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Geometry of the Lagrangian Grassmannian $\mathbf{LG}(3,6)$ with Applications to Brill–Noether Loci

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Dedicated to Andrei Nikolaevich Tyurin

1. Introduction

A beautiful theorem of Mukai is his interpretation of the general complete intersection of the Lagrangian Grassmannian $\mathbf{LG}(3,6) \subset \mathbf{P}^{13}$ in its Plücker embedding with a linear subspace $\mathbf{P}^{10}$ as a non-abelian Brill–Noether locus of vector bundles on a plane quartic curve [28]. This quartic curve can in a natural way be interpreted as the orthogonal plane section of the dual variety to $\mathbf{LG}(3,6)$ in $\check{\mathbf{P}}^{13}$ (cf. [11]). In this paper we consider, very much in the spirit of [27], general linear sections of $\mathbf{LG}(3,6) \subset \mathbf{P}^{13}$ of various dimensions, and we show that the orthogonal linear section of the dual variety $\check{F}$ of $\mathbf{LG}(3,6)$ has an interpretation as a moduli space of vector bundles on the original linear section. A similar study of linear sections of the 10-dimensional spinor variety or orthogonal Grassmannian $\mathbf{OG}(5,10) \subset \mathbf{P}^{15}$ is taken up by the first author and Markushevich in [12].

The moduli spaces of stable vector bundles on curves is by now a classical subject dating back to the 1960s and the fundamental work of Narasimhan, Seshadri and Tyurin (see [30; 36]). More recently, the subvarieties of these moduli spaces representing bundles with many sections has attracted the attention from many authors [2; 3; 9; 27; 31; 34]. The corresponding theory for vector bundles on $K3$ surfaces becomes particularly nice as explained in Mukai's fundamental paper [20]. Our purpose here is to present examples in this theory where the moduli spaces are complete linear sections of the dual variety of $\mathbf{LG}(3,6)$ in $\check{\mathbf{P}}^{13}$. A general tangent hyperplane section of $\mathbf{LG}(3,6)$ is nodal; that is, it has a unique tangency point with a quadratic singularity. We construct a rank-2 vector bundle on a nodal hyperplane section of $\mathbf{LG}(3,6)$ blown up in the node. This construction is key to the proof of Theorem 3.3.4 and Corollary 3.3.10, which are summarized in the following theorem.

Theorem 1.1. *The projection of a nodal hyperplane section of* $\mathbf{LG}(3,6)$ *from the node is a complete* 5*-dimensional linear section of a Grassmannian variety* $\mathbf{Gr}(2,6)$. *This linear section contains a* 4*-dimensional quadric, and the general* 5*-dimensional linear section of* $\mathbf{Gr}(2,6)$ *that contains a* 4*-dimensional quadric appears this way.*

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We expect similar results to hold for the homogeneous varieties whose general curve section are canonical curves of smaller genus. In particular, we expect (i) the projection from the node of the general nodal hyperplane section of $\mathbf{Gr}(2,6)$ to be a 7-fold linear section of the spinor variety S_{10} and (ii) the projection from the node of the general nodal 5-fold linear section of S_{10} to be the complete intersection of a Grassmannian variety $\mathbf{Gr}(2,5)$ and a quadric. These lower-genus cases will not be treated here.

Given a smooth linear section X of $\mathbf{LG}(3,6)$ of dimension at most 4, each nodal hyperplane section that contains X gives rise to an embedding of X into a $\mathbf{Gr}(2,6)$. In particular, it gives rise to a rank-2 vector bundle on X with a 6-space of global sections and determinant equal to the restriction of the Plücker divisor on $\mathbf{LG}(3,6)$. Furthermore, this vector bundle is stable, and when X is at least 2-dimensional we show that the set $\check{F}(X)$ of nodal hyperplane sections of $\mathbf{LG}(3,6)$ that contain X forms a component of the corresponding moduli space of stable rank-2 vector bundles on X (Theorem 3.4.8, Proposition 3.4.10). When X is a curve, $\check{F}(X)$ forms a component of the corresponding Brill–Noether locus in the moduli space of stable rank-2 vector bundles on X (Theorem 3.4.7).

The paper is organized as follows. Section 2 is devoted to the geometry of $\mathbf{LG}(3,6)$ and contains a number of results that we think are interesting on their own. In particular, we describe geometrically the cycles of Lagrangian planes that contain a given point or intersect a given plane. The Lagrangian Grassmannian $\mathbf{LG}(3,6)$ is the minimal orbit of an irreducible representation of the symplectic group $\mathrm{Sp}_6(\mathbf{C})$. We recall the four orbits of this group and the singular hyperplane sections corresponding to the four orbits of the group in the dual projective space; $\mathbf{LG}(3,6) \subset \mathbf{P}^{13}$ parameterizes the 6-fold of Lagrangian planes in $\mathbf{P}^5$ with respect to a given nondegenerate 2-form. In Section 3 we show that a hyperplane section that is singular at a point $p \in \mathbf{LG}(3,6)$ defines naturally a conic in the Lagrangian plane represented by p. Furthermore, this conic parameterizes smooth quadric 3-folds contained in the hyperplane section. The correspondence between singular hyperplane sections and conics in Lagrangian planes manifests itself in various ways and is the key to the main results of this paper. In particular, the conic corresponding to a nodal hyperplane section is the crucial ingredient in the construction of a rank-2 vector bundle with a 6-space of global sections on the nodal hyperplane section blown up in the node. The application to moduli spaces of vector bundles and Brill–Noether loci occupies the last part of Section 3.

NOTATION. We will use $\mathbf{Gr}(k,n)$ to denote not only the Grassmannian of rank-k subspaces of an n-dimensional vector space when $k < n$ but also the Grassmanian of rank-n quotient spaces of a k-dimensional vector space when $n < k$.

2. Geometry of the Lagrangian Grassmannian

2.1. *The Group* $\mathrm{Sp}_6(\mathbf{C})$ *and Its Homogeneous Space* $\mathbf{LG}(3,6)$

Let $V = \mathbf{C}^6$ be a 6-dimensional complex vector space, and let $\alpha\colon V \times V \to \mathbf{C}$, $\alpha\colon (v,v') \mapsto \alpha(v,v')$ be a symplectic form on V. It follows that α is bilinear, skew symmetric, and nondegenerate (i.e., $\alpha(v \times V) = 0$ implies $v = 0$). We may choose a basis $\{e_1, \ldots, e_6\}$ of V in which the Gram matrix

$$J = (\alpha(e_i, e_j)) = \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix},$$

where I_3 is the unit 3×3 matrix. The complex symplectic group $\mathrm{Sp}_6(\mathbf{C})$ has a natural embedding in $\mathrm{SL}(6, \mathbf{C})$ as the subgroup of all the complex rank-6 matrices that leave the matrix J invariant:

$$\mathrm{Sp}_6(\mathbf{C}) = \{Z \in \mathrm{SL}(6, \mathbf{C}) : {}^tZJZ = J\}.$$

A subspace $U \subset V$ is called *isotropic* if $\alpha(U, U) = 0$. The maximal dimension of an isotropic subspace in V is 3, and in this case it is called *Lagrangian*. Any isotropic subspace is contained in some Lagrangian subspace.

By definition, the complex Lagrangian Grassmannian $\mathbf{LG}(3, V)$ is the set of Lagrangian subspaces of $V = \mathbf{C}^6$. The group $\mathrm{Sp}_6(\mathbf{C})$ acts on the set of Lagrangian subspaces by $U \mapsto A \cdot U$; it is easy to check that this action is transitive, so $\mathbf{LG}(3, V)$ is a homogeneous space under $\mathrm{Sp}_6(\mathbf{C})$. More precisely, $\mathbf{LG}(3, V)$ is a smooth complex 6-fold that admits a representation as a homogeneous space $\mathrm{Sp}_6(\mathbf{C})/\mathrm{St}$, where St is the stabiliser group St_U of any Lagrangian subspace $U \subset V$. In our fixed base the subspace $U_0 = \langle e_1, e_2, e_3 \rangle \subset V$ is Lagrangian and so $\mathrm{St} = \mathrm{St}_{U_0} \subset \mathrm{Sp}_6(\mathbf{C})$ consists of all the matrices of the form $\left(\begin{smallmatrix} A & B \\ 0 & {}^tA^{-1} \end{smallmatrix}\right)$, where A and B are complex 3×3 matrices such that $A \cdot {}^tB = B \cdot {}^tA$.

The Lagrangian Grassmannian has an alternative representation as a quotient for the compact group $\mathrm{Sp}(3) = \mathrm{Sp}_6(\mathbf{C}) \cap U(6) \subset \mathrm{Sp}_6(\mathbf{C})$ by the subgroup $\mathrm{St}_{U_0} \cap U(6) \subset \mathrm{Sp}(3)$. It is easy to see that $\mathrm{St}_{U_0} \cap U(6)$ consists of all the 6×6 matrices of the form $\left(\begin{smallmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{smallmatrix}\right)$, where $A \in U(3)$ is a unitary 3×3 matrix. Now one can see that the Lagrangian Grassmannian $\mathbf{LG}(3, V)$ is diffeomorphic to $\mathrm{Sp}(3)/U(3)$ (cf. [35, Sec. 17]; see also [29]).

2.2. *Representations and Plücker Embedding*

From now on we fix the form α, the basis $\{e_1, \ldots, e_6\}$ for V, and a dual basis $\{x_1, \ldots, x_6\}$ for V^*. With respect to this basis for V, the matrix of the form α is

$$J = (\alpha(e_i, e_j)) = \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix},$$

where I_3 is the unit 3×3 matrix. In the basis $\{x_i \wedge x_j \mid 1 \le i < j \le 6\}$ for $\wedge^2 V^*$, the symplectic form α has the following expression:

$$\alpha = x_1 \wedge x_4 + x_2 \wedge x_5 + x_3 \wedge x_6.$$

It defines a correlation

$$L_\alpha \colon V \to V^*, \quad v \mapsto \alpha(v, -),$$

which is nonsingular because α is nondegenerate. This correlation induces isomorphisms that we also denote by L_α,

$$L_\alpha \colon \wedge^k V \cong \wedge^k V^*$$

for $k = 1, \dots, 6$. Consider the representations $\wedge^k V$ and $\wedge^k V^*$ of $\mathrm{SL}(6, \mathbf{C})$ and the induced representation of $\mathrm{Sp}_6(\mathbf{C}) \subset \mathrm{SL}(6, \mathbf{C})$. The isomorphisms $\wedge^k V \cong \wedge^k V^*$ induced by α are clearly isomorphisms of these $\mathrm{Sp}_6(\mathbf{C})$-representations.

Consider the $\mathrm{SL}(6, \mathbf{C})$ representation $\wedge^3 V$ restricted to the subgroup $\mathrm{Sp}_6(\mathbf{C}) \subset \mathrm{SL}_6(C)$. The form α defines naturally a contraction

$$\alpha \colon \wedge^3 V \to V.$$

The representation therefore decomposes as

$$\wedge^3 V = V(14) \oplus V(6),$$

where

$$V(14) = \{w \in \wedge^3 V \mid \alpha(w) = 0\}$$

and

$$V(6) = \{w \in \wedge^3 V \mid L_\alpha(w) \in \alpha \wedge V^*\}$$

are irreducible representations of $\mathrm{Sp}_6(\mathbf{C})$ that have dimensions 14 and 6, respectively (cf. [8, p. 258]). Furthermore,

$$V(14)^* := L_\alpha(V(14)) = \{\omega \in \wedge^3 V^* \mid \omega \wedge \alpha = 0\} \subset \wedge^3 V^*$$

whereas

$$V(6)^* = \alpha \wedge V^*.$$

For an explicit description of $V(14)^* \subset \wedge^3 V^*$, consider the decomposition of V in two Lagrangian subspaces $U_0 = \langle e_1, e_2, e_3 \rangle$ and $U_1 = \langle e_4, e_5, e_6 \rangle$. We denote by $U_0^\perp$ the Lagrangian subspace

$$L_\alpha(U_0) = \langle x_4, x_5, x_6 \rangle \subset V^*,$$

and likewise

$$U_1^\perp = L_\alpha(U_1) = \langle x_1, x_2, x_3 \rangle.$$

The decomposition

$$V^* = U_1^\perp \oplus U_0^\perp$$

induces a decomposition of $\wedge^3 V^*$:

$$\wedge^3 V^* = \wedge^3 U_1^\perp \oplus (\wedge^2 U_1^\perp \otimes U_0^\perp) \oplus (U_1^\perp \otimes \wedge^2 U_0^\perp) \oplus \wedge^3 U_0^\perp.$$

The exterior products

$$e_{ijk} = e_i \wedge e_j \wedge e_k, \quad 1 \le i < j < k \le 6,$$

form a basis of $\wedge^3 V$, with dual basis $(x_{ijk} = x_i \wedge x_j \wedge x_k)_{1 \le i,j,k \le 6}$. Thus we interpret the basis for $\wedge^3 V$ as coordinates on $\wedge^3 V^*$. According to the previous decomposition, the coordinates of a 3-form $\omega \in \wedge^3 V^*$ may therefore be organized in matrices:

$$u^* := e_{123}, \qquad X^* = (x^*_{ab}) := \begin{pmatrix} e_{423} & e_{143} & e_{124} \\ e_{523} & e_{153} & e_{125} \\ e_{623} & e_{163} & e_{126} \end{pmatrix},$$

$$Y^* = (y^*_{ab}) := \begin{pmatrix} e_{156} & e_{416} & e_{451} \\ e_{256} & e_{426} & e_{452} \\ e_{356} & e_{436} & e_{453} \end{pmatrix}, \qquad z^* := e_{456}.$$

The component of ω given by the matrix X^* defines a bilinear form on U_0. If

$$a_1e_1 + a_2e_2 + a_3e_3, b_1e_1 + b_2e_2 + b_3e_3 \in U_0$$

then the form defined by X becomes

$$(a_1, a_2, a_3) \cdot \begin{pmatrix} e_{423} & e_{143} & e_{124} \\ e_{523} & e_{153} & e_{125} \\ e_{623} & e_{163} & e_{126} \end{pmatrix} \cdot (b_1, b_2, b_3)^t.$$

In these coordinates, the subspace $V(14)^* \subset \wedge^3 V^*$ has a simple interpretation:

$$V(14)^* = \{\omega \in \wedge^3 V^* \mid e_{i14} + e_{i25} + e_{i36} = 0,\ 1 \le i \le 6\}.$$

For the decomposition $\omega = [u^*, X^*, Y^*, z^*]$ we have that

$$\omega \in V(14)^* \iff X^* \text{ and } Y^* \text{ are symmetric } 3 \times 3 \text{ matrices.}$$

In particular, the bilinear form defined by the component X^* is symmetric on U_0.

Therefore, we have shown the following lemma.

LEMMA 2.2.1. *There are natural isomorphisms*

$$q(U_0)\colon V(14)^* \cap U_1^{\perp} \otimes \wedge^2 U_0^{\perp} \to \operatorname{Sym}^2 U_0^*$$

and

$$q(U_0^{\perp})\colon V(14) \cap U_1 \otimes \wedge^2 U_0 \to \operatorname{Sym}^2 (U_0^{\perp})^*.$$

For $\omega \in \wedge^3 U_0^{\perp} \oplus U_1^{\perp} \otimes \wedge^2 U_0^{\perp}$, we denote the quadratic form associated to the projection on the second factor by $q_\omega(U_0)$ (or just by q_ω, if U_0 is understood from the context), and by abuse we sometimes use the same notation for the conic in $\mathbf{P}(U_0)$ that the quadratic form defines. Similarly, $w \in \wedge^3 U_0 \oplus U_1 \otimes \wedge^2 U_0$ defines a conic

$$q_w(U_0^{\perp}) \subset \mathbf{P}(U_0^{\perp}).$$

The Plücker embedding $\Sigma := \mathbf{LG}(3, V) \subset \mathbf{Gr}(3, V) \subset \mathbf{P}(\wedge^3 V)$ of the Lagrangian Grassmannian $\mathbf{LG}(3, V)$ is the intersection of $\mathbf{Gr}(3, V)$ with $\mathbf{P}(V(14))$; that is,

$$\Sigma = \mathbf{P}(V(14)) \cap \mathbf{Gr}(3, V) \subset \mathbf{P}(\wedge^3 V).$$

2.3. Orbits and Pivots

Recall from [15, Sec. 9] that the action ρ of the group $\mathrm{Sp}_6(\mathbf{C})$ on $\mathbf{P}^{13} = \mathbf{P}(V(14))$ has precisely four orbits:

$$\mathbf{P}^{13} \setminus F,\ F \setminus \Omega,\ \Omega \setminus \Sigma,\ \Sigma.$$

The dual action $\check{\rho}$ is equivalent to ρ induced by L_α, so it has four corresponding orbits,

$$\check{\mathbf{P}}^{13} \setminus \check{F},\ \check{F} \setminus \check{\Omega},\ \check{\Omega} \setminus \check{\Sigma},\ \check{\Sigma},$$

in $\check{\mathbf{P}}^{13} = \mathbf{P}(V(14)^*)$. In this section we give a geometric characterization of these orbits.

The smallest orbit Σ, and the only closed one, is the Lagrangian Grassmannian itself. The closure of the orbit $F \setminus \Omega$ is the union of the projective tangent spaces to Σ, and it forms a hypersurface $F \subset \mathbf{P}^{13}$. Similarly, the dual variety to Σ (the variety of tangent hyperplanes to Σ) is a hypersurface $\check{F} \subset \check{\mathbf{P}}^{13}$ that is isomorphic to F. By [33, p. 108] the equation defining F is

$$f(w) = (uz - \operatorname{tr} XY)^2 + 4u \det Y + 4z \det X - 4\Sigma_{ij} \det(X_{ij}) \cdot \det(Y_{ij}),$$

where X_{ij} and Y_{ij} are the complementary matrices to the elements x_{ij} and y_{ij} (see also [33, p. 83]). Here Ω is the singular locus of F defined by the Jacobian ideal of f. A simple computation in MACAULAY (see [1]) shows that the ideal I_Ω of Ω in $S = \operatorname{Sym}(V(14)^*)$ has a resolution

$$0 \leftarrow I_\Omega \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_4 \leftarrow 0$$

with $F_i = \bigoplus_{j \in \mathbf{Z}} \beta_{ij} S(-j)$ and Betti numbers β_{ij}:

$$\begin{matrix} \beta_{00} & \beta_{11} & \beta_{22} & \beta_{33} & \beta_{44} \\ \beta_{01} & \beta_{12} & \beta_{23} & \beta_{34} & \beta_{45} \\ \beta_{02} & \beta_{13} & \beta_{24} & \beta_{35} & \beta_{46} \\ \beta_{03} & \beta_{14} & \beta_{25} & \beta_{36} & \beta_{47} \end{matrix} = \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 14 & 21 & 0 & 0 \\ 0 & 0 & 0 & 14 & 6 \end{matrix}.$$

The invariants of the closures of the orbits are as follows.

PROPOSITION 2.3.1.

(i) $\dim F = 12$, $\deg F = 4$, $K_F = \mathcal{O}_F(-10)$, *and F has quadratic singularities along* $\Omega \setminus \Sigma$.
(ii) $\dim \Omega = 9$ *and* $\deg \Omega = 21$.
(iii) $\dim \Sigma = 6$, $\deg \Sigma = 16$, *and* $K_\Sigma = \mathcal{O}_\Sigma(-4)$.

Proof. For F it remains to check the statements on singularities. Let f be the polynomial defining F. The singularities of f along the subscheme defined by the partials are quadratic if and only if the subscheme is smooth. But the subscheme defined by the partials of f is exactly Ω, which is smooth outside Σ. The invariants of Ω follow from the Betti numbers of the foregoing resolution. To compute the invariants of Σ we consider the universal exact sequence of vector bundles on $G = \mathbf{Gr}(3,6)$:

$$0 \to U \to V \otimes \mathcal{O}_G \to Q \to 0,$$

where U is the universal subbundle. The 2-form α restricts naturally to U—that is, to a section α_U of $(\wedge^2 U)^* \cong \wedge^2 U^*$. The variety $\Sigma \subset G$ of Lagrangian subspaces of V with respect to α is therefore nothing but the 0-locus $Z(\alpha_U)$ of this section, and the class $[\Sigma] = c_3(\wedge^2 U^*) \cap G = (c_1(U^*)c_2(U^*) - c_3(U^*)) \cap G$.

Therefore, we have $\deg \Sigma = c_1^6(U^*) \cap \Sigma = 16$, and the canonical divisor $K_\Sigma = K_G|_\Sigma + c_1(\wedge^2 U^*) \cap \Sigma = -4c_1(U^*) \cap \Sigma$. In particular, Σ is a Fano 6-fold of index 4. □

Landsberg and Manivel prove the following theorem in [16] (see also [6, Chap. 3]).

THEOREM 2.3.2. *The representation* ρ *of* $\mathrm{Sp}_6(\mathbf{C})$ *on* $\mathbf{P}^{13} = \mathbf{P}(V(14))$ *has four orbits,*

$$\mathbf{P}^{14} = \Sigma \cup (\Omega \setminus \Sigma) \cup (F \setminus \Omega) \cup (\mathbf{P}^{13} \setminus F),$$

where:

(i) Σ *is the* 6*-fold Lagrangian Grassmannian* $\mathbf{LG}(3,6)$;
(ii) $w \in \Omega$ *if and only if there exist infinitely many tangent lines to* Σ *passing through* w;
(iii) $w \in F \setminus \Omega$ *iff there is a unique tangent line* L_w *to* Σ *through* w; *and*
(iv) $w \in \mathbf{P}^{13} \setminus F$ *iff there exists a unique secant line* L_w *to* Σ *through* w.

We adopt Donagi's notation and let the *pivots* of $w \in \mathbf{P}^{13} \setminus \Omega$ be the intersection points $\{a, b\} = L_w \cap \Sigma$, where L_w is the unique secant line through w. Similarly, if $w \in F \setminus \Omega$ (the case when $a = b$) then we call a the pivot of w. When $w \in \Omega$, a pivot of w is any point $u \in \Sigma$ such that w lies on a tangent line through u (we will see in Proposition 2.5.1 that the set of pivots to a point $w \in \Omega \setminus \Sigma$ form a smooth quadric surface).

The restriction of the universal exact sequence on G to Σ becomes

$$0 \to U \to V \otimes \mathcal{O}_\Sigma \to Q \to 0.$$

The correlation $L_\alpha \colon V \to V^*$ ($v \mapsto \alpha(v, -)$) sets up a natural isomorphism: $Q \cong U^*$, so the universal sequence becomes

$$0 \to U \to V \otimes \mathcal{O}_\Sigma \to U^* \to 0. \tag{1}$$

It follows that the tangent bundle T_Σ is a subbundle of $\mathcal{H}om(U, U^*) = U^* \otimes U^*$. In fact, it is the subbundle consisting of symmetric tensors:

$$T_\Sigma = \mathrm{Sym}^2 U^*.$$

In the coordinates $[u : X : Y : z]$ around the point $u = [1, 0, 0, 0]$ on Σ, the tangent space $T_u\Sigma$ at u to Σ is defined by

$$T_u\Sigma = \mathbf{P}^6_u = \langle \{[u : X : 0 : 0] : {}^tX = X\} \rangle.$$

In particular, we reinterpret the quadratic forms of Section 2.2: If $w \in T_u\Sigma$, then the quadratic form defined by the symmetric matrix X coincides with $q_w(U_0^\perp)$.

The following result is the $\mathrm{Sp}_6(\mathbf{C})$ analogue of [6, Lemma 3.4] (see also [16]).

PROPOSITION 2.3.3. *Let* $u \in \Sigma$ *and, as before, let* $\mathbf{P}^6_u$ *be the tangent projective space to* $\Sigma \subset \mathbf{P}^{13}$ *at* u. *If* $w \in \mathbf{P}^6_u$, *then* q_w *has rank* 0, 1, 2, *or* 3 *when* $w = u$, $w \in \Sigma \setminus u$, $w \in \Omega \setminus \Sigma$, *or* $w \in \mathbf{P}^6_u \setminus \Omega$, *respectively. Moreover*:

(i) $C_u := \Sigma \cap \mathbf{P}^6_u$ *is a cone over the Veronese surface with a vertex* u; *and*
(ii) $\Omega \cap \mathbf{P}^6_u$ *is a cubic hypersurface.*

Proof. The tangent cone $C_u = \Sigma \cap \mathbf{P}^6_u$ is defined by $\mathrm{rk}\, X \le 1$ (i.e., the equations of a Veronese surface), so $C_u = \Sigma \cap \mathbf{P}^6_u$ is a cone over a Veronese surface with a vertex at u. The secant lines to this Veronese surface fill the determinantal cubic hypersurface $\det X = 0$ in $\mathbf{P}^6_u$. Since the points on this hypersurface lie on infinitely many tangent lines of the Veronese surface, it follows by Theorem 2.3.2 that $\Omega \cap \mathbf{P}^6_u$ must coincide with this cubic hypersurface. □

2.4. *Quadric* 3*-Folds in* Σ

Consider the restriction of the universal exact sequence on G to Σ as in (1). If $\tau_i = c_i(U^*)$, then

$$H^*(\Sigma) \cong Z[\tau_1, \tau_2, \tau_3]/(\tau_1^2 - 2\tau_2, \tau_2^2 - 2\tau_1\tau_3, \tau_3^2).$$

Thus the Betti numbers are:

$$(b_0(\Sigma), b_2(\Sigma), \dots, b_{12}(\Sigma)) = (1, 1, 1, 2, 1, 1, 1)$$

(see [32, Sec. 6]). Naturally, the classes $\tau_i \cap \Sigma$ are represented by cycles that are restrictions of special Schubert cycles on G to Σ.

For $v \in V$, let $V_v = \ker L_\alpha(v)$.

LEMMA 2.4.1. *For every point $p = \langle v \rangle \in \mathbf{P}(V)$, the variety Q_p of Lagrangian planes that contain p is a 3-dimensional smooth quadric in Σ. It is isomorphic to the Lagrangian Grassmannian $\mathbf{LG}(2,4)$ of Lagrangian subspaces of $V_v/\langle v \rangle$ with respect to the 2-form α_v on $V_v/\langle v \rangle$ induced by α.*

Proof. The cycle Q_p on Σ represents the class $\tau_3 \cap \Sigma$ and has degree 2. Any Lagrangian 3-space that contains v is itself contained in the 5-space V_v. The restriction of α to V_v has kernel v, so we may identify Q_p with the Lagrangian Grassmannian with respect to the nondegenerate 2-form α_v induced by α on $V_v/\langle v \rangle$. This is nothing but a smooth hyperplane section of a $\mathbf{Gr}(2,4)$, that is, a smooth quadric 3-fold. □

TWO INCIDENCE VARIETIES. The span of the quadric Q_p is a projective 4-space $\mathbf{P}^4_p \subset \mathbf{P}(V(14))$. Consider the incidence variety

$$I_Q = \{([\mathbf{P}(U)], p) \mid p \in \mathbf{P}(U)\} \subset \Sigma \times \mathbf{P}(V).$$

By Lemma 2.4.1, I_Q is a quadric bundle over $\mathbf{P}(V)$. Now Σ spans $\mathbf{P}(V(14))$, and each $\mathbf{P}^4_p$ is contained in this span, so we may also consider the incidence

$$I_P = \{(q, p) \mid q \in \mathbf{P}^4_p\} \subset \mathbf{P}(V(14)) \times \mathbf{P}(V).$$

The variety I_P is a $\mathbf{P}^4$-bundle over $\mathbf{P}(V)$, which has been studied by Decker, Manolache, and Schreyer [5]. Its associated rank-5 bundle is self-dual, so we describe a construction that is dual to theirs. Consider the third exterior power of the Euler sequence on $\mathbf{P}(V)$ twisted by $\mathcal{O}_{\mathbf{P}(V)}(2)$,

$$0 \to \wedge^2 T_{\mathbf{P}(V)}(-1) \to \wedge^3 V \otimes \mathcal{O}_{\mathbf{P}^5}(2) \to \wedge^3 T_{\mathbf{P}(V)}(-1) \to 0,$$

and restrict the contraction $\alpha\colon \wedge^3 V \to V$, defined by

$$u \wedge v \wedge w \mapsto \alpha(u \wedge v)w + \alpha(v \wedge w)u + \alpha(w \wedge u)v$$

over $\mathbf{P}(V)$, to the subbundle $\wedge^2 T_{\mathbf{P}(V)}(-1)$. On the one hand this restriction,

$$\alpha_U\colon \wedge^2 T_{\mathbf{P}(V)}(-1) \to V \otimes \mathcal{O}_{\mathbf{P}(V)}(2),$$

is nothing but the form α restricted over each point $p = \langle v\rangle$ to 3-dimensional subspaces in V that contain v. On the other hand, L_α defines a map

$$L_{\alpha_U}\colon V \otimes \mathcal{O}_{\mathbf{P}(V)}(2) \to \mathcal{O}_{\mathbf{P}(V)}(3)$$

by $L_{\alpha_U}(u)(v) = \alpha(u \wedge v)$. The composition

$$L_{\alpha_U} \circ \alpha_U\colon \wedge^2 T_{\mathbf{P}(V)}(-1) \to V \otimes \mathcal{O}_{\mathbf{P}(V)}(2) \to \mathcal{O}_{\mathbf{P}(V)}(3)$$

is zero, since

$$\alpha(u \wedge v)\alpha(w \wedge u) + \alpha(v \wedge w)\alpha(u \wedge u) + \alpha(w \wedge u)\alpha(v \wedge u) = 0.$$

The kernel $\ker L_{\alpha_U}$ is the rank-5 bundle $\Omega_{\mathbf{P}(V)}(3)$, so α_U defines a bundle map

$$\alpha_U\colon \wedge^2 T_{\mathbf{P}(V)}(-1) \to \Omega_{\mathbf{P}(V)}(3).$$

It is easy to check that this map is surjective as soon as α is nondegenerate. Denote by E the rank-5 kernel bundle $\ker(\alpha_U)$. If U is a Lagrangian 3-space that contains v, then $\wedge^3 U$ clearly is contained in the fiber E_p over the point $p = \langle v\rangle$. Thus $\mathbf{P}(E_p)$ coincides with the fiber of the incidence I_P over p.

PROPOSITION 2.4.2 [5, Props. 1.2 & 1.3]. *Let E be the rank-5 kernel bundle of the natural surjective map*

$$\alpha_U\colon \wedge^2 T_{\mathbf{P}(V)}(-1) \to \Omega_{\mathbf{P}(V)}(3)$$

as before.

Then E has Chern polynomial $c_t(E) = 1 + 5t + 12t^2 + 16t^3 + 8t^4$, and $H^0(\mathbf{P}(V), E) \cong V(14)^*$. Furthermore, E is the rank-5 bundle associated to the $\mathbf{P}^4$-bundle I_P over $\mathbf{P}(V)$, and the projection of I_P into the first factor $\mathbf{P}(V(14))$ is Ω.

Proof. We need only compute the Chern polynomial, but this is straightforward from the construction. The twisted bundle $E(-1)$ coincides with the dual of the bundle $\mathcal{B}$ defined in [5], where it is shown that B is self-dual. Thus, the invariants of the bundle also follow from the results of [5]. □

2.5. Singular Hyperplane Sections

Landsberg and Manivel describe the hyperplane sections of Σ corresponding to the different $\mathrm{Sp}_6(\mathbf{C})$-orbits in $\mathbf{P}(V(14)^*)$ [16]. We recall their results and apply them together with the incidences of Section 2.4 to describe the set of quadric 3-folds contained as subvarieties of singular hyperplane sections. These incidences are crucial, not only in the analysis of the vector bundles constructed in Section 3 but also in establishing the relation between linear sections of Σ and orthogonal sections of the dual variety $\check{F}$ described at the end of this section.

Toward these ends, the following four maps provide useful notation. The first is the basic correlation

$$L\colon \Sigma \to \check{\Sigma}, \quad \mathbf{P}(U) \mapsto \mathbf{P}(L_\alpha(U)).$$

The second map is the involution

$$\iota_\alpha\colon \mathbf{Gr}(3, V) \to \mathbf{Gr}(3, V).$$

The third map, which we call the *vertex map,* is

$$v\colon \Omega \setminus \Sigma \to \mathbf{P}(V), \quad q \mapsto \pi_2\pi_1^{-1}(q),$$

where π_1 and π_2 are the projections from the incidence

$$I_P = \{(q, p) \mid q \in \mathbf{P}^4_p\} \subset \mathbf{P}(V(14)) \times \mathbf{P}(V).$$

The point $v(w) \in \mathbf{P}(V)$ is called the *vertex* of $w \in \Omega$. The fourth map is the pivot map:

$$\mathrm{piv}\colon F \setminus \Omega \to \Sigma, \quad p \mapsto \text{the (unique) pivot of } p.$$

The corresponding maps on the dual space are marked with a $*$. Notice that we have the following relations:

$$L^{-1} \circ \mathrm{piv}^* = \mathrm{piv} \circ L^{-1}\colon \check{F} \setminus \check{\Omega} \to \Sigma$$

and

$$L^{-1} \circ v^* = v \circ L^{-1}\colon \check{\Omega} \setminus \check{\Sigma} \to \mathbf{P}(V).$$

The first of these will be denoted by u,

$$u = L^{-1} \circ \mathrm{piv}^*\colon \check{F} \setminus \check{\Omega} \to \Sigma,$$

whereas (by abuse of notation) the second one will be denoted by v,

$$v = L^{-1} \circ v^*\colon \check{\Omega} \setminus \check{\Sigma} \to \mathbf{P}(V).$$

For the point $\omega \in \mathbf{P}(V(14)^*)$, denote by $\mathbf{P}^{12}_\omega \subset \mathbf{P}(V(14))$ the hyperplane defined by ω and denote by $H_\omega = \mathbf{P}^{12}_\omega \cap \Sigma$ the corresponding hyperplane section of Σ.

Landsberg and Manivel prove the following [16, Prop. 8.2].

PROPOSITION 2.5.1. (i) *If $\omega \in \mathbf{P}(V(14)^*) \setminus \check{F}$, then $H_\omega = \mathbf{P}^{12}_\omega \cap \Sigma$ is smooth.*

(ii) *If $\omega \in \check{F} \setminus \check{\Omega}$, then H_ω has a unique quadratic singularity at the point $u = u(\omega) = L(\mathrm{piv}^*(\omega))$.*

(iii) *If $\omega \in \check{\Omega} \setminus \check{\Sigma}$, then $H_\omega = \Sigma_{P_1} = \Sigma_{P_2}$ for an involutive pair of planes $P_1 = P_1(\omega)$ and $P_2 = P_2(\omega) := \iota_\alpha(P_1(\omega))$. Furthermore, H_ω has quadratic singularities along a smooth quadric surface Q_ω in $P_{v(\omega)}$ that parameterizes the set of Lagrangian planes passing through $v(\omega)$ and intersecting P_1 and P_2 in a line.*

(iv) *If $\omega \in \check{\Sigma}$, then H_ω is singular along a cone C_u over a Veronese surface, with vertex $u = L^{-1}(\omega)$.*

SCHUBERT HYPERPLANE SECTIONS. The singular hyperplane sections of points on $\check{\Sigma}$ and $\check{\Omega}$ have a natural description as restrictions of Schubert cycles on $\mathbf{Gr}(3, 6)$. The isomorphism $L_\alpha\colon \wedge^3 V \cong \wedge^3 V^*$ induces an isomorphism $\mathbf{Gr}(3, V) \to \mathbf{Gr}(V, 3)$, which composed with the natural isomorphism $\mathbf{Gr}(V, 3) \to \mathbf{Gr}(3, V)$, $\omega \mapsto \ker(\omega)$, defines an involution

$$\iota_\alpha \colon \mathbf{Gr}(3, V) \to \mathbf{Gr}(3, V).$$

The fixed point locus of this involution is Σ, the set of Lagrangian planes. Moreover, every involutive pair $\{P, \iota_\alpha(P)\}$ of planes has a common point $P \cap \iota_\alpha(P) = \ker(\alpha|_P)$ and is, of course, contained in the corresponding correlated hyperplane.

Let $P \subset \mathbf{P}(V)$ be any plane. Then the restriction of the Schubert cycle $\sigma_1(P)$ in $\mathbf{Gr}(3, V)$ is clearly a hyperplane section of Σ. Thus we have shown our next proposition.

PROPOSITION 2.5.2. *If $P \subset \mathbf{P}(V)$ is a non-Lagrangian plane, then the variety Σ_P of Lagrangian planes that intersect P coincides with the variety $\Sigma_{\iota_\alpha(P)}$ of Lagrangian planes that intersect $\iota_\alpha(P)$, and they both define a hyperplane section of Σ.*

PROPOSITION 2.5.3. *For a Lagrangian plane $\mathbf{P}(U) \subset \mathbf{P}(V)$ corresponding to $u \in \Sigma$: the variety Σ_u of Lagrangian planes that intersect $\mathbf{P}(U)$ is the hyperplane section defined by $L_\alpha(u) \in \mathbf{P}(V(14)^*)$; whereas the variety of Lagrangian planes that intersect $\mathbf{P}(U)$ in a line is a cone C_u over a Veronese surface of degree 4 with vertex at u in $\mathbf{P}^6 = \mathbf{P}^6_u$, the projective tangent space to Σ at u.*

Proof. Observe that Σ_u is a hyperplane section with $\mathbf{P}(U)$ invariant under the involution ι_α. The lines in Σ through u represent precisely the pencils of Lagrangian planes that contain $\mathbf{P}(U)$. But these lines generate precisely the tangent cone C_u to Σ at u, so the result follows from Proposition 2.3.3. □

Recall the decomposition $V = U_0 \oplus U_1$ from Section 2.2. The restriction of the global sections $V(14)^*$ of the vector bundle E of Proposition 2.4.2 to the Lagrangian plane $P = \mathbf{P}(U_0)$ decomposes as the restriction of the decomposition

$$\wedge^3 V^* = \wedge^3 U_1^\perp \oplus (\wedge^2 U_1^\perp \otimes U_0^\perp) \oplus (U_1^\perp \otimes \wedge^2 U_0^\perp) \oplus \wedge^3 U_0^\perp$$

to $V(14)^*$. By the natural map

$$q(U_0) \colon \wedge^3 U_0^\perp \oplus U_1^\perp \otimes \wedge^2 U_0^\perp \to \mathrm{Sym}^2 U_0^*$$

this decomposition becomes

$$H^0(P, E|_P) \cong \wedge^3 U_1^\perp \oplus U' \oplus \mathrm{Sym}^2 U_0^*,$$

where the first summand consists of the constant forms, the second summand is $V(14)^* \cap (\wedge^2 U_1^\perp \otimes U_0^\perp)$, and the last summand consists of the quadratic forms on P. The restriction of the vector bundle E to P therefore decomposes as a sum of two line bundles and a rank-3 bundle. We thus have the following.

PROPOSITION 2.5.4. *For a Lagrangian plane $P = \mathbf{P}(U) \subset \mathbf{P}(V)$, the restriction of the vector bundle E to P is $E|_P = \mathcal{O}_P \oplus \mathcal{O}_P(2) \oplus E_P$, where E_P is a rank-3 vector bundle with Chern polynomial $c_t(E_P) = 1 + 3t + 6t^2$.*

Proof. The Chern polynomial follows by a direct calculation. □

QUADRIC 3-FOLDS IN SINGULAR HYPERPLANE SECTIONS. Consider the involutive pair of planes $P_1(\omega)$, $P_2(\omega)$ that appears in Proposition 2.5.1. It follows from part (iii) that the union of the planes $P_1(\omega) \cup P_2(\omega)$ is the set of points q in $\mathbf{P}(V)$ such that $\mathbf{P}^4_q \subset \mathbf{P}^{12}_\omega$ (or, equivalently, quadric 3-folds $Q_q \subset H_\omega$). This fact fits in the description of the incidence

$$J = \{(p, w) \in \mathbf{P}(V) \times \mathbf{P}(V(14)^*) \mid \mathbf{P}^4_p \subset \mathbf{P}^{12}_\omega\} \subset \mathbf{P}(V) \times \mathbf{P}(V(14)^*).$$

Let $\pi : J \to \mathbf{P}^5 = \mathbf{P}(V)$ and $\psi : J \to \check{\mathbf{P}}^{13} = \mathbf{P}(V(14)^*)$ be the two projections of J.

PROPOSITION 2.5.5. *The image of the second projection is precisely the quartic $\check{F} \subset \check{\mathbf{P}}^{13}$. The fiber of the second projection over a point $\omega \in \check{F} \setminus \check{\Omega}$ is the smooth conic $q_\omega(U) \subset \mathbf{P}(U)$, where $\mathbf{P}(U)$ is the plane of the pivot $u(\omega)$. The fiber of ψ over a point ω on $\check{\Omega} \setminus \check{\Sigma}$ is the union of the two planes $P_1(\omega) \cup P_2(\omega)$. The fiber of ψ over a point $\omega \in \check{\Sigma}$ is the Lagrangian plane $\mathbf{P}(U)$ of $u = L^{-1}(\omega)$.*

Proof. The last statement follows from Proposition 2.5.3, while the case $\omega \in \check{\Omega} \setminus \check{\Sigma}$ follows (as explained previously) from Proposition 2.5.1(iii). The remainder of the proposition is an immediate consequence of the next lemma. □

LEMMA 2.5.6. *Let $\omega \in \mathbf{P}(V(14)^*)$, and let $p \in \mathbf{P}(V)$. Then*

$$\mathbf{P}^4_p \subset \mathbf{P}^{12}_\omega \iff \omega \in T^{\perp}_{[U]} = T_{[U^\perp]}$$

for some Lagrangian subspace U with $p \in \mathbf{P}(U)$ and $q_\omega(U)(p) = 0$.

Proof. By abuse of notation we do not distinguish between $\omega \in \mathbf{P}(V(14)^*)$ and any nonzero vector in $V(14)^*$ representing it. Thus we consider ω as a section of the vector bundle E and then analyze its restriction to Lagrangian planes.

Let $P = \mathbf{P}(U) \subset \mathbf{P}(V)$ be a Lagrangian plane and let $u \in \Sigma$ be the corresponding point. According to Proposition 2.5.4, the restriction of the vector bundle E to P decomposes into three direct summands:

$$E|_P = \mathcal{O}_P \oplus \mathcal{O}_P(2) \oplus E_P,$$

where E_P is a rank-3 vector bundle with Chern polynomial $c_t(E_P) = 1 + 3t + 6t^2$. Therefore, the restriction ω_P of ω to P decomposes into $\omega_P = a \oplus b \oplus c$, where a is a constant, b is a quadratic form, and c is a section of the rank-3 bundle E_P.

LEMMA 2.5.7.

(i) *If $\omega \in \check{\Sigma}$ such that $L_\alpha(u) = \omega$, then $a(\omega) = b(\omega) = c(\omega) = 0$.*
(ii) *$a(\omega) = c(\omega) = 0$ if and only if $\omega \in \check{F}$ and $L_\alpha(u)$ is a pivot of ω—that is, iff the line $\langle L_\alpha(u), \omega\rangle$ is tangent at $L_\alpha(u)$.*
(iii) *$u \in \mathbf{P}^{12}_\omega$ iff $a = 0$.*

Proof. Let $u \in \Sigma$ with corresponding Lagrangian plane $P = \mathbf{P}(U)$. By Proposition 2.5.3, the hyperplane $\mathbf{P}^{12}_\omega$ contains Σ_u iff $a = b = c = 0$, so (i) follows. The hyperplane $\mathbf{P}^{12}_\omega$ contains the tangent cone C_u at u (the cone over a Veronese surface) iff the restriction $\omega_P \in \mathrm{Sym}^2(U)^*$ (i.e., iff $a(\omega_P) = c(\omega_P) = 0$), so (ii) follows. Finally, $a = 0$ if and only if the $\mathbf{P}^{12}_\omega$ passes through the vertex u of Σ_u. $\square$

If $\omega \in \check{F}$ and u is a dual pivot, then the quadratic form $b(\omega_P)$ is nothing but the quadratic form $q_\omega(U)$. Let $p \in P = \mathbf{P}(U)$. Then $\omega(p) = 0$ iff $\omega \in \check{F}$ with a dual pivot u, and $q_\omega(U)(p) = 0$ by Lemma 2.5.7(ii). On the other hand, $\omega(p) = 0$ if and only if $\mathbf{P}^4_p \subset \mathbf{P}^{12}_\omega$, so Lemma 2.5.6 follows. $\square$

Linear Sections and Sp(3)-Dual Sections. We end this section by describing the relations between linear sections of Σ and the orthogonal linear sections of its dual variety $\check{F}$.

For $2 \le k \le 5$, let $\mathbf{P}^{13-k} \subset \mathbf{P}(V(14))$ be a general linear subspace of codimension k, and let $\Pi^{k-1} = (\mathbf{P}^{13-k})^\perp \subset \mathbf{P}(V(14)^*)$ be the $(k-1)$-dimensional orthogonal subspace of hyperplanes that pass through $\mathbf{P}^{13-k}$. Let

$$X = \Sigma \cap \mathbf{P}^{13-k} \quad \text{and} \quad \check{F}(X) = \Pi^{k-1} \cap \check{F}$$

and let

$$\check{\Omega}(X) = \check{F}(X) \cap \check{\Omega} = \Pi^{k-1} \cap \check{\Omega}.$$

We call $\check{F}(X)$ the Sp(3)-*dual* section to X.

We restrict our attention to general linear subspaces; more precisely, we will assume that $X = \Sigma \cap \mathbf{P}^{13-k}$ is a $(6-k)$-dimensional smooth variety. Obviously, there are similar results for singular linear sections.

Lemma 2.5.8. *Let $P \subset \mathbf{P}(V(14))$ be a linear subspace and let $\omega \in P^\perp \cap \check{F}$ be such that a dual pivot $u \in \Sigma$ of ω lies in P. Then $u \in \mathrm{Sing}(P \cap \Sigma)$.*

Proof. Let $H_\omega = \Sigma \cap \mathbf{P}^{12}_\omega$ be the hyperplane section of Σ defined by ω. By Proposition 2.5.1, H_ω is singular at the pivot u. Since $\omega \in P^\perp$, the variety $P \cap \Sigma \subset H_\omega$ is a complete intersection of H_ω and hyperplanes that pass through the singular point u. Therefore $P \cap \Sigma$ is singular at u. $\square$

It follows from Lemma 2.5.8 that, if $X = \Sigma \cap \mathbf{P}^{13-k}$ is smooth, then $u(\omega) \notin \mathbf{P}^{13-k}$ for any $\omega \in \check{F}(X) \setminus \check{\Omega}(X)$. Combined with Proposition 2.3.1, this yields the following.

Proposition 2.5.9. *Let X be a smooth $(6-k)$-dimensional linear section of Σ.*

If $2 \le k \le 4$, then $\check{\Omega}(X) = \emptyset$ and $\check{F}(X) \subset \Pi^{k-1}$ is a smooth quartic $(k-2)$-fold (e.g., if $k = 2$ then $\check{F}(X)$ is a set of four points each with multiplicity 1); and if X is a curve, then $\mathrm{Sing}\,\check{F}(X) = \check{\Omega}(X) = \{\omega_1, \dots, \omega_{21}\}$ is a set of 21 ordinary double points (nodes) of the quartic 3-fold $\check{F}(X)$.

The linear section X is subcanonical. More precisely, $K_X = (-4+k)H$ where H is the class of the hyperplane section. When $k \le 4$ and X is general, $\mathbf{Pic}(X) = \mathbf{Z}[H]$.

Proof. Only the last statement remains to be shown, but this follows from [19] (see also [24]). $\square$

3. Rank-2 Vector Bundles on Linear Sections

We now turn to the main application of our study of Σ. On each nodal hyperplane section of Σ we will construct a rank-2 vector bundle with a 6-space of global sections.

As before, we use the following notation. For a point $\omega \in \check{\mathbf{P}}^{13}$ we consider the hyperplane $\mathbf{P}^{12}_{\omega}$ and the hyperplane section $H_{\omega} = \mathbf{P}^{12}_{\omega} \cap \Sigma$. If $\omega \in \check{F} \setminus \check{\Omega} \subset \check{\mathbf{P}}^{13}$ then $u(\omega) \in \Sigma$ is the pivot of ω on Σ, and if $\omega \in \check{\Omega} \setminus \check{\Sigma} \subset \check{\mathbf{P}}^{13}$ then $Q_{\omega} \subset \Sigma$ is the smooth quadric surface of pivots of ω on Σ and is also the singular locus of H_{ω}.

3.1. The Projection of H_{ω} from the Pivot $u(\omega)$

Let $\mathbf{P}^{12}_{\omega} \subset \mathbf{P}^{13}$ be the hyperplane defined by $\omega \in \check{F} \setminus \check{\Omega} \subset \check{\mathbf{P}}^{13}$, let $\pi_u : \mathbf{P}^{12}_{\omega} \dashrightarrow \bar{\mathbf{P}}^{11}_{\omega}$ be the projection from $u = u(\omega) \in \Sigma$, and let the variety $\bar{H}_{\omega} \subset \bar{\mathbf{P}}^{11}_{\omega}$ be the proper π_u-image of the hyperplane section $H_{\omega} = \Sigma \cap \mathbf{P}^{12}_{\omega} \subset \mathbf{P}^{12}_{\omega}$.

Let $\sigma : H'_{\omega} \to H_{\omega}$ be the blowup of $u \in H_{\omega}$, and let $\psi : H'_{\omega} \to \bar{H}_{\omega}$ be the projection into $\bar{\mathbf{P}}^{11}_{\omega}$. By Proposition 2.5.1(i), $u = u(\omega)$ is an ordinary double point of H_{ω} and so the exceptional divisor $Q' = \sigma^{-1}(u) \subset H'_{\omega}$ of σ is isomorphic to a smooth 4-dimensional quadric; that is, $Q' \cong \mathbf{Gr}(2, \mathbf{C}^4)$.

The projection π_u contracts the tangent cone $C_u \subset \mathbf{P}^6_u = T_u\Sigma$ at u to a Veronese surface S_u (cf. Proposition 2.3.3):

$$
\begin{tikzcd}
Q' \arrow[d] & \subset & H'_{\omega} \arrow[d, "\sigma"'] \arrow[dr, "\psi"] & \supset & C'_u \arrow[drrr] & & & \\
u & \in & H_{\omega} \arrow[r, "\pi_u"] & \bar{H}_{\omega} & \supset & \bar{Q} & \supset & S_u . \\
 & & \cup & & & & & \\
 & & C_u & & & & &
\end{tikzcd}
$$

Since the exceptional divisor $Q' \subset H'_{\omega}$ is isomorphic to the projectivized tangent cone to H_{ω} at $u = u(\omega)$, it follows that the strict transform C'_u of C_u in H'_{ω} intersects Q' in a Veronese surface. By Proposition 2.5.1(i), Q' is isomorphic to a smooth 4-dimensional quadric; hence the isomorphic image $\bar{Q} = \psi(Q') \subset \bar{H}_u$ is a smooth 4-dimensional quadric containing the surface S_u.

Since $\bar{H}_{\omega}$ is a birational projection of H_{ω} from its double point $u = u(c)$, the degree $\deg \bar{H}_{\omega} = \deg H_{\omega} - 2 = 14$. Let L be the hyperplane divisor on H_{ω}. We denote by L also the pullback $\sigma^* L$ and let L' be the strict transform of the general hyperplane divisor that passes through the point u (i.e., $L' \equiv L - Q'$). Next we summarize some further properties of the morphism ψ.

LEMMA 3.1.1. *The 5-fold $\bar{H}_{\omega} \subset \bar{\mathbf{P}}^{11}_{\omega}$ has singularities at most on the surface $S_u \subset \bar{Q}$, while H'_{ω} is a smooth 5-fold. Furthermore, the morphism $\psi : H'_{\omega} \to \bar{H}_{\omega}$ contracts the codimension-2 subvariety C'_u to the surface $S_u \subset \bar{Q}$ and is an isomorphism outside C'_u. In particular, $\bar{H}_{\omega}$ has singularities at most on the surface S_u.*

The canonical divisor on H'_ω is $K_{H'_\omega} = -3L'$, where L' is the pullback of a hyperplane divisor on $\bar{H}_\omega$.

Proof. Since ψ induces the projection from u, the divisor $L' = L - Q'$ is the pullback of a hyperplane divisor on $\bar{H}_\omega$. It remains only to compute the canonical divisor. The canonical divisor on Σ is $K_\Sigma \equiv -4H$, so by adjunction the canonical divisor $K_{H_\omega} = -3L$. Since $\sigma : H'_\omega \to H_\omega$ blows up the double point $u \in H_\omega$, the canonical divisor

$$K_{H'_\omega} \equiv \sigma^* K_{H_\omega} + (\dim H_\omega - 2)Q' \equiv \sigma^*(-3L) + 3Q' \equiv -3L'. \qquad \square$$

We shall see in Theorem 3.3.4 that the 5-fold $\bar{H}_\omega$ is in fact a linear section of the Grassmannian $\mathbf{Gr}(2,6) \subset \mathbf{P}^{14}$ with a special codimension-3 subspace in $\mathbf{P}^{14}$.

Let $X = \mathbf{P}^{13-k} \cap \Sigma$ be a smooth $(6-k)$-dimensional linear section of Σ as in Section 2.5. Let $\omega \in \check{F}(X) - \check{\Omega}(X)$. Then, by Lemma 2.5.8, the pivot $u(\omega)$ is not contained in $\mathbf{P}^{13-k}$. Let $\mathbf{P}^{14-k}_\omega$ be the subspace of $\mathbf{P}^{13} = \mathbf{P}(V(14))$ spanned by $\mathbf{P}^{13-k}$ and $u(\omega)$, and let $\Pi^{k-2}_\omega = (\mathbf{P}^{14-k}_\omega)^\perp$. Clearly Π^{k-2}_ω is a linear subspace of Π^{k-1} of codimension 1.

Denote by W_ω the intersection

$$W_\omega = \Sigma \cap \mathbf{P}^{14-k}_\omega.$$

Since the $(6-k)$-fold X is a proper linear section of Σ and a linear section of W_ω with the codimension-1 subspace $\mathbf{P}^{13-k} \subset \mathbf{P}^{14-k}_\omega$, the dimension $\dim W_\omega = 7-k$. Furthermore, by Lemma 2.5.8, the pivot $u(\omega)$ is a singular point of W_ω.

Consider now the projection $\pi_{u(\omega)} : \mathbf{P}^{12}_\omega \to \bar{\mathbf{P}}^{11}_\omega$ from the pivot $u(\omega)$. Since $u(\omega) \notin \mathbf{P}^{13-k} = \langle X \rangle$, the restriction of $\pi_{u(\omega)}$ to $\mathbf{P}^{13-k}$ is a projective-linear isomorphism onto $\bar{\mathbf{P}}^{13-k}_\omega := \pi_{u(b)}(\mathbf{P}^{13-k})$; in particular, $\pi_{u(b)} : X \to \bar{X}_\omega = \pi_{u(b)}(X) \subset \bar{\mathbf{P}}^{13-k}_b$ is a projective-linear isomorphism.

Since $\mathbf{P}^{13-k} \subset \mathbf{P}^{14-k}_\omega$ is a hyperplane and since $u(\omega) \in \mathbf{P}^{14-k}_\omega$, the projection $\pi_{u(\omega)}$ maps $\mathbf{P}^{14-k}_\omega$ onto $\bar{\mathbf{P}}^{13-k}_\omega$. The pivot point $u(\omega)$ is a quadratic singularity of W_ω, so the proper $\pi_{u(\omega)}$-image $\bar{W}_\omega$ of W_ω will contain a quadric $\bar{Q}_\omega \subset \bar{Q}$ of dimension $6-k$ under the condition $2 \le k \le 5$.

We will show in Theorem 3.3.4 that the projection $\pi_{u(\omega)}$ sends the hyperplane section $H_\omega = \Sigma \cap \mathbf{P}^{12}_\omega$ to a codimension-3 linear section $\bar{H}_\omega = \bar{\mathbf{P}}^{11}_\omega \cap \mathbf{Gr}(2,\mathbf{C}^6)$ containing a smooth 4-fold quadric $\bar{Q} = \mathbf{Gr}(2,\mathbf{C}^4)$ for some $\mathbf{C}^4 \subset \mathbf{C}^6$. Thus, $\bar{X}_\omega$ is a subvariety of the linear section $\bar{W}_\omega$ of $\mathbf{Gr}(2,\mathbf{C}^6)$, a linear section that contains a $(6-k)$-dimensional quadric.

These observations lie behind our subsequent description of a family of embeddings of linear sections of Σ into $\mathbf{Gr}(2,\mathbf{C}^6)$ or (what amounts to the same) a description of a family of rank-2 vector bundles on linear sections of Σ with a 6-space of global sections.

3.2. Del Pezzo and Segre 3-Folds

Here we define special Del Pezzo 3-folds and identify them with projections of Segre 3-folds. Later we show that these are subcanonical varieties on $\bar{H}_\omega$ and thus are zero loci of sections of a rank-2 vector bundle (via the Serre construction).

Let $\mathbf{Gr}(2,5) \subset \mathbf{P}^9 = \mathbf{P}(\wedge^2\mathbf{C}^5)$ be the Grassmannian of lines in $\mathbf{P}^4 = \mathbf{P}(\mathbf{C}^5)$, and let $\mathbf{Gr}(5,2) \subset \check{\mathbf{P}}^9 = \mathbf{P}(\wedge^2\check{\mathbf{C}}^5)$ be the Grassmannian of lines in the dual space $\check{\mathbf{P}}^4 = \mathbf{P}(\check{\mathbf{C}}^5)$. Any plane $\Pi \subset \check{\mathbf{P}}^9$ is the plane of linear equations of its 0-space $\mathbf{P}^6_\Pi \subset \mathbf{P}^9$. The group $\mathrm{GL}(5,\mathbf{C})$ acts on $\check{\mathbf{P}}^9$ via its linear representation on $\wedge^2\check{\mathbf{C}}^5$ and therefore also on $\mathbf{Gr}(3,\wedge^2\check{\mathbf{C}}^5)$, the Grassmannian of planes in $\check{\mathbf{P}}^9$. We denote this action by ρ_5.

Let $U_0 \subset \mathbf{Gr}(3,\check{\mathbf{C}}^5)$ be the open set of these planes $\Pi \subset \check{\mathbf{P}}^9$ such that $\Pi \cap \mathbf{Gr}(5,2) = \emptyset$. The linear section $V_\Pi = \mathbf{P}^6_\Pi \cap \mathbf{Gr}(2,5)$ is singular if and only if it is contained in a tangent hyperplane, since the contact locus of a tangent hyperplane is a plane. Therefore, $\Pi \in U_0$ if and only if $V_\Pi = \mathbf{P}^6_\Pi \cap \mathbf{Gr}(2,5)$ is a smooth Fano 3-fold of degree 5 and index 2 (i.e., $K_{V_\Pi} \equiv \mathcal{O}_{V_\Pi}(-2)$). The action ρ_5 is transitive on U_0; in other words, U_0 is an orbit of ρ_5 [33, Sec. 5, Prop. 14]. Hence all the V_Π, $\Pi \in U_0$, are conjugate to each other by the action $\wedge^2$ of $\mathrm{GL}(5,\mathbf{C})$ on $\mathbf{P}^9$ (see also [13, Sec. 6.5]).

Let $U_{xxx} \subset \mathbf{Gr}(3,\wedge^2\check{\mathbf{C}}^5)$ be the subset of planes $\Pi \subset \check{\mathbf{P}}^9$ such that Π theoretically intersects the $\mathbf{Gr}(5,2)$ scheme in exactly three points. These three points of intersection cannot be collinear because $\mathbf{Gr}(5,2)$ is an intersection of quadrics.

As before, U_{xxx} is an orbit of ρ_5, and all the V_Π ($\Pi \in U_{xxx}$) are conjugate to each other by the action $\wedge^2$ of $\mathrm{GL}(5,\mathbf{C})$ on $\mathbf{P}^9$. We call the unique 3-fold V_Π ($\Pi \in U_{xxx}$) the *Del Pezzo 3-fold of type xxx*.

Now we turn to Segre 3-folds. For this we first make a slight detour to 2-forms on even-dimensional spaces and prove the following result.

PROPOSITION 3.2.1. *Let $V = \mathbf{C}^{2n}$ and let $\alpha, \alpha' \in \wedge^2 V$ be two general 2-vectors. Then there exists a unique (up to scalars) n-tuple $\gamma_1, \dots, \gamma_n$ of 2-vectors of rank 2 such that both α and α' are linear combinations of the γ_i.*

REMARK 3.2.2. The proposition may be reformulated in terms of multisecant spaces to the Grassmannian $\mathbf{Gr}(2,2n)$ of lines in $\mathbf{P}^{2n-1}$ embedded in a Plücker space: A general line in the Plücker space is contained in a unique n-secant $(n-1)$-space to $\mathbf{Gr}(2,2n)$.

Proof of Proposition 3.2.1. First we prove the uniqueness. Because the pair α, α' of 2-vectors is general, we may suppose that they both have rank $2n$ and that

$$\alpha = \sum_{i=1}^{n} \gamma_i \quad \text{and} \quad \alpha' = \sum_{i=1}^{n} \lambda_i \gamma_i,$$

where the λ_i are pairwise distinct coefficients. Let

$$\beta_i = \lambda_i \alpha - \alpha', \quad i = 1, \dots, n.$$

Then the β_i are precisely the 2-vectors of the pencil generated by α and α' that have rank less than $2n$. Furthermore, their rank is exactly $2n-2$ since $\lambda_i \neq \lambda_j$ for $i \neq j$. Thus each $\beta_i \in \wedge^2 V_i$ for a unique rank-$(2n-2)$ subspace $V_i \subset V$. Let $U_j = \bigcap_{i \neq j} V_i$. Then U_j is 2-dimensional and γ_j is a nonzero 2-vector that generates the subspace $\wedge^2 U_j \in \wedge^2 V$, so the 2-vectors γ_i are determined uniquely by the pencil generated by α and α'.

For the existence we use a dimension argument. On the one hand, the Grassmannian of lines in the Plücker space $\mathbf{P}(\wedge^2 V)$ has dimension $2(n(2n-1)-2) = 4n^2 - 2n - 4$. On the other hand, the Grassmannian $\mathbf{Gr}(2, 2n)$ has dimension $4n-4$, so the family of lines contained in n-secant $(n-1)$-spaces to $\mathbf{Gr}(2, 2n)$ has dimension at most $(4n-4)n + 2(n-2) = 4n^2 - 2n - 4$. By the uniqueness argument, such a line in general lies in a unique n-secant $(n-1)$-space, so the two dimensions actually coincide. Since there is an obvious inclusion of the latter into the former and since the former is irreducible, we may conclude. $\square$

This result leads to a simple description of Segre n-folds in the Grassmannian $\mathbf{Gr}(2, 2n)$: embeddings

$$s\colon X_n = \mathbf{P}^1 \times \mathbf{P}^1 \times \cdots \times \mathbf{P}^1 \hookrightarrow \mathbf{P}^{\binom{2n}{2}-1},$$

which factors through the Segre embedding $s_n\colon X_n \to \mathbf{P}^{2^n-1}$, and a linear map $\mathbf{P}^{2^n-1} \hookrightarrow \mathbf{P}^{\binom{2n}{2}-1}$ and where $s(X_n) \subset \mathbf{Gr}(2, 2n) \subset \mathbf{P}^{\binom{2n}{2}-1}$.

PROPOSITION 3.2.3. *Let $V = \mathbf{C}^{2n}$ and let $\alpha, \alpha' \in \wedge^2 V^*$ be two general 2-forms on V. Then the set of common Lagrangian n-spaces of V with respect to the forms α and α' form a Segre n-fold $\mathbf{P}^1 \times \mathbf{P}^1 \times \cdots \times \mathbf{P}^1$ in the Grassmannian $\mathbf{Gr}(2, 2n)$. Let $\gamma_1, \dots, \gamma_n$ be the unique n-tuple of 2-forms of rank 2 such that both α and α' are linear combinations of the γ_i, and let $U_i \subset V$ be the $(2n-2)$-dimensional kernel of γ_i. Then the common Lagrangian n-spaces U with respect to the forms α and α' are precisely the n-spaces that intersect each U_i in an $(n-1)$-space. Equivalently, if $W_i = \bigcap_{i \neq j} U_j$ then W_i is 2-dimensional, and an n-space is Lagrangian with respect to α and α' if and only if it has a nontrivial intersection with each W_i.*

Proof. Clearly, the two characterizations of Lagrangian n-spaces are equivalent. Furthermore, the last one describes a family of n-spaces that form a Segre n-fold, since every n-space intersects each line $\mathbf{P}(W_i)$ in a unique point.

An n-space U that intersects each U_i in an $(n-1)$-space is clearly isotropic with respect to each 2-form γ_i and is thus also Lagrangian for α and α'.

On the other hand, it is a straightforward exercise in Schubert calculus to show that the set of common Lagrangian n-spaces for α and α' is n-dimensional of degree $n!$—that is, the degree of the Segre n-fold. Since we have an inclusion, the result follows. $\square$

We now return to the case $n = 3$.

LEMMA 3.2.4. *Let $X = X_3 \subset \mathbf{P}^7$ be a Segre 3-fold, and let $u \in X$. Then the projection $\bar{V}$ of X from u is a Del Pezzo 3-fold of type xxx.*

Conversely, the Del Pezzo 3-fold $\bar{V}$ of type xxx is a projection of the Segre 3-fold X from a point $u \in X$.

Proof. Consider the blowup $X' \to X$ centered at u, and let E denote the exceptional divisor. Let L_i be the pullback to X of $\mathcal{O}_{\mathbf{P}^1}(1)$ on each factor of X, and let $F_X = L_1 \oplus L_2 \oplus L_3$. Let s_u be the unique global section of F_X whose zero locus is u, and let $F_{X'}$ be the pullback of F_X to X'. The pullback of s_u to $F_{X'}$ vanishes on E and corresponds to the unique nonvanishing section s' of $F_{X'}(-E)$. The exterior multiplication with s' defines a surjective map $\wedge^2 F_{X'}(-E) \to \wedge^3 F_{X'}(-2E)$ that fits into an exact sequence

$$0 \to F_0 \to \wedge^2 F_{X'}(-E) \to \wedge^3 F_{X'}(-2E) \to 0,$$

where F_0 is a rank-2 vector bundle on X'. Notice that $\wedge^3 F_{X'}(-2E)$ is the line bundle $\mathcal{O}_{X'}(H_X - 2E)$, where H_X is the pullback to X' of the hyperplane class on X in the Segre embedding. Furthermore, $c_1(\wedge^2 F_{X'}(-E)) = 2H_X - 3E$, so $c_1(F_0) = H_X - E$. Similarly, one computes $c_2(F_0) = (H_X - E)^2$. On the other hand, $h^0(\mathcal{O}_{X'}(H_X - 2E)) = 4$ while $h^0(\wedge^2 F_{X'}(-E)) = 9$, so $h^0(F_0) \geq 5$. Hence the morphism defined by $H_X - E$ (i.e., the projection of X from u) maps X' into $\mathbf{Gr}(2,5)$; it is a 3-fold of degree 5 that spans a $\mathbf{P}^6$, so it is a linear section of $\mathbf{Gr}(2,5)$.

The 3-fold X contains three quadric surfaces that meet pairwise along a line through u. Thus the projection $\bar{X}$ (i.e., the image of X') contains three planes that meet pairwise in three points. Evidently these points are precisely the singularities of $\bar{X}$, so $\bar{X}$ is a Del Pezzo 3-fold of type xxx.

The converse is clear by the transitivity of ρ_5. □

LEMMA 3.2.5. *Let $X = X_3 \subset \Sigma$ be a Segre 3-fold, and let H be a hyperplane section of Σ that contains X. Then H is singular—that is, a tangent hyperplane section to Σ—and the point of tangency of H lies in X.*

Proof. For each point u on X there is a $\mathbf{P}^2$ of hyperplanes tangent to Σ at u that contain X. Because such a hyperplane is in general tangent to Σ at u only, there is altogether a 5-dimensional family of tangent hyperplanes that contain X. But X spans a $\mathbf{P}^7$, so there is a $\mathbf{P}^5$ of hyperplanes that contain X. Therefore, the two sets must coincide and the lemma follows. □

3.3. A Rank-2 Vector Bundle on Singular Hyperplane Sections

Recall the universal sequence of vector bundles on Σ, the restriction of the universal sequence on $G = \mathbf{Gr}(3, V)$:

$$0 \to U \to V \otimes \mathcal{O}_\Sigma \to Q \to 0;$$

here U is the universal subbundle and $U^* \cong Q$ by the natural map induced by α. Any global section of the rank-3 bundle $\wedge^2 U^*$ comes from a 2-form $\alpha' \in \wedge^2 V^*$. In the previous section we saw that if the 2-form α' is general then the zero locus $X = Z(\alpha')$ is a Segre 3-fold. In fact, the characterization in Proposition 3.2.3 yields a straightforward argument that any Segre 3-fold in Σ is the zero locus of a section of $\wedge^2 U^*$.

From Lemma 3.2.5 we know that any hyperplane that contains the Segre 3-fold X is tangent to Σ. So we fix an $\omega \in \check{F}$ such that the hyperplane $\mathbf{P}^{12}_\omega$ contains X and assume that it is tangent to Σ only at $u \in X$. Thus $u = u(\omega)$ in the notation of

Section 2.5. Then X has codimension 2 in this hyperplane section, and it is still the zero locus of the 2-form α' restricted to the hyperplane.

Consider the blowup H'_ω of the hyperplane section $H_\omega = \Sigma \cap \mathbf{P}^{12}_\omega$ in the singular point u. Let Q'_ω be the exceptional divisor on H'_ω. It is isomorphic to a 3-dimensional quadric (when ω is general).

In the notation of Lemma 3.1.1, the canonical bundle $K_X = \mathcal{O}_X(-2L)$. On the strict transform X_ω of X, the canonical bundle is therefore

$$K_{X_\omega} = \mathcal{O}_{X_\omega}(-2L + 2Q'_\omega) = \mathcal{O}_{X_\omega}(-2L'),$$

so X_ω is subcanonical with respect to the hyperplane line bundle L' induced by the projection from u. Hence, by the Serre construction, X_ω is the zero locus of a rank-2 vector bundle on H'_ω. The aim of this section is to identify this vector bundle. To construct it one may apply the Serre construction starting with X_ω. We choose a different and more direct argument similar to the one used in the proof of Lemma 3.2.4.

Let

$$\wedge^2 U^*_\omega = \wedge^2 U^* \otimes \mathcal{O}_{H'_\omega},$$

and consider the twisted bundle

$$\wedge^2 U^*_\omega(-Q'_\omega).$$

Observe that the section of $\wedge^2 U^*_\omega$, given by the restriction and pullback of the section α', vanishes on Q'_ω; it therefore corresponds to a section α'_ω of $\wedge^2 U^*_\omega(-Q'_\omega)$. The vector bundle $\wedge^2 U^*_\omega(-Q'_\omega)$ has rank 3, while the zero locus X_ω of the section α'_ω has codimension 2. We will show that α'_ω is a section of a rank-2 subbundle of $\wedge^2 U^*_\omega(-Q'_\omega)$. Toward this end we consider vector bundle maps:

$$\wedge^2 U^*_\omega(-Q'_\omega) \to \mathcal{O}_{H'_\omega}(L - 2Q'),$$

where L is the pullback of the hyperplane divisor on H_ω. Let $U \subset V$ be the Lagrangian 3-space represented by $u \in \Sigma$ and let $U^\perp = L_\alpha(U) \subset V^*$. Then any element $x \in U^\perp$ induces (by an exterior multiplication) such a map:

$$m_x \colon \wedge^2 U^*_\omega(-Q'_\omega) \to \mathcal{O}_{H'_\omega}(L - 2Q'_\omega).$$

The kernel of this map, which we denote by E'_x, is of course a torsion-free sheaf. If the map m_x is surjective then the kernel is even a vector bundle of rank 2. Therefore, E'_x is our candidate for a rank-2 vector bundle. If we look at the stalks, we see that the multiplication by x is surjective outside the zero locus of x—that is, outside the strict transform $C(x)$ on H'_ω of the quadric cone $Q_x \cap H_\omega$ with vertex at u. Since Q_x is 3-dimensional, $C(x)$ is a (rational) surface scroll whose image in $\bar{H}_\omega$ is a conic on the Veronese surface S_u. Thus we have an exact sequence

$$0 \to E'_x \to \wedge^2 U^*_\omega(-Q'_\omega) \to \mathcal{O}_{H'_\omega}(L - 2Q'_\omega) \to \mathcal{O}_{C(x)}(L - 2Q'_\omega) \to 0.$$

Outside $C(x)$, the kernel sheaf E'_x is a rank-2 vector bundle. This will be enough for our purposes at this point, but eventually we will show that E'_x is a subsheaf of a bundle E_x that coincides with E'_x outside C_x.

The problem is to get $h^0(E'_x) = 6$. The sections of $\mathcal{O}_\Sigma(L)$ that are singular at u can naturally be identified with

$$V(14)^* \cap (\wedge^2 U^\perp \otimes U_1^\perp \oplus \wedge^3 U^\perp) = \mathrm{Sym}^2 U^* \oplus \wedge^3 U^\perp$$

(see Section 2.2). By restriction and pullback from Σ, we therefore have a natural surjection of sections

$$r_\omega \colon V(14)^* \cap (\wedge^2 U^\perp \otimes U_1^\perp \oplus \wedge^3 U^\perp) = \mathrm{Sym}^2 U^* \oplus \wedge^3 U^\perp \\ \to H^0(\mathcal{O}_{H'_\omega}(L - 2Q'_\omega)).$$

The kernel of this map is generated by $q_\omega \in \mathrm{Sym}^2 U^*$, so $h^0(\mathcal{O}_{H'_\omega}(L - 2Q'_\omega) = 6$.

Similarly, there is a natural surjection

$$U^\perp \otimes U_1^\perp \oplus \wedge^2 U^\perp \to H^0(\wedge^2 U^*_\omega(-Q'_\omega)).$$

Here the kernel is generated by α, and $U^\perp \otimes U_1^\perp \oplus \wedge^2 U^\perp$ is 12-dimensional so $h^0(\wedge^2 U^*_\omega(-Q'_\omega)) = 11$. Thus $h^0(E'_x) = 6$ only if the map m_x is not surjective on global sections.

We now consider more carefully the image of the map m_x on global sections. Notice that $r_\omega(\eta)$ for a form

$$\eta \in V(14)^* \cap (\wedge^2 U^\perp \otimes U_1^\perp \oplus \wedge^3 U^\perp)$$

is in the image of m_x if and only if there exist a 2-form $\beta \in U^\perp \otimes U_1^\perp \oplus \wedge^2 U^\perp$ and a 1-form $y \in U^\perp$ such that

$$\eta = \alpha \wedge y + \beta \wedge x.$$

The subspace of 3-forms of this kind in $\wedge^2 U^\perp \otimes U_1^\perp \oplus \wedge^3 U^\perp$ has dimension 9 (i.e., codimension 1): the 3-forms of the kind $\alpha \wedge y$ form a 3-dimensional space, whereas the 3-forms of the kind $\beta \wedge x$ (where β varies) form a subspace of dimension 7. Since these two subspaces intersect each other in $\langle \alpha \wedge x \rangle$, the dimension of their sum is 9. The intersection with $V(14)^*$ has codimension 3 as defined by the symmetrizer relations (see Section 2.2), so the subspace

$$U_x = \{\eta = \alpha \wedge y + \beta \wedge x \mid \eta \wedge \alpha = 0\} \subset V(14)^* \cap (\wedge^2 U^\perp \otimes U_1^\perp \oplus \wedge^3 U^\perp)$$

has dimension 6. The image of the map m_x on global sections is just the projection of U_x from the form ω. Thus we have shown our next lemma.

LEMMA 3.3.1. *The exterior multiplication*

$$m_x \colon \wedge^2 U^*_\omega(-Q'_\omega) \to \mathcal{O}_{H'_\omega}(L - 2Q'_\omega)$$

is not surjective on global sections if and only if ω is an element of U_x.

Let $p = \langle v \rangle \in \mathbf{P}(U)$ and $x = L_\alpha(v) \in U^\perp$. Let q_ω be the quadratic form defined by ω on U (see Section 2.2).

LEMMA 3.3.2. *Let $\omega \in V(14)^* \cap (\wedge^2 U^\perp \otimes U_1^\perp \oplus \wedge^3 U^\perp)$. Then $\omega \in U_x$ if and only if $q_\omega(v) = 0$.*

Proof. First we assume that

$$\omega = \alpha \wedge y + \beta \wedge x.$$

The common zero locus of the 2-forms α and β is then contained in H_ω. Hence we may choose a basis and coordinates (e_i, x_i) on V such that $U = U_0 = \langle e_1, e_2, e_3\rangle$ and $U_1 = \langle e_4, e_5, e_6\rangle$, and we assume that

$$\beta = sx_{14} + tx_{25} + ux_{36}.$$

Since $\alpha \wedge \omega = (x_{14} + x_{25} + x_{36}) \wedge \omega = 0$, it follows that

$$\omega = b(x_{145} + x_{356}) - c(x_{416} + x_{256}) - a(x_{452} + x_{436})$$

for suitable scalar coefficients a, b, c.

The quadratic form q_ω on U is then (see Section 2.2)

$$q_\omega = bx_1x_3 - cx_1x_2 - ax_2x_3.$$

In the expression

$$\omega = \beta \wedge x + \alpha \wedge y,$$

it is clear that

$$x, y \in \langle x_4, x_5, x_6\rangle = L_\alpha(U).$$

Thus we may write $x = \beta_4x_4 + \beta_5x_5 + \beta_6x_6$ and $y = \alpha_4x_4 + \alpha_5x_5 + \alpha_6x_6$. A straightforward calculation gives the following solutions:

$$\alpha_4 = a\frac{u+t}{u-t}, \quad \alpha_5 = b\frac{u+s}{u-s}, \quad \alpha_6 = c\frac{t+s}{t-s},$$

and

$$\beta_4 = \frac{a}{u-t}, \quad \beta_5 = \frac{b}{u-s}, \quad \beta_6 = \frac{c}{t-s};$$

thus

$$v = L_\alpha^{-1}(x) = \frac{a}{u-t}e_1 + \frac{b}{u-s}e_2 + \frac{c}{t-s}e_3,$$

and $q_\omega(v) = 0$.

Conversely, assume $q_\omega(v) = 0$. Let X be a Segre 3-fold through $u = u(\omega)$ contained in H_ω. Then we may assume that X is the zero locus of a 2-form β. Coordinates may therefore be chosen as before, and $q_\omega(v) = 0$ implies that

$$v = \frac{a}{u-t}e_1 + \frac{b}{u-s}e_2 + \frac{c}{t-s}e_3.$$

With

$$y = a\frac{u+t}{u-t}x_4 + b\frac{u+s}{u-s}x_5 + c\frac{t+s}{t-s}x_6,$$

we obtain

$$\omega = \alpha \wedge y + \beta \wedge x. \qquad \square$$

Corollary 3.3.3. *Let $v \in U$ and let $x = L_\alpha(v) \in U^\perp$. Let E'_x on H'_ω be the kernel sheaf of the map m_x described previously. Then $h^0(E'_x) = 6$ iff $q_\omega(v) = 0$, where H'_ω is tangent at $u \in \Sigma$ and q_ω is the quadratic form defined by ω on U.*

Proof. Since $\wedge^2 U^*_\omega(-Q'_\omega)$ has eleven sections, it follows that $h^0(E'_x) = 6$ if and only if m_x is not surjective on global sections—that is, iff $\omega = \alpha \wedge y + \beta \wedge x$ for some 1-form y and 2-form β. Hence the corollary follows from Lemma 3.3.2. □

For each $v \in U$ such that $\{q_\omega(v) = 0\}$ we have constructed a sheaf E'_x with $x = L_\alpha(v)$, locally free of rank 2 outside $C(x)$ on H'_ω, and with $h^0(E'_x) = 6$. Each sheaf gives rise to a rational map of H'_ω into the Grassmannian $\mathbf{Gr}(2,6)$. This map is defined by sections of the determinant line bundle of E'_x, whose first Chern class is given by

$$\begin{aligned} c_1(E_x) &= c_1(\wedge^2 U^*_\omega(-Q'_\omega)) - c_1(\mathcal{O}_{\Sigma_\omega}(L - 2Q'_\omega)) \\ &= (2L - 3Q'_\omega) - (L - 2Q'_\omega) = L - Q'_\omega = L'. \end{aligned}$$

If the natural map

$$\wedge^2 H^0(E'_x) \to H^0\mathcal{O}_{H'_\omega}(L')$$

is surjective, then the map to $\mathbf{Gr}(2,6)$ is nothing but the projection of H'_ω from its singular point $u = u(\omega)$. If it is not surjective, then the image of H'_ω in $\mathbf{Gr}(2,6)$ spans at most a $\mathbf{P}^{10}$. Since H'_ω is not a cone, the image is 5-dimensional and the intersection of its span with $\mathbf{Gr}(2,6)$ is not proper; that is, it contains a variety of codimension 3. Now the class of a codimension-3 subvariety in $\mathbf{Gr}(2,6)$ is a non-negative linear combination of Schubert cycles of degrees 4 and 5. A codimension-3 cycle of degree 5 is a hyperplane section of a $\mathbf{Gr}(2,5) \subset \mathbf{Gr}(2,6)$, whereas a codimension-3 cycle of degree 4 is the variety of lines in $\mathbf{Gr}(2,6)$ that meet a fixed line. Thus multiples of the latter are $\mathbf{P}^4$-bundles. The general hyperplane section of H'_ω has Picard number 1, so it cannot have an image in a $\mathbf{P}^4$-bundle. Furthermore, the vector bundle E_x has six sections, so the image of H'_ω cannot be contained in a $\mathbf{Gr}(2,5)$, either. Now, since codimension-3 subvarieties in $\mathbf{Gr}(2,6)$ of degree 9 and 13 are not contained in a $\mathbf{P}^{10}$, we may conclude that the image of H'_ω spans $\mathbf{P}^{11}$. Hence the image of H'_ω in $\mathbf{Gr}(2,6)$ is precisely the projection $\bar{H}_\omega$. Moreover, this map is independent of x.

THEOREM 3.3.4. *The projection of H_ω from its singular point is a linear section of the Grassmannian* $\mathbf{Gr}(2,6)$.

Proof. What remains is to show that the image $\bar{H}_\omega$ of H'_ω under the projection is a linear section of $\mathbf{Gr}(2,6)$. Because Σ has degree 16 and sectional genus 9, the projection of H_ω must have degree 14 and sectional genus 8; it is 5-dimensional, spans a $\mathbf{P}^{11}$, and is contained in $\mathbf{Gr}(2,6)$. Also it contains a 4-dimensional quadric, the image of the exceptional divisor on H'_ω. Thus $\bar{H}_\omega$ has the same degree, sectional genus, and codimension as $\mathbf{Gr}(2,6)$, so if the intersection

$$\mathbf{Gr}(2,6) \cap \langle \bar{H}_\omega \rangle$$

is different from $\bar{H}_\omega$, then this intersection is 6-dimensional and not proper. But the only codimension-2 varieties in $\mathbf{Gr}(2,6)$ that are contained in a $\mathbf{P}^{11}$ are those representing special Schubert cycles of codimension 2: one is represented by the subvariety of rank-2 subspaces that intersect a given rank-3 subspace, and the other is represented by the Grassmannian $\mathbf{Gr}(2,5)$. The latter does not span a $\mathbf{P}^{11}$ but the former do. In the former case, $\bar{H}_\omega$ is contained in a variety that is a $\mathbf{P}^4$ scroll parameterized by a $\mathbf{P}^2$. Each $\mathbf{P}^4$ must intersect $\bar{H}_\omega$ in a 3-fold. But H_ω is cut out by quadrics and contains only a 1-parameter family of threefold hypersurfaces (in fact, the linear 3-spaces that appear as projections of $Q_p \subset H'_\omega$ in Proposition 2.5.5), so we have a contradiction and the theorem follows. □

The restriction and pullback to H'_ω of the universal rank-2 quotient bundle on $\mathbf{Gr}(2,6)$ is clearly a rank-2 vector bundle, which we denote by E_ω. This immediately yields the following corollary.

COROLLARY 3.3.5. *The sheaf E'_x is a subsheaf of the restriction and pullback E_ω to H'_ω of the universal rank-2 quotient bundle on $\mathbf{Gr}(2,6)$. The bundle E_ω is independent of x; we have $h^0(E_\omega) = 6$ and $\det E_\omega = \mathcal{O}_{H'_\omega}(L')$, and the zero scheme of E_ω's general section is isomorphic to the strict transform of a Segre 3-fold that passes through the singular point of H_ω.*

Proof. Outside $C(x)$ the two sheaves E'_x and E_ω coincide. The zero scheme of a general section of E_ω is precisely the zero scheme of a 2-form on Σ: the strict transform on H'_ω of a Segre 3-fold that passes through the singular point u. Since $C(x)$ has codimension 3, the corollary follows. □

Clearly $\bar{H}_\omega$ is a special linear section of $\mathbf{Gr}(2,6)$ because it contains a 4-dimensional quadric, but a natural question arises: Is a general $\mathbf{P}^{11}$-section of $\mathbf{Gr}(2,6)$ that contains a 4-dimensional quadric the projection of a singular section of the Lagrangian Grassmannian Σ?

Now we prove that this is the case and give another characterization of these linear sections of $\mathbf{Gr}(2,6)$. We set

$$Z = \bar{H}_\omega,$$

and observe that the projection of H'_ω is an isomorphism on the exceptional quadric and outside the tangent cone at u. Thus it is singular at most along the image of the tangent cone, that is, a Veronese surface (inside the 4-dimensional quadric).

First, since $\check{F} \setminus \check{\Omega}$ is an orbit of ρ, all the $\bar{H}_\omega$ are projectively equivalent to the same 5-fold Z. To fix the notation, let $V \cong \mathbf{C}^6$ and let $\mathbf{P}^{14} = \mathbf{P}(\wedge^2 V)$. Let $\mathbf{Gr}(2,V)$ be the Grassmannian of 2-dimensional subspaces $U \subset V$, and let

$$\mathbf{Gr}(2,V) \to \mathbf{P}^{14}, \quad U \mapsto \mathbf{P}(\wedge^2 U),$$

be the Plücker embedding.

Let $\check{\mathbf{P}}^{14} = \mathbf{P}(\wedge^2 V^*)$ be the dual space to $\mathbf{P}^{14}$. The space $\wedge^2 V^*$ is isomorphic to the space $\mathrm{Alt}(V, V^*)$ of skew-symmetric linear maps $A\colon V \to V^*$. Recall that the rank of A is even. The rank stratification is given by the inclusions

$$\mathbf{Gr}(V,2) = \mathbf{Gr}(2,V^*) \subset \check{P}f \subset \check{\mathbf{P}}^{14},$$

that is, by the Grassmannian variety parameterizing $\mathbf{C}^*$-classes of maps $A \neq 0$ such that $\operatorname{rank}(A) = 2$ and by the Pfaffian cubic hypersurface parameterizing $\mathbf{C}^*$-classes of maps A such that $\operatorname{rank}(A) \leq 4$.

Let $\Pi^2 \subset \check{\mathbf{P}}^{14}$ be the plane of linear equations that define $\mathbf{P}^{11} \subset \mathbf{P}^{14}$, so that the 5-fold $Z' = \mathbf{P}^{11} \cap \mathbf{Gr}(2, V)$ contains a smooth 4-dimensional quadric Q. Obviously, $Q = \mathbf{Gr}(2, W) \subset \mathbf{Gr}(2, V)$ for some 4-dimensional subspace $W \subset V$.

LEMMA 3.3.6. $\Pi^2 \subset \check{P}f$.

Proof. The set of forms in V^* that vanish on the 4-dimensional subspace $W \subset V$ is a rank-2 subspace $W^\perp \subset V^*$, so

$$(\wedge^2 W)^\perp = V^* \wedge W^\perp \subset \wedge^2 V^*.$$

Therefore, any $A \in (\wedge^2 W)^\perp$ is of rank at most 4; that is,

$$\mathbf{P}((\wedge^2 W)^\perp) \subset \check{P}f.$$

Since $Q = \mathbf{Gr}(2, W) \subset Z'$, the lemma follows. □

Fix a 4-dimensional subspace $W \subset V$, and let $\mathbf{P}^8_W = \mathbf{P}((\wedge^2 W)^\perp) \subset \check{P}f$. Then $\mathbf{P}^8_W$ intersects the Grassmannian $\mathbf{Gr}(V, 2)$ along the 5-fold Schubert cycle $Y_W := \sigma_{30}(W^\perp)$ of 2-dimensional subspaces of V^* that intersect the rank-2 subspace $W^\perp$ nontrivially. Therefore, the general plane in $\mathbf{P}^8$ does not intersect Y_W.

When A has rank 4, the kernel is a rank-2 subspace $U_A \subset V$. Hence there is a natural kernel map,

$$\text{pker}\colon \check{P}f \setminus \mathbf{Gr}(V, 2) \to \mathbf{Gr}(2, V), \quad [A] \mapsto [U_A].$$

This map can also be seen as the map

$$\wedge^2 V^* \to \wedge^4 V^* \cong \wedge^2 V, \quad \alpha \mapsto \alpha \wedge \alpha,$$

so it is quadratic in the coordinates. Now, the hyperplane section $H_A \cap \mathbf{Gr}(2, V)$ is singular precisely in $\mathbf{P}(\wedge^2 U_A)$. On the other hand, $\mathbf{P}(\wedge^2 U_A) \in \mathbf{Gr}(2, W')$ for any 4-dimensional subspace $W' \subset V$ that contains U_A. Clearly $U_A \subset W$ for any $A \in \wedge^2 W^\perp$. Therefore,

$$Z' = \bigcap_{A \in \Pi^2} (H_A \cap \mathbf{Gr}(2, V))$$

contains the image of

$$s\colon \Pi^2 \to \mathbf{Gr}(2, V), \quad A \mapsto \mathbf{P}(\wedge^2 U_A).$$

Since $\mathbf{P}(\wedge^2 U_A)$ is a singular point in $H_A \cap \mathbf{Gr}(2, V)$, the image of s is contained in the singular locus of Z'.

LEMMA 3.3.7. $\operatorname{Sing}(Z')$ *is contained in a Veronese surface if and only if*

$$\Pi^2 \cap \mathbf{Gr}(V, 2) = \emptyset.$$

Proof. Indeed, the preceding map s is defined everywhere on Π^2 only if

$$\Pi^2 \cap \mathbf{Gr}(V,2) = \emptyset,$$

and in this case the image is clearly a Veronese surface. On the other hand, if $A \in \Pi^2 \cap \mathbf{Gr}(V,2)$ and $U_A \subset V$ is the kernel of A (regarded as a skew-symmetric map as before), then the hyperplane section $H_A \subset \mathbf{Gr}(2,V)$ defined by A is singular along a 4-fold quadric $Q_A = \mathbf{Gr}(2,U_A) \subset \mathbf{Gr}(2,V)$. Thus

$$Z' = \bigcap_{A \in \Pi^2} (H_A \cap \mathbf{Gr}(2,V))$$

is singular at least along a codimension-2 linear section of a 4-fold quadric Q_A, which is clearly not contained in a Veronese surface. □

PROPOSITION 3.3.8. *The linear section $Z = \bar{H}_\omega$ of $\mathbf{Gr}(2,V)$ is defined by a plane Π^2 of linear equations in $\check{P}f \setminus \mathbf{Gr}(V,2)$. The singular locus of Z is a Veronese surface.*

Proof. We have noted that the singular locus of Z is contained in a Veronese surface. It follows from Lemmas 3.3.6 and 3.3.7 (respectively) that the singular locus of Z is a Veronese surface and that the orthogonal plane does not intersect $\mathbf{Gr}(2,V^*)$. □

The following result of Sato and Kimura on the orbits of the $\mathrm{GL}(V)$-action on $\check{\mathbf{P}}^{14}$ fits well with the foregoing remark.

PROPOSITION 3.3.9 [33, p. 94]. *The action $\wedge^2$ of* $\mathrm{GL}(V)$ *on* $\check{\mathbf{P}}^{14}$ *is transitive on the set of planes* $\Pi^2 \subset \check{P}f \setminus \mathbf{Gr}(V,2)$.

Propositions 3.3.8 and 3.3.9 immediately imply the following corollary and the second part of Theorem 1.1.

COROLLARY 3.3.10. *Let $Z \subset \mathbf{Gr}(2,V)$ be a 5-fold linear section. Then the following statements are equivalent.*

(i) *Z contains a 4-dimensional smooth quadric, and* $\mathrm{Sing}\, Z$ *is a Veronese surface.*
(ii) *The orthogonal complement $\langle Z\rangle^\perp$ is contained in $\check{P}f \setminus \mathbf{Gr}(V,2)$.*
(iii) *Z is the projection of a nodal hyperplane section of $\Sigma = \mathbf{LG}(3,6) \subset \mathbf{P}^{13}$ from its node.*

Let $S = Z \cap \mathbf{P}^8$ be a general linear surface section of Z. Then, clearly, S is a $K3$ surface with a conic C.

COROLLARY 3.3.11. *The Picard group of a general linear surface section S of Z has rank 2 and is generated by the class of a hyperplane and the class of the conic on S.*

Proof. According to the refined versions of Mukai's linear section theorem [21; 25], the general $K3$ surface S—with Picard group generated by a very ample line bundle $\mathcal{O}_S(H)$ of degree $H^2 = 14$ and the line bundle $\mathcal{O}_S(C)$ of a rational curve C with $C \cdot H = 2$—is a linear section of $\mathbf{Gr}(2,6)$. The conic C lies then in a unique 4-dimensional quadric $\mathbf{Gr}(2,4)$ inside $\mathbf{Gr}(2,6)$, and S is therefore a linear section of a subvariety Z. □

3.4. Stable Rank-2 Vector Bundles on Linear Sections

Theorem 3.3.4 allows us to construct families of rank-2 vector bundles on linear sections of Σ as promised at the end of Section 3.1. Let $1 < k < 6$, and consider a $(6-k)$-dimensional smooth linear section $X = \mathbf{P}^{13-k} \cap \Sigma$ and its Sp(3)-dual linear section $\check{F}(X)$ of the quartic $\check{F}$. Then, to each point $\omega \in \check{F}(X) \setminus \check{\Omega}(X)$ we may associate a rank-2 vector bundle $E_{\omega,X}$ on X with Chern classes $c_1(E_{\omega,X}) = H$ and $c_2(E_{\omega,X}) = \sigma_X$, where σ_X is the class of a codimension-$(k-1)$ linear section of a Segre 3-fold. In case X is a curve, the class $c_2(E_{\omega,X}) = 0$ but the vector bundle is special with $h^0(E_{\omega,X}) \geq 6$.

In fact, if $\omega \in \check{F}(X) \setminus \check{\Omega}(X)$, then by Theorem 3.3.4 the projection $\pi_{u(\omega)}$ from the pivot $u(\omega) = L(\mathrm{piv}^*(\omega))$ sends the hyperplane section $H_\omega = \Sigma \cap \mathbf{P}^{12}_\omega$ to the linear section $\bar{H}_\omega = \mathbf{Gr}(2,6) \cap \bar{\mathbf{P}}^{11}_\omega$. Now $\omega \in \check{F}(X) \subset (\mathbf{P}^{13-k})^\perp$, so $H_\omega \supset X$. By assumption X is smooth, so Lemma 2.5.8 implies that $u(\omega) \notin \mathbf{P}^{13-k} = \langle X \rangle$. Therefore, the projection $\pi_{u(\omega)} \colon \mathbf{P}^{12}_\omega \to \bar{\mathbf{P}}^{11}_\omega$ restricts to a linear embedding of $X \subset \mathbf{P}^{13-k}$. If E_ω is the rank-2 vector bundle on H'_ω constructed in Corollary 3.3.5, then the restriction $E_{\omega,X} = E_\omega|_X$ is a rank-2 vector bundle on X. Via the linear embedding $X \subset \mathbf{Gr}(2,6)$, we obtain that $E_{\omega,X}$ is the pullback of the universal rank-2 quotient bundle. Hence $h^0(E_{\omega,X}) \geq 6$, and $c_1(E_{\omega,X}) = H_X$. By Corollary 3.3.5, the general section of E_ω vanishes on the projection of a Segre 3-fold inside H_ω through $u(\omega)$. Since X does not pass through $u(\omega)$, the restriction to X is that of a codimension-$(k-1)$ linear section of this Segre 3-fold, so $c_2(E_{\omega,X}) = \sigma_X$.

Denote by $\mathcal{M}_X(2,H,\sigma_X)$ the moduli space of stable rank-2 vector bundles on X with Chern classes $c_1(E) = H$ and $c_2(E) = \sigma_X$. This moduli space exists as a quasiprojective variety (see also [17; 18; 34]).

PROPOSITION 3.4.1. *Let $1 < k < 6$. Let $X = \Sigma \cap \mathbf{P}^{13-k}$ be a smooth linear section of Σ without nontrivial automorphisms and let $\check{F}(X)$ be its* Sp(3)*-dual linear section of the quartic $\check{F}$. Then there is a natural map*

$$e_X \colon \check{F}(X) \setminus \check{\Omega}(X) \to \mathcal{M}_X(2,H,\sigma_X), \quad \omega \mapsto [E_{\omega,X}],$$

where $[E_{\omega,X}]$ is the isomorphism class of $E_{\omega,X}$ and where $\sigma_X = 0$ if X is a curve. Furthermore, the map is injective on the set where $H^0(E_{\omega,X}) = 6$.

Proof. First we prove stability. Recall that a rank-2 vector bundle E is *Takemoto–Mumford stable* (resp. *semistable*) with respect to the polarization H if, for each line subbundle L, the inequality

$$2H^i \cdot L < H^i \cdot c_1(E) \quad (\text{resp. } 2H^i \cdot L \leq H^i \cdot c_1(E))$$

holds, where $i = \dim X - 1$.

It is clear from the definion of Takemoto–Mumford that it is enough in our case to check stability when X is a curve; instability in the other cases would imply instability by restriction to a curve section of X.

So let $C = X \subset \Sigma$ be a smooth linear curve section, and let $\omega \in \check{F}(C) \setminus \check{\Omega}(C)$. We may assume that C has no automorphisms and that C has no g^1_5 (see Theorem 3.4.5). Consider the vector bundle $E_{\omega,C}$. By assumption it is the pullback of the universal rank-2 quotient bundle on $\mathbf{Gr}(2,6)$, so the associated map of the $\mathbf{P}^1$-bundle

$$\mathbf{P}(E_{\omega,C}) \to \mathbf{P}^5$$

is a morphism. Let $S_C \subset \mathbf{P}^5$ be the image ruled surface of this morphism. The curve C is contained in a 5-dimensional linear section $Z \subset \mathbf{Gr}(2,6)$ that contains a 4-dimensional quadric. The linear span of C intersects Z in a surface, which by Corollary 3.3.11 is a $K3$-surface section Y containing a unique conic section that does not intersect C. Assume now that $E_{\omega,C}$ has a subbundle of degree $d \geq 6$. Then S_C has d members of the ruling contained in a hyperplane H. In particular, $C \subset \mathbf{Gr}(2,6)$ meets the Grassmannian $\mathbf{Gr}(2,H)$ in d points. But $\mathbf{Gr}(2,H)$ has degree 5, so $\mathbf{Gr}(2,H)$ must intersect the surface Y in at least a curve. Since Y is an irreducible surface, the intersection must be a curve that spans at most a $\mathbf{P}^4$. Since $d \geq 6$, the corresponding divisor on C of degree d spans at least a $\mathbf{P}^4$. Therefore this curve has degree 5. But by Corollary 3.3.11 the Picard group of Y is generated by the class of H and the class of the unique conic, so in particular every curve on it has even degree—a contradiction. Therefore $E_{\omega,C}$ is stable.

For injectivity, note that for a given embedding of $X \in \mathbf{Gr}(2,6)$ the linear span of X cuts the Grassmannian in a variety Y of dimension $\dim X + 1$. Given two elements ω and ω' in $\check{F}(X) \setminus \check{\Omega}(X)$, the vector bundles $E_{\omega,X}$ and $E_{\omega',X}$ are isomorphic only if the two linear sections Y_ω and $Y_{\omega'}$ are projectively equivalent. In fact, the global sections of $E_{\omega,X}$ define the map into $\mathbf{Gr}(2,6)$, and the linear span of the image defines Y_ω. Now Y_ω is the projection from $u(\omega)$ of the linear section $\tilde{Y}_\omega$ of Σ defined by the span of X and $u(\omega)$. Therefore, Y_ω and $Y_{\omega'}$ are projectively equivalent if and only if $\tilde{Y}_\omega$ and $\tilde{Y}_{\omega'}$ are projectively equivalent. Mukai proves in [29, Thm. B] that two linear curve sections of Σ are projectively equivalent if and only if they lie in the same orbit of the group action ρ. This clearly extends to surface sections, so the linear span of X and $u(\omega)$ and the linear span of X and $u(\omega')$ are in the same orbit under the action ρ. As soon as X has no nontrivial automorphisms, this can no longer happen. □

We next analyze the image of the map e_X, starting with the curve case. The general stable rank-2 vector bundle with canonical determinant on $C = X$ has no sections. The subset of $M_C(2;K)$ corresponding to vector bundles with a given number of sections has the structure of a subvariety, which has been studied by several authors [3; 27; 28; 31]. Following their notation, we define the Brill–Noether locus $M_C(2;K,k)$ to be the subvariety of $M_C(2;K)$ corresponding to vector bundles with at least $k+2$ sections. Let E be a rank-2 vector bundle on a general linear curve section $C \subset \Sigma$ with $[E] \in M_C(2;K,4)$. Assume that E is generated by global sections, and let

$$\wedge^2 H^0(C,E) \to H^0(C,K)$$

be the natural map. If this map is surjective, then E clearly is in the image of the map e_C as soon as the induced image $C \subset \mathbf{Gr}(2, H^0(C,K))$ is contained in a subvariety isomorphic to Z. By Corollary 3.3.10 this condition is satisfied as soon as the orthogonal complement of the span of C in $\mathbf{P}(\wedge^2 H^0(C,K))$ does not meet $\mathbf{Gr}(H^0(C,K),2)$. Thus we have our next lemma.

LEMMA 3.4.2. *The isomorphism class* $[E] \in M_C(2;K,4)$ *fails to be in the image of* e_C *only if*

(i) $\wedge^2 H^0(C,E) \to H^0(C,K)$ *is not surjective, or*
(ii) *the orthogonal complement of the span of* C *meets* $\mathbf{Gr}(4, H^0(C,K))$, *or*
(iii) E *is not globally generated.*

We will not show that the cases of the lemma do not occur; we simply note that they represent closed subvarieties of $M_C(2;K,4)$. A more general result is as follows.

PROPOSITION 3.4.3 ([28, Thm. 4] or [3, pp. 260–261]). *Let* C *be a curve of genus* 9 *with no* g^1_5. *If* $M_C(2;K,5) = \emptyset$, *then* $M_C(2;K,4)$ *is smooth and of dimension* 3 *precisely at the points representing bundles* E *for which the Petri map* $\mu\colon \mathrm{Sym}^2 H^0(C,E) \to H^0(C, \mathrm{Sym}^2 E)$ *is injective.*

The injectivity of the Petri map is shown by Bertram and Feinberg [3] for $g(C) \geq 2$ and for any stable rank-2 vector bundle with canonical determinant and $h^0(C,E) \leq 5$. The same line of argument yields the following lemma.

LEMMA 3.4.4. *Let* C *be a curve of genus* 9 *as before. Then the Petri map*

$$\mu\colon \mathrm{Sym}^2 H^0(C,E) \to H^0(C, \mathrm{Sym}^2 E)$$

is injective for any stable bundle $E \in M_C(2;K,4)$.

Proof. Let S be the scroll in $\mathbf{P}(H^0(C,E)^*)$ defined via mapping $\mathbf{P}(E)$ by the global sections of E. Then an element in the kernel of the Petri map

$$\mu\colon \mathrm{Sym}^2 H^0(C,E) \to H^0(C, \mathrm{Sym}^2 E)$$

defines a quadric hypersurface containing S (see [3, p. 267]). So the Petri map is injective if and only if S is not contained in any quadric Q. Since $\det E$ is the canonical line bundle, the degree of S is 16. Because C is a section of Σ, it has no g^1_5, g^2_7, g^3_9, or g^4_{11}. Furthermore, since E is stable, it has no line subbundle of degree greater than 7, or (equivalently) no section of E vanishes in a divisor of degree 8. In particular, no $\mathbf{P}^4$ contains eight lines on S, and no plane intersects S in a section. For the latter, it is clear that a plane section of degree at most 7 corresponds to a linear series on C of dimension 2 and degree at most 7; whereas, for a plane section of degree at least 8, the residual net of hyperplanes defines a linear series of dimension 2 and degree at most 8. Equality corresponds to a semistable bundle E.

Following [3, Sec. 4], we regard separately the cases $1 \le \operatorname{rank} Q \le 6$ as follows.

If $\operatorname{rank} Q = 6$, then the family $F(Q) \subset \mathbf{Gr}(2,6)$ of lines on the smooth quadric Q is isomorphic to the 5-fold

$$F(Q) = \mathbf{P}(T_{\mathbf{P}^3}(-1)) \subset \mathbf{P}^3 \times \check{\mathbf{P}}^3,$$

which is the incidence variety between points and planes in $\mathbf{P}^3$. The quadric Q is then interpreted as the Grassmannian of lines in this $\mathbf{P}^3$. Denote the pullbacks of the hyperplane divisors in $\mathbf{P}^3$ and $\check{\mathbf{P}}^3$ to C by h and h'. Then h and h' are complementary divisors in a canonical divisor; that is, $h + h' \equiv K_C$.

Assume that one of the two projections maps C to a line. Then this line will intersect all lines in $\mathbf{P}^3$ that are parameterized by $S \subset Q$. But this means that S is contained in a tangent $\mathbf{P}^4$ to Q, a contradiction. Thus the linear series defined by h and h' both have dimension at least 2. Since C has no g^2_7, the degree of both h and h' is at least 8. Since they are complementary in a canonical divisor, this happens only if they both define g^2_8s. This corresponds to a semistable vector bundle E.

If $\operatorname{rank} Q = 5$, then Q is a cone with vertex a point, and the planes in Q all pass through the vertex and are parameterized by $\mathbf{P}^3$. Therefore $F(Q) \subset \mathbf{Gr}(2,6)$ is a $\mathbf{P}^2$-bundle over a $\mathbf{P}^3$. Since S is not a cone, only finitely many lines of S, say d lines, pass through the vertex of Q. Let P be the $\mathbf{P}^4$ of lines in $\mathbf{Gr}(2,6)$ passing through the vertex, and let $p\colon C \to \mathbf{Gr}(2,5)$ be the projection from P. Then p corresponds to the projection of S from the vertex of Q, and p maps C into the double Veronese embedding of $\mathbf{P}^3$ in $\mathbf{Gr}(2,5)$. Thus the canonical linear series has a decomposition as a sum $K_C = D + 2L$, where $D = C \cap P$ is a divisor of even degree d. Since D spans at most a $\mathbf{P}^4$ in the canonical embedding of C, the degree $d \le 6$. If $d \le 2$, then $p(C)$ spans at least a $\mathbf{P}^6$ and so L is of degree at most 8 and dimension 3, contrary to the assumption for C. If $d = 4$ or $d = 6$, then L is a g^2_6 (resp., a g^1_5)—also a contradiction.

If $\operatorname{rank} Q = 4$, then Q contains two pencils of $\mathbf{P}^3$s. The restriction of these pencils to S defines two pencils of curves $|D|$ and $|D'|$ on S such that $D + D'$ is a hyperplane section. We may assume that D is a section of S and that D' is the pullback of a divisor on C. Thus $\deg D' \ge 6$ and $\deg D \le 10$. Since C has no g^1_5, only the equality is possible. In this case the decomposition $D + D'$ of a hyperplane section of S corresponds to an exact sequence

$$0 \to \mathcal{O}_C(D'_C) \to E \to \mathcal{O}_C(K_C - D'_C) \to 0.$$

The assumption on C actually implies this sequence is exact on global sections—in other words, that the connecting homomorphism $\delta_{[E]}\colon H^0(\mathcal{O}_C(K_C - D'_C)) \to H^1(\mathcal{O}_C(D'_C))$ is zero. Therefore $E = \mathcal{O}_C(D'_C) \oplus \mathcal{O}_C(K_C - D'_C)$; and since $\deg(K_C - D'_C) = 10 > 8$, the bundle E is not even semistable.

If $\operatorname{rank} Q = 3$, then Q is a cone with vertex a plane $\mathbf{P}^2$ over a smooth plane conic q. Thus the hyperplane divisor H on S decomposes to $H = 2D + D_0$, where $D_0 = S \cap \mathbf{P}^2$ is the intersection of S with the vertex of Q. Note that D_0 must be a curve, for otherwise $H = 2D$ contradicts the fact that H is a section of S over

C. Hence D_0 must be a section of S, but we have already seen that no section of S lies in a plane and so this case is impossible.

If rank $Q = 2$ or $Q = 1$, then the scroll S spans at most a $\mathbf{P}^4$, contrary to the assumption. □

Our next theorem restates one part of Mukai's famous linear section theorem.

THEOREM 3.4.5 [29]. *Any smooth curve C of genus 9 with no g^1_5 is isomorphic to a linear section X of $\Sigma = \mathbf{LG}(3,6)$.*

On the other hand, we have the following lemma.

LEMMA 3.4.6. *No smooth linear curve section C of Σ has a g^1_5.*

Proof. Consider the curve C as a subvariety of $\mathbf{Gr}(3,6)$, and let D be a member of a g^1_5 on C. Then D spans a $\mathbf{P}^3$ and must therefore correspond to five planes in a $\mathbf{P}^4$; that is, $D \subset \mathbf{Gr}(3,5) \subset \mathbf{Gr}(3,6)$. The intersection of Σ with any $\mathbf{Gr}(3,5)$ is, however, always a $\mathbf{P}^4$-section of a Grassmannian quadric. Therefore the intersecton with the span of D must be a quadric surface, contradicting the fact that C is a linear section. □

Therefore, if we combine Proposition 3.4.3 and Lemma 3.4.4 and the injectivity of the map e_C, we recover Mukai's result on the Brill–Noether locus as follows.

THEOREM 3.4.7 [28, p. 17]. *For a smooth linear curve section C of Σ, the quartic 3-fold $\check{F}(C) \setminus \check{\Omega}(C)$ is a connected component of the Brill–Noether locus $M_C(2;\, K, 4)$. The 21 double points $\check{\Omega}(C) \subset \check{F}(C)$ in the boundary correspond to semistable vector bundles that are not stable.*

Proof. It only remains to check the semistable boundary. The *semistable boundary* $\delta_{\mathrm{ss}} M_C(2;\, K)$ of $M_C(2;\, K)$ is the image of $\mathrm{Pic}^{g-1}(C)$ under the map

$$j\colon \mathrm{Pic}^{g-1}(C) \to M_C(2;\, K), \quad L \mapsto L \oplus K \otimes L^{-1}$$

(see [31, Sec. 1]). The *semistable boundary* of the locus $M_C(2;\, K, 4) \subset M_C(2;\, K)$ is the intersection $\delta_{\mathrm{ss}} M_C(2;\, K, 4) = \delta_{\mathrm{ss}} M_C(2;\, K) \cap M_C(2;\, K, 4)$. Therefore,

$$\begin{aligned}\delta_{\mathrm{ss}} M_C(2;\, K, 4) &= \{L \in \mathrm{Pic}^8(C) : L \oplus K \otimes L^{-1} \in M_C(2;\, K, 4)\}\\ &= \{L \in \mathrm{Pic}^8(C) : h^0(L \oplus K \otimes L^{-1}) \geq 6\}.\end{aligned}$$

Since $C \subset \mathbf{LG}(3,6)$ has no g^1_5 it has no g^3_8, so any line bundle L (and likewise any $K \otimes L^{-1}$) such that $h^0(L \oplus K \otimes L^{-1}) \geq 6$ must be a g^2_8. Let $W^r_d(C) \subset \mathrm{Pic}^d(C)$ be the Brill–Noether locus of all the invertible sheaves L of degree d on C such that $h^0(C, L) \geq r$. Since C is general of genus $g = 9$, it follows that the fundamental class of $W^r_d(C)$ in $\mathrm{Pic}^d(C) \cong J(C)$ is

$$[W^r_d] = \frac{r! \cdot (r-1)! \cdots 0!}{(g+2r-d)! \cdots (g+r-d)!}\, \Theta^{(r+1)(g+r-d)},$$

where $(J(C), \Theta)$ is the principally polarized Jacobian of C (see [9, Chap. 2, Sec. 7]). In particular, dim $W^2_8(C) = 0$, and since $\deg(\Theta^9/9!) = 1$ we have

$$\deg W^2_8(C) = \frac{2! \cdot 1! \cdot 0!}{5! \cdot 4! \cdot 3!} 9! = 42.$$

Therefore, on the general curve C of genus 9, there are exactly 42 line bundles L such that $\deg(L) = 8$ and $h^0(C, L) = 3$. Moreover, since $K \otimes L^{-1}$ also has degree 8 and three sections, the map

$$\bar{\ }: W^2_8(C) \to W^2_8(C), \quad L \mapsto \bar{L} = K \otimes L^{-1},$$

is an involution of $W^2_8(C)$. The fixed points, if any exist, of the involution $\bar{\ }$ are these L such that $L^{\otimes 2} = K_C$ (i.e., such that L is a theta-characteristic of C) and for which $h^0(C, L) = 3$. But since C is general, it follows that $h^0(C, L) \leq 1$ for any theta-characteristic of C; that is, $\bar{\ }$ has no fixed points.

On the general curve C of genus 9, we get exactly 21 (nonordered) pairs $(L_i, \bar{L}_i)$, $1 \leq i \leq 21$, of line bundles such that

$$\deg L_i = \deg \bar{L}_i = 8, \qquad h^0(C, L_i) = h^0(C, \bar{L}_i) = 3,$$

and $L_i \otimes \bar{L}_i = K_C$. Hence the semistable boundary $\delta_{\mathrm{ss}} M_C(2; K, 4)$ of $M_C(2; K, 4)$ is a finite set of 21 points representing the 21 rank-2 vector bundles $E_i = L_i \oplus \bar{L}_i$, $1 \leq i \leq 21$.

In our setting, if C is a general linear section of Σ then the Sp(3)-dual $\check{F}(C)$ is a quartic 3-fold that intersects $\check{\Omega}$ in 21 nodes. Since this number fits with the number of semistable vector bundles just computed, we try to extend the map e_C described previously to $\check{\Omega}(C)$. This is possible, as we now demonstrate.

Let $\omega \in \check{\Omega}$, let u be a dual pivot of ω, and consider the blowup of Σ at u. Let Q'_ω be the strict transform of the exceptional divisor on H'_ω as before. Consider the exact sequence

$$0 \to E'_x \to \wedge^2 U^*_\omega(-Q'_\omega) \to \mathcal{O}_{H'_\omega}(L') \to N_x \to 0,$$

where the cokernel sheaf $N_x = \mathcal{O}_{C(x)}(L')$ is the restriction of the line bundle $\mathcal{O}_{\Sigma_\omega}(L')$ to the zero scheme of x. By Lemma 3.3.2 the kernel sheaf E'_x has a 6-space of sections as soon as $\mathbf{P}^4_v \subset \mathbf{P}^{12}_\omega$, where $v = L^{-1}(x)$. So v belongs to one of the planes in the involutive pair P_1 and P_2 corresponding to ω (see Proposition 2.5.5). Furthermore, the zero scheme of x has codimension 2 and so does not intersect a general curve section $C \subset H_\omega$. Therefore, the restriction of E'_x to C becomes a rank-2 vector bundle E_ω with canonical determinant and six sections. On the other hand, for any line l in one of the planes P_i, the subvariety Σ_l of Lagrangian planes that meet l is a Weil divisor on H_ω. Thus there are two nets of Weil divisors on H_ω. When restricted to the curve C these divisors become Cartier divisors, and the vector bundle E_ω splits as the sum of the corresponding line bundles. This yields the desired semistable vector bundle corresponding to the point $\omega \in \check{\Omega}(C)$. □

Finally, we study the image of the map e_X from Proposition 3.4.1 in the cases where X is a surface, a 3-fold, and a 4-fold. Thus we recover and generalize the results by Mukai that initiated this investigation.

Let (S,h) be a polarized $K3$ surface of genus $g = 2n+1$, and let s be an integer such that $s \le n$. By [20, Sec. 10] or [22, Sec. 3], each component of the moduli space of stable rank-2 vector bundles E on S,

$$M_S(2,h,s) = \{E \mid E \text{ is stable, } c_1(E) = h, \text{ and } \chi(S,E) = s+2\}/(\mathrm{iso}),$$

is a nonsingular symplectic variety of dimension $2(g-2s)$. In particular, if $s = n = (g-1)/2$ then each connected component of $\hat{S} := M_S(2,h,s)$ is a $K3$ surface. See [9] for a proof of irreducibility that includes references to other proofs.

Let $S = X$ be a general $K3$ surface of genus 9 embedded as a linear section of Σ by a codimension-4 subspace $\mathbf{P}^9 \subset \mathbf{P}^{13}$, and let H be the hyperplane class of $S \subset \mathbf{P}^9$. Then the moduli space $M_S(2,H,\sigma_S)$ coincides with $M_S(2,H,4)$, since $\chi(S,E) = 4+2 = c_2(E) = \deg \sigma_S$. When combined with Proposition 3.4.1, this yields our last theorem as follows.

THEOREM 3.4.8. *For the general linear surface section $S = X \subset \Sigma$, the $K3$ surface $\hat{S} = M_S(2,H,4)$ is isomorphic to the* $\mathrm{Sp}(3)$*-dual quartic surface $\check{F}(S)$.*

Proof. The map e_X is injective, so $\check{F}(S)$ is a subvariety of $M_S(2,H,4)$. On the other hand, $M_S(2,H,4)$ is a $K3$ surface, so they must coincide. □

If we compare this result with Proposition 3.4.1, we see that, in fact, the map e_S for linear surface sections $S \subset \Sigma$ is surjective.

PROPOSITION 3.4.9. *Let X be a general smooth* 3*-fold or* 4*-fold linear section of Σ, and let E be a stable rank-*2 *vector bundle on X with $h^0(X,E) = 6$ and* $\det E = \mathcal{O}_X(H)$. *Assume that the natural map $\wedge^2 H^0(X,E) \to H^0(\mathcal{O}_X(H))$ is surjective. Then E is in the image of e_X.*

Proof. Assume that E is a rank-2 vector bundle on X that satisfies the conditions of the proposition. Then the surjection $\wedge^2 H^0(X,E) \to H^0(\mathcal{O}_X(H))$ defines an embedding $X \subset \mathbf{Gr}(2, H^0(X,E)^*)$. Clearly, E is in the image of e_X if and only if there is a $\mathbf{P}^{11}$ such that $X \subset Z_X = \mathbf{P}^{11} \cap \mathbf{Gr}(2, H^0(X,E)^*)$ for some $Z_X \cong Z$ as in Proposition 3.3.8.

Assume that $\dim X \ge 3$. Since e_S is surjective for any general surface section, it follows that each surface section S intersects the Grassmannian in a 3-fold that is a linear section of a variety $Z_S \cong Z$. Hence, the 4-dimensional quadric $Q \subset Z_S$ intersects the linear span of S in a quadric surface. If two surface sections S and S' of X give rise to subvarieties Z_S and $Z_{S'}$ with distinct quadrics Q and Q', then these quadrics are Grassmannians $\mathbf{Gr}(2,W)$ and $\mathbf{Gr}(2,W)'$ for 4-dimensional subspaces W and W' of $H^0(X,E)^*$. So Q and Q' have a plane or a point in common. But S and S' are linear sections of X, and the corresponding quadric surfaces may be chosen to be smooth with exactly a conic in common—a contradiction. Therefore, the subvarieties Z for distinct surface sections of X contain the same 4-dimensional quadric Q.

The linear span of X intersects the quadric Q in a quadric of dimension $\dim X - 1$. Therefore the linear span of $X \cup Q$ has dimension 11 and cuts $\mathbf{Gr}(2,6)$ in a subvariety that is projectively equivalent to Z. □

By [23], any smooth Fano 3-fold X of degree 16, with rank Pic $= 1$, and of index 1 (called a *prime* Fano 3-fold of degree 16) is a linear 3-fold section of Σ, and the hyperplane class H of $X = \Sigma \cap \mathbf{P}^{10}$ is the ample generator of Pic X over $\mathbf{Z}$.

If X is general, then its Sp(3)-dual linear section $\check{F}(X)$ is a smooth plane quartic curve that does not intersect $\check{\Omega}$.

PROPOSITION 3.4.10. *Let $X = X_{16} \subset \mathbf{P}^{10}$ be a general prime Fano 3-fold of degree 16. Then the* Sp(3)*-dual to X, the plane quartic curve $\check{F}(X)$, is isomorphic to an irreducible component of the moduli space $M_X(2; H, \sigma_X)$ of stable rank-2 vector bundles on X with $c_1 = [h]$ and $c_2 = \sigma_X$, where $[h]$ is the class of the hyperplane section and σ_X is the class of an elliptic sextic curve on X.*

Proof. The condition in Proposition 3.4.9 is certainly an open one, so the complement of the image of the map e_X is closed. Since e_X is injective (by Proposition 3.4.1) and its image is closed, the proposition follows. □

We end with an easy corollary in the 4-fold case.

COROLLARY 3.4.11. *For a general linear 4-fold section $X \subset \Sigma$, the* Sp(3)*-dual $\check{F}(X)$ consists of four points. Whenever X has no automorphisms, these four points define precisely the four isomorphism classes of stable rank-2 vector bundles E on X with $c_1(E) = H$ and $c_2(E) = \sigma_X$, where σ_X is the class of a Del Pezzo surface of degree 6 on X such that the natural map $\wedge^2 H^0(X, E) \to H^0(\mathcal{O}_X(H))$ is surjective.*

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