Michigan Math. J. 53 (2005)

Solving the Schröder Equation at the Boundary in Several Variables

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0. Prologue

Let D be a domain of $\mathbb{C}$ containing the origin 0 and let $f \in \mathrm{Hol}(D, \mathbb{C})$ be a given holomorphic function defined on D with $f(0) = 0$ and $f'(0) \neq 0$. The classical Schröder equation [22] is the functional equation

$$\sigma \circ f = \lambda\sigma, \tag{0.1}$$

where σ is an unknown function, called a *Schröder map* for f, and λ is an unknown complex number.

This equation has a solution if, for instance, $|f'(0)| < 1$ (see Königs [16]). In particular, if $f \in \mathrm{Hol}(\Delta, \Delta)$ is a self-map (not an automorphism) of the unit disc $\Delta \subset \mathbb{C}$, then there exists a solution σ of (0.1) that is defined and holomorphic on Δ, and this solution is unique if the value of $\sigma'(0)$ is chosen. This last fact can be stated as the possibility of "representing" the given $f \in \mathrm{Hol}(\Delta, \Delta)$ by means of the linear part of its expansion at the fixed point 0.

If $f \in \mathrm{Hol}(\Delta, \Delta)$ has no fixed points in Δ, then the situation becomes more complicated and the existence of a solution σ of (0.1) depends on the "dynamics" of f. It is well known (see e.g. [1]) that, if $f \in \mathrm{Hol}(\Delta, \Delta)$ has no fixed points in Δ, then there exists a unique point $\tau \in \partial\Delta$ such that f has nontangential limit τ at τ—that is, τ is a boundary fixed point for f—and f' has nontangential limit $f'(\tau) = \alpha_\tau \in (0, 1]$ at τ. The real number α_τ is called the *boundary dilatation coefficient* of f at τ. If $\alpha_\tau < 1$ then the function f is called *hyperbolic,* otherwise it is called *parabolic.* If f is hyperbolic then (0.1) has a solution. The proof of this fact dates back to the times of Valiron [23]. If f is parabolic then there are no injective solutions of (0.1), and one is led to solve the so-called Abel's equation [3; 11; 19].

The Schröder (and Abel) equation can be written in a more general framework as follows. Let $f \in \mathrm{Hol}(\Delta, \Delta)$ and consider the equation

$$\sigma \circ f = \Phi_f \circ \sigma, \tag{0.2}$$

where Ω is a complex manifold, $\sigma : \Delta \to \Omega$ is an unknown holomorphic function (called an *intertwining map*), and Φ_f is a biholomorphism of Ω. A solution

Received March 29, 2004. Revision received September 21, 2004.
Partially supported by Progetto MIUR Proprietà geometriche delle varietà reali e complesse and by GNSAGA of INdAM.

of (0.2) gives rise to a "representation model" for f, described by the following commuting diagram:

$$\begin{tikzcd} \Delta \arrow[r, "f"] \arrow[d, "\sigma"'] & \Delta \arrow[d, "\sigma"] \\ \Omega \arrow[r, "\Phi_f"] & \Omega. \end{tikzcd}$$

The more information one has on the σ, Ω, Φ_f appearing in this diagram, the more useful the model turns out to be. Among many interesting papers that have appeared on the subject we recall here Pommerenke [19], Baker and Pommerenke [3], Cowen [11], Bourdon and Shapiro [6], and Poggi Corradini [18]. In this last paper a "reversed" Schröder equation is solved.

The motivations—aside from the natural beauty contained in the problem itself—for all such efforts to solve the Schröder (and Abel) equation in its form (0.1), or in a more general setting as in (0.2), have to be found in the number of questions that can be affirmatively solved by means of a solution of (0.1) (and (0.2)). By means of a Schröder map, for instance, one can study the dynamics of holomorphic self-maps of Δ (see e.g. [1]) or questions concerning the composition operators associated to a given map [6; 13]. Other interesting problems that can be attacked by means of "representation models" of type (0.2) concern the study of families of commuting mappings [1; 9; 12; 20] and their boundary properties [4; 10]. These last results state that, roughly speaking, one can characterize the families of holomorphic maps of Δ that commute under composition as those families of mappings that "share" a common Schröder map.

In the case of several complex variables, one can in principle follow the one-dimensional approach of Cowen [11] in order to discover an "abstract" representation model. Namely, arguing as in [11] (with some additional hypotheses), one can effectively solve (0.2) for a given holomorphic self-map f of the open unit $(n+1)$-ball. In this approach the domain Ω turns out to be an abstract, noncompact, simply connected complex manifold, so the lack of a uniformization theorem in several complex variables makes such a model not particularly interesting.

Recently there have been several attempts to construct useful representation models in the unit ball of $\mathbb{C}^{n+1}$. Until now, only the cases of linear fractional mappings and of an "isolated inner fixed point" have been solved. In fact, in [15] Khatskevich, Reich, and Shoikhet deal with functional equations for linear fractional maps in (even infinite-dimensional) Banach spaces, while a paper by Cowen and MacCluer [14] solves the Schröder equation for holomorphic self-maps of the open unit $(n+1)$-ball having an isolated inner fixed point (see also [5], where a different approach to this problem is presented). Cowen and McCluer study the equation

$$\sigma \circ F = M\sigma, \tag{0.3}$$

where F is a holomorphic self-map of the open unit ball in $\mathbb{C}^{n+1}$ fixing O, M is an $(n+1) \times (n+1)$ matrix, and the unknown σ is a (vector) map from the ball to $\mathbb{C}^n$.

Under the condition that dF_O is semisimple, invertible, and has no eigenvalues of modulus 1, Cowen and MacCluer characterize the existence of a Schröder map σ, holomorphic in the ball, by means of the semisimplicity of the matrix associated to the composition operator induced by F. In [14] (and [5]) the problem of uniqueness for the Schröder map is investigated. It is shown how (contrary to the one-dimensional case) the Schröder map is no longer unique in general—even when $d\sigma_O$ is given—but rather depends on the presence of resonances among the eigenvalues of dF_O. Later, MacCluer [17] used the aforementioned results to characterize families of commuting holomorphic self-mappings of the open unit ball fixing an isolated point.

In this paper we solve the Schröder equation in several variables for holomorphic mappings with "no inner fixed points".

We actually deal with (0.3) for a given holomorphic self-map F of $\mathbb{H} := \{(z,w) \in \mathbb{C}^n \times \mathbb{C} : \operatorname{Im} w > \|z\|^2\}$ (the unbounded Siegel realization of the Euclidean unit ball of $\mathbb{C}^{n+1}$) with no fixed points in $\mathbb{H}$. As in the one-dimensional case, if $F \in \operatorname{Hol}(\mathbb{H}, \mathbb{H})$ has no inner fixed points then there exists a unique point $P \in \partial\mathbb{H} \cup \{\infty\}$—which we may assume to belong to $\partial\mathbb{H}$, the *Wolff point* of F, such that F has nontangential limit P at P and the normal (with respect to $\partial\mathbb{H}$ at P) component of dF has nontangential limit $\alpha_P \in (0,1]$ at P (see e.g. [1; 21]). If $\alpha_P < 1$ then we call F a hyperbolic map; otherwise F is called parabolic.

For a hyperbolic map having a "good" expansion at its Wolff point, we prove that the Schröder map exists and is unique in an appropriate class of maps.

In order to give a better explanation of our hypotheses on the "good" expansibility of the map, we need a few definitions. For $\varepsilon > 0$, we say that H is *expandable* of order $1+\varepsilon$ at $P \in \partial\mathbb{H}$—and we write $H \in \mathcal{E}^{1+\varepsilon}(P)$—provided H, holomorphic on $\mathbb{H}$ with values in $\mathbb{C}^{n+1}$, has a first-order expansion at P whose remainder satisfies certain "directional bounds" with respect to the normal direction to $\partial\mathbb{H}$ at P. Roughly speaking, a map H belongs to $\mathcal{E}^{1+\varepsilon}(P)$ if it "follows up to the order $1+\varepsilon$" the behavior "prescribed" by the Julia–Wolff–Carathéodory theorem for a holomorphic self-map of $\mathbb{H}$ at its Wolff point (see Section 1 for precise definitions and comments). We also define the linear part A_P of $H \in \mathcal{E}^{1+\varepsilon}(P)$ as the $(n+1) \times (n+1)$ matrix given by the linear part of the expansion of H at P. This linear part A_P can be thought of as the differential of H at P, and indeed this is exactly what it is in most cases. However we warn the reader that, for a map, being expandable of order $1+\varepsilon$ does not imply being differentiable, and neither would the converse statement hold. The reason for considering the class of expandable maps instead of the class of differentiable maps becomes clear when one tries to follow the approach of Valiron or Bourdon–Shapiro to define the Schröder map in the one-dimensional case (see Section 1).

If $\varphi \in \mathcal{E}^{1+\varepsilon}(P) \cap \operatorname{Hol}(\mathbb{H}, \mathbb{H})$ then its linear part A_P defines a natural action, which we indicate by τ_φ, on the complex tangent space to $\partial\mathbb{H}$ at P. For instance, if φ is differentiable at P then one can think of τ_φ as the action of $d\varphi_P$ on $T_P^{\mathbb{C}}\partial\mathbb{H}$. We say that the action τ_φ is normal provided it is so with respect to the hermitian structure of $T_P^{\mathbb{C}}\partial\mathbb{H}$ induced by that of $\mathbb{C}^{n+1}$.

With this notation, our main result is as follows.

THEOREM 0.1. *For some* $P \in \partial\mathbb{H}$, *let* $\varphi \in \mathcal{E}^{1+\varepsilon}(P) \cap \mathrm{Hol}(\mathbb{H}, \mathbb{H})$. *Let* $\alpha_P > 0$ *be the boundary dilatation coefficient of* φ *at* P *and let* A_P *be the linear part of* φ *at* P. *Suppose that* $\alpha_P < 1$, *the action* $\tau_\varphi \colon T_P^{\mathbb{C}}\partial\mathbb{H} \to T_P^{\mathbb{C}}\partial\mathbb{H}$ *is normal, and its eigenvalues are all of modulus strictly greater than* $\alpha_P^{(1+\varepsilon)/2}$. *Then there exists a* $\sigma \in \mathcal{E}^{1+\varepsilon}(P)$ *with invertible linear part at* P *such that*

$$\sigma \circ \varphi = A_P \sigma. \tag{0.4}$$

Moreover, $\tilde{\sigma} \in \mathcal{E}^{1+\varepsilon}(P)$ *with invertible linear part is another solution of* (3.1) *if and only if there exists an* $(n+1) \times (n+1)$ *invertible matrix* E *such that* $[A_P, E] = 0$ *and* $\tilde{\sigma} = E\sigma$.

Thus—unlike in the case of inner fixed points—for expandable hyperbolic maps having a boundary fixed point one can select a “special”, essentially unique, solution of the Schröder equation.

At this point we apply our construction to the study of commuting holomorphic maps with no fixed points in $\mathbb{H}$. We prove that, even in the case of a boundary fixed point, families of maps that commute under composition can be characterized as those families of maps that “share” a common Schröder map (see Theorems 5.1 and 5.2).

The plan of the paper is as follows. In the first section we discuss differentiable versus expandable maps and set up the basic environment to prove our results. Section 2 is devoted to our proof of the main theorem for hyperbolic maps with Wolff point at infinity. In Section 3 the proof is extended to the case of any other fixed point of $\partial\mathbb{H}$. Section 4 deals with resonances and explains the reason for the uniqueness of the Schröder map. Finally, in Section 5 we study and characterize families of commuting holomorphic maps.

In a forthcoming paper we shall discuss Abel's equation in several complex variables.

The authors thank the referee for many useful comments.

1. The Class of Differentiable Maps and the Class of Expandable Maps

In this section we define the classes of maps we are working with and study their properties of regularity.

From now on we tacitly assume that all vectors introduced are column vectors. However, to simplify notation we forgo the symbol of transposition whenever we need to write a column vector as a row vector in the text. Also, if $v = (v_1, \dots, v_n, v_{n+1}) \in \mathbb{C}^{n+1}$ then we use the symbol v' to denote the first n components of v, namely $v' = (v_1, \dots, v_n) \in \mathbb{C}^n$. Sometimes we decompose $\mathbb{C}^{n+1} \ni Z = (z, w) = (z_1, \dots, z_n, w) \in \mathbb{C}^n \times \mathbb{C}$; that is, $z = Z'$. Moreover, we denote by $\langle \cdot, \cdot \rangle$ the canonical hermitian product in $\mathbb{C}^{n+1}$. If $P \in \partial\mathbb{H}$ then we denote by $T_P^{\mathbb{C}}\partial\mathbb{H}$ the complex tangent space to $\partial\mathbb{H}$ at P. When we refer to metric properties we will always tacitly assume $T_P^{\mathbb{C}}\partial\mathbb{H}$ to be endowed with the standard hermitian structure induced by $\mathbb{C}^{n+1}$.

DEFINITION 1.1. Let $P \in \partial\mathbb{H}$, and let $\varphi\colon \mathbb{H} \to \mathbb{C}^{n+1}$ be holomorphic. We say that φ *belongs to the class* $\mathcal{D}^1(P)$ if

$$\varphi(Z) - P = A_P(Z - P) + \gamma(Z), \tag{1.1}$$

where A_P is an $(n+1) \times (n+1)$ matrix and $\|\gamma(Z)\| = \|(\gamma'(Z), \gamma_{n+1}(Z))\|) = o(\|Z - P\|)$ for Z near P.

We let $\mathcal{D}^1_{\mathbb{H}}(P) = \mathcal{D}^1(P) \cap \mathrm{Hol}(\mathbb{H}, \mathbb{H})$.

Therefore $\varphi \in \mathcal{D}^1(P)$ if and only if φ is differentiable at P and $\varphi(P) = P$, and if φ is given by (1.1) then A_P is its differential at P. In particular, if φ extends C^1 to $\mathbb{H} \cup \{P\}$ and $\varphi(P) = P$ then $\varphi \in \mathcal{D}^1(P)$.

By conjugating $\varphi \in \mathcal{D}^1_{\mathbb{H}}(P)$ with suitable automorphisms of $\mathbb{H}$ of the form $(z, w) \mapsto (z + a', w + a_{n+1} + 2i\langle z, a'\rangle)$ with $(a', a_{n+1}) \in \partial\mathbb{H}$, one can always suppose without loss of generality that $P = O$.

We begin with the following simple result.

LEMMA 1.2. *Let $\varphi \in \mathcal{D}^1(O)$ be given by* (1.1). *Then φ is continuous on $\mathbb{H} \cup \{O\}$, $\varphi(O) = O$, and $d\varphi_Z \to A_O$ for $Z \to O$ nontangentially. In particular, if A_O is invertible, then for any nontangential region V with vertex O there exists a $T = T(V) > 0$ such that φ is injective on $V \cap \{Z \in \mathbb{H} : \|Z\| < T\}$.*

Moreover, if $\varphi \in \mathcal{D}^1_{\mathbb{H}}(O)$ (i.e., $\varphi(\mathbb{H}) \subseteq \mathbb{H}$) then

$$A_O = \begin{pmatrix} A'_O & B_O \\ O & \alpha_O \end{pmatrix} \tag{1.2}$$

for some $n \times n$ matrix A'_O, $B \in \mathbb{C}^n$, and $\alpha_O \in (0, 1]$. In particular, A_O is invertible if and only if A'_O is.

Proof. The first assertion is obvious. Let $\{Z_m\} \subset \mathbb{H}$ be a sequence converging to O nontangentially. One may use the Cauchy representation formula to show that, if $\delta_m > 0$ is such that the ball of radius δ_m and center Z_m is contained in $\mathbb{H}$, then

$$\|d\varphi_{Z_m} - A_O\| \le \frac{1}{2\pi} \max_{\|X - Z_m\| = \delta_m} \frac{\|\gamma(Z_m)\|}{\delta_m}.$$

Since $Z_m \to O$ nontangentially, one can choose $\delta_m = O(|w_m|)$ (where w_m is the $(n+1)$th component of Z_m) and then $d\varphi_{Z_m} \to A_O$ as desired.

If A_O is invertible then—because φ is C^1 on the closure of every nontangential region of vertex O—A_O is injective on V for $\|Z\|$ small enough.

The form (1.2) in case $\varphi(\mathbb{H}) \subseteq \mathbb{H}$ follows directly from Rudin's version of the Julia–Wolff–Carathéodory theorem for the unit ball (see [1, Thm. 2.2.29] or [21, Thm. 8.5.6]). □

REMARK 1.3. As the referee pointed out, it would be interesting to see whether one can add some tangential directions to the nontangential region V for which the conclusions of Lemma 1.2 hold. For our purposes, however, the nontangential approach is enough.

It is often useful to move the base point O to infinity. Toward this end one can exploit the following transformation:

$$G(z,w) = \left(\frac{z}{w}, -\frac{1}{w}\right). \tag{1.3}$$

It follows easily that G is an automorphism of $\mathbb{H}$ with inverse $G^{-1}(z,w) = (-z/w, -1/w)$. Furthermore, G "maps" O to ∞; that is,

$$\lim_{\mathbb{H}\ni(z,w)\to O} \|G(z,w)\| = \infty.$$

If $\varphi \in \mathcal{D}^1_{\mathbb{H}}(O)$, let $F = G \circ \varphi \circ G^{-1}$. A straightforward calculation shows that

$$\begin{aligned} &F(z,w) \\ &= \left(\frac{A'_O z + B - w\gamma'(-z/w, -1/w)}{\alpha_O - w\gamma_{n+1}(-z/w, -1/w)}, \frac{w}{\alpha_O - w\gamma_{n+1}(-z/w, -1/w)}\right). \end{aligned} \tag{1.4}$$

We know that $|\gamma_{n+1}| = o(\|(z/w, 1/w)\|)$ but, in several variables, this does not mean in general that $|w\gamma_{n+1}| < 1$. Suppose now that $\lim_{w\to\infty}|w\gamma_{n+1}| = 0$—that is, $|\gamma_{n+1}(v', v_{n+1})| = o(|v_{n+1}|)$ as $v \to 0$. Then one can expand the denominator of (1.4) as

$$\frac{1}{\alpha_O - w\gamma_{n+1}} = \frac{1}{\alpha_O}\left[1 + \frac{w\gamma_{n+1}}{\alpha_O}(1 + S(z,w))\right],$$

where $S(z,w) \to 0$ for $w \to \infty$. The first n components of F then become

$$F'(z,w) = \frac{1}{\alpha_O} A'_O z + \Gamma'(z,w),$$

where $\|\Gamma'(z,w)\| = o(\|z\|) = o(|w|^{1/2})$ for $\|z\|^2 < \operatorname{Im} w < |w|$ in $\mathbb{H}$ and $\|(z,w)\| \to \infty$. Note that this allows the "linear part" $A'_\infty = A'_O/\alpha_O$ to be well-defined. For the $(n+1)$th component of F we have

$$F_{n+1}(z,w) = \frac{1}{\alpha_O} w + \Gamma_{n+1}(z,w),$$

where this time $|\Gamma_{n+1}(z,w)| = o(|w|)$ for $|w| \to \infty$. Note that, since $\|z\| = o(|w|)$ for $\|(z,w)\| \to \infty$, it follows that $\Gamma_{n+1}(z,w)$ also contains any term in z. In other words, the "linear part" in z does not appear in the last component. Summing up, we have

$$F(z,w) = (A'_\infty z + \Gamma'(z,w), \alpha_\infty w + \Gamma_{n+1}(z,w)), \tag{1.5}$$

where $\alpha_\infty = 1/\alpha_O$, $A_\infty = A'_O/\alpha_O$, $\|\Gamma'\| = o(|w|^{1/2})$, and $|\Gamma_{n+1}| = o(|w|)$ for $|w| \to \infty$.

On the other hand, starting from a map F of the form (1.5), a calculation analogous to the previous one shows that $\varphi = G^{-1} \circ F \circ G$, for G as in (1.3), is given by

$$\varphi(z,w) = (A'_O z + \gamma'(z,w), \alpha_O w + \gamma_{n+1}(z,w)), \tag{1.6}$$

where $\alpha_O = 1/\alpha_\infty$, $A'_O = A'_\infty/\alpha_\infty$, $\|\gamma'\| = o(|w|^{1/2})$, and $|\gamma_{n+1}| = o(|w|)$ for $|w| \to 0$. Notice that $|w| = o(|w|^{1/2})$ for $(z,w) \to 0$ and thus "linear terms in w", if well-defined, are contained entirely in γ'.

Therefore, we are led to define a new class $\mathcal{E}^1$ as follows.

Definition 1.4. Let $F : \mathbb{H} \to \mathbb{C}^{n+1}$ be holomorphic. We write $F \in \mathcal{E}^1(\infty)$ (resp. $F \in \mathcal{E}^1(O)$) if F has an expansion as in (1.5) (resp. as in (1.6)), where $\|\Gamma'(z,w)\| = o(|w|^{1/2})$ and $|\Gamma_{n+1}(z,w)| = o(|w|)$ for $\|(z,w)\| \to \infty$ (resp., $\|\gamma'(z,w)\| = o(|w|^{1/2})$ and $|\gamma_{n+1}(z,w)| = o(|w|)$ for $(z,w) \to O$). We let $\mathcal{E}^1_{\mathbb{H}}(\infty) = \mathcal{E}^1(\infty) \cap \mathrm{Hol}(\mathbb{H},\mathbb{H})$ and $\mathcal{E}^1_{\mathbb{H}}(O) = \mathcal{E}^1(O) \cap \mathrm{Hol}(\mathbb{H},\mathbb{H})$.

The foregoing considerations lead to our next result.

Proposition 1.5. *There exists a one-to-one correspondence between $\mathcal{E}^1_{\mathbb{H}}(O)$ and $\mathcal{E}^1_{\mathbb{H}}(\infty)$. The correspondence is obtained by conjugation using the automorphism G of $\mathbb{H}$ given by* (1.3). *Moreover, if a map $F \in \mathcal{E}^1_{\mathbb{H}}(\infty)$ given by* (1.5) *corresponds to $\varphi \in \mathcal{E}^1_{\mathbb{H}}(O)$ given by* (1.6), *then $\alpha_\infty = 1/\alpha_O$ and $A'_\infty = A'_O/\alpha_\infty$.*

Remarks 1.6. (1) In the one-dimensional case (i.e., for $n = 0$) $\mathcal{D}^1_{\mathbb{H}}(O) = \mathcal{E}^1_{\mathbb{H}}(O)$, whereas this is not the case in several variables. For instance, the map $\mathbb{C}^2 \ni (z,w) \mapsto (0, w + z^2) \in \mathbb{C}^2$ is in $\mathcal{D}^1_{\mathbb{H}}(O)$ but not in $\mathcal{E}^1_{\mathbb{H}}(O)$. Actually, if $\varphi \in \mathcal{D}^1_{\mathbb{H}}(O)$ has an expansion at O up to the second order, then $\varphi \in \mathcal{E}^1_{\mathbb{H}}(O)$ if and only if $\gamma_{n+1}(z,w) = Kzw + bw^2 + o(\|(z,w)\|^2)$ for some $n \times n$ matrix K and $b \in \mathbb{C}$. Namely, γ_{n+1} does not contain pure terms in $z_j z_k$.

(2) In [6, p. 50] the class $\mathcal{D}^1_{\mathbb{H}}(\infty) = \mathcal{E}^1_{\mathbb{H}}(\infty)$ for $n = 0$ is denoted by $C^1(\infty)$. In fact, in that paper it is claimed that $\varphi \in C^1(\infty)$ if and only if φ' extends continuously to $\mathbb{H} \cup \{\infty\}$. This assertion is unfortunately not true. Indeed, $\varphi(w) = aw + i + e^{iw}$ ($a \geq 0$) belongs to $\mathcal{E}^1_{\mathbb{H}}(\infty)$, but the limit of $\varphi'(w)$ does not exist if $w \to \infty$ tangentially.

(3) The class $\mathcal{E}^1_{\mathbb{H}}(O)$ (resp. $\mathcal{E}^1_{\mathbb{H}}(\infty)$) is closed under conjugation with automorphisms of $\mathbb{H}$ fixing O (resp. ∞). This fact follows easily from the explicit form of such automorphisms (see e.g. [1, Chap. 2.2.1] or [21, Chap. 2]).

(4) If $\varphi \in \mathcal{D}^1_{\mathbb{H}}(O)$ (resp. $\varphi \in \mathcal{E}^1_{\mathbb{H}}(O)$) then the number α_O appearing in (1.2) (resp. (1.6)) is such that $\alpha_O > 0$, and it coincides with the boundary dilatation coefficient of φ at O. See [8] for results on this matter.

(5) By the Julia–Wolff–Carathéodory theorem for the ball (see [1, Thm. 2.2.29] or [21, Thm. 8.5.6]) it follows that—if $\varphi \in \mathrm{Hol}(\mathbb{H},\mathbb{H})$, φ extends C^1 on the boundary, and O is a fixed point for φ—then φ has an expression as in (1.6) with $\|\gamma'\| = o(|w|^{1/2})$ and $|\gamma_{n+1}| = o(|w|)$. Thus, in a sense, the class $\mathcal{E}^1(O)$ contains maps that well approximate the boundary behavior of self-maps of $\mathbb{H}$ near boundary fixed points.

More generally, we give the following definition.

Definition 1.7. We say that $\varphi \in \mathcal{E}^1(P)$ for $P \in \partial\mathbb{H}$ if there exists an automorphism η of $\mathbb{H}$ mapping O to P and fixing ∞ such that $\eta^{-1} \circ \varphi \circ \eta \in \mathcal{E}^1(O)$. As usual, we let $\mathcal{E}^1_{\mathbb{H}}(P) = \mathcal{E}^1(P) \cap \mathrm{Hol}(\mathbb{H},\mathbb{H})$.

Recall that if η is an automorphism of $\mathbb{H}$ fixing ∞ then it has one of the following forms:

$$\mathbb{C}^n \times \mathbb{C} \ni (z,w) \mapsto (Uz + c', w + c_{n+1} + 2i\langle Uz, c'\rangle) \tag{1.7}$$

for $U \in \mathrm{U}(n)$ and $c = (c', c_{n+1}) \in \partial\mathbb{H}$; or

$$\mathbb{C}^n \times \mathbb{C} \ni (z,w) \mapsto (kUz, k^2 w) \tag{1.8}$$

for $U \in \mathrm{U}(n)$ and $k > 0$.

From (1.7) and (1.8) one sees that, given any $P \in \partial\mathbb{H}$, there always exists an automorphism η of $\mathbb{H}$ that fixes ∞ and maps O to P. Moreover, even if $\varphi\colon \mathbb{H} \to \mathbb{C}^{n+1}$ is such that $\varphi(\mathbb{H}) \not\subseteq \mathbb{H}$, the composition $\eta^{-1} \circ \varphi \circ \eta$ appearing in Definition 1.7 is well-formed. Finally, Remark 1.6(3) assures that Definition 1.7 is well posed—that is, $\varphi \in \mathcal{E}^1(P)$ if and only if $\eta^{-1} \circ \varphi \circ \eta \in \mathcal{E}^1(O)$ for every automorphism η of $\mathbb{H}$ fixing ∞.

REMARK 1.8. It is a direct but tedious calculation to see that $\varphi \in \mathcal{E}^1(P)$ if and only if φ has an expression as in (1.1) with $|\langle \gamma, \nu_P\rangle| = o(|\langle Z - P, \nu_P\rangle|)$, where ν_P is the complex normal to $T_P^{\mathbb{C}}\partial\mathbb{H}$ and $|\langle \gamma, \tau\rangle| = o(|\langle Z - P, \nu_P\rangle|^{1/2})$ for any $\tau \in T_P^{\mathbb{C}}\partial\mathbb{H}$.

Let $\varphi \in \mathcal{E}^1_{\mathbb{H}}(O)$ and define an action τ_φ of φ on $T_O^{\mathbb{C}}\partial\mathbb{H}$ by

$$\tau_\varphi\colon T_O^{\mathbb{C}}\partial\mathbb{H} \ni (v', 0) \mapsto (A'_O v', 0) \in T_O^{\mathbb{C}}\partial\mathbb{H}.$$

If η is an automorphism of $\mathbb{H}$ and $\eta(O) = P$, then the action $\tau_{\tilde\varphi}$ of $\tilde\varphi = \eta\circ\varphi\circ\eta^{-1} \in \mathcal{E}^1_{\mathbb{H}}(P)$ on $T_P^{\mathbb{C}}\partial\mathbb{H}$ can be defined by $\tau_{\tilde\varphi} = d\eta_O \circ \tau_\varphi \circ d\eta_P^{-1}$, since (as is well known) $d\eta_O$ and $d\eta_P^{-1}$ are isometries of the complex tangent spaces to $\partial\mathbb{H}$.

DEFINITION 1.9. Let $\varphi \in \mathcal{E}^1_{\mathbb{H}}(P)$. We call the induced action τ_φ of φ on $T_P^{\mathbb{C}}\partial\mathbb{H}$ a *normal* action if τ_φ commutes with the adjoint action τ_φ^*.

Note that the normality of a map's action depends only on the map's class of conjugacy. For a map $\varphi \in \mathcal{E}^1_{\mathbb{H}}(O)$ of the form (1.6), it follows that τ_φ is normal if and only if $A'_O(A'_O)^* = (A'_O)^*A'_O$.

Let us define the boundary complex tangent space $T_\infty^{\mathbb{C}}\partial\mathbb{H}$ of $\partial\mathbb{H}$ at ∞ by

$$T_\infty^{\mathbb{C}}\partial\mathbb{H} := \{v \in \mathbb{C}^{n+1} : v_{n+1} = 0\},$$

and let us give it a hermitian structure by restricting to $T_\infty^{\mathbb{C}}\partial\mathbb{H}$ the classical hermitian structure of $\mathbb{C}^{n+1}$.

If $F \in \mathcal{E}^1_{\mathbb{H}}(\infty)$ then the action τ_F of F on $T_\infty^{\mathbb{C}}\partial\mathbb{H} := \{v \in \mathbb{C}^{n+1} : v_{n+1} = 0\}$ given by

$$\tau_F\colon T_\infty^{\mathbb{C}}\partial\mathbb{H} \ni (v', 0) \mapsto (v'A'_\infty, 0)$$

is well-defined, and τ_F is conjugated by an isometry of $T_\infty^{\mathbb{C}}\partial\mathbb{H}$ to $\tau_{\tilde F}$ if $\tilde F$ is conjugated to F by means of an automorphism fixing ∞. By Proposition 1.5, if F is conjugated to $\varphi \in \mathcal{E}^1_{\mathbb{H}}(O)$, then τ_F is normal if and only if τ_φ is normal.

When the action is normal, one can "diagonalize" F at the first order as follows.

LEMMA 1.10. *Let $F \in \mathcal{E}^1_{\mathbb{H}}(\infty)$. If τ_F is normal then, up to conjugations with automorphisms of $\mathbb{H}$ fixing ∞, we can write*

$$F(z_1,\dots,z_n,w)$$
$$= (\mu_1^\infty z_1 + \Gamma_1(z,w),\dots,\mu_n^\infty z_n + \Gamma_n(z,w), \alpha_\infty w + \Gamma_{n+1}(z,w)), \quad (1.9)$$

where $\mu_j^\infty \in \mathbb{C}$ *are the eigenvalues of* A'_∞ *and* $|\mu_j^\infty| \le \sqrt{\alpha_\infty}$ *for* $j = 1,\dots,n$.

Proof. Since A'_∞ commutes with $(A'_\infty)^*$, up to conjugation with an automorphism of type $(z,w) \mapsto (Uz,w)$ for some $U \in \mathrm{U}(n)$ we may assume that F is given by (1.5) with A'_∞ a diagonal matrix with entries $\mu_1^\infty,\dots,\mu_n^\infty \in \mathbb{C}$. Since $F(\mathbb{H}) \subseteq \mathbb{H}$, for any $(z,w) = (z_1,\dots,z_n,w) \in \mathbb{H}$ with $\|z\|$ sufficiently large we have

$$\alpha_\infty \operatorname{Im} w + |\Gamma_{n+1}| \ge \alpha_\infty \operatorname{Im} w + \operatorname{Im} \Gamma_{n+1} > \|A'_\infty z + \Gamma'\|^2 \ge (\|A'_\infty z\| - \|\Gamma'\|)^2;$$

that is,

$$\alpha_\infty \operatorname{Im} w + |\Gamma_{n+1}| > (\|A'_\infty z\| - \|\Gamma'\|)^2. \quad (1.10)$$

Let $\delta > 0$. Setting $(z,w) = (z, \|z\|^2 + i(1+\delta)\|z\|^2) \in \mathbb{H}$, dividing (1.10) by $\|z\|^2$, letting $z_2 = \cdots = z_n = 0$, and taking the limit for $|z_1| \to \infty$ (taking into account that $\|\Gamma'\| = o(|w|^{1/2})$), we obtain $\alpha_\infty(1+\delta) \ge |\mu_1^\infty|^2$. Since this holds for any $\delta > 0$ we have $\alpha_\infty \ge |\mu_1^\infty|^2$, and similarly for $\mu_2^\infty,\dots,\mu_n^\infty$. □

Lemma 1.10 and Proposition 1.5 imply the following.

Corollary 1.11. *Let* $\varphi \in \mathcal{E}^1_{\mathbb{H}}(O)$ *be given by* (1.6) *and assume that* τ_φ *is normal. Moreover, assume that* φ *corresponds by conjugation to* $F \in \mathcal{E}^1_{\mathbb{H}}(\infty)$ *given by* (1.5). *Let* μ_j^O *be the eigenvalues of* A'_O *and* μ_j^∞ *the eigenvalues of* A'_∞ *for* $j = 1,\dots,n$. *Then, up to a reordering, we have* $\mu_j^O = \mu_j^\infty/\alpha_\infty$ *for* $j = 1,\dots,n$. *In particular,* $|\mu_j^O| \le \sqrt{\alpha_O}$.

In our construction we need some more regularity on the remainder of the expansion of $F \in \mathcal{E}^1(\infty)$.

Definition 1.12. Let $F \in \mathcal{E}^1(\infty)$ be given by

$$F(z,w) = (A'_\infty z + \Gamma'(z,w), \langle z, B_\infty\rangle + \alpha_\infty w + \Gamma_{n+1}(z,w)), \quad (1.11)$$

with $B_\infty \in \mathbb{C}^n$. We say that $F \in \mathcal{E}^{1+\varepsilon}(\infty)$ if $|\Gamma_{n+1}(Z)| \le M|w|^{1-\varepsilon}$ and $\|\Gamma'(Z)\| \le M|w|^{(1-\varepsilon)/2}$ for some $M > 0$, $\varepsilon > 0$, and $\|Z\| \to \infty$. We also set $\mathcal{E}^{1+\varepsilon}_{\mathbb{H}}(\infty) = \mathcal{E}^{1+\varepsilon}(\infty) \cap \operatorname{Hol}(\mathbb{H},\mathbb{H})$.

Remark 1.13. Since $|\langle z, B_\infty\rangle| \le \|B_\infty\|\|z\| \le \|B_\infty\||w|^{1/2}$, the term $\langle z, B_\infty\rangle$ in (1.11) is actually contained in Γ_{n+1} if $\varepsilon \le 1/2$, whereas it is well-defined for $\varepsilon > 1/2$. Therefore, from now on we tacitly set $B_\infty = 0$ whenever $\varepsilon \le 1/2$.

Using (1.7) and (1.8), one can check that $\mathcal{E}^{1+\varepsilon}_{\mathbb{H}}(\infty)$ is closed under conjugation with automorphisms of $\mathbb{H}$ fixing ∞.

Definition 1.14. Let $F \in \mathcal{E}^{1+\varepsilon}(\infty)$ be given by (1.11). We call the matrix

$$A_\infty = \begin{pmatrix} A'_\infty & O \\ B_\infty & \alpha_\infty \end{pmatrix} \quad (1.12)$$

the *linear part of* F *at* ∞. We say that the linear part is *invertible* provided $\alpha_\infty \det(A'_\infty) \ne 0$.

We could define the class $\mathcal{E}^{1+\varepsilon}_{\mathbb{H}}(O)$ by conjugating $\mathcal{E}^{1+\varepsilon}_{\mathbb{H}}(\infty)$ using the automorphism G given by (1.3). A straightforward computation then shows that $\varphi \in \mathcal{E}^{1+\varepsilon}_{\mathbb{H}}(O)$ if and only if

$$\varphi(z,w) = (A'_O z + Cw + \gamma'(z,w), \alpha_O w + \gamma_{n+1}(z,w)), \tag{1.13}$$

where $C \in \mathbb{C}^n$ (we set $C = 0$ if $\varepsilon \le 1/2$) and where $\|\gamma'\| \le M'|w|^{(1+\varepsilon)/2}$ and $|\gamma_{n+1}| \le M'|w|^{1+\varepsilon}$ for some $M' > 0$. Using this last fact, we define the class $\mathcal{E}^{1+\varepsilon}(O)$. Again, for $\varphi \in \mathcal{E}^{1+\varepsilon}(O)$ given by (1.13), we call the matrix

$$A_O = \begin{pmatrix} A'_O & C \\ O & \alpha_O \end{pmatrix} \tag{1.14}$$

the linear part of φ at O. The linear part is invertible provided $\alpha_O \det(A'_O) \neq 0$.

One can similarly define the classes $\mathcal{E}^{1+\varepsilon}(P)$ and $\mathcal{E}^{1+\varepsilon}_{\mathbb{H}}(P)$. One can also define the linear part A_P at $P \in \partial\mathbb{H}$ for a map $\varphi \in \mathcal{E}^{1+\varepsilon}(P)$. We leave the details to the reader.

In the hyperbolic case (i.e., $\alpha_\infty > 1$) one can always dispose of the term B_∞ in (1.11), as our next result shows.

PROPOSITION 1.15. *Let $F \in \mathcal{E}^{1+\varepsilon}_{\mathbb{H}}(\infty)$ be given by* (1.11). *Assume $\alpha_\infty > 1$ and τ_F normal. Then, up to conjugation with automorphisms of $\mathbb{H}$, we can write*

$$\begin{aligned} &F(z_1, \ldots, z_n, w) \\ &\quad = (\mu_1^\infty z_1 + \Gamma_1(z,w), \ldots, \mu_n^\infty z_n + \Gamma_n(z,w), \alpha_\infty w + \Gamma_{n+1}(z,w)), \end{aligned} \tag{1.15}$$

where $\mu_j^\infty \in \mathbb{C}$ are the eigenvalues of A'_∞, $|\mu_j^\infty| \le \sqrt{\alpha_\infty}$ for $j = 1, \ldots, n$, and $\|\Gamma'\| \le M^{1/2}|w|^{(1-\varepsilon)/2}$ and $|\Gamma_{n+1}| \le M|w|^{1-\varepsilon}$ for some $M > 0$.

Proof. If $\varepsilon \le 1/2$ then the result is contained in Lemma 1.10 (and there is no need of the condition on α_∞).

Suppose $\varepsilon > 1/2$. By Lemma 1.10 we can assume A'_∞ to be diagonal with entries μ_j^∞ ($j = 1, \ldots, n$) satisfying the appropriate bounds. In particular, if $\alpha_\infty > 1$ then $\mu_j^\infty \neq \alpha_\infty$ for all j. Suppose now that $B_\infty \neq 0$ and let η be an automorphism of type (1.7) (with $U = \mathrm{id}$ since A'_∞ is already diagonal). A direct calculation shows that

$$\begin{aligned} &\eta^{-1} \circ F \circ \eta(z,w) \\ &\qquad = (A'_\infty z + \Gamma', \alpha_\infty w + \langle z, B_\infty\rangle + 2i\langle (A'_\infty - \alpha_\infty \,\mathrm{id})z, c'\rangle + \Gamma_{n+1}), \end{aligned}$$

where (with some abuse of notation) we have also denoted by Γ the remainder of $\eta \circ F \circ \eta^{-1}$. Now consider the following equation in the unknown c':

$$\langle z, B_\infty\rangle + 2i\langle (A'_\infty - \alpha_\infty \,\mathrm{id})z, c'\rangle \equiv 0 \quad \forall z \in \mathbb{C}^n. \tag{1.16}$$

Since $\det(A'_\infty - \alpha_\infty \,\mathrm{id}) \neq 0$, it follows that (1.16) has a solution $c'_o \in \mathbb{C}^n$. Let $c^o_{n+1} \in \mathbb{C}$ be such that $(c'_o, c^o_{n+1}) \in \partial\mathbb{H}$. If we set $c' = c'_o$ and $c_{n+1} = c^o_{n+1}$ in (1.7), then by (1.16) we see that $\eta \circ F \circ \eta^{-1}$ has the desired form (1.15). $\square$

2. The Schröder Equation at Infinity

In this section we solve the Schröder equation for hyperbolic elements of the class $\mathcal{E}_{\mathbb{H}}^{1+\varepsilon}(\infty)$ whose eigenvalues satisfy some relations. To simplify the notation, in this section we may omit the "∞" subscripts and superscripts; that is, we set $\alpha = \alpha_\infty$ and $\mu_j = \mu_j^\infty$.

Recall that a horosphere of center ∞ and radius $R > 0$ in $\mathbb{H}$ is given by

$$\mathbb{H}_R := \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : \operatorname{Im} w > \|z\|^2 + R\}.$$

By the well-known Wolff lemma (see e.g. [1]), for any $F \in \mathcal{E}_{\mathbb{H}}^1(\infty)$ given by (1.5) and $R > 0$ we have

$$F(\mathbb{H}_R) \subseteq \mathbb{H}_{\alpha R}. \tag{2.1}$$

By (2.1), it follows that if F is hyperbolic (i.e., $\alpha > 1$) then F has no fixed points in $\mathbb{H}$ and ∞ is its Wolff point—in particular, $F^k(Z) \to \infty$ as $k \to \infty$ for any $Z \in \mathbb{H}$.

Proposition 2.1. *Let $F \in \mathcal{E}_{\mathbb{H}}^{1+\varepsilon}(\infty)$ be given by* (1.15). *Assume $\alpha > 1$ and $|\mu_j| > \alpha^{(1-\varepsilon)/2}$ for $j = 1, \dots, n$. Then there exists a* (*nonconstant*) *holomorphic map $\sigma = (\sigma^1, \dots, \sigma^n, \sigma^{n+1}) \in \mathcal{E}^{1+\varepsilon}(\infty)$ such that, for any $(z, w) \in \mathbb{H}$,*

$$\sigma(F(z, w)) = (\mu_1 \sigma^1(z, w), \dots, \mu_n \sigma^n(z, w), \alpha \sigma^{n+1}(z, w)) \tag{2.2}$$

and

$$\sigma(z, w) = (z + \Lambda'(z, w), w + \Lambda_{n+1}(z, w)). \tag{2.3}$$

Moreover, σ is the unique solution of (2.2) *in the class $\mathcal{E}^{1+\varepsilon}(\infty)$ with the asymptotic expansion* (2.3).

Remark 2.2. For $n = 0$ (i.e., F a holomorphic self-map of the unit disc of $\mathbb{C}$) we recover Bourdon–Shapiro's theorem [6, Thm. 4.9, p. 58] under weaker hypotheses. In fact, in Proposition 2.2—and contrary to what happens in Bourdon–Shapiro's theorem—we do not require that the boundary $\partial\mathbb{H}$ be mapped to $\mathbb{H}$ by F nor that F be continuous on $\partial\mathbb{H}$ and univalent in $\mathbb{H}$.

For the rest of this section we assume F as in the hypotheses of Proposition 2.1. Moreover, we write the kth iterate of F as

$$F^k(z, w) = (f_k^1(z, w), \dots, f_k^n(z, w), g_k(z, w))$$

and set $f_k = (f_k^1, \dots, f_k^n)$.

We begin with the following lemma.

Lemma 2.3. *For the k*th *iterate of F, we have*

$$\begin{aligned} f_k^h(z, w) &= \mu_h^k z + \sum_{j=0}^{k-1} \mu_h^{k-1-j} \Gamma_h(F^j) \quad \text{for } h = 1, \dots, n \quad \text{and} \\ g_k(z, w) &= \alpha^k w + \sum_{j=0}^{k-1} \alpha^{k-1-j} \Gamma_{n+1}(F^j). \end{aligned} \tag{2.4}$$

Furthermore,

$$|g_k(z,w)| \leq |w| \prod_{j=0}^{k-1} \left(\alpha + \frac{M}{|g_j(z,w)|^{\varepsilon}} \right). \tag{2.5}$$

Proof. Equation (2.4) follows from (1.15) by a simple induction. We point out that

$$|g(z,w)| \leq |w|(\alpha + M|w|^{-\varepsilon})$$

and then, noting that $g_k = g_{k-1} \circ F$, (2.5) follows by induction. □

Now we need the following estimates on the images of compact subsets of $\mathbb{H}$ under iterates of F.

LEMMA 2.4. *Let $t \in \mathbb{R}$ be such that $1 < t < \alpha$. Then there exists $\delta = \delta(t) > 0$ such that*

$$|g_k(z,w)| \geq \delta t^k \tag{2.6}$$

for any $k \in \mathbb{N}$ and $(z,w) \in E_\delta$, where $E_\delta := \{(z,w) \in \mathbb{H} : |w| > \delta\}$. Moreover, $F(E_\delta) \subset E_\delta$.

Proof. By the triangle inequality we have

$$|g(z,w)| \geq |w|(\alpha - M|w|^{-\varepsilon}).$$

In particular, for $\alpha > t > 1$ there exists a $\delta > 0$ such that, for any $|w| > \delta$,

$$|g(z,w)| > t|w| > t\delta > \delta. \tag{2.7}$$

Thus $F(E_\delta) \subset E_\delta$ and, by iterating such inequality, we obtain

$$|g_k(z,w)| > t^k|w| > t^k\delta \tag{2.8}$$

for any $k \in \mathbb{N}$ and $(z,w) \in E_\delta$. □

LEMMA 2.5. *Let $K \subset\subset \mathbb{H}$ be a compact set. Let $\alpha > t > 1$ and $\delta > 0$ as in Lemma 2.4. Then there exists a $k_0 = k_0(K) \in \mathbb{N}$ such that, for any $(z,w) \in K$ and $k \geq k_0$,*

$$|g_k(z,w)| \geq \delta t^{k-k_0}. \tag{2.9}$$

Proof. If $K \subset E_\delta$ then the statement is proved by setting $k_0 = 0$ and invoking Lemma 2.4. So, assume $K \not\subset E_\delta$. We claim there exists a $k_0 \in \mathbb{N}$ such that

$$F^j(z,w) \in E_\delta \quad \forall (z,w) \in K,\ j \geq k_0. \tag{2.10}$$

Given (2.10), it follows by (2.8) that for any $k \in \mathbb{N}$ we have

$$|g_{k+k_0}(z,w)| = |g_k(F^{k_0}(z,w))| > t^k\delta.$$

Hence we are left to prove (2.10). Since K is compact, there exists an $R > 0$ such that $K \subset \mathbb{H}_R$. By (2.1), $F(\mathbb{H}_R) \subseteq \mathbb{H}_{\alpha R}$ and $F^j(\mathbb{H}_R) \subseteq \mathbb{H}_{\alpha^j R}$. Then, taking $k_0 \in \mathbb{N}$ such that $\alpha^{k_0} R \geq \delta$ we have, for any $(z,w) \in K$,

$$|g_{k_0}(z,w)| \geq \operatorname{Im} g_{k_0}(z,w) > \|f_{k_0}(z,w)\|^2 + \delta \geq \delta$$

and thus (2.10) follows. □

Now we are in a position to prove the existence of σ.

Proof of Proposition 2.1 (*existence*). For $k \in \mathbb{N}$, let $\sigma_k(z,w) = (\sigma_k^1(z,w), \dots, \sigma_k^{n+1}(z,w)) \in \mathbb{C}^{n+1}$ be defined as

$$\sigma_k^h(z,w) = \frac{f_k^h(z,w)}{\mu_h^k} \quad \text{for } h = 1,\dots,n \quad \text{and}$$
$$\sigma_k^{n+1}(z,w) = \frac{g_k(z,w)}{\alpha^k}.$$

A direct computation shows that

$$\sigma_{k-1} \circ F = (\mu_1 \sigma_k^1, \dots, \mu_n \sigma_k^n, \alpha \sigma_k^{n+1})$$

and thus, if $\{\sigma_k\}$ converges to a nonconstant holomorphic map σ, then σ verifies the functional equation (2.2). We prove that $\{\sigma_k\}$ converges uniformly on compacta. Fix K a compact subset of $\mathbb{H}$ and let $(z,w) \in K$. We start looking at σ_k^h, $h = 1,\dots,n$. By (2.4),

$$\sigma_k^h(z,w) = \frac{f_k^h}{\mu_h^k} = z + \sum_{j=1}^{k} \mu_h^{-j} \Gamma_h(F^{j-1}(z,w)).$$

We want to estimate the series on the right-hand side. By (2.5) we have

$$\begin{aligned}
&\sum_{j=1}^{k} |\mu_h|^{-j} |\Gamma_h(F^{j-1}(z,w))| \\
&\quad \le \sum_{j=1}^{k} |\mu_h|^{-j} \|\Gamma'(F^{j-1}(z,w))\| \le \sum_{j=1}^{k} |\mu_h|^{-j} M^{1/2} |g_{j-1}(z,w)|^{(1-\varepsilon)/2} \\
&\quad \le M^{1/2} |w|^{(1-\varepsilon)/2} \sum_{j=1}^{k} |\mu_h|^{-j} \prod_{l=0}^{j-1} \left(\alpha + \frac{M}{|g_l(z,w)|^{\varepsilon}} \right)^{(1-\varepsilon)/2} \\
&\quad = M^{1/2} |w|^{(1-\varepsilon)/2} \sum_{j=1}^{k} \left(\frac{\alpha^{(1-\varepsilon)/2}}{|\mu_h|} \right)^j \prod_{l=0}^{j-1} \left(1 + \frac{M}{\alpha |g_l(z,w)|^{\varepsilon}} \right)^{(1-\varepsilon)/2}.
\end{aligned}$$

Now observe that the product $\prod_{l=0}^{j-1}(1 + M/\alpha|g_l(z,w)|^{\varepsilon})$ converges for $j \to \infty$ if $\sum_{l=0}^{\infty} 1/|g_l(z,w)|^{\varepsilon} < \infty$. Let k_0, t, and δ be as in Lemma 2.5; then

$$\sum_{l=k_0}^{\infty} \frac{1}{|g_l(z,w)|^{\varepsilon}} \le \frac{1}{\delta} \sum_{l=k_0}^{\infty} t^{-(l-k_0)\varepsilon} < \infty, \tag{2.11}$$

and the product is actually converging uniformly on K. Since $\alpha^{(1-\varepsilon)/2} < |\mu_h|$ (by hypothesis), the previous argument implies that there exists a constant $C_h > 0$, depending on K, such that

$$\sum_{j=1}^{\infty} |\mu_h|^{-j} |\Gamma_h(F^{j-1}(z,w))| \le C_h |w|^{(1-\varepsilon)/2}, \tag{2.12}$$

yielding that σ_k^h converges uniformly on compacta for $k \to \infty$, $h = 1, \dots, n$. By Lemma 2.4, the previous argument works for any $(z, w) \in E_\delta$ and thus there exists a $C_h > 0$ such that (2.12) is valid on E_δ. In particular this shows that $\sigma^h = z + \Lambda_h$, with the remainder $\Lambda_h = o(|w|^{1/2})$ for any $\varepsilon > 0$. Since $z = O(|w|^{1/2})$, it follows that σ^h is not constant and has the claimed asymptotic expansion for any $\varepsilon > 0$.

Arguing similarly, we can show that for

$$\sigma_k^{n+1}(z, w) = \frac{g_k}{\alpha^k} = w + \sum_{j=1}^{k} \alpha^{-j} \Gamma_{n+1}(F^{j-1}(z, w))$$

we have

$$\sum_{j=1}^{k} \alpha^{-j} |\Gamma_{n+1}(F^{j-1}(z, w))| \le M|w|^{1-\varepsilon} \sum_{j=1}^{k} \alpha^{-\varepsilon j} \prod_{l=0}^{j-1} \left(1 + \frac{M}{\alpha |g_l(z, w)|^\varepsilon}\right)^{1-\varepsilon},$$

and hence there exists a $C > 0$ such that

$$\sum_{j=1}^{\infty} \alpha^{-j} |\Gamma_{n+1}(F^{j-1}(z, w))| \le C|w|^{1-\varepsilon}. \tag{2.13}$$

In particular, σ^{n+1} is nonconstant and has the required expansion. □

Our next aim is to show the uniqueness of the solution of equation (2.2). Since the solutions of (2.2) form a $\mathbb{C}$-vector space, the uniqueness claimed in Proposition 2.1 will follow at once from the following lemma.

LEMMA 2.6. *Let $R \ge 0$, and let $h = (h_1, \dots, h_{n+1}) \colon \mathbb{H}_R \to \mathbb{C}^{n+1}$ be a holomorphic solution of* (2.2).

(1) *If $|h_{n+1}(z, w)| = o(|w|)$ for $(z, w) \to \infty$, then $h_{n+1} \equiv 0$.*
(2) *If there exists an $M > 0$ such that $|h_j(z, w)| \le M|w|^{(1-\varepsilon)/2}$ for $(z, w) \to \infty$ and for some $j = 1, \dots, n$, then $h_j \equiv 0$.*

Proof. Assume $|h_{n+1}(z, w)| = o(|w|)$. We want to show that $h_{n+1} \equiv 0$. If not, then by (2.2) and (2.1) we have that, for any $(z, w) \in \mathbb{H}_R$,

$$h_{n+1}(F^m(z, w)) = \alpha^m h_{n+1}(z, w) \tag{2.14}$$

for all $m \in \mathbb{N}$. Dividing both sides of (2.14) by $g_m(z, w)$ and taking the limit as $m \to \infty$, we have

$$\lim_{m \to \infty} \frac{|h_{n+1}(F^m(z, w))|}{|g_m(z, w)|} = \lim_{m \to \infty} \frac{\alpha^m |h_{n+1}(z, w)|}{|g_m(z, w)|}. \tag{2.15}$$

Since $|g_m(z, w)| \to \infty$ by (2.1), the left-hand side of (2.15) is such that

$$\lim_{m \to \infty} \frac{|h_{n+1}(F^m(z, w))|}{|g_m(z, w)|} = \lim_{m \to \infty} \frac{o(|g_m(z, w)|)}{|g_m(z, w)|} = 0.$$

As for the right-hand side of (2.15), by (2.4) we have

$$\lim_{m\to\infty}\frac{\alpha^m|h_{n+1}(z,w)|}{|g_m(z,w)|}=|h_{n+1}(z,w)|\lim_{m\to\infty}\frac{1}{\left|w+\sum_{j=0}^{m-1}\alpha^{-(1+j)}\Gamma_{n+1}(F^j(z,w))\right|}$$
$$\geq\frac{|h_{n+1}(z,w)|}{|w|+\sum_{j=0}^{\infty}\alpha^{-(1+j)}|\Gamma_{n+1}(F^j(z,w))|}$$
$$\geq\frac{|h_{n+1}(z,w)|}{|w|+C|w|^{1-\varepsilon}}>0,$$

where in the last line we used (2.13). Thus $h_{n+1}\equiv 0$.

Now assume $|h_j(z,w)|\leq M|w|^{(1-\varepsilon)/2}$ for some $j=1,\dots,n$. Let $\beta>(1-\varepsilon)/2$ be such that $|\mu_j|>\alpha^\beta$, and let $(z,w)\in\mathbb{H}_R$. From (2.2) we have

$$\lim_{m\to\infty}\frac{|h_j(F^m(z,w))|}{|g_m(z,w)|^\beta}=\lim_{m\to\infty}\frac{|\mu_j|^m|h_j(z,w)|}{|g_m(z,w)|^\beta}. \tag{2.16}$$

The left-hand side of (2.16) is zero for $|h_j(F^m(z,w))|=o(|g_m(z,w)|^\beta)$. As for the right-hand side, if $h_j\not\equiv 0$ then by (2.5) we have

$$\lim_{m\to\infty}\frac{|\mu_j|^m|h_j(z,w)|}{|g_m(z,w)|^\beta}\geq\lim_{m\to\infty}\frac{|h_j(z,w)|}{|w|^\beta|\mu_j|^{-m}\prod_{h=0}^{m-1}\left(\alpha+\frac{M}{|g_h(z,w)|^\varepsilon}\right)^\beta}$$
$$=\lim_{m\to\infty}\frac{|h_j(z,w)|}{|w|^\beta\left(\frac{\alpha^\beta}{|\mu_j|}\right)^m\prod_{h=0}^{m-1}\left(1+\frac{M}{\alpha|g_h(z,w)|^\varepsilon}\right)^\beta}=+\infty$$

because the infinite product is converging, as follows from (2.11). Therefore $h_j\equiv 0$, and this completes the proof. □

Now we collect some properties of the solution of (2.2) given by Proposition 2.1.

PROPOSITION 2.7. *Let σ be the solution of* (2.2) *given by Proposition 2.1. Then the following statements hold.*

(1) $\operatorname{Im}\sigma^{n+1}(z,w)>0$ *for any* $(z,w)\in\mathbb{H}$.
(2) *If* $|\mu_j|=\sqrt{\alpha}$ *for* $j=1,\dots,n$, *then* $\sigma(\mathbb{H})\subseteq\mathbb{H}$. *In particular, in this case* $\sigma\in\mathcal{E}_{\mathbb{H}}^{1+\varepsilon}(\infty)$.
(3) *If* $\mu_j=\mu_k$ *then* $\sigma^j=\sigma^k\circ U_{j,k}$, *where* $U_{j,k}$ *is a unitary matrix that swaps* z_j *for* z_k *and keeps fixed the other variables.*

Proof. For any $k\in\mathbb{N}$,

$$\operatorname{Im}\sigma_k^{n+1}=\frac{\operatorname{Im}g_k(z,w)}{\alpha^k}>\frac{\|f_k(z,w)\|^2}{\sqrt{\alpha^{2k}}}>0$$

and thus $\operatorname{Im}\sigma^{n+1}>0$. Moreover, if $|\mu_1|=\cdots=|\mu_n|=\sqrt{\alpha}$ then we have $\|f_k(z,w)\|^2/\sqrt{\alpha^{2k}}=\|(\sigma_k^1,\dots,\sigma_k^n)\|^2$ and $\sigma_k(\mathbb{H})\subseteq\mathbb{H}$, from which part (2) follows. Part (3) follows easily from the uniqueness of σ. □

Looking at the proof of Proposition 2.1, without any assumption on the μ_j one can build a one-dimensional model along the normal direction. Note that, in the proof of Lemma 2.6, the uniqueness along the normal direction is obtained in a slightly larger class. Thus we have our next result.

PROPOSITION 2.8. *Let $F \in \mathcal{E}_{\mathbb{H}}^{1+\varepsilon}(\infty)$ be given by* (1.15) *and assume that $\alpha > 1$. Then there exists a $\theta\colon \mathbb{H} \to \{\zeta \in \mathbb{C} : \operatorname{Im} \zeta > 0\}$ holomorphic such that*

$$\theta(F(z,w)) = \alpha\theta(z,w) \tag{2.17}$$

and $\theta(z,w) = w + \Lambda(z,w)$ with $|\Lambda(z,w)| \le M|w|^{1-\varepsilon}$ for some $M > 0$. Moreover, the map θ is the unique solution of (2.17) *such that $\theta(z,w) - w = o(|w|)$.*

We end this section by finding all the (regular) solutions to the Schröder equation at ∞.

PROPOSITION 2.9. *Let $F \in \mathcal{E}_{\mathbb{H}}^{1+\varepsilon}(\infty)$ be given by* (1.11). *Suppose τ_F is normal and $\alpha_\infty > 1$. Let $\mu_1^\infty, \dots, \mu_n^\infty$ denote the eigenvalues of A'_∞ and suppose that $|\mu_j^\infty| > \alpha_\infty^{(1-\varepsilon)/2}$ for every j. Then there exists a $\sigma_0 \in \mathcal{E}^{1+\varepsilon}(\infty)$ with invertible linear part at ∞ that solves the Schröder equation*

$$\sigma \circ F = A_\infty \sigma.$$

If $\tilde\sigma \in \mathcal{E}^{1+\varepsilon}(\infty)$ with invertible linear part at ∞ is a solution of the same Schröder equation, then there exists an $(n+1) \times (n+1)$ invertible matrix E such that $[A_\infty, E] = 0$ and $\tilde\sigma = E\sigma_0$.

Proof. By Proposition 1.15, there exists an automorphism η of $\mathbb{H}$ fixing ∞ such that $\tilde F = \eta \circ F \circ \eta^{-1}$ is given by (1.15). Then Proposition 2.1 gives a solution σ_D of the diagonal Schröder equation $\sigma_D \circ \tilde F = D\sigma_D$, where D is a diagonal matrix with entries $\mu_1, \dots, \mu_n, \alpha_\infty$. Note that, even if (in general) $\sigma_D(\mathbb{H}) \not\subset \mathbb{H}$, the map $\sigma_0 = \eta^{-1} \circ \sigma_D \circ \eta$ is well-defined (recall the possible forms (1.7) and (1.8) for η). Since $\eta^{-1} D\eta = A_\infty$, it is easy to see that $\sigma_0 \in \mathcal{E}^{1+\varepsilon}(\infty)$ is one solution of the Schröder equation. Using the conjugation with η, we see that any solution of the Schröder equation corresponds to a solution of (2.2). From this and Lemma 2.6, the assertion follows. $\square$

3. The Schröder Equation at a Finite Boundary Point

THEOREM 3.1. *Let $\varphi \in \mathcal{E}_{\mathbb{H}}^{1+\varepsilon}(P)$ for some $P \in \partial\mathbb{H}$. Let $\alpha_P > 0$ be the boundary dilatation coefficient of φ at P and let A_P be the linear part of φ at P. Suppose that $\alpha_P < 1$, the action $\tau_\varphi\colon T_P^{\mathbb{C}}\partial\mathbb{H} \to T_P^{\mathbb{C}}\partial\mathbb{H}$ is normal, and its eigenvalues are all of modulus strictly greater than $\alpha_P^{(1+\varepsilon)/2}$. Then there exists a $\sigma \in \mathcal{E}^{1+\varepsilon}(P)$ with invertible linear part at P such that*

$$\sigma \circ \varphi = A_P\sigma. \tag{3.1}$$

Moreover, $\tilde{\sigma} \in \mathcal{E}^{1+\varepsilon}(P)$ *with invertible linear part is another solution of* (3.1) *if and only if there exists an* $(n+1) \times (n+1)$ *invertible matrix* E *such that* $[A_P, E] = 0$ *and* $\tilde{\sigma} = E\sigma$.

Proof. Let G be an automorphism of $\mathbb{H}$ that maps P to ∞ and let $F = G \circ \varphi \circ G^{-1}$. Then $F \in \mathcal{E}_{\mathbb{H}}^{1+\varepsilon}(\infty)$ and one can easily check that it satisfies all the hypotheses of Proposition 2.9; thus we have a solution σ_0 of the Schröder equation for F at ∞. Since $\sigma = G^{-1} \circ \sigma_0 \circ G$ is well-defined by Proposition 2.7(1), it turns out that σ has all the required properties. It is obvious that if $[E, A_P] = 0$ then $E\sigma \in \mathcal{E}^{1+\varepsilon}(P)$ is still a solution of (3.1). However, one cannot use directly Proposition 2.9 for concluding that all the (regular) solutions of (3.1) are of that form $E\sigma$. The point is that if $\tilde{\sigma} \in \mathcal{E}^{1+\varepsilon}(P)$ is a solution of (3.1) then it is possible that $\tilde{\sigma}(\mathbb{H})$ contains the indeterminacy locus of any automorphism of $\mathbb{H}$ that maps P to ∞. However, using linear automorphisms as in (1.7) and (1.8), one can assume $P = O$. Write $\tilde{\sigma}_{n+1}(z, w) = tw + \kappa_{n+1}$ with $t \in \mathbb{C}$ and $|\kappa_{n+1}| = o(|w|)$ for $(z, w) \to O$. Note that $\tilde{\sigma}$ has invertible linear part at O and so $t \neq 0$; also, by the triangle inequality, there exists a $\delta > 0$ such that for $|w| < \delta$ we have

$$|\tilde{\sigma}_{n+1}(z, w)| > r|w|$$

for some $r > 0$. Thus, for $(z, w) \in \mathbb{H}$ with $|w| < \delta$, it follows that $\tilde{\sigma}_{n+1} \neq 0$. Hence there exists an $R > 0$ such that $G \circ \tilde{\sigma} \circ G^{-1}$ is a well-defined solution of the Schröder equation for F at ∞ on $\mathbb{H}_R$ with the appropriate bound on the remainder. Then an application of Lemma 2.6 shows that $\tilde{\sigma} = E\sigma$ for some E commuting with A_O, as desired. □

REMARK 3.2. There exists a $\sigma \in \mathcal{E}^{1+\varepsilon}(P)$, unique in the class $\mathcal{E}^{1+\varepsilon}(P)$, that is a solution of (3.1) with linear part at P equal to the identity. Indeed, if $\sigma_0 \in \mathcal{E}^{1+\varepsilon}(P)$ is any solution of (3.1) with invertible linear part E then clearly $[E, A_p] = 0$, and thus $\sigma = E^{-1}\sigma_0$ is the Schröder map we seek.

4. Resonances

Let $\varphi \in \mathcal{E}_{\mathbb{H}}^{1+\varepsilon}(O)$ be given by (1.13) and assume that φ satisfies the hypotheses of Theorem 3.1. Let $\mu_1, \ldots, \mu_n$ be the eigenvalues of A'_O (we omit the superscript ^{O}). Recall that, according to Poincaré–Dulac (see e.g. [2]), the eigenvalue μ_j for $j = 1, \ldots, n$ is resonant if there exist $k_1, \ldots, k_n, s \in \mathbb{N}$ with $k_1 + \cdots + k_n + s \geq 2$ such that

$$\mu_j = \mu_1^{k_1} \cdots \mu_n^{k_n} \alpha_O^s. \tag{4.1}$$

However, since $\alpha_O^{(1+\varepsilon)/2} < |\mu_j| \leq \alpha_O^{1/2}$ and $\alpha_O < 1$ by hypothesis, it follows that μ_j cannot be resonant for $j = 1, \ldots, n$. Indeed, if (4.1) were true then we would have

$$|\mu_j| = |\mu_1|^{k_1} \cdots |\mu_n|^{k_n} \alpha_O^s \leq \alpha_O^{(k_1+\cdots+k_n)/2+s} \leq \alpha_O^{(1+\varepsilon)/2}.$$

Therefore, under our hypothesis on the μ_j, the eigenvalues μ_j are not resonant.

Now we examine the eigenvalue α_O. Writing the resonant condition $\alpha_O = \mu_1^{k_1} \cdots \mu_n^{k_n} \alpha_O^s$ for $k_1 + \cdots + k_n + s \geq 2$ and arguing as before, it follows that α_O is resonant if and only if $s = 0$ and there exist $j_1, j_2 \in \{1, \ldots, n\}$ (possibly $j_1 = j_2$) such that $|\mu_{j_1} \mu_{j_2}| = \alpha_O$. Hence the resonances of α_O are all of degree 2. Let σ be one of the solutions of (3.1) given by Theorem 3.1. For $c_{jk} \in \mathbb{C}$, let us consider a map of the form

$$g(z, w) = \left(z, w + \sum_{j,k=1}^{n} c_{jk} z_j z_k \right),$$

where $c_{j,k} = 0$ if $|\mu_j \mu_k| < \alpha_O$. Clearly, $g \circ \sigma$ is still a solution of (3.1); but $g \circ \sigma \notin \mathcal{E}^1(O)$. This means that, contrary to what happens to the case of an inner fixed point (see [14]), there exists (up to multiplication with matrices) a unique special solution of the Schröder equation.

5. Applications to Commuting Mappings

In this section we apply the theory developed so far to the study of commuting holomorphic mappings without fixed points in the ball. We will find results analogous to those stated in [17] for the case of maps with inner fixed points.

By [7], two commuting holomorphic maps of the ball have generically the same Wolff point. That is to say, two commuting holomorphic self-maps of the ball have the same Wolff point unless they fix (as a set) the complex slice joining the two different Wolff points and are hyperbolic automorphisms of such a slice.

THEOREM 5.1. *Let $f, g \in \mathcal{E}_{\mathbb{H}}^{1+\varepsilon}(P)$ for some $P \in \partial\mathbb{H}$. Assume that $f \circ g = g \circ f$ and that P is the Wolff point of both f and g. Moreover, suppose that the boundary coefficient $\alpha_P(f)$ of f at P is strictly smaller than* 1, *the action $\tau_f \colon T_P^{\mathbb{C}} \partial\mathbb{H} \to T_P^{\mathbb{C}} \partial\mathbb{H}$ is normal, and its eigenvalues are all of modulus strictly greater than $\alpha_P(f)^{(1+\varepsilon)/2}$. If g has invertible linear part at P, then f, g share a common Schröder map with invertible linear part at P.*

Proof. Let A_f and A_g denote (respectively) the linear part of f and g at P. Let $\sigma \in \mathcal{E}^{1+\varepsilon}(P)$ be the solution of the Schröder equation $\sigma \circ f = A_f \sigma$ given by Remark 3.2. Then

$$(\sigma \circ g) \circ f = \sigma \circ f \circ g = A_f (\sigma \circ g).$$

Hence $\sigma \circ g \in \mathcal{E}^{1+\varepsilon}(P)$ is another solution of the Schröder equation of f. Moreover, the linear part of $\sigma \circ g$ at P is A_g, which is invertible. Thus, by the uniqueness statement of Theorem 3.1, it follows that $\sigma \circ g = A_g \sigma$ as desired. □

The converse of Theorem 5.1 holds in the following form.

THEOREM 5.2. *Let $f, g \in \mathcal{E}_{\mathbb{H}}^1(P)$, and let A_f and A_g be the linear part of f and g, respectively. Assume that $[A_f, A_g] = 0$. If there exists a $\sigma \in \mathcal{D}^1(P)$ with invertible linear part at P such that both $\sigma \circ f = A_f \sigma$ and $\sigma \circ g = A_g \sigma$ hold, then $f \circ g = g \circ f$.*

Proof. Since

$$\sigma \circ (f \circ g) = A_f(\sigma \circ g) = A_f A_g \sigma = A_g A_f \sigma = A_g(\sigma \circ f) = \sigma \circ (g \circ f),$$

we have $\sigma \circ (f \circ g)(Z) = \sigma \circ (g \circ f)(Z)$ for any $Z \in \mathbb{H}$. By Lemma 1.2, the map σ is eventually injective in any nontangential region at P. Since $f, g \in \mathcal{E}^1_{\mathbb{H}}(P)$, it follows that $f \circ g$ has finite boundary dilatation coefficient at P and (nontangentially) fixes P (see [1; 8]). Therefore, by the Julia-type lemma for the ball (see [1; 21]) we can take an open set $W \subset \mathbb{H}$ such that $(f \circ g)(W)$ is contained in one region of injectivity of σ, and thus $f \circ g = g \circ f$. □

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