Rotation of Wedges of Extendability for Tubelike CR Manifolds of CR Dimension 1

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1. Introduction

The concept of wedge extendability, as described for example in the work of Tumanov [T] and Baouendi and Rothschild [BaR], has been a major focus of research on CR extension. (See e.g. [BaER] for a general treatment of these and related topics.) We briefly describe this concept. Consider a smooth, generic CR submanifold $M$ of $\mathbb{C}^n$ with codimension $d$ and CR dimension $n - d$. This means that if $p \in M$, there exists an open neighborhood $U$ of $p$ in $\mathbb{C}^n$ and a smooth function $\rho = (\rho_1, \ldots, \rho_d): U \to \mathbb{R}^d$ such that

$$M = \{ z \in U \mid \rho(z) = 0 \}$$

and $\partial \rho_1 \wedge \cdots \wedge \partial \rho_d \neq 0$. We then define a wedge as follows.

**Definition 1.1.** Let $\Gamma$ be an open, convex cone in $\mathbb{R}^d$ with vertex at the origin, and let $\mathcal{V} \subset \mathbb{C}^n$ be an open neighborhood of the point $p$. Then the wedge $\mathcal{W}(\Gamma, \mathcal{V}, p)$ with edge $M$ centered at $p$ is

$$\{ z \in \mathcal{V} \mid \rho(z) \in \Gamma \}. \quad (1.1)$$

A number of additional hypotheses on $M$ will guarantee the existence of a common wedge to which all CR functions on a fixed open set on the manifold extend holomorphically. The weakest such hypothesis is minimality. The submanifold $M$ is said to be *minimal* at $p$ if there is no submanifold $S \subset M$ through $p$ with strictly smaller real dimension but the same CR dimension as $M$. To be precise, the following theorem holds.

**Theorem 1.2 [T; BaR].** If $M$ is minimal at $p$ then, for every open neighborhood $U$ of $p$ in $M$, there exists an open neighborhood $\mathcal{V}$ of $p$ in $\mathbb{C}^n$ with $M \cap \mathcal{V} \subset U$ and a wedge $\mathcal{W} = \mathcal{W}(\Gamma, \mathcal{V}, p)$ such that every continuous CR function on $U$ extends holomorphically to $\mathcal{W}$. Conversely, if $M$ is not minimal at $p$ then there exists a continuous CR function defined in some neighborhood of $p$ in $M$ that does not extend holomorphically to any wedge with edge $M$ centered at $p$.

Sufficiency was established by Tumanov, and necessity was established by Baouendi and Rothschild.

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Wedges do not have some of the characteristics one might reasonably expect of the regions described by a general theory of CR extension. To begin with, if \( U \) and \( V \) are open sets in \( M \) and if \( V \subseteq U \), then the region of extendability for CR functions on \( V \) should be contained in the region of extendability for CR functions on \( U \). Boggess, Glenn, and Nagel [BoGN] show that wedges do not have this property. They give an example of a manifold for which, given any fixed wedge of extendability for CR functions on an open neighborhood \( U \) of the origin, one can find a sufficiently small open neighborhood \( V \) of the origin such that every wedge of extendability for \( V \) is disjoint from the fixed wedge for \( U \). In this example, there is no fixed direction contained in the region of extendability associated with every neighborhood of the origin. We refer to this phenomenon as \textit{wedge rotation}.

The goal of this paper is to characterize those points on a generic tubelike CR submanifold of \( \mathbb{C}^n \) of CR dimension 1 at which wedge rotation occurs. After making precise what it means to say that the region of extendability for any neighborhood of a point contains a fixed direction, we present an alternative approach to the example considered by Boggess, Glenn, and Nagel that gives some insight into why wedges fail to have the desired containment property. We then give a necessary and sufficient condition for the region of extendability for a tubular neighborhood of a point of finite type to contain a fixed direction in terms of the Hörmander numbers at that point. This theorem thus leads to a necessary condition for an \textit{arbitrary} neighborhood of a point of finite type to contain a fixed direction, and it shows that the phenomenon of wedge rotation is common. This provides a strong argument for the development of an alternative to wedges in the theory of CR extension.

2. Definitions and Statement of the Theorem

Let \( M \) be a \((2n - d)\)-dimensional smooth \((C^\infty)\) real submanifold of \( \mathbb{C}^n = \mathbb{R}^{2n} \). Let \( p \in M \); denote by \( T_p(M) \) the real tangent space to \( M \) at \( p \) and by \( H_p(M) \) the maximal complex subspace of \( T_p(M) \). Then

\[
H_p(M) = T_p(M) \cap J(T_p(M)),
\]

where \( J \) is the usual complex structure map on \( \mathbb{R}^{2n} \). We say that \( M \) is a CR \textit{submanifold} of \( \mathbb{C}^n \) of \textit{CR dimension} \( k \) if \( \dim_{\mathbb{R}} H_p(M) = 2k \) for all \( p \in M \). Then

\[
2n - 2d \leq 2k \leq 2n - d.
\]

We say that \( M \) is \textit{generic} if \( 2k = 2n - 2d \), that is, if \( k \) is minimal. In this paper we consider the special case in which \( M \) is a tubelike, generic CR submanifold of \( \mathbb{C}^n \) of CR dimension 1. For such a manifold, one can find local coordinates in which \( p \) is the origin and \( M \) is given near the origin by

\[
\{(z = x + iy, w_1, \ldots, w_{n-1}) \in \mathbb{C}^n \mid \Re(w_j) = \phi_j(x), 1 \leq j \leq n - 1\}. \quad (2.1)
\]

Since each \( \phi_j \) is \( C^\infty \) it follows that

\[
\phi_j(x) = \sum_{\ell=2}^{m} a_{j,\ell}x^\ell + E^j_m(x), \quad (2.2)
\]
where $E^j_m(x) = x^{m+1}c_j(x)$ with each $c_j$ a $C^\infty$-function in a neighborhood of the origin.

Denote the complexifications of $T_0(M)$ and $H_0(M)$ by $T^C_0(M)$ and $H^C_0(M)$, respectively. The complexified holomorphic tangent bundle $H^C(M)$ is spanned near the origin by the vector fields

$$
\bar{L} = \frac{\partial}{\partial \bar{z}} + \sum_{j=1}^{n-1} \phi'_j(x) \frac{\partial}{\partial \bar{w}_j}, \\
L = \frac{\partial}{\partial z} + \sum_{j=1}^{n-1} \phi'_j(x) \frac{\partial}{\partial w_j}.
$$

Let $w_j = u_j + iv_j$. We may write $L = \frac{1}{2}(X - iY)$ and $\bar{L} = \frac{1}{2}(X + iY)$, where

$$
X = \frac{\partial}{\partial x} + \sum_{j=1}^{n-1} \phi'_j(x) \frac{\partial}{\partial u_j}, \\
Y = \frac{\partial}{\partial y} + \sum_{j=1}^{n-1} \phi'_j(x) \frac{\partial}{\partial v_j}.
$$

Then $H^C_0(M)$ is spanned by $\{X_0 = \frac{\partial}{\partial x}, Y_0 = \frac{\partial}{\partial y}\}$. We consider commutators of $X$ and $Y$ at the origin as follows:

$$
[X, Y]_0 = \sum_{j=1}^{n-1} \phi''_j(0) \frac{\partial}{\partial v_j} = 2 \sum_{j=1}^{n-1} a_{j,2} \frac{\partial}{\partial v_j}.
$$

More generally, define

$$
[X, Y]^{(p,0)}_0 = [X, [X, \ldots, [X, Y]]], \quad p \geq 0,
$$

to be the $(p + 1)$th-order commutator involving $[X, Y]$ and $p$ additional copies of the vector field $X$. It is clear that

$$
[X, Y]^{(p,0)}_0 = \sum_{j=1}^{n-1} \phi^{(p+2)}_j(0) \frac{\partial}{\partial v_j} = (p + 2)! \sum_{j=1}^{n-1} a_{j,p+2} \frac{\partial}{\partial v_j}, \quad (2.3)
$$

Thus we may identify $[X, Y]^{(p,0)}_0$ with the vector $(a_{1,p+2}, \ldots, a_{n-1,p+2}) \in \mathbb{R}^{n-1}$. Observe that, for any $p \geq 0$, $[X, Y]^{(p,0)}_0 = 0$ and so the only nonvanishing commutators of $X$ and $Y$ at the origin are the $[X, Y]^{(p,0)}_0$. 

Let $\mathcal{L}_0^0(M)$ be the subspace of $T_0^0(M)$ spanned by $X_0$ and $Y_0$, and for $p \geq 2$ let $\mathcal{L}_0^p(M)$ be the subspace of $T_0^p(M)$ spanned by $X_0$, $Y_0$, and the $\{X, Y\}^q_{0,0}$ for $0 \leq q \leq p - 2$. We define the Hörmander numbers $(m_1, \ldots, m_{n-1})$ at the origin by

$$m_r = \inf\{ p \mid \dim_C \mathcal{L}_0^p(M) \geq r + 2 \}.$$ 

We say that $M$ is of finite type $m$ at the origin if $m_{n-1} = m$. This is equivalent to saying that $\mathcal{L}_0^m(M) = T_0^0(M)$ but $\mathcal{L}_0^{m-1}(M)$ is a proper subspace of $T_0^0(M)$. It is clear that $M$ is of type $m$ if $\{(a_1, \ell, \ldots, a_{n-1}, \ell) \mid 2 \leq \ell \leq m\}$ spans $\mathbb{R}^{n-1}$.

Let $M^\epsilon = \{(z, w_1, \ldots, w_{n-1}) \in M \mid |\Re(z)| < \epsilon\}$ and

$$\gamma_M^\epsilon = \{(x, \phi_1(x), \ldots, \phi_{n-1}(x)) \mid |x| < \epsilon\}.$$ 

Denote the convex hull of $\gamma_M^\epsilon$ by $\text{ch}(\gamma_M^\epsilon)$. At the origin, the tangent space to $\gamma_M^\epsilon$ is spanned by $(1, 0, \ldots, 0)$. We call the orthogonal complement in $\mathbb{R}^n$ to this tangent space the space normal to the curve, denoting the slice of $\text{Int}(\text{ch}(\gamma_M^\epsilon))$ in the normal space at the origin by $N_0(\gamma_M^\epsilon)$.

We then have the following special case of the theorem of Boivin and Dwilewicz.

**Theorem 2.1** [BoiD]. Every continuous CR function on $M^\epsilon = \gamma_M^\epsilon + i\mathbb{R}^n$ can be continuously extended to a function on $\text{Int}(\text{ch}(\gamma_M^\epsilon)) + i\mathbb{R}^n$ that is holomorphic on $\text{Int}(\text{ch}(\gamma_M^\epsilon)) + i\mathbb{R}^n$.

We conclude this section with a definition and the statement of our main theorem.

**Definition 2.2.** We say that $N_0(\gamma_M^\epsilon)$ contains a fixed direction for every $\epsilon > 0$ if there is a vector $v$ in $\{(x, u_1, \ldots, u_{n-1}) \mid x = 0\}$ such that, for every $\epsilon > 0$, there exists a $c = c(\epsilon) > 0$ such that $cv \in N_0(\gamma_M^\epsilon)$.

**Theorem 2.3.** For every $\epsilon > 0$, $N_0(\gamma_M^\epsilon)$ contains a fixed direction if and only if $M$ has at most one even Hörmander number. More specifically:

(i) if all of the Hörmander numbers are odd, then $\text{Int}(\text{ch}(\gamma_M^\epsilon))$ contains a full neighborhood of the origin;

(ii) if there is exactly one even Hörmander number, $m_K$, then there exists a $c = c(\epsilon) > 0$ such that $-cJ([X, Y]_{0,0}^{\ell,m-2,0}) \in N_0(\gamma_M^\epsilon)$; and

(iii) if there are two or more even Hörmander numbers at the origin, then $N_0(\gamma_M^\epsilon)$ does not contain a fixed direction for every $\epsilon > 0$.

The theorem of Boivin and Dwilewicz (our Theorem 2.1) describes exactly the regions of extendability for tubular neighborhoods of a point. In general, it is much harder to describe precisely the region of extendability for an arbitrary neighborhood of a point. However, an arbitrary open neighborhood of a point is contained in a tubular neighborhood, so if the regions of extendability for tubular neighborhoods of a point fail to contain a fixed direction as the neighborhoods shrink then the same will be true of arbitrary neighborhoods. Hence this theorem immediately implies that, if $p$ is a point of finite type on $M$ and if $\{U^\gamma\}$ is any family of open
3. An Example

The following example illustrates what can happen to wedges as the open set $U$ shrinks about the origin.

**Example.** Let

$$M = \{(z = x + iy, w_2 = u_2 + iv_2, w_4 = u_4 + iv_4) \in \mathbb{C}^3 \mid u_2 = x^2, u_4 = x^4\}.$$  

The Hörmander numbers at the origin are $(2, 4)$. Let

$$\gamma'_M = \{(x, x^2, x^4) \mid |x| < \varepsilon\}$$

and

$$\gamma'_4 = \{(x, x^2, x^3, x^4) \mid |x| < \varepsilon\}.$$  

If $\Pi$ is the projection map

$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_4)$$

then, since $\Pi$ is linear,

$$\text{ch}(\gamma'_4) = \Pi(\text{ch}(\gamma'_M)).$$

The classical theory of power moment sequences gives a precise description of $\text{ch}(\gamma'_4)$ (see for example [KNu, Chap. III, Thm. 2.3]). A point $(u_1, u_2, u_3, u_4)$ is in $\text{Int}(\text{ch}(\gamma'_4))$ if and only if the quadratic forms

$$\sum_{j,k=0}^2 u_{j+k} \xi_j \xi_k \quad \text{and} \quad \sum_{j,k=0}^1 (\varepsilon^2 u_{j+k} - u_{j+k+2}) \xi_j \xi_k, \quad u_0 = 1, \quad (3.1)$$

are positive. Thus we see that $(0, u_2, u_3, u_4) \in N_0(\gamma'_4)$ if and only if the matrices

$$\begin{bmatrix} 1 & 0 & u_2 \\ 0 & u_2 & u_3 \\ u_2 & u_3 & u_4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \varepsilon^2 - u_2 & -u_3 \\ -u_3 & \varepsilon^2 u_2 - u_4 \end{bmatrix}$$

are positive definite. By the signature rule of Jacobi, this is equivalent to the condition that the successive principal minors are positive. That is, $(0, u_2, u_3, u_4) \in N_0(\gamma'_4)$ if and only if the $u_j$ satisfy

$$u_2 > 0,$$

$$u_2(u_4 - u_2^2) - u_3^2 > 0,$$

$$\varepsilon^2 - u_2 > 0,$$

$$(\varepsilon^2 - u_2)(\varepsilon^2 u_2 - u_4) - u_3^2 > 0.$$
Since $\Pi$ maps the space normal to $\gamma_4^c$ at the origin onto the space normal to $\gamma_M^c$ at the origin, we need only consider the image of this set under $\Pi$. This image certainly contains
\[ \{(0, u_2, u_4) \mid 0 < u_2 < \epsilon^2, u_2^2 < u_4 < \epsilon^2 u_2 \}, \] (3.2)
and if $c \neq 0$ then the projection under $\Pi$ of the level set where $u_3 = c$ is contained in the set in (3.2). Therefore, this set is equal to $N_0(\gamma_M^c)$.

Suppose now that $W$ is a fixed wedge to which CR functions on $M^\epsilon$ extend. (See Definition 1.1 for the precise definition of a wedge.) The existence of such a wedge follows from Tumanov’s theorem (our Theorem 1.2) because $M$ is of finite type—and hence minimal—at every point. As a result, there exist $\delta_1$ and $\delta_2$ positive such that if $(x + iy, u_2 + iv_2, u_4 + iv_4) \in W$ then
\[ \delta_1 u_2 < u_4 < \delta_2 u_2, \]
where $\delta_2 \leq \epsilon^2$. Consider now any wedge of extendability for CR functions on $M^{\epsilon_0}$, where $\epsilon_0 = \sqrt{\delta_1/2}$. There exist $\eta_1$ and $\eta_2$ positive such that a point $(x + iy, u_2 + iv_2, u_4 + iv_4)$ in this wedge satisfies
\[ \eta_1 u_2 < u_4 < \eta_2 u_2 \leq \epsilon_0^2 u_2 = \frac{\delta_1}{2} u_2. \]
Therefore, any wedge of extendability for $M^{\epsilon_0}$ is disjoint from $W$. Observe that in this example there is no fixed direction contained in $N_0(\gamma_M^c)$ for every $\epsilon > 0$.

4. Proof of Theorem 2.3

We first make a change of coordinates. Let $\tilde{A}$ be the nonsingular linear transformation that reduces the matrix $(a_{j,\ell})$ for $1 \leq j \leq n - 1$ and $2 \leq \ell \leq m$ to its corresponding row-echelon form. Define a biholomorphic change of variables on $\mathbb{C}^n$ by setting
\[ z = z \quad \text{and} \quad \zeta = \tilde{A}w, \]
where $\zeta = (\xi_1, \ldots, \xi_{n-1}) \in \mathbb{C}^{n-1}$ and $w = (w_1, \ldots, w_{n-1})$. As before, let $z = x + iy$ and set $\zeta_j = t_j + is_j, 1 \leq j \leq n - 1$. In terms of these new coordinates our manifold becomes
\[ \tilde{M} = \{(z, \zeta) \in \mathbb{C}^n \mid t_j = \psi_j(x) = x^{m_j} + \mathcal{E}_j(x), 1 \leq j \leq n - 1\}, \] (4.1)
where $\mathcal{E}_j(x) = x^{m_j+1}c_j(x)$ for a function $c_j$ that is bounded in a neighborhood of the origin. Observe that, if $\{e_0, \ldots, e_{n-1}\}$ denotes the standard basis for $\mathbb{R}^n$, then the transformation $\tilde{A}$ maps $-J([X, Y]_0^{m_j-2, 0})$ to a vector in the $e_j$ direction. Also, in the new coordinates, the lowest-degree term in the $j$th defining function for $\tilde{M}$ is $x^{m_j}$, where $m_j$ is the $j$th Hörmander number at the origin.

To prove the theorem, we establish three lemmas. Lemma 4.3 implies that the slice of the convex hull of $\gamma$ in the normal space at the origin is contained in the convex hull of the projection of $\gamma$ onto this same space. Although we do not have the reverse containment, Lemma 4.2 can be interpreted roughly as stating that
the slice of the convex hull in the normal space to the curve includes (the projections of) all directions realized in the convex hull itself. We begin with several definitions.

**Definition 4.1.** The hyperplane

\[ H = \left\{ (\xi_0, \xi_1, \ldots, \xi_{n-1}) \left| \sum_{j=0}^{n-1} \alpha_j \xi_j = a \right. \right\} \]

cuts the set \( E \) if there exist \( y \) and \( z \) in \( E \) such that \( \sum_{j=0}^{n-1} \alpha_j y_j < a \) and \( \sum_{j=0}^{n-1} \alpha_j z_j > a \). We say that \( H \) supports \( E \) if, for all \( x \in E \), either \( \sum_{j=0}^{n-1} \alpha_j x_j \geq a \) or \( \sum_{j=0}^{n-1} \alpha_j x_j \leq a \).

**Lemma 4.2.** If the plane

\[ \left\{ (\xi_0, \xi_1, \ldots, \xi_{n-1}) \left| \sum_{j=1}^{n-1} \alpha_j \xi_j = 0 \right. \right\} \]

cuts \( \gamma_{M} \) at \( (x_0, \psi_1(x_0), \ldots, \psi_{n-1}(x_0)) \), then it also cuts \( N_0(y_{M}^{\varepsilon}) \) for \( \varepsilon > |x_0| \).

**Proof.** We show first that if \( (x_0, \psi_1(x_0), \ldots, \psi_{n-1}(x_0)) \in \gamma_{M} \) then, given \( \eta > 0 \), there exists a \( \lambda > 0 \) such that

\[ \{ (0, t_1, \ldots, t_{n-1}) \ | \ |t_j - \lambda \psi_j(x_0)| < \lambda \eta \} \]

contains a point of \( N_0(y_{M}^{\varepsilon}) \). Geometrically, this means that every neighborhood of the projection onto the space normal to the curve of a point on the curve contains a dilate of some point in \( N_0(y_{M}^{\varepsilon}) \).

To prove the claim, let \( (x_0, \psi_1(x_0), \ldots, \psi_{n-1}(x_0)) \) and \( (y, \psi_1(y), \ldots, \psi_{n-1}(y)) \) be two points on \( \gamma_{M} \), and let \( 0 < \lambda < 1 \). A convex linear combination of the two points on the curve is in \( N_0(y_{M}^{\varepsilon}) \) if \( \lambda x_0 + (1 - \lambda) y = 0 \) or (equivalently)

\[ y = -\frac{\lambda}{1 - \lambda} x_0, \]

and if \( \lambda < 1/2 \) then the point corresponding to \( y \) is in \( y_{M}^{\varepsilon} \). The \( j \)th coordinate of the corresponding point in \( N_0(y_{M}^{\varepsilon}) \) is

\[ t_j = \lambda [x_0^{m_j} + \varepsilon_j(x_0)] + (1 - \lambda) \left[ \left( \frac{\lambda}{1 - \lambda} x_0 \right)^{m_j} + \varepsilon_j \left( \frac{\lambda}{1 - \lambda} x_0 \right) \right]. \]

Then

\[ |t_j - \lambda \psi_j(x_0)| = (1 - \lambda) \left| \left( \frac{\lambda}{1 - \lambda} x_0 \right)^{m_j} + \varepsilon_j \left( \frac{\lambda}{1 - \lambda} x_0 \right) \right| \]

\[ \leq \lambda \left[ |x_0|^{m_j} \left( \frac{\lambda}{1 - \lambda} \right)^{m_j - 1} + C|x_0|^{m_j + 1} \left( \frac{\lambda}{1 - \lambda} \right)^{m_j} \right]. \quad (4.2) \]

Since \( \lambda < 1/2 \) and \( m_j > 2 \), it follows from (4.2) that

\[ |t_j - \lambda \psi_j(x_0)| \leq \lambda \left[ \lambda 2^m |x_0|^{m_j} + C|x_0|^{m_j + 1} \right]. \quad (4.3) \]
The quantity in braces can be made arbitrarily small by taking \( \lambda \) sufficiently small. This establishes the claim.

Now, if the plane cuts \( y_M^j \) at \((x_0, \psi_1(x_0), \ldots, \psi_{n-1}(x_0))\), then given \( \delta > 0 \) there exist \( x_1 \) and \( x_2 \) in \((x_0 - \delta, x_0 + \delta)\) such that \( \sum_{j=1}^{n-1} \alpha_j \psi_j(x_1) < 0 \) and \( \sum_{j=1}^{n-1} \alpha_j \psi_j(x_2) > 0 \). Then, by (4.3), given \( \eta > 0 \) there exist a \( \lambda > 0 \) and a point \((0, t_1, \ldots, t_{n-1}) \in N_0(y_M^{x_1})\) with
\[
|t_j - \lambda \psi_j(x_1)| < \lambda \eta,
\]
and thus
\[
|\alpha_j t_j - \lambda \alpha_j \psi_j(x_1)| < \lambda \eta |\alpha_j|.
\]
Hence
\[
\alpha_j t_j < \lambda \alpha_j \psi_j(x_1) + \lambda \eta |\alpha_j|.
\]
Summing over all \( j \), we obtain
\[
\sum_{j=1}^{n-1} \alpha_j t_j < \lambda \left( \sum_{j=1}^{n-1} \alpha_j \psi_j(x_1) + \eta \sum_{j=1}^{n-1} |\alpha_j| \right).
\]
Since \( \sum_{j=1}^{n-1} \alpha_j \psi_j(x_1) < 0 \), by taking \( \eta \) sufficiently small we can find a point in \( N_0(y_M^{x_1}) \) for which \( \sum \alpha_j t_j < 0 \). A similar argument shows that we can find \((0, t'_1, \ldots, t'_{n-1}) \in N_0(y_M^{x_2})\) for which \( \sum \alpha_j t'_j > 0 \). The plane \( \sum_{j=1}^{n-1} \alpha_j \xi_j = 0 \) thus cuts \( N_0(y_M^\varepsilon) \) for any \( \varepsilon \geq \max\{|x_1|, |x_2|\} \). Because \( x_1 \) and \( x_2 \) can be taken to be as close to \( x_0 \) as we like, this proves the lemma.

Let \( \Pi: \mathbb{R}^n \to S_p \) be the orthogonal projection of \( \mathbb{R}^n \) with standard basis \( \{e_0, \ldots, e_{n-1}\} \) onto the subspace of \( \mathbb{R}^n \) spanned by \( \{e_{\ell_1}, \ldots, e_{\ell_p}\} \), where \( p \leq n \) and \( \ell_1 < \ell_2 < \cdots < \ell_p \). Then
\[
[\Pi(\xi_0, \ldots, \xi_{n-1})] = \begin{cases} \xi_j & \text{if } j = \ell_k, \\ 0 & \text{otherwise}. \end{cases}
\]

**Lemma 4.3.** Let \( \gamma \) be a curve in \( \mathbb{R}^n \). Then
\[
\text{ch}(\gamma) \cap S_p \subseteq \text{ch}(\Pi(\gamma)).
\]

**Proof.** Let \( H = \{\xi \in \mathbb{R}^p \mid \sum_{k=1}^p \alpha_k \xi_k = a\} \) be a hyperplane in \( S_p \) supporting \( \Pi(\gamma) \). Define \( \tilde{H} \) to be the corresponding hyperplane in \( \mathbb{R}^n \) given by \( \sum_{j=0}^{n-1} \tilde{\alpha}_j \xi_j = a \), where \( \tilde{\alpha}_j = \alpha_k \) if \( j = \ell_k \) and \( \tilde{\alpha}_j = 0 \) otherwise. Then \( \tilde{H} \) supports \( \gamma \), hence \( \text{ch}(\gamma) \cap S_p \), and therefore \( \text{ch}(\gamma) \cap S_p \) lies on the same side of \( H \) as \( \Pi(\gamma) \). That is: \( H \) supports \( \text{ch}(\gamma) \cap S_p \); since \( \Pi(\gamma) \) is the intersection of all such supporting half-spaces, the lemma is established.

**Lemma 4.4.** If \( \{m_{\ell_1}, \ldots, m_{\ell_p}\} \) are odd Hörmander numbers for \( \tilde{M} \) at the origin, then any hyperplane of the form
\[
\alpha_0 \xi_0 + \sum_{k=1}^p \alpha_k \xi_k = 0
\]
cuts \( y_M^\varepsilon \) for any \( \varepsilon > 0 \).
Proof. Let $x \in (-\varepsilon, \varepsilon)$. Then

$$\alpha_0 x + \sum_{k=1}^{p} \alpha_{jk} \psi_{jk} (x) = \alpha_0 x + \sum_{k=1}^{p} \alpha_{jk} x^{m_{jk}} [1 + c_{jk} (x) x]$$

$$= x \left\{ \alpha_0 + \sum_{k=1}^{p} \alpha_{jk} x^{m_{jk} - 1} [1 + c_{jk} (x) x] \right\}, \quad (4.4)$$

and if $\alpha_0 \neq 0$ then the expression on the right changes sign at the origin. If $\alpha_0 = 0$, suppose $K$ is the first integer for which $\alpha_{jK} \neq 0$. Then the right-hand side of (4.4) reduces to

$$\sum_{k=K}^{p} \alpha_{jk} x^{m_{jk}} [1 + c_{jk} (x) x] = x^{m_{K}} \sum_{k=K}^{p} \alpha_{jk} x^{m_{jk} - m_{K}} [1 + c_{jk} (x) x].$$

Since $m_{jk}$ is odd and $\alpha_{jk} \neq 0$, this expression changes sign at the origin. Therefore, the hyperplane cuts the curve $\gamma_{\hat{M}}^\varepsilon$ for any $\varepsilon > 0$.

With these lemmas established, we now complete the proof of Theorem 2.3.

Proof. Statement (i) follows immediately from Lemma 4.4, for if all of the Hörmander numbers are odd then the lemma states that, for any $\varepsilon > 0$, any plane through the origin cuts $\gamma_{\hat{M}}^\varepsilon$ and hence the origin is an interior point of the convex hull of $\gamma_{\hat{M}}^\varepsilon$.

We now establish (ii). Toward this end, suppose that $\hat{M}$ has exactly one even Hörmander number $m_K$ at the origin. By Lemma 4.4, any plane of the form

$$\sum_{j=1}^{n-1} \alpha_j \xi_j = 0$$

with $\alpha_K = 0$ cuts $\gamma_{\hat{M}}^\varepsilon$ for every $\varepsilon > 0$. Thus, by Lemma 4.2, any such hyperplane cuts the convex set $N_0(\gamma_{\hat{M}}^\varepsilon)$. Since $m_K$ is even, there exists an $\varepsilon_1 > 0$ such that, for all $x \in (-\varepsilon_1, \varepsilon_1)$ with $x \neq 0$, we have $\psi(x) > 0$. Let $\mathcal{R}^{\varepsilon_1} = N_0(\gamma_{\hat{M}}^\varepsilon) \cap \{ \xi \in \mathbb{R}^n \mid \xi_K > 0 \}$. This set is convex because it is the intersection of two convex sets, and the only point possibly in $N_0(\gamma_{\hat{M}}^\varepsilon)$ that is not in $\mathcal{R}^{\varepsilon_1}$ is the origin. Consider $\Pi(\mathcal{R}^{\varepsilon_1})$, the projection of $\mathcal{R}^{\varepsilon_1}$ onto the plane on which $\xi_K = 0$. Every hyperplane in this linear subspace passing through the origin cuts $\Pi(\mathcal{R}^{\varepsilon_1})$ and so the origin is an interior point of this set. But the origin is the projection of some point on the $\xi_K$ axis, so there is some point on this axis in $\mathcal{R}^{\varepsilon_1}$ and hence in $N_0(\gamma_{\hat{M}}^\varepsilon)$, as desired.

Finally, we prove (iii). Suppose there are two even Hörmander numbers, $m_K$ and $m_J$, with $K < J$. Then there exists an $\varepsilon_1 > 0$ such that, if $x \in (-\varepsilon, \varepsilon_1) \cup \{ \xi \in \mathbb{R}^n \mid \xi_K, \xi_J > 0 \}$. Consider the projection of the curve $\gamma_{\hat{M}}^{\varepsilon_1}$ onto the subspace $\mathcal{S}_p$ spanned by $\{e_K, \ldots, e_{n-1}\}$. Then consider the angle made with $e_K$ by the vector from the origin to the projected point. This angle satisfies
cos θ = \frac{\psi_K(x)}{\sqrt{[\psi_K(x)]^2 + \cdots + [\psi_{n-1}(x)]^2}} = \frac{x^mK(1 + xcK(x))}{x^mK \sqrt{1 + xd(x)}}.

Let η(ε₁) be the maximum angle that the projection of a point of γ_M^(ε₁) onto S_p makes with e_0. Since cos θ → 1 as x → 0, it follows that η(ε₁) → 0 as ε₁ → 0. Let E^η(ε₁) be the set of all points in S_p that make an angle of less than η(ε₁) with the ξ_K-axis. This set is convex, and since it contains \Pi(γ_M^(ε₁)) it also contains ch[\Pi(γ_M^(ε₁))]. Hence, by Lemma 4.3,

\text{ch}(γ_M^(ε₁)) \cap S_p \subseteq \text{ch}[\Pi(γ_M^(ε₁))] \subseteq E^η(ε₁).

The only points contained in E^η(ε) for every ε > 0 are those on the ξ_K-axis. Thus, if there is a scalar multiple of a fixed vector (0, a₁, ..., a_n−1) in the normal space for all ε, then the vector must satisfy a_{K+1} = \cdots = a_{n-1} = 0.

On the other hand, since m_J is even there exist 0 < ε_2 < ε_1 such that, if x ∈ (−ε_2, ε_2) and x ≠ 0, then \psi_J(x) > 0. Hence, the normal space contains no point in the plane ξ_J = 0 other than the origin and so contains no vector in that direction. Since we have already shown that any vector in N_0(γ_M^(ε)) for all ε, must satisfy a_j = 0 for j ≥ K + 1 and since J > K, it follows that a_J = 0. This shows that no such vector exists, completing our proof of the theorem.

References


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