 4.4]). Let f be as in (2.2), and suppose that $\delta(B'_t) < a$. Then

$$\alpha(B'_t) \leq b(a), \tag{2.4}$$

where b is an increasing function depending only on m, n, and K_O and where $b(a) \to 1$ as $a \to 0$.

Again, let f be as before. For each $t \in (0,1)$, the point symmetrizations of the set B'_t and its closure will be the open ball $B(0, \alpha^{-1/n}|B'_t|^{1/n})$ and its closure, respectively. Hence the symmetrization of each S'_t will be a sphere enclosing a ball with the same volume as the set enclosed by S'_t. We define a function $p\colon (0,1) \to (0,\infty)$ by setting $p(t) = \alpha^{-1/n}|B'_t|^{1/n}$. Thus, the image of the set S'_t under point symmetrization is the sphere $S_{p(t)}$. Note that p is strictly increasing. We will use the following notation. If $s = p(t)$, we denote the isoperimetric defect and the linear distortion of B'_t by

$$\delta_s := \delta(B'_t) \quad \text{and} \quad \alpha_s := \alpha(B'_t),$$

respectively. The following estimate is proved in [3, Lemma 5.2]: Suppose that $I \subset (0,1)$ is a Borel measurable set and

$$\Lambda = \{S'_t : t \in I\};$$

then

$$M_S(\Lambda) \leq \omega_{n-1}^{-1/(n-1)} \int_{p(I)} \frac{1-\delta_s}{s}\, ds. \tag{2.5}$$

Finally, we will make use of the notion of infinitesimal space. Let $f\colon \Omega \to \mathbb{R}^n$ be a nonconstant quasiregular mapping and let $x_0 \in \Omega$. Set $\rho_0 = \operatorname{dist}(x_0, \partial\Omega)$ and $R(\rho) = \rho_0/\rho$ for all positive ρ. Define $F_\rho\colon B_{R(\rho)} \to \mathbb{R}^n$ by setting

$$F_\rho(x) = \frac{f(x_0+\rho x) - f(x_0)}{r(x_0, f, \rho)}, \tag{2.6}$$

where

$$r(x_0, f, \rho) = \alpha_n^{-1/n} |f(B(x_0, \rho))|^{1/n}.$$

Then the infinitesimal space of f at x_0 is defined as the set of all nonconstant mappings $F\colon \mathbb{R}^n \to \mathbb{R}^n$ with the property that F is the limit under locally uniform convergence of some sequence (F_{ρ_j}), where $\rho_j \to 0$. By combining the Arzela–Ascoli theorem and Reshetnyak's compactness theorem, one can prove that the infinitesimal space is always nonempty; see [1]. Also,

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^n} K_I(x, F) \leq \operatorname*{ess\,sup}_{x\in\Omega} K_I(x, f)$$

and

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^n} K_O(x, F) \leq \operatorname*{ess\,sup}_{x\in\Omega} K_O(x, f).$$

3. Proofs of Theorems 1.1 and 1.4

Proof of Theorem 1.4. Without loss of generality, we may assume that $x_0 = 0 = f(x_0)$. Let (r_j) be a decreasing sequence converging to zero such that (a) $r_1 < s_0$, where s_0 is as in Section 2, and (b) also the rest of the assumptions needed for the results in Section 2 to hold true are satisfied. Then, (2.3) holds in particular. Also, by the continuity estimate of [5, III, Thm. 4.7], the function $r \to \sigma_f^m(0, r)$ is bounded. Assume that

$$\frac{\sigma_f^m(0)}{\sigma_f^m(0, r_j)} \geq 1 - \frac{1}{j} \quad \forall j \in \mathbb{N} \tag{3.1}$$

and that $\sigma_f^m(0, r_j) \geq \sigma_f^m(0, r_k)$ for $j < k$. Notice that the assumption $\sigma_f^m(0) > 0$ is used here. For $r < r_1$, set $r' := \sup_{x\in B'_r} |x|$. For $j, k \in \mathbb{N}$ $(j < k)$, the inequality (3.1) yields

$$\begin{aligned}(mK_O)^{1/(n-1)} \log\left(\frac{(1-1/j)r'_j}{r'_k}\right) &\leq (mK_O)^{1/(n-1)} \log\left(\frac{\sigma_f^m(0, r_k) r'_j}{\sigma_f^m(0, r_j) r'_k}\right) \\ &\leq \log \frac{r_j}{r_k}.\end{aligned} \tag{3.2}$$

Set $\Lambda_{jk} := \{S_t : r_k < t < r_j\}$ and $\Lambda'_{jk} := \{S'_t : r_k < t < r_j\}$. Then, by (2.2), we have $M_S\Lambda_{jk} \le (mK_O)^{1/(n-1)} M_S\Lambda'_{jk}$. Furthermore, by (2.5),

$$M_S\Lambda'_{jk} \le \omega_{n-1}^{-1/(n-1)} \int_{p(r_k)}^{p(r_j)} \frac{1-\delta_s}{s}\, ds.$$

Recall that

$$p(r) = \alpha_n^{-1/n} |U(0, f, r)|^{1/n}.$$

Now

$$\begin{aligned}\omega_{n-1}^{-1/(n-1)} \log \frac{r_j}{r_k} &= M_S\Lambda_{jk} \le (mK_O)^{1/(n-1)} M_S\Lambda'_{jk} \\ &\le \left(\frac{mK_O}{\omega_{n-1}}\right)^{1/(n-1)} \int_{p(r_k)}^{p(r_j)} \frac{1-\delta_s}{s}\, ds, \end{aligned} \tag{3.3}$$

and combining (3.2) and (3.3) yields

$$\log\left(\frac{(1-1/j)r'_j}{r'_k}\right) \le \int_{p(r_k)}^{p(r_j)} \frac{1-\delta_s}{s}\, ds. \tag{3.4}$$

Now fix $a > 0$, and denote $I = \{s \in (p(r_k), p(r_j)) : \delta_s > a\}$ and $J = (p(r_k), p(r_j)) \setminus I$. Then we have

$$\begin{aligned}\int_{p(r_k)}^{p(r_j)} \frac{1-\delta_s}{s}\, ds &= \int_J \frac{1-\delta_s}{s}\, ds + \int_I \frac{1-\delta_s}{s}\, ds \le \mu(J) + (1-a)\mu(I) \\ &= \log \frac{p(r_j)}{p(r_k)} - a\mu(I), \end{aligned} \tag{3.5}$$

where μ denotes the logarithmic measure. We have $p(r_j) \le r'_j$ and $p(r_k) \ge r'_k/C$, where C is the constant in (2.3). Hence (3.4) and (3.5) give

$$\log\left(\frac{(1-1/j)r'_j}{r'_k}\right) \le \log\left(\frac{Cr'_j}{r'_k}\right) - a\mu(I);$$

that is,

$$\mu(I) \le \frac{1}{a} \log \frac{C}{1-1/j}. \tag{3.6}$$

Notice that this estimate does not depend on k, and thus (3.6) holds with

$$I = \{s \in (0, p(r_j)) : \delta_s > a\}.$$

Next, fix $\varepsilon > 0$ and consider the intervals $A_i := [(1+\varepsilon)^{-i-1}, (1+\varepsilon)^{-i})$ for $i \ge i_0$, where i_0 satisfies $(1+\varepsilon)^{-i_0} < p(r_j)$. Since

$$\mu(A_i) = \log(1+\varepsilon),$$

(3.6) implies that there exist $i(j)$ such that $A_i \cap J \ne \emptyset$ for all $i \ge i(j) - 1$. Now, for all $r < r_{i(j)}$ we find r^1 and r^2 so that, if $p(r) \in A_L$, then $p(r^1) \in A_{L+1}$ and $p(r^2) \in A_{L-1}$ and $\delta_{r^1}, \delta_{r^2} < a$. By (2.4), $\alpha_{r^1}, \alpha_{r^2} < b(a)$, where $b(a) \to 1$ as $a \to 0$. Suppose that

$$S'_{r^2} \subset B(z, R_1) \setminus B(z, R_2) \quad \text{and} \quad S'_{r^1} \subset B(w, R_3) \setminus B(w, R_4),$$

so that $\alpha_{r^2} = R_1/R_2$ and $\alpha_{r^1} = R_3/R_4$. Then, since

$$|B'_{r^2}| = \alpha_n p(r^2)^n \le \alpha_n (1+\varepsilon)^{n(-L+2)}$$

and

$$|B'_{r^1}| = \alpha_n p(r^1)^n \ge \alpha_n (1+\varepsilon)^{n(-L-1)},$$

we have

$$\begin{aligned} R_1 \le b(a)R_2 \le b(a)\alpha_n^{-1/n}|B'_{r^2}|^{1/n} &\le (1+\varepsilon^3)b(a)\alpha_n^{-1/n}|B'_{r^1}|^{1/n} \\ &\le b(a)(1+\varepsilon)^3 R_3 \le b(a)^2(1+\varepsilon)^3 R_4. \end{aligned}$$

Observe that

$$B(w, R_4) \subset S'_{r^1} \subset S'_{r^2} \subset B(z, R_1).$$

Since there exists a ball $B(z, R_4 - 2(R_1 - R_4)) \subset B(w, R_4)$ (we may assume a and ε to be so small that $R_4 - 2(R_1 - R_4) > 0$), we further have

$$S'_r \subset B(z, R_1) \setminus B(w, R_4) \subset B(z, R_1) \setminus B(z, R_4 - 2(R_1 - R_4)),$$

and so

$$\alpha(B'_r) \le \frac{R_1}{R_4 - 2(R_1 - R_4)} = \frac{R_1}{3R_4 - 2R_1} \le \frac{1}{3C(a,\varepsilon) - 2},$$

where $C(a,\varepsilon) \to 1$ as $a, \varepsilon \to 0$. We conclude that for all $a, \varepsilon > 0$ there exist $r(a,\varepsilon)$ such that, for all $r < r(a,\varepsilon)$,

$$\alpha(B'_r) \le \frac{1}{3C(a,\varepsilon) - 2}$$

and hence

$$\limsup_{r\to 0} \alpha(B'_r) = 1.$$

Let us next define a variant of σ_f^m; set

$$\phi_f^m(0,\delta) := \delta^{-(mK_O)^{-1/(n-1)}} p(\delta).$$

Then, by (2.3), $\phi_f^m(0,\delta) \ge (1/C)\sigma_f^m(0,\delta)$ and, in particular, we can choose a decreasing sequence (r_j), converging to zero, such that $\phi_f^m(0,r_j) > \phi_f^m(0,r_k)$ for $j < k$ and

$$\frac{\phi_f^m(0)}{\phi_f^m(0,r_j)} \ge 1 - \frac{1}{j} \quad \forall j \in \mathbb{N},$$

where

$$\phi_f^m(0) = \limsup_{\delta\to 0} \phi_f^m(0,\delta) \ge \frac{1}{C}\sigma_f^m(0) > 0.$$

We next show that actually

$$\phi_f^m(0) = \lim_{\delta\to 0} \phi_f^m(0,\delta). \tag{3.7}$$

Fix $j \in \mathbb{N}$, and consider $r < r_j$ and $r_k < r$. Denote $\Lambda_j = \{S_t : t \in (r, r_j)\}$, $\Lambda'_j = \{S'_t : t \in (r, r_j)\}$, $\Lambda_k = \{S_t : t \in (r_k, r)\}$, and $\Lambda'_k = \{S'_t : t \in (r_k, r)\}$. As before, we have

$$\omega_{n-1}^{-1/(n-1)} \log \frac{r_j}{r} = M_S \Lambda_j \leq (mK_O)^{1/(n-1)} M \Lambda'_j \\ \leq \left(\frac{mK_O}{\omega_{n-1}}\right)^{1/(n-1)} \int_{p(r)}^{p(r_j)} \frac{1-\delta_s}{s}\, ds \\ \leq \left(\frac{mK_O}{\omega_{n-1}}\right)^{1/(n-1)} \log \frac{p(r_j)}{p(r)} \tag{3.8}$$

and

$$\omega_{n-1}^{-1/(n-1)} \log \frac{r}{r_k} = M_S \Lambda_k \leq (mK_O)^{1/(n-1)} M \Lambda'_k \\ \leq \left(\frac{mK_O}{\omega_{n-1}}\right)^{1/(n-1)} \int_{p(r_k)}^{p(r)} \frac{1-\delta_s}{s}\, ds \\ \leq \left(\frac{mK_O}{\omega_{n-1}}\right)^{1/(n-1)} \log \frac{p(r)}{p(r_k)}. \tag{3.9}$$

Combining (3.8) and (3.9) with the definition of ϕ_f^m then yields

$$\phi_f^m(0, r_k) \leq \phi_f^m(0, r) \leq \phi_f^m(0, r_j).$$

We conclude that the function $\delta \to \phi_f^m(0, \delta)$ is increasing, and (3.7) holds in particular.

In order to complete the proof of Theorem 1.4, we need to show that the balls defining the $\alpha(B'_r)$ must be centered sufficiently near the origin when $r \to 0$. Toward this end, we use the following auxiliary result (cf. [3, Lemma 6.1]). Let $B(x, r) \subset B(y, R)$, and denote by Λ the family of all sets separating $B(x, r)$ and $\mathbb{R}^n \setminus B(y, R)$. Then

$$M_S(\Lambda) \leq \left(1 - A\left(\frac{R}{r}, |x - y|\right)\right) \omega_{n-1}^{-1/(n-1)} \log \frac{R}{r}, \tag{3.10}$$

where $t \to A(R/r, t)$ is continuous and strictly increasing, and $A(R/r, 0) = 0$.

Now, fix R so that $\alpha(B'_r) < 1 + 1/k$ for all $r < R$ and so that

$$\frac{\phi_f^m(0)}{\phi_f^m(0, r)} \geq 1 - \frac{1}{k} \quad \text{for all } r < R. \tag{3.11}$$

Denote $R_i := (1/i)R$, and set

$$\Lambda_i := \{S_t : t \in (R_i, R)\}, \qquad \Lambda'_i := \{S'_t : t \in (R_i, R)\}.$$

Suppose m_i is the radius of the largest ball $B(x_i, m_i)$ with the property $B(x_i, m_i) \subset B'_{R_i}$ for some $x_i \in \mathbb{R}^n$ that is a minimizing point in the definition of $\alpha(B'_{R_i})$. Also, let M_i be the smallest radius such that $B'_R \subset B(y_i, M_i)$ for some $y_i \in \mathbb{R}^n$ a minimizing point in the definition of $\alpha(B'_R)$. Denote by Λ_i^* the family of all sets separating $B(x_i, m_i)$ and $\mathbb{R}^n \setminus B(y_i, M_i)$. We have

$$m_i \geq \frac{1}{\alpha(B'_{R_i})} p(R_i) \geq \frac{k}{k+1} p(R_i), \tag{3.12}$$

and similarly

$$M_i \le \Big(1+\frac{1}{k}\Big)p(R). \tag{3.13}$$

On the other hand, (3.11) gives

$$\frac{p(R_i)}{p(R)} = \frac{\phi_f^m(0,R_i)}{\phi_f^m(0,R)}\Big(\frac{R_i}{R}\Big)^{(mK_O)^{-1/(n-1)}} \ge \Big(1-\frac{1}{k}\Big)i^{-(mK_O)^{-1/(n-1)}}. \tag{3.14}$$

Furthermore,

$$\begin{aligned}\omega_{n-1}^{-1/(n-1)}\log i &= \omega_{n-1}^{-1/(n-1)}\log\frac{R}{R_i}\\ &= M_S\Lambda_i \le (mK_O)^{1/(n-1)}M\Lambda_i' \le (mK_O)^{1/(n-1)}M\Lambda_i^* \end{aligned}\tag{3.15}$$

and, by (3.10),

$$M\Lambda_i^* \le (1-A(i,|x_i-y_i|))\omega_{n-1}^{-1/(n-1)}\log\frac{M_i}{m_i}. \tag{3.16}$$

By (3.12), (3.13), and (3.14),

$$\log\frac{M_i}{m_i} \le \log\Big(\frac{p(R)(k+1)^2}{p(R_i)k^2}\Big) \le 3\log\Big(\frac{k}{k-1}\Big)+(mK_O)^{-1/(n-1)}\log i. \tag{3.17}$$

Combining (3.15), (3.16), and (3.17) then yields

$$\log i \le (1-A(i,|x_i-y_i|))\Big(3(mK_O)^{1/(n-1)}\log\Big(\frac{k}{k-1}\Big)+\log i\Big)$$

and so

$$A(i,|x_i-y_i|) \le \frac{3(mK_O)^{1/(n-1)}\log\big(\frac{k}{k-1}\big)}{\log i} =: \Phi(k,i).$$

If we denote the inverse of $t\to A(i,t)$ at a point T by $A^{-1}(i,T)$, we see that

$$\begin{aligned} H^*(0,f,R) &\le \frac{|y_i+M_i|}{M_i/\alpha(B_R')-|y_i|}\\ &\le \frac{|x_i-y_i|+\alpha(B_{R_i}')p(R_i)+\big(1+\frac{1}{k}\big)p(R)}{\frac{k-1}{k}p(R)-|x_i-y_i|-\alpha(B_{R_i}')p(R_i)}\\ &\le \frac{A^{-1}(i,\Phi(k,i))+\big(1+\frac{1}{k}\big)(p(R_i)+p(R))}{\frac{k-1}{k}p(R)-A^{-1}(i,\Phi(k,i))-\big(1+\frac{1}{k}\big)p(R_i)}. \end{aligned}\tag{3.18}$$

We may take $k\to\infty$ as $R\to 0$, and so (3.18) yields

$$H^*(0,f) \le \frac{p(R)+p(R_i)}{p(R)-p(R_i)}$$

(notice that $A^{-1}(i,\Phi(k,i)) \to 0$ as $k\to\infty$). On the other hand, $p(R_i) \le \varphi(i)p(R)$, where φ depends only on i and $\varphi(i)\to 0$ as $i\to\infty$. Hence, letting $i\to\infty$ gives $H^*(0,f)=1$ and so

$$\lim_{\delta\to 0}\sigma_f^m(0,\delta)=\lim_{\delta\to 0}\phi_f^m(0,\delta)=\phi_f^m(0)=\sigma_f^m(0).$$

The proof of Theorem 1.4 is complete. □

Proof of Theorem 1.1. We again assume that $x_0 = 0 = f(x_0)$. By applying Theorem 1.4 to f^{-1}, we see that $H(0, f) = 1$ and $\omega_f(0) = \lim_{\delta\to 0}\omega_f(0,\delta)$. Let g be the locally uniform limit of a converging sequence (F_ρ) as in (2.6). For each fixed r and each $x \in R(\rho)$ such that $|x| = r$ and $R(\rho) > r$, we have

$$|F_\rho(x)|=\frac{\alpha^{1/n}|f(\rho x)|}{|f(B_\rho)|^{1/n}}\le\frac{(r\rho)^{K_I^{-1/(n-1)}}\omega_f(0,r\rho)H(0,f,\rho)}{\rho^{K_I^{-1/(n-1)}}\omega_f(0,\rho)}\xrightarrow{\rho\to 0} r^{K_I^{-1/(n-1)}}$$

and

$$|F_\rho(x)|\ge\frac{(r\rho)^{K_I^{-1/(n-1)}}\omega_f(0,r\rho)}{\rho^{K_I^{-1/(n-1)}}\omega_f(0,\rho)H(0,f,r\rho)}\xrightarrow{\rho\to 0} r^{K_I^{-1/(n-1)}}.$$

Hence $|g(x)| = |x|^{K_I^{-1/(n-1)}}$ for all $x \in \mathbb{R}^n$.

Recall that $K_I(x, g) \le K_I$ for almost every $x \in \mathbb{R}^n$. We next claim that

$$g(tx)=t^{K_I^{-1/(n-1)}}g(x)\quad\text{for all } t>0 \text{ and } x\in\mathbb{R}^n. \tag{3.19}$$

If this is not the case, then there exists a ray $L_y = \{ty : t > 0\}$ for some $y \in S_1$ such that $g^{-1}(L_y)$ is not a ray of the form $\{tx : t > 0\}$ for any $x \in S_1$. Furthermore, by continuity of g, there exist an $\varepsilon > 0$, a spherical cap $C^\varepsilon \subset S_1$, and a $T > 0$ such that (denoting $A_T = B_{2T} \setminus B_T$), for all $y \in C^\varepsilon$,

$$\int_{g^{-1}(L_y)\cap A_T} G\,ds > 1+\varepsilon, \tag{3.20}$$

where $G(x) = \frac{1}{|x|}\log^{-1} 2$. Denote by Γ the family of all line segments of the form

$$\{ty : t\in (T^{K_I^{-1/(n-1)}},(2T)^{K_I^{-1/(n-1)}})\}.$$

Then, since for

$$F=\bigcup_{y\in C^\varepsilon}(g^{-1}(L_y)\cap A_T)$$

we have $|F| > 0$ and since, by (3.20), we can choose a test function G' for $M(g^{-1}(\Gamma))$ as

$$G'(x)=\begin{cases}\frac{1}{|x|}\log^{-1}2, & x\in A_T\setminus F,\\ \frac{1}{1+\varepsilon}\frac{1}{|x|}\log^{-1}2, & x\in F,\end{cases}$$

it follows that

$$\begin{aligned}K_I\omega_{n-1}\log^{1-n}2&=\omega_{n-1}\log^{1-n}\left(\frac{2T}{T}\right)^{K_I^{-1/(n-1)}}=M\Gamma\le K_I M(g^{-1}\Gamma)\\&\le K_I\int_{A_T}G'(x)^n\,dx<K_I\omega_{n-1}\log^{1-n}2.\end{aligned}$$

This is a contradiction and hence (3.19) holds true.

We have shown that g is of the form $g(x) = |x|^{K_I^{-1/(n-1)}} h(x/|x|)$, where $h\colon S_1 \to S_1$ is a homeomorphism. To finish the proof, we need to show that $\mathcal{H}^{n-1}(h(V)) = \mathcal{H}^{n-1}(V)$ for all Borel sets $V \in S_1$. If this is not true, then there exists a Borel set $W \subset S_1$ such that

$$\mathcal{H}^{n-1}(h(W)) > \mathcal{H}^{n-1}(W). \tag{3.21}$$

Denote

$$\Gamma_W := \bigcup_{w \in W} \{tw : t \in (1,2)\}.$$

Then the paths in $g(\Gamma_W)$ are line segments of the form

$$\{th(w) : t \in (1, 2^{K_I^{-1/(n-1)}})\},$$

and we have

$$\begin{aligned} \omega_{n-1}\mathcal{H}^{n-1}(h(W))K_I \log^{1-n} 2 = M(g\Gamma_W) &\le K_I M(\Gamma_W) \\ &= \omega_{n-1}\mathcal{H}^{n-1}(W)K_I \log^{1-n} 2, \end{aligned}$$

which contradicts (3.21). The proof of Theorem 1.1 is complete. □

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