Michigan Math. J. 53 (2005)

Linearity of Sets of Strange Functions

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1. Introduction

In analysis, sometimes very strange phenomena appear. For instance, one should mention continuous nowhere differentiable functions, everywhere divergent Fourier series of functions in $L^1(\mathbb{T})$, or universal Taylor series. By experience, it is known that as soon as such a pathological example is exhibited, it is most often generic in the sense of Baire's categories. Namely, in a well-chosen topological space, all elements of a dense G_δ set share this pathological behavior.

More recently, the algebraic structure of these sets has been investigated (see e.g. [Ro] or [AGM]). Let us recall the following definition (introduced in [GuQ]).

Definition 1. A set M in a linear topological space X is said to be *spaceable* if $M \cup \{0\}$ contains a closed infinite-dimensional subspace of X.

In this paper, we give several examples of sets of functions with irregular behavior that are spaceable. Our main tool is the use of basic sequences, a technique initiated in this context by Bernal-Gonzalez and Montes-Rodriguez [BeMo; Mo] in the particular case of hypercyclic vectors. We recall some basic definitions and results, which are taken from [Di]. A sequence $(x_n)_{n\geq 1}$ of a Banach space X is called a *basic sequence* if, for each x belonging to $X_0 = \overline{\text{span}}(x_n : n \geq 1)$, there exists a unique sequence of scalars (α_n) such that $x = \sum_{n=1}^{+\infty} \alpha_n x_n$. The coefficient functionals are defined by $x_k^*\bigl(\sum_{n=1}^{+\infty} \alpha_n x_n\bigr) = \alpha_k$. They are continuous on X_0 and can be extended to X by the Hahn–Banach theorem. Two basic sequences (x_n) and (y_n) are *equivalent* if the convergence of $\sum \alpha_n x_n$ is equivalent to the convergence of $\sum \alpha_n y_n$. We will intensively use the following result (see [Di, Thm. 9]).

Lemma 1. *Let (x_n) be a basic sequence in X, and let (y_n) be a sequence in X satisfying $\sum \|x_n^*\| \|x_n - y_n\| < 1$. Then (y_n) is a basic sequence equivalent to (x_n).*

This lemma explains our strategy for building large subspaces of functions with strange behavior. First, we exhibit in the space a basic sequence of functions with a very regular behavior. Next, we slightly disturb these functions, so that the new functions behave very irregularly, yet with the new sequence remaining a basic sequence. Finally, we show that a good choice of the perturbations ensures that the irregular behavior transfers to the subspace generated by the basic sequence. This

Received February 12, 2004. Revision received October 8, 2004.

strategy has been proved to be efficient in many different settings (we refer to the following sections for precise statements and necessary definitions):

- in geometry of Banach spaces, to build subspaces of Gâteaux differentiable functions whose derivatives are far away from each other;
- in complex analysis, and more precisely in the theory of universal Taylor series;
- in harmonic analysis, we prove that the set of functions of $L^1(\mathbb{T})$ with everywhere divergent Fourier series is spaceable;
- in operator theory, the set of supercyclic operators is spaceable in $\mathcal{L}(H)$ for H a Hilbert space.

2. Gâteaux-Smooth Functions

If X is a real Banach space and f is a real-valued function defined on X and everywhere Frechet differentiable, Maly's theorem (see [Ma]) asserts that the range of f' is connected. On the other hand, Deville and Hayek [DH] have demonstrated the existence of a continuous function f from ℓ^1 to $\mathbb{R}^2$ that is everywhere Gâteaux differentiable and whose derivatives are far away from each other: $\|f'(x) - f'(y)\|_{\mathcal{L}(l^1,\mathbb{R}^2)} \geq 1$ for every $x \neq y \in \ell^1$. Moreover, they prove that it is possible to choose such a function from ℓ^p to c_0. We improve their result and build, for any Banach space X, a closed subspace of $C_b(X, c_0)$ (the set of bounded continuous functions from X to c_0 equipped with the supremum norm) of such functions.

THEOREM 1. *Given any real separable Banach space X, the set of Lipschitz continuous functions from X to c_0 that are Gâteaux differentiable at each point of X, and such that the range of the Gâteaux derivative consists of isolated points, is spaceable in $C_b(X, c_0)$.*

We begin with several lemmas; the first two can be found in [DH]. Observe that the norm on $\mathbb{R}^2$ is the norm induced by c_0.

LEMMA 2. *Given $\Delta = (a', a, b, b') \in \mathbb{R}^4$ with $a' < a < b < b'$ and $\eta > 0$, there exists a $\mathcal{C}^\infty$-function $\varphi = \varphi_{\Delta,\eta} : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

(i) $\|\varphi(\alpha, \beta)\| \leq \eta$ *for all* $(\alpha, \beta) \in \mathbb{R}^2$,
(ii) $\varphi(\alpha, \beta) = 0$ *whenever* $\alpha \notin [a', b']$,
(iii) $\left\|\frac{\partial\varphi}{\partial\alpha}(\alpha, \beta)\right\| \leq \eta$ *for all* $(\alpha, \beta) \in \mathbb{R}^2$,
(iv) $\left\|\frac{\partial\varphi}{\partial\beta}(\alpha, \beta)\right\| = 1$ *whenever* $\alpha \in [a, b]$, *and*
(v) $\left\|\frac{\partial\varphi}{\partial\beta}(\alpha, \beta)\right\| \leq 1$ *for all* $(\alpha, \beta) \in \mathbb{R}^2$.

In particular, this $\mathcal{C}^\infty$-function is $(1 + \eta)$-Lipschitz.

LEMMA 3. *Let X and Y be separable real Banach spaces, and for all n let $f_n : X \to Y$ be Gâteaux differentiable mappings. Assume that $\left(\sum f_n\right)$ converges pointwise on X and that, for all h,*

$$\left(\sum \frac{\partial f_n}{\partial h}(x)\right) \textit{ converges uniformly with respect to } x.$$

Then the mapping $f = \sum_{n\geq 1} f_n$ *is Gâteaux differentiable on* X *and one has* $f' = \sum_{n\geq 1} f_n'$, *where the convergence holds in* $\mathcal{L}(X,Y)$ *for the strong operator topology.*

The third lemma deals with biorthogonal systems: this is a useful tool for replacing bases in the Banach space setting.

LEMMA 4 [OP]. *If* X *is an infinite-dimensional separable Banach space, then there is a sequence* $(u_p)_{p\geq 1}$ *of* X *and a sequence* $(\delta_q)_{q\geq 1}$ *of* X^* *such that*:

(i) $\delta_q(u_p) = 1$ *if* $p = q$, 0 *otherwise*;
(ii) $\overline{\text{span}}(u_p : p \geq 1) = X$;
(iii) $\delta_q(x) = 0$ *for any* q *implies that* $x = 0$;
(iv) $\|u_p\| = 1$ *for any* p *and* $\sup_q \|\delta_q\| = C < +\infty$.

Observe in particular that (δ_q) is weak-star convergent to 0.

We are now able to prove Theorem 1. We denote by (e_n) the canonical basis of c_0 and by (u_p, δ_q) a biorthogonal system of $X \times X^*$.

Step 1: A basic sequence of good functions in $C_b(X, c_0)$.

We define G_j on X by $G_j(x) = e_j$. Of course, G_j is everywhere differentiable, with $G_j' = 0$. Moreover, observe that for any $\alpha_1, \ldots, \alpha_n$ we have

$$\|\alpha_1 G_1 + \cdots + \alpha_n G_n\|_\infty = \sup_{1\leq j\leq n} |\alpha_j|.$$

Therefore, $(G_j)_{j\geq 1}$ is a basic sequence in $C_b(X, c_0)$ that is equivalent to the canonical basis of c_0.

Step 2: A perturbation of these functions.

Let us fix $\Delta_k = (a_k', a_k, b_k, b_k')$ an enumeration of all quadruples of rational numbers, with $a_k' < a_k < b_k < b_k'$. By $m_{j,k,n}$ we denote, for $j, k, n \geq 1$, the integer $m_{j,k,n} = 2^k p_{j+1}^n$; here p_{j+1} is the $(j+1)$th prime number. Observe that

$$m_{j,k,n} \geq \max(j,k,n) \quad \text{and}$$
$$m_{j,k,n} = m_{j',k',n'} \iff (j,k,n) = (j',k',n').$$

Let $(\varepsilon_j)_{j\geq 1}$ be a sequence of positive numbers with $\sum \varepsilon_j < 1/2$ and let $(\eta_{j,k,n})_{j,k,n\geq 1}$ be another sequence of positive numbers with $\sum_{k,n\geq 1} \eta_{j,k,n} < \varepsilon_j$. For j, k, n greater than unity, $T_{j,k,n}$ is the linear map from $\mathbb{R}^2$ to c_0 defined by

$$T_{j,k,n}(\alpha, \beta) = \alpha e_{2m_{j,k,n}} + \beta e_{2m_{j,k,n}+1}.$$

These functions have disjointly supported ranges in c_0. Finally we set, for $x \in X$,

$$\begin{aligned} F_j(x) &= G_j(x) + \sum_{k,n\geq 1} T_{j,k,n} \circ \varphi_{\Delta_k, \eta_{j,k,n}}(\delta_n(x), \delta_{m_{j,k,n}}(x)) \\ &:= G_j(x) + \sum_{k,n\geq 1} f_{j,k,n}(x), \end{aligned}$$

where the function $\varphi_{j,k,n} = \varphi_{\Delta_k,\eta_{j,k,n}}$ is given by Lemma 2. According to condition (i) of Lemma 2, $\|f_{j,k,n}\| \leq \eta_{j,k,n}$ and so F is well-defined. In addition, one has $\|F_j - G_j\| \leq \varepsilon_j$, and by Lemma 1 it follows that (F_j) is a basic sequence in $C_b(X, c_0)$ that is equivalent to the canonical basis of c_0. Let us denote by E the closed linear span in $C_b(X, c_0)$ generated by the sequence (F_j). Observe that each F_j is Lipschitz continuous because each $f_{j,k,n}$ is Lipschitz with constant $C(1+\eta_{j,k,n}) \leq 2C$, where C is the constant that appears in Lemma 4, and because these functions have disjointly supported ranges in c_0. Therefore,

$$\begin{aligned}\|F_j(x) - F_j(y)\| &\leq \sup_{k,n} \|f_{j,k,n}(x) - f_{j,k,n}(y)\| \\ &\leq 2C\|x-y\|.\end{aligned}$$

Step 3: Any function F in $E \setminus \{0\}$ is Lipschitz continuous and Gâteaux differentiable, and the application $x \mapsto F'(x)$ has discrete range in $\mathcal{L}(X, c_0)$.

Let $F = \sum_{j\geq 1} \alpha_j F_j$ be such a function, with $\alpha_j \to 0$. Let us first prove that F is Gâteaux differentiable on X. Fix $x, h \in X$. Then

$$\begin{aligned}\frac{\partial f_{j,k,n}}{\partial h}(x) &= \delta_n(h) T_{j,k,n}\left(\frac{\partial \varphi_{j,k,n}}{\partial \alpha}(\delta_n(x), \delta_{m_{j,k,n}}(x))\right) \\ &\quad + \delta_{m_{j,k,n}}(h) T_{j,k,n}\left(\frac{\partial \varphi_{j,k,n}}{\partial \beta}(\delta_n(x), \delta_{m_{j,k,n}}(x))\right) \\ &:= \delta_n(h) w_{j,k,n}(x) + \delta_{m_{j,k,n}}(h) v_{j,k,n}(x), \end{aligned} \tag{1}$$

where $\|w_{j,k,n}(x)\| \leq \eta_{j,k,n}$ (see Lemma 2(iii)) and $\|v_{j,k,n}(x)\| \leq 1$ and where $w_{j,k,n}(x)$ and $v_{j,k,n}(x)$ are supported by $\operatorname{span}(e_{2m_{j,k,n}}, e_{2m_{j,k,n}+1})$. Clearly, the series $\sum_{j,k,n} \alpha_j \delta_n(h) w_{j,k,n}(x)$ is normally convergent on X. Let us prove that the series $\sum_{j,k,n} \alpha_j \delta_{m_{j,k,n}}(h) v_{j,k,n}(x)$ is uniformly convergent on X because it satisfies the uniform Cauchy condition. Indeed, if A is any finite subset of $\mathbb{N}^3$ then

$$\left\| \sum_{j,k,n\in A} \alpha_j \delta_{m_{j,k,n}}(h) v_{j,k,n}(x) \right\| \leq \|\alpha\|_\infty \max_{j,k,n\in A} |\delta_{m_{j,k,n}}(h)|.$$

Since (δ_q) is weak-star convergent to 0, this becomes very small as soon as A does not intersect a well-chosen finite subset of $\mathbb{N}^3$. Therefore, the uniform convergence is proved.

Next, Lemma 3 implies that F is Gâteaux differentiable on X. Moreover, since the ranges of the functions $f_{j,k,n}$ are disjointly supported, it is easy to check that F is Lipschitz continuous (with constant $\leq 2C\|\alpha\|_\infty$). We achieve the proof by showing that, if $x \neq y \in X$, then

$$\|F'(x) - F'(y)\| \geq \|\alpha\|_\infty \left(1 - 2\sum_l \varepsilon_l\right).$$

By Lemma 4, there exists an integer n with $\delta_n(x) \neq \delta_n(y)$. Let k be such that

$$\delta_n(x) \in [a_k, b_k] \quad \text{and} \quad \delta_n(y) \notin [a'_k, b'_k].$$

Take j with $|\alpha_j| = \|\alpha\|_\infty$. It is plain that

$$\|F'(x) - F'(y)\| \geq \left\| \frac{\partial F}{\partial u_{m_{j,k,n}}}(x) - \frac{\partial F}{\partial u_{m_{j,k,n}}}(y) \right\|.$$

Now, the following equalities and inequalities hold.

- For any $l \geq 1$,

$$\frac{\partial G_l}{u_{m_{j,k,n}}}(x) = \frac{\partial G_l}{u_{m_{j,k,n}}}(y) = 0, \quad \left\| \frac{\partial f_{j,k,n}}{\partial u_{m_{j,k,n}}}(x) \right\| = 1, \quad \left\| \frac{\partial f_{j,k,n}}{\partial u_{m_{j,k,n}}}(y) \right\| = 0;$$

  this follows from Lemma 2(ii) and (iv), Lemma 4(i), and (1).
- For $(l, q, r) \neq (j, k, n)$,

$$\left\| \frac{\partial f_{l,q,r}}{\partial u_{m_{j,k,n}}}(x) \right\| \leq \eta_{l,q,r} \quad \text{and} \quad \left\| \frac{\partial f_{l,q,r}}{\partial u_{m_{j,k,n}}}(y) \right\| \leq \eta_{l,q,r}.$$

  Indeed, in this situation $\delta_{m_{l,q,r}}(u_{m_{j,k,n}}) = 0$, as follows from Lemma 2(iii).

Combining these results yields

$$\begin{aligned} \|F'(x) - F'(y)\| &\geq |\alpha_j| - 2\|\alpha\|_\infty \sum_{l,q,r} \eta_{l,q,r} \\ &\geq \|\alpha\|_\infty \left(1 - 2\sum_l \varepsilon_l\right). \end{aligned}$$

REMARKS. (a) Since the ranges of the derivatives of the functions $f_{j,k,n}$ are disjointly supported in c_0, it follows for any x in X that

$$\left\| \sum_{j=1}^{n} \alpha_j F_j'(x) \right\| = \max_{1 \leq j \leq n} |\alpha_j| \|f_{j,k,n}'(x)\|.$$

Now, by (1),

$$\|f_{j,k,n}'(x)\| = \sup_{\|h\|=1} \left\| \frac{\partial f_{j,k,n}}{\partial h}(x) \right\| \leq 2C.$$

On the other hand, we also have

$$\|f_{j,k,n}'(x)\| \geq \left\| \frac{\partial f_{j,k,n}}{\partial u_{m_{j,k,n}}}(x) \right\| \geq 1.$$

Therefore, E is also a closed subspace of bounded functions from X to c_0, with bounded Gâteaux derivative and equipped with the norm $\|F\| + \|F'\|$.

(b) A careful look at the proof shows that the following result is actually true: Given any $0 < k < 1$, there exists a closed infinite-dimensional subspace of $C_b(X, c_0)$ such that, for any nonzero function F in this subspace, (i) F is Lipschitz continuous and Gâteaux-differentiable and (ii) if $L(F)$ is the Lipschitz constant of F then, for any $x \neq y$ in X,

$$\|F'(x) - F'(y)\| \geq k \frac{L(F)}{C},$$

where C is a constant that depends only on the geometry of the Banach space.

3. Taylor Series

Let $H(\mathbb{D})$ be the space of holomorphic functions in the unit disk $\mathbb{D}$, endowed with the topology of uniform convergence on compacta. For K a subset of $\mathbb{C}$ and for F a continuous function on K, we use $\|F\|_{C(K)}$ to denote the sup norm of F on K.

Nestoridis proved in 1996 (see [N]) the existence of a function $f = \sum_{n\geq 0} a_n z^n$ in $H(\mathbb{D})$ that is universal in the following sense: Given any compact set $K \subset \mathbb{C}\setminus\mathbb{D}$ with connected complement and any function g continuous on K and holomorphic inside K, there exists a subsequence of the Taylor series of f that converges uniformly to g on K. Actually, this is true for all functions (except 0) in a closed subspace of $H(\mathbb{D})$.

THEOREM 2. *There exists a closed infinite-dimensional subspace F of $H(\mathbb{D})$ such that, for any $f = \sum_{n=0}^{+\infty} a_n z^n \in F \setminus \{0\}$, for every compact set K in $\mathbb{C} \setminus \mathbb{D}$ with connected complement, and for any function g continuous on K and holomorphic in the interior of K, there exists an increasing sequence (n_k) of positive integers such that*

$$S_{n_k}(f)(z) = \sum_{n=0}^{n_k} a_n z^n \to g(z) \text{ uniformly on } K.$$

Proof. We begin with a refinement of Mergelyan's theorem that essentially says we can choose a polynomial with arbitrarily high valuation.

LEMMA 5. *Let K be a compact set in $\mathbb{C} \setminus \mathbb{D}$ with connected complement, L a compact subset of $\mathbb{D}$, and g a function continuous on K and analytic inside K. For any $\varepsilon \geq 0$ and $N \geq 1$, there exists a polynomial $P(z) = \sum_{n=N}^{q} a_n z^n$ such that*:

$$\|P\|_{C(L)} < \varepsilon;$$
$$\|P - g\|_{C(K)} < \varepsilon.$$

Proof. Set $R > 1$ and $0 < r < 1$ such that $K \subset B(0, R)$ and $L \subset B(0, r)$. By Mergelyan's theorem, there exists a polynomial $Q(z) = \sum_0^q b_n z^n$ such that:

$$\|Q\|_{C(\bar{B}(0,r))} < \frac{\varepsilon}{2N} \times \frac{r^N}{R^N};$$
$$\|Q - g\|_{C(K)} < \frac{\varepsilon}{2}.$$

By Cauchy's estimates, for $k \leq N$ one has $|b_k| \leq \varepsilon/2NR^k$. We set $P(z) = \sum_{n=N}^{q} b_n z^n$. Observe that, for $z \in B(0, R)$,

$$|P(z) - Q(z)| \leq \sum_{0}^{N-1} |b_k| R^k$$
$$\leq \frac{\varepsilon}{2}.$$

Therefore, P satisfies the conclusion of Lemma 5. □

We now proceed with the proof of Theorem 2. Fix a sequence (K_m) of compact subsets of $\mathbb{C} \setminus \mathbb{D}$ with connected complement such that, for every compact set $K \subset \mathbb{C} \setminus \mathbb{D}$ with connected complement, there exists an $m \in \mathbb{N}$ with $K \subset K_m$ (see [N]). Let (Q_l) be an enumeration of all polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$ and let $\varphi, \psi : \mathbb{N}^* \to \mathbb{N}^*$ be two functions such that, given any couple (m,l) in $\mathbb{N}^* \times \mathbb{N}^*$, there exist infinitely many j with $(\varphi(j), \psi(j)) = (m,l)$. Let us finally fix (r_j) an increasing sequence in $]1/2, 1[$ that is converging to 1. We use induction to build sequences of polynomials $(f_{j,k})$ for $k \leq j$.

Step 1. Define $g_{1,1}(z) = 2z + P(z)$, where P is given by Lemma 5 with $K = K_{\varphi(1)}$, $L = \bar{B}(0, r_1)$, $g = Q_{\psi(1)}$, $N = 2$, and $\varepsilon = 1/2^3$. The Taylor series of $g_{1,1}$ approaches $Q_{\psi(1)}$ on $K_{\varphi(1)}$. We now correct its value on $K_{\varphi(2)}$ for further expansions by setting $f_{1,1}(z) = g_{1,1}(z) + Q(z)$, where Q is given by Lemma 5 with $K = K_{\varphi(2)}$, $L = \bar{B}(0, r_1)$, $g = -g_{1,1}$, $N = \deg(P) + 1$, and $\varepsilon = 1/2^3$.

Step $j-1$ to step j. Assume that polynomials $(f_{j-1,k})$ have been built in the previous step for $k \leq j-1$. Let $N_{j,1} = \max_{1 \leq k \leq j-1} \deg(f_{j-1,k}) + 1$. We define an intermediate polynomial $g_{j,1}$ by setting $g_{j,1}(z) = f_{j-1,1}(z) + P(z)$, where P is given by Lemma 5 for $K = K_{\varphi(j)}$, $g = Q_{\psi(j)} - f_{j-1,1}$, $L = \bar{B}(0, r_j)$, $N = N_{j,1}$, and $\varepsilon = 1/2^{j+2}$. In order to prepare step $j+1$, we now correct the value of the Taylor series by setting

$$f_{j,1}(z) = g_{j,1}(z) + Q(z),$$

where Q is given by Lemma 5 for $K = K_{\varphi(j+1)}$, $g = -g_{j,1}$, $L = \bar{B}(0, r_j)$, $N = \deg(P) + 1$, and $\varepsilon = 1/2^{j+2}$. Therefore, the following inequalities are satisfied:

$$\|f_{j,1} - f_{j-1,1}\|_{C(\bar{B}(0,r_j))} \leq \frac{1}{2^{j+1}}; \tag{2}$$

$$\|f_{j,1}\|_{C(K_{\varphi(j+1)})} \leq \frac{1}{2^{j+1}}. \tag{3}$$

Moreover, $S_n(f_{j,1}) = S_n(f_{j-1,1})$ for $n < N_{j,1}$. Taking $N_{j,2} = \deg(f_{j,1}) + 1$, we apply the same construction to deduce $f_{j,2}$ from $f_{j-1,2}$, and inductively we build polynomials $f_{j,k}$ $(1 \leq k \leq j-1)$ satisfying inequalities similar to (2) and (3). Finally, if $N_{j,j}$ is an integer greater than the degree of all polynomials $f_{j,k}$ $(1 \leq k \leq j-1)$, then $f_{j,j}$ is deduced from $2^{N_{j,j}} z^{N_{j,j}}$ by following the same process.

Condition (2) ensures that the sequence $(f_{j,k})_{j \geq k}$ converges uniformly on any compact subset of $\mathbb{D}$ to a function $f_k \in H(\mathbb{D})$. Let E be the vector space consisting of all series $\sum_1^\infty \alpha_k f_k$ that converge uniformly on compacta of $\mathbb{D}$, and let F be the closure of E in $H(\mathbb{D})$. Clearly F is a closed infinite-dimensional subspace of $H(\mathbb{D})$, and we need only prove that it consists of universal functions.

We begin with the case of a series $h = \sum_1^\infty \alpha_k f_k$ in E, where $\sum |\alpha_k|^2 < +\infty$. Take $\varepsilon > 0$, K_m a compact set in the family described at the beginning of the proof, and (Q_l) a polynomial with coefficients in $\mathbb{Q} + i\mathbb{Q}$. Without loss of generality, assume that $\alpha_1 = 1$ (if $\alpha_1 = 0$, we take the least integer k such that $\alpha_k \neq 0$; the proof is exactly the same). Let j be such that $(\varphi(j), \psi(j)) = (m,l)$, and let N be the degree of the polynomial $g_{j,1}$ built at step j. Then, for $z \in K_m$,

$$
\begin{aligned}
|S_N(h)(z) - Q_l(z)| &\leq |g_{j,1}(z) - Q_l(z)| + \left|\sum_{k=1}^{j-1} \alpha_k f_{j-1,k}(z)\right| \\
&\leq \frac{1}{2^j} + \left(\sum_1^{+\infty} |\alpha_k|^2\right)^{1/2} \left(\sum_1^{j-1} |f_{j-1,k}(z)|^2\right)^{1/2} \\
&\leq \frac{1}{2^j} + \|\alpha\|_2 \times \frac{j^{1/2}}{2^j}
\end{aligned}
$$

(the last inequality follows from (3)). Letting j go to infinity, one obtains

$$\|S_N(h) - Q_l\|_{C(K_m)} \leq \varepsilon.$$

This shows that $\sum_1^{+\infty} \alpha_k f_k$ is universal in the sense of Nestoridis. Note that the integer N does not depend specifically on h; it depends only on $\|\alpha\|_2$ and the condition $\alpha_1 = 1$.

In order to transfer the universal behavior to all elements of F, we use the theory of basic sequences as in [BeMo]. First of all, let $H = L^2\left(\frac{1}{2}\mathbb{T}\right)$. If g belongs to $H(\mathbb{D})$ then it clearly belongs to H with $\|g\|_H \leq \|g\|_{C(\bar{B}(0,1/2))}$. Hence, by construction,

$$
\begin{aligned}
\|f_k - 2^{N_{k,k}} z^{N_{k,k}}\|_H^2 &\leq \sum_{j>k} \frac{1}{4^j} \\
&\leq \frac{1}{3 \times 4^k}.
\end{aligned}
$$

Therefore (f_j) is a basic sequence of $L^2\left(\frac{1}{2}\mathbb{T}\right)$ that is equivalent to the canonical basis of ℓ^2. Now, if h belongs to F then there is a sequence (h^r) of E that converges to h. By continuity of $\|\cdot\|_H$ with respect to the maximum norm on $\bar{B}(0,1/2)$, this sequence of series converges also to h in $L^2\left(\frac{1}{2}\mathbb{T}\right)$. Therefore, h has a representation as a series $\sum_{k=1}^{+\infty} \alpha_k f_k$ in $L^2\left(\frac{1}{2}\mathbb{T}\right)$, perhaps not convergent in $H(\mathbb{D})$. Let us write $h^r = \sum_{k=1}^{+\infty} \alpha_k^r f_k$, with $\alpha^r \in \ell^2$ and $\alpha^r \to \alpha$ in ℓ^2. Take $\varepsilon > 0$, K_m, and Q_l as before. We suppose again that $\alpha_1 = 1$, and it is obvious that we may consider that $\alpha_1^r = 1$ for all $r \in \mathbb{N}$. Since (α^r) is convergent in ℓ^2, the sequence of its norms is bounded by M. If j is such that $(\varphi(j), \psi(j)) = (m, l)$, then our previous calculation yields

$$\exists N_j : \forall r \in \mathbb{N},\ \forall z \in K_m, \quad |S_{N_j}(h_r)(z) - Q_l(z)| \leq \frac{1}{2^j} + M \times \frac{j^{1/2}}{2^j}.$$

We definitively fix j such that $1/2^j + M \times j^{1/2}/2^j \leq \varepsilon/2$. Now, by taking the limit in the (finite) sum of the Taylor series, one obtains r with

$$\forall z \in K_m, \quad |S_{N_j}(h)(z) - S_{N_j}(h_r)(z)| \leq \frac{\varepsilon}{2}.$$

This completes the proof that h is universal (Theorem 2). □

4. Divergence

4.1. Fourier Series

The existence of integrable functions with everywhere divergent Fourier series is a pathological but generic phenomenon. It is also algebraically generic in the sense of spaceability.

THEOREM 3. *Let M be the set of functions of $L^1(\mathbb{T})$ whose Fourier series diverges everywhere on $\mathbb{T}$. Then M is spaceable.*

Proof. Let $u_k(t) = e^{i2^k t}$. By Paley's inequality (see e.g. [Du]), (u_k) is a basic sequence of $L^1(\mathbb{T})$ that is equivalent to the canonical basis of ℓ^2. We build by induction a sequence of "bad polynomials" whose spectra are mutually disjoint and fill the holes between the 2^k. More precisely, we build two sequences of integers $n(j,k)$ and $m(j,k)$, a sequence of trigonometric polynomials $R(j,k)$ defined for $j \geq 1$ and $1 \leq k \leq j$, and an increasing sequence of positive integers $(l_j)_{j\geq 1}$ such that, for any $1 \leq k < k' \leq j$,

$$2^{l_j} < n(j,k) \leq m(j,k) < n(j,k') \leq m(j,k') < 2^{l_j+1}, \tag{4}$$

$$\mathrm{Sp}(R(j,k)) \subset [n(j,k), m(j,k)], \tag{5}$$

$$\|R(j,k)\|_1 \leq \frac{1}{2^{j+2}}, \tag{6}$$

$$\forall t \in [0,2\pi], \qquad \sup_{n(j,k)\leq n\leq m(j,k)} |S_n(R(j,k),t)| \geq j, \tag{7}$$

where $S_n\left(\sum_{l\in\mathbb{Z}} a_l e^{ilt}\right) = \sum_{|l|\leq n} a_l e^{ilt}$ and $\mathrm{Sp}\left(\sum_{l\in\mathbb{Z}} a_l e^{ilt}\right) = \{l \in \mathbb{Z} : a_l \neq 0\}$.

Indeed, for $j = 1$, let Q be a trigonometric polynomial such that:

- $\|Q\|_1 \leq 1/4$;
- for all $t \in [0,2\pi]$, $\sup_N |S_N(Q,t)| \geq 1$.

Such a polynomial appears in the course of the proof of Kolmogorov's theorem. If $\mathrm{Sp}(Q) \subset [-a,a]$, we fix l_1 large enough so that $2^{l_1} > 2a+1$. Now the polynomial $R(1,1)(t) = \exp[i(2^{l_1}+a+1)t]Q(t)$ works, with $n(1,1) = 2^{l_1+1}$ and $m(1,1) = 2^{l_1+2a+1} < 2^{2l_1}$. Given the construction through rank $j-1$, let us fix Q a trigonometric polynomial and a an integer such that:

- $\|Q\|_1 \leq 1/2^{j+2}$;
- for all $t \in E$, $\sup_N |S_N(Q,t)| \geq j$;
- $\mathrm{Sp}(Q) \subset [-a,a]$.

Let l_j be a sufficiently large integer such that $l_j > l_{j-1}$ and $2^{l_j} > j(2a+1)$. We then define $R(j,k)$, $n(j,k)$, and $m(j,k)$ by setting

- $R(j,1)(t) = \exp[i(2^{l_j}+a+1)t]Q(t)$,
- $n(j,1) = 2^{l_j}+1$, and
- $m(j,1) = 2^{l_j}+2a+1$

if $k = 1$ and, if $2 \leq k \leq j$, by using induction to set

- $R(j,k)(t) = \exp[i(m(j,k-1)+1)t]Q(t)$,
- $n(j,k) = m(j,k-1)+1$,
- $m(j,k) = m(j,k-1)+2a+1$.

Then $m(j,j) \leq 2^{l_j} + j(2a+1) < 2^{l_j+1}$ and so conditions (4)–(7) are fulfilled. Finally, we set:

$$v_k = u_k + \sum_{j\geq k} R(j,k)$$

and observe that

$$\|v_k - u_k\| \leq \sum_{j\geq k} \|R(j,k)\| \leq \frac{1}{2^{k+1}}.$$

Now let (u_k^*) be the coefficient functionals corresponding to the basic sequence (u_k). Clearly, one has $\|u_k^*\| \leq 1$ and therefore $\sum_{k=1}^{+\infty}\|u_k^*\|\|u_k - v_k\| \leq \sum_{k=1}^{+\infty} < 1$. Thus, (v_k) is a basic sequence. Denote by F the closed linear span generated by (v_k) and pick any $f \in F \setminus \{0\}$, which may be written $f = \sum_{k=1}^{+\infty}\alpha_k v_k$. Since f is not equal to zero, there exists a coefficient α_k that is not zero. Next observe that, for $k \neq k'$ and $j \geq k$,

$$\operatorname{Sp}(v_{k'}) \cap [n(j,k), m(j,k)] = \emptyset.$$

By (7), for $j \geq k$ and $t \in [0, 2\pi]$ we have

$$\begin{aligned}\sup_{n(j,k)\leq n\leq m(j,k)} |S_n(f,t) - S_{n(j,k)-1}(f,t)| &= |\alpha_k| \sup_{n(j,k)\leq n\leq m(j,k)} |S_n(R(j,k),t)| \\ &\geq |\alpha_k| j.\end{aligned}$$

Hence, the Fourier series $(S_n(f,t))_{n\geq 0}$ is everywhere unboundedly divergent. □

REMARK. This method of proof applies to many examples where we have divergent series.

(1) Let $E \subset \mathbb{T}$ be a set of Lebesgue measure 0. The set of functions of $C(\mathbb{T})$ whose Fourier series is everywhere unboundedly divergent on E is spaceable. This is reminiscent of Du Bois–Reymond's example, and the proof is almost the same. The theory of Sidon sets (see [LR]) replaces Paley's inequality to prove that $e^{i2^k t}$ is a basic sequence in $C(\mathbb{T})$ that is equivalent to the canonical basis of ℓ^1. The following lemma from [KaK] is used to build the sequence of bad polynomials: *Given F and a finite union of intervals of $\mathbb{T}$ with Lebesgue measure a $(0 < a < 1/\pi)$, there exists a trigonometric polynomial Q with norm 1 in $C(\mathbb{T})$ and such that*

$$\sup_{n\in\mathbb{N}} |S_n(Q,t)| \geq \frac{1}{\pi}\log\frac{1}{a\pi} \quad \textit{when } t \in F.$$

Since E is negligible, we can find a sequence of sets (F_j) such that: F_j is a finite union of intervals; its measure a_j satisfies

$$\frac{1}{\pi}\log\frac{1}{a_j\pi} \geq 2^{j+1}j;$$

and each point of E belongs to infinitely many F_j. The trigonometric polynomial Q used at step j is now given by the previous lemma, with $F = F_j$.

(2) The same holds for Fefferman's example of a function in $L^2(\mathbb{T}^2)$ whose Fourier series is everywhere divergent on $\mathbb{T}^2$.

4.2. *Dirichlet Series*

Let us now turn to the case of $\mathcal{H}^\infty$, the space of Dirichlet series $f(s) = \sum_1^\infty a_n n^{-s}$ with convergence and boundedness of f in the half-plane $\mathbb{C}_+ = \{s \in \mathbb{C} : \Re(s) > 0\}$; $\mathcal{H}^\infty$ is a Banach space with the norm

$$\|f\|_\infty = \sup\{|f(s)| : s \in \mathbb{C}_+\}.$$

Recall that a Dirichlet polynomial is a Dirichlet series with a finite number of nonzero coefficients $\sum_1^N a_n n^{-s}$ and that, if $f(s) = \sum_{n\geq 1} a_n n^{-s}$, then the spectrum of f is defined by

$$\mathrm{Sp}(f) = \{n \geq 1 : a_n \neq 0\}.$$

In [BKoQ], it is proved that there exists a Dirichlet series $f(s) = \sum_1^\infty a_n n^{-s} \in \mathcal{H}^\infty$ such that $\sum_1^\infty a_n n^{it}$ diverges for each $t \in \mathbb{R}$. In the course of the proof, we build a sequence of Dirichlet polynomials $Q_k(s) = \sum_{n=1}^{N_k} a_n(k) n^{-s}$ and a sequence of intervals X_k such that

$$\|Q_k\|_\infty \xrightarrow{k\to\infty} 0,$$

$$X_k \subset X_{k+1}, \qquad \bigcup_{k=1}^{+\infty} X_k = \mathbb{R},$$

$$\forall t \in X_k, \qquad \sup_{1\leq l\leq N_k} \left| \sum_{n=1}^{l} a_n(k) n^{it} \right| \geq \delta;$$

here $\delta > 0$ is an absolute constant. The techniques used in the previous section can be developed here to prove that there exists a closed subspace F of $\mathcal{H}^\infty$ such that, for each $f = \sum_{n\geq 1} a_n n^{-s} \in F\setminus\{0\}$, the series $\sum_{n\geq 1} a_n n^{it}$ is everywhere divergent. In this context Paley's inequality is replaced by Bohr's inequality, which we recall here as a lemma (see [Q] for a proof).

LEMMA 6. *Let $(q_j)_{j\geq 1}$ be a sequence of distinct prime numbers. Then, for any $N \geq 1$ and any $a_1, \ldots, a_N \in \mathbb{C}$,*

$$\left\| \sum_{n=1}^{N} a_n q_n^{-s} \right\|_\infty = \sum_{n=1}^{N} |a_n|.$$

THEOREM 4. *Let M be the subset of $\mathcal{H}^\infty$ of the Dirichlet series*

$$f(s) = \sum_{n=1}^{+\infty} a_n n^{-s}$$

such that the series $\sum_{n=1}^{+\infty} a_n n^{it}$ is everywhere divergent on $\mathbb{R}$. Then M is spaceable.

Proof. We just sketch the proof. As in the proof of Theorem 3, one may build by induction on j a sequence $R(j,k)$ of bad "Dirichlet polynomials", two sequences of integers $n(j,k)$ and $m(j,k)$, and an increasing sequence (q_j) of prime numbers such that

$$q_j < n(j,k) \le m(j,k) < n(j,k') \le m(j,k') < q_{j+1},$$
$$\mathrm{Sp}(R(j,k)) \subset [n(j,k), m(j,k)],$$
$$\|R(j,k)\|_\infty \le \frac{1}{2^{j+2}},$$
$$\forall t \in [-j, j], \qquad \sup_{n(j,k)\le n\le m(j,k)} |S_n(R(j,k),t)| \ge j,$$

where in this context $S_n f(s)$ means $\sum_1^n a_k k^{-s}$ once f has been given by $f(s) = \sum_1^\infty a_k k^{-s}$. If we use v_k to denote $q_k^{-s} + \sum_{j\ge k} R(j,k)$, then Lemmas 1 and 6 ensure that (v_k) is a basic sequence of $\mathcal{H}^\infty$. The rest of the proof follows exactly that of Theorem 3. □

5. Supercyclic Operators

If X is a Banach space, then an operator T on X is called:

- *hypercyclic* provided there exists an $x \in X$ such that $\{T^n x : n \ge 0\}$ is dense in X;
- *supercyclic* provided there exists an $x \in X$ such that $\{\lambda T^n x : n \ge 0, \lambda \in \mathbb{C}\}$ is dense in X.

Because a hypercyclic operator has norm greater than 1, the set of hypercyclic operators is never spaceable. On the other hand, the following theorem holds.

THEOREM 5. *The set of supercyclic operators on a separable Hilbert space H is spaceable.*

Proof. We may assume that $H = H^2(\mathbb{D})$. By a theorem of Mazur (see e.g. [Di]), we can find a basic sequence $(\varphi_k)_{k\ge1}$ in $H_0^\infty(\mathbb{D}) = \{f \in H^\infty(\mathbb{D}) : f(0) = 0\}$.

For $\varphi \in H_0^\infty(\mathbb{D})\setminus\{0\}$, denote by M_φ the induced multiplier on $H^2(\mathbb{D})$—namely, $M_\varphi(f) = \varphi f$. One may find $\lambda > 0$ such that $\lambda\varphi(\mathbb{D}) \cap \mathbb{T} \ne \emptyset$. By a result of Godefroy and Shapiro [GoS], the operator $(\lambda M_\varphi)^* = M^*_{\lambda\varphi}$ is hypercyclic on $H^2(\mathbb{D})$. Therefore, the operator M^*_φ is supercyclic. Observe now that, for any $N \ge 1$ and any complex numbers $a_1, \dots, a_n$,

$$\left\|\sum_{k=1}^N a_k M^*_{\varphi_k}\right\| = \left\|M_{\sum_{k=1}^N \overline{a_k}\varphi_k}\right\| = \left\|\sum_{k=1}^N \overline{a_k}\varphi_k\right\|_\infty.$$

This implies that $(M_{\varphi_k^*})$ is a basic sequence in $\mathcal{L}(H^2)$, and clearly the closed Banach space generated by the sequence $(M^*_{\varphi_k})$ solves the problem. □

REMARKS. (1) The proof actually gives a slightly stronger result: We have proved that the set of operators for which there exists a scalar multiple that is hypercyclic is spaceable (there exist supercyclic operators whose scalar multiples are never hypercyclic).

(2) The result stated here holds for a complex Hilbert space. We do not know if it remains true for a real Hilbert space or for an arbitrary Banach space.

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