

Loop Structures on the Homotopy Type of S^3 Revisited

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1. Introduction

In an attempt to understand Lie groups from a homotopy theory point of view, Rector suggested studying Lie groups through their classifying spaces. Using S^3 as a test case, he proved in his pioneering paper [8] that there are uncountably many homotopically distinct deloopings of S^3 . These deloopings form the so-called genus of the classifying space BS^3 . To be more precise, for a nilpotent finite type space X , the *genus* of X is defined to be the set of homotopy types of nilpotent finite type spaces Y such that the p -completions of X and Y are homotopy equivalent for each prime p and also their rationalizations are homotopy equivalent. When considering genus, one often ignores the difference between a homotopy type and a space with that homotopy type.

Rector actually provided a complete list of classification invariants, defined by using integral and mod p cohomology, for the genus of BS^3 , which we now call the Rector invariants. Briefly, the Rector invariants of a space X in the genus of BS^3 are signs, $(X/p) \in \{\pm 1\}$, one for each prime p . Two spaces in the genus of BS^3 are homotopy equivalent if and only if they have the same corresponding Rector invariants, and any such sequence does occur for some space. Also, BS^3 itself has 1 as its Rector invariant for each prime.

Generalizing the approach used by Rector, Møller [6] showed that this property of having a huge genus is not restricted to BS^3 . In fact Møller proved that, whenever G is a compact connected non-abelian Lie group, the genus of its classifying space BG is uncountably large. There is, however, no explicit list of classification invariants (like the Rector invariants) in this general setting. A remarkable result of Notbohm [7] showed that K -theory ring together with the λ -operations classify the genus of BG , provided G is a simply connected compact Lie group. In other words, two spaces in the genus of BG are homotopy equivalent if and only if their K -theory are isomorphic as λ -rings. Notbohm's proof does not involve computing the K -theory of these spaces, and consequently we do not know how they are mutually nonisomorphic. This prompts the question of *how* these uncountably many λ -rings, the K -theory of the spaces in the genus of a given classifying space BG , are mutually nonisomorphic.

Ideally, one would like to compute explicitly these K -theory λ -rings, at least partially. Then one can use this knowledge to show that these λ -rings are mutually

nonisomorphic by doing pure algebra. Any work in this direction would shed new light into the results just mentioned of Notbohm, Møller, and Rector.

The purpose of this paper is to do just that in the case BS^3 . In fact, the author has already shown in [10] that spaces in the genus of BS^3 all have isomorphic K -theory rings, namely, the power series ring $\mathbf{Z}[[x]]$ with x in filtration 4. (The result in [10] actually shows, more generally, that the underlying K -theory ring is constant throughout the genus of BG as long as BG has torsion-free integral cohomology.) We will use both KO -theory and K -theory, since in order to define $(X/2)$ one must consider either symplectic or orthogonal K -theory. Our actual procedure is to first observe that the Rector invariants show up in the Adams operations in (both unitary and orthogonal) K -theory. Then we show that these λ -rings are pairwise nonisomorphic by doing simple algebra about power series.

In addition to providing some new insights about the spaces in the genus of BS^3 , the author also hopes that this paper will attract more attention to the general problem of determining classification invariants for the genus of a classifying space BG . Such classification invariants (if available), together with the author's result in [10], will probably allow one to show explicitly how the K -theory λ -rings of the spaces in the genus of BG differ from one another.

In the next section we first recall some basic definitions concerning λ -rings and Adams operations; those who are familiar with λ -rings can safely skip this section. Our main results are presented in the last section, which begins with the construction of the Rector invariants.

2. λ -RINGS and Adams Operations

In this section, we recall the definitions of a (filtered) λ -ring and of Adams operations. The reader is referred to Atiyah and Tall [2] or Knutson [4] for more information about λ -rings.

2.1. λ -RINGS. A λ -ring is a commutative ring R with unit together with functions

$$\lambda^i: R \rightarrow R \quad (i = 0, 1, 2, \dots)$$

such that, for any elements r and s in R :

- $\lambda^0(r) = 1$;
- $\lambda^1(r) = r$ and $\lambda^n(1) = 0$ for all $n > 1$;
- $\lambda^n(r + s) = \sum_{i=0}^n \lambda^i(r)\lambda^{n-i}(s)$;
- $\lambda^n(rs) = P_n(\lambda^1(r), \dots, \lambda^n(r); \lambda^1(s), \dots, \lambda^n(s))$;
- $\lambda^n(\lambda^m(r)) = P_{n,m}(\lambda^1(r), \dots, \lambda^m(r))$.

Here the P_n and $P_{n,m}$ are certain universal polynomials with integer coefficients. Note that the terminology "special" λ -ring is often used in the literature (see e.g. Atiyah and Tall [2]).

A λ -ring map $f: R \rightarrow S$ between two λ -rings is a ring map between the underlying rings that respects the operations λ^i : $f\lambda^i = \lambda^i f$ ($i \geq 0$).

2.2. FILTERED λ -RINGS. By a filtered ring we mean a commutative ring R with unit together with a decreasing sequence of ideals,

$$R = I^0 \supset I^1 \supset I^2 \supset \dots,$$

such that $I^n I^m \subset I^{n+m}$ for any n, m . A map of filtered rings is a map of rings that respects the filtrations in the sense that the image of the n th filtration is inside the n th filtration of the target. (Note that this is a stronger condition than just saying that the ring map is continuous under the filtration topology.)

A *filtered λ -ring* is a filtered ring $R = (R, \{I^n\})$ which is also a λ -ring such that the ideals I^n are all closed under the operations λ^i ($i > 0$). A filtered λ -ring map is a λ -ring map which is also a filtered ring map.

2.3. ADAMS OPERATIONS. Given a λ -ring R , the *Adams operations*

$$\psi^k : R \rightarrow R$$

for $k = 1, 2, \dots$ are defined inductively by the Newton formula:

$$\psi^k(a) - \lambda^1(a)\psi^{k-1}(a) + \dots + (-1)^{k-1}\lambda^{k-1}(a)\psi^1(a) = (-1)^{k-1}k\lambda^k(a).$$

The Adams operations satisfy the following properties:

- (1) all the $\psi^k : R \rightarrow R$ are λ -ring maps;
- (2) $\psi^1 = \text{Id}$ and $\psi^k \psi^l = \psi^{kl}$ for any $k, l \geq 1$;
- (3) $\psi^p(a) \equiv a^p \pmod{pR}$ for each prime p and element a in R .

If R is a filtered λ -ring, then the Adams operations are all filtered λ -ring maps. Observe from the second property that the ψ^p for p primes determine all the Adams operations.

3. Rector Invariants and the Genus of BS^3

Let X be a space in the genus of BS^3 . Using the proof of the Main Theorem in [10] with KO -theory instead of K -theory, one can show that

$$KO^*(X) \cong KO^*[[x]]$$

as filtered rings, where $x \in KO_4^4(X)$ is a representative of an integral generator $x_4 \in H^4(X; \mathbf{Z}) = E_2^{4,0}$ in the KO^* -Atiyah-Hirzebruch spectral sequence for X . Here $KO_b^a(X)$ denotes the subgroup of $KO^a(X)$ consisting of elements u that restrict to 0 under the natural map

$$KO^a(X) \rightarrow KO^a(X_{b-1}),$$

where X_{b-1} denotes the $(b - 1)$ -skeleton of X . Such an element u is said to be in *degree a and filtration b* .

Now we recall the relevant notation, definitions, and results regarding Rector's classification of the genus of BS^3 [8]. Let $\xi \in \pi_{-4}KO$ and $b_R \in \pi_{-8}KO$ be the generators so that $\xi^2 = 4b_R$. Rector observed in [9, Sec. 4] that there exists an integer a , depending on the choice of the representative x , such that the following three statements hold.

- (1) $\psi^2(\xi x) = 4\xi x + 2ab_R x^2 \pmod{KO_9^0(X)}$.
- (2) The integer a is well-defined $\pmod{24}$. This means that if $x' \in KO_4^4(X)$ is another representative of x_4 with corresponding integer a' , then $a \equiv a'$

(mod 24). Furthermore, if x_4 is replaced with $-x_4$, then a will be replaced with $-a$. We can (and we will) therefore write $a(X)$ for a .

(3) $a(X) \equiv \pm 1, \pm 5, \pm 7, \text{ or } \pm 11 \pmod{24}$.

Condition (3) follows from the examples constructed by Rector in [9, Sec. 5] and James’s result [3] which states that there are precisely eight homotopy classes of homotopy associative multiplications on S^3 . These eight classes can be divided into four pairs, with each pair consisting of a homotopy class of multiplication and its inverse.

Rector’s invariant (X/p) for p an odd prime is defined as follows [8]. In the mod p Steenrod algebra, the Adem relation $P^1P^1 = 2P^2$ implies that

$$P^1\bar{x}_4 = \pm 2\bar{x}_4^{(p+1)/2}$$

in $H^*(X; \mathbf{Z}/p)$, where \bar{x}_4 is the mod p reduction of the integral generator x_4 . Then $(X/p) \in \{\pm 1\}$ is defined as the sign on the right-hand side of this equation. Because replacing x_4 by $-x_4$ (the other integral generator) will change all the signs (X/p) simultaneously, one must also specify an orientation of the integral generator in order for the (X/p) to be well-defined.

The Rector invariant $(X/2)$ and a canonical choice of an orientation for the integral generator are given as follows. Using the (mod 24) integer $a(X)$, define

$$((X/2), (X/3)) \stackrel{\text{def}}{=} \begin{cases} (1, 1) & \text{if } a(X) \equiv \pm 1 \pmod{24}, \\ (1, -1) & \text{if } a(X) \equiv \pm 5 \pmod{24}, \\ (-1, 1) & \text{if } a(X) \equiv \pm 7 \pmod{24}, \\ (-1, -1) & \text{if } a(X) \equiv \pm 11 \pmod{24}. \end{cases} \quad (3.1)$$

The orientation of the integral generator is then chosen so that $(X/3)$ is as given in (3.1). This definition of Rector’s invariants coincides with the original one (cf. [5, Sec. 9]). In particular, specifying the Rector invariants of a space X is equivalent to specifying the (mod 24) integer $a(X)$ and the (X/p) for p primes, $p > 2$. Observe from this discussion that $a(X)$ appears in the Adams operation ψ^2 in KO -theory.

Now we can recall Rector’s classification theorem of the genus of BS^3 [8].

THEOREM 3.1 (Rector). *The (X/p) for p primes provide a complete list of classification invariants for the genus of BS^3 . Any combination of values of the (X/p) can occur. If X is BS^3 , then $(X/p) = 1$ for every prime p .*

It should be remarked that Rector also proved that BS^3 can be distinguished from the rest of the spaces in its genus by considering certain maps—the so-called maximal torus—from infinite complex projective space. However, that result will not be needed here.

We have already seen how the (mod 24) integer $a(X)$ appears in the Adams operation ψ^2 in KO -theory. Now we will see how the Rector invariants (X/p) for p odd appear in the Adams operations ψ^p in K -theory.

PROPOSITION 3.2. *Let p be a fixed odd prime and let X be a space in the genus of BS^3 , so that $K^*(X)$ can be written as $K^*[[u]]$ for some representative $u \in K_4^4(X)$ of the integral generator $x \in H^4(X; \mathbf{Z}) = E_2^{4,0}$ in the K^* -Atiyah-Hirzebruch spectral sequence. Then there exist some elements $w \in K_{2p+3}^0(X)$ and $x_0 \in K_4^0(X)$ such that*

$$\psi^p(b^2u) = (b^2u)^p + 2(X/p)p(b^2u)^{(p+1)/2} + pw + p^2x_0,$$

where $b \in \pi_{-2}K$ is the Bott element.

Proof. To see this, note that since $b^2u \in K_4^0(X)$ is in filtration 4, it follows from Atiyah's theorem [1, Prop. 5.6] that

$$\psi^p(b^2u) = (b^2u)^p + px_1 + p^2x_0$$

for some $x_i \in K_{4+2i(p-1)}^0(X)$ ($i = 0, 1$). Moreover, one has

$$\bar{x}_1 = P^1\bar{b^2u},$$

where \bar{z} is the mod p reduction of an element z and P^1 is the Steenrod operation of degree $2(p-1)$ in mod p cohomology. Thus, to prove the proposition it is enough to show that

$$x_1 = 2(X/p)(b^2u)^{(p+1)/2} + w + pz \tag{3.2}$$

for some $w \in K_{2p+3}^0(X)$ and some $z \in K_{2p+2}^0(X)$. Now in $H^*(X; \mathbf{Z}) \otimes \mathbf{Z}/p = H^*(X; \mathbf{Z}/p)$ we have

$$\begin{aligned} \bar{x}_1 &= P^1\bar{b^2u} \\ &= P^1\bar{x} \\ &= 2(X/p)\bar{x}^{(p+1)/2} \\ &= 2(X/p)\bar{b^2u}^{(p+1)/2}. \end{aligned}$$

The proposition follows from this because the mod p cohomology of X is the mod p associated graded ring of its K -theory. □

Equipped with the knowledge of how the Rector invariants appear in K -theory, we now show that the K -theory of the spaces in the genus of BS^3 can be distinguished by doing some rather elementary algebra.

THEOREM 3.3. *Let X and Y be spaces in the genus of BS^3 . If there exists a filtered λ -ring isomorphism $\sigma : KO^*(X) \xrightarrow{\cong} KO^*(Y)$, then*

$$a(X) \equiv \pm a(Y) \pmod{24}$$

and

$$(X/p) = (Y/p)$$

for each odd prime p . In this case, by Rector's classification theorem it follows that X and Y have the same homotopy type.

Proof. We have already explained that $KO^*(X) = KO^*[[x]]$ and $KO^*(Y) = KO^*[[y]]$, where $x \in KO_4^4(X)$ and $y \in KO_4^4(Y)$ represent the integral generators $x_4 \in H^4(X; \mathbf{Z})$ and $y_4 \in H^4(Y; \mathbf{Z})$, respectively.

Since σ is a ring isomorphism, we have

$$\sigma(\xi x) = \varepsilon \xi y + \sigma_2 b_R y^2 \pmod{KO_9^0(Y)}$$

for some $\varepsilon \in \{\pm 1\}$ and integer σ_2 . Computing modulo $KO_9^0(X)$, we have

$$\begin{aligned} 4\sigma(b_R x^2) &= \sigma(\xi x)^2 \\ &= \xi^2 y^2 \\ &= 4b_R y^2. \end{aligned}$$

Therefore,

$$\sigma(b_R x^2) = b_R y^2 \pmod{KO_9^0(Y)}.$$

First we claim that there is an equality

$$a(X) = 6\sigma_2 + \varepsilon a(Y). \tag{3.3}$$

To prove (3.3), we compute both sides of the equality

$$\sigma \psi^2(\xi x) = \psi^2 \sigma(\xi x) \pmod{KO_9^0(Y)}.$$

Working modulo $KO_9^0(Y)$ we have, on the one hand,

$$\begin{aligned} \sigma \psi^2(\xi x) &= \sigma(4\xi x + 2a(X)b_R x^2) \\ &= 4(\varepsilon \xi y + \sigma_2 b_R y^2) + 2a(X)b_R y^2 \\ &= 4\varepsilon \xi y + (4\sigma_2 + 2a(X))b_R y^2. \end{aligned}$$

On the other hand, still working modulo $KO_9^0(Y)$, we have

$$\begin{aligned} \psi^2 \sigma(\xi x) &= \varepsilon \psi^2(\xi y) + \sigma_2 \psi^2(b_R y^2) \\ &= \varepsilon(4\xi y + 2a(Y)b_R y^2) + \sigma_2(2^4 b_R y^2) \\ &= 4\varepsilon \xi y + (16\sigma_2 + 2\varepsilon a(Y))b_R y^2. \end{aligned}$$

Equation (3.3) now follows by equating the coefficients of $b_R y^2$.

In view of (3.3), finishing the proof of our assertion about the mod 24 integer a requires only that we establish

$$\sigma_2 \equiv 0 \pmod{4}. \tag{3.4}$$

To prove (3.4), note that since σ is a KO^* -module map we have

$$\xi \sigma(x) = \sigma(\xi x).$$

Since σ is a ring isomorphism, we also have

$$\sigma(x) = \varepsilon' y + \sigma_2' \xi y^2 \pmod{KO_9^4(Y)}$$

for some $\varepsilon' \in \{\pm 1\}$ and integer σ_2' . Therefore, working modulo $KO_9^0(Y)$ yields

$$\begin{aligned} \xi\sigma(x) &= \varepsilon'\xi y + \sigma_2'\xi^2 y^2 \\ &= \varepsilon'\xi y + 4\sigma_2'b_R y^2 \\ &= \varepsilon\xi y + \sigma_2 b_R y^2. \end{aligned}$$

In particular, by equating the coefficients of $b_R y^2$ we obtain

$$\sigma_2 = 4\sigma_2',$$

thereby proving (3.4).

This finishes the proof of our assertion that $a(X) \equiv \pm a(Y) \pmod{24}$. We still need to prove the assertion about the Rector invariants for odd primes p .

Let, then, p be a fixed odd prime. Just as before we have

$$K^*(X) \cong K^*[[u_x]],$$

with $u_x \in K_4^4(X)$ a representative of the integral generator $x_4 \in H^4(X; \mathbf{Z}) = E_2^{4,0}$ in the K^* -Atiyah-Hirzebruch spectral sequence. Moreover, we may choose u_x so that

$$c(x) = u_x,$$

where

$$c: KO^*(X) \rightarrow K^*(X)$$

is the complexification map. Similar remarks apply to Y , so that

$$K^*(Y) \cong K^*[[u_y]].$$

The λ -ring isomorphism σ induces (via c) a λ -ring isomorphism

$$\sigma_c: K^*(X) \xrightarrow{\cong} K^*(Y).$$

By composing σ_c with a suitable λ -ring automorphism of $K^*(Y)$ if necessary, we obtain a λ -ring isomorphism

$$\alpha: K^*(X) \xrightarrow{\cong} K^*(Y)$$

with the property that

$$\alpha(b^2 u_x) = b^2 u_y + [\text{higher terms in } b^2 u_y]. \tag{3.5}$$

Using Proposition 3.2 and (3.5), one infers that

$$\alpha\psi^p(b^2 u_x) = 2(X/p)p(b^2 u_y)^{(p+1)/2} \pmod{K_{2p+3}^0(Y) \text{ and } p^2} \tag{3.6}$$

and

$$\psi^p\alpha(b^2 u_x) = 2(Y/p)p(b^2 u_y)^{(p+1)/2} \pmod{K_{2p+3}^0(Y) \text{ and } p^2}. \tag{3.7}$$

Since $\alpha\psi^p = \psi^p\alpha$, it follows from (3.6) and (3.7) that

$$2(X/p)p \equiv 2(Y/p)p \pmod{p^2}$$

or, equivalently,

$$2(X/p) \equiv 2(Y/p) \pmod{p}.$$

But p is odd, and so

$$(X/p) \equiv (Y/p) \pmod{p}.$$

Hence $(X/p) = (Y/p)$, as desired. This finishes the proof of Theorem 3.3. \square

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