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Introduction to the Toric Mori Theory

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1. Introduction

The main purpose of this paper is to give a simple and noncombinatorial proof of the toric Mori theory. Here, the toric Mori theory means the (log) minimal model program (MMP, for short) for toric varieties.

In his famous and beautiful paper [R], Reid carried out the toric Mori theory under the assumption that the variety is *complete*. His arguments are combinatorial. Thus, it is not so obvious whether we can remove the completeness assumption from his paper. We quote his idea from [R].

> (0.3) *Remarks.* The hypothesis that A is complete is not essential; it can be reduced to the projective case, or possibly eliminated by a careful (and rather tedious) rephrasing of the arguments of §§1–3. The projectivity hypothesis on f is needed in order for the statement of (0.2) to make sense, since without projectivity the cone $\mathrm{NE}(V/A)$ will usually not have any extremal rays.

We prefer not to simply rephrase his approach, which entails tedious combinatorial arguments. Instead, our proof (which is independent of Reid's proof) heavily relies on the general machinery of the MMP and the special properties of toric varieties. Thus, our proof works without the completeness assumption.

For the details of the toric Mori theory, see [O, Sec. 2.5; KMM, Sec. 5.2; OP; L; I2; Ma, Ch. 14; W; Fj1]. Matsuki [Ma] corrected some minor errors in [R] and pointed out some ambiguities in [R] and [KMM]; see Remarks 14-1-3(ii), 14-2-3, and 14-2-7 in [Ma]. We believe that these remarks help the reader to understand [R]. We recommend that the reader compare this paper with [Ma, Ch. 14]. Shokurov treats the MMP for toric varieties in a noncombinatorial way (see [Sh, Ex. 3]). His arguments are quite different from ours. For the more advanced topics of toric Mori theory, see [Fj2]. For the outline of the general MMP, which is still conjectural in dimension ≥ 4, see [KMM, Introduction] or [KoMo, 2.14, Sec. 3.7].

We note that the Zariski decomposition on toric varieties has already been treated by various researchers. The reason we treat it here is to show that the Zariski

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decomposition on toric varieties is an easy consequence of the toric Mori theory. The partial resolutions of nondegenerate hypersurface singularities were treated by Ishii [I1], who divided the cone by the data of the Newton polytope. One of her proofs contains a gap (see Remark 6.6), but the results themselves are correct. Her results also become easy consequences of the toric Mori theory.

Note that we cannot recover combinatorial aspects of [R] and [I1] by our method. So, this paper does not depreciate [R] and [I1].

Almost all the results in this paper are more or less known to the experts. However, some had not been stated explicitly before. Also, some of the proofs that we give in this paper are new and much simpler than the known ones. We hope that this paper will help the reader to understand the toric Mori theory.

In Section 2, we fix the notation and collect basic results. Section 3 explains the toric Mori theory. Section 4 is the main part of this paper, where we give a simple and noncombinatorial proof of toric Mori theory. In Section 5, we consider the Zariski decomposition on toric varieties. In Section 6, we apply the toric Mori theory to the study of the partial resolutions of nondegenerate hypersurface singularities and also reprove Ishii's results.

NOTATION. Here is a list of some of the standard notation we use.

(1) For a real number d, its *round down* is the largest integer $\leq d$, which is denoted by $\llcorner d \lrcorner$. If $D = \sum d_i D_i$ is a divisor with real coefficients and the D_i are distinct prime divisors, then we define the *round down* of D as $\llcorner D \lrcorner := \sum \llcorner d_i \lrcorner D_i$.
(2) Let $f : X \to Y$ be a proper birational morphism between normal varieties. Then, f is said to be *small* if f is an isomorphism in codimension 1.
(3) The symbol $\mathbb{Z}_{\geq 0}$ (resp. $\mathbb{Q}_{\geq 0}, \mathbb{R}_{\geq 0}$) denotes the set of nonnegative integers (resp. rational numbers, real numbers).

Throughout, we will work over an algebraically closed field k. The characteristic of k is arbitrary from Section 2 to Section 5 (unless otherwise stated); in Section 6, we assume that k is the complex number field $\mathbb{C}$.

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2. Preliminaries

2.1. Toric Varieties

In this section we recall the basic notion of toric varieties and fix notation. For basic results about toric varieties, see [Ke+; D; O; Fl].

2.1. Let $N \simeq \mathbb{Z}^n$ be a lattice of rank n. A toric variety $X(\Delta)$ is associated to a *fan* Δ, a finite collection of convex cones $\sigma \subset N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ that satisfy the following conditions.

(i) Each convex cone $\sigma \in \Delta$ is rational polyhedral in the sense that there are finitely many $v_1, \ldots, v_s \in N \subset N_{\mathbb{R}}$ such that

$$\sigma = \{r_1 v_1 + \cdots + r_s v_s;\ r_i \in \mathbb{R}_{\geq 0} \text{ for all } i\}$$

and is strongly convex in the sense that

$$\sigma \cap (-\sigma) = \{0\}.$$

(ii) Each face τ of a convex cone $\sigma \in \Delta$ again belongs to Δ.

(iii) The intersection of two cones in Δ is a face of each.

DEFINITION 2.2. The *dimension* $\dim \sigma$ of σ is the dimension of the linear space $\mathbb{R} \cdot \sigma = \sigma + (-\sigma)$ spanned by σ.

We define the sublattice N_σ of N generated (as a subgroup) by $\sigma \cap N$ as follows:

$$N_\sigma := \sigma \cap N + (-\sigma \cap N).$$

The *star* of a cone τ can be defined abstractly as the set of cones σ in Δ that contain τ as a face. For any such cone σ, the image

$$\bar{\sigma} = (\sigma + (N_\tau)_{\mathbb{R}})/(N_\tau)_{\mathbb{R}} \subset N(\tau)_{\mathbb{R}}$$

by the projection $N \to N(\tau) := N/N_\tau$ is a cone in $N(\tau)$. These cones $\{\bar{\sigma};\ \tau \prec \sigma\}$ form a fan in $N(\tau)$, and we denote this fan by $\mathrm{Star}(\tau)$. We set $V(\tau) = X(\mathrm{Star}(\tau))$. It is well known that $V(\tau)$ is an $(n-k)$-dimensional closed toric subvariety of $X(\Delta)$, where $\dim \tau = k$. If $\dim V(\tau) = 1$ (resp. $n-1$), then we call $V(\tau)$ a *torus invariant curve* (resp. *torus invariant divisor*). For details on the correspondence between τ and $V(\tau)$, see for example [Fl, 3.1, Orbits].

DEFINITION 2.3 ($\mathbb{Q}$-Cartier Divisor and $\mathbb{Q}$-Factoriality). Let $D = \sum d_i D_i$ be a $\mathbb{Q}$-divisor on a normal variety X; that is, $d_i \in \mathbb{Q}$ and D_i is a prime divisor on X for every i. Then D is $\mathbb{Q}$-*Cartier* if there exists a positive integer m such that mD is a Cartier divisor. A normal variety X is said to be $\mathbb{Q}$-*factorial* if every prime divisor D on X is $\mathbb{Q}$-Cartier.

The next lemma is well known (see e.g. [R, (1.9)] or [Ma, Lemma 14-1-1]).

LEMMA 2.4. *A toric variety $X(\Delta)$ is $\mathbb{Q}$-factorial if and only if each cone $\sigma \in \Delta$ is simplicial.*

The following remarks are easy but important.

REMARK 2.5. Let D be a Cartier (resp. $\mathbb{Q}$-Cartier) divisor on a toric variety. Then D is linearly (resp. $\mathbb{Q}$-linearly) equivalent to a torus invariant divisor (resp. $\mathbb{Q}$-divisor).

REMARK 2.6 (cf. [R, (4.1)]). Let X be a toric variety and D the complement of the big torus regarded as a reduced divisor. Then $K_X + D \sim 0$.

2.7 (Kleiman–Mori Cone). Let $f\colon X \to Y$ be a proper morphism between normal varieties X and Y; a 1-cycle of X/Y is a formal sum $\sum a_i C_i$ with complete curves C_i in the fibers of f and with $a_i \in \mathbb{Z}$. We put

$$Z_1(X/Y) := \{\text{1-cycles of } X/Y\}$$

and

$$Z_1(X/Y)_{\mathbb{Q}} := Z_1(X/Y) \otimes \mathbb{Q}.$$

There is a pairing

$$\mathrm{Pic}(X) \times Z_1(X/Y)_{\mathbb{Q}} \longrightarrow \mathbb{Q}$$

defined by $(\mathcal{L}, C) \mapsto \deg_C \mathcal{L}$, extended by bilinearity. Define

$$N^1(X/Y) := (\mathrm{Pic}(X) \otimes \mathbb{Q})/\equiv,$$
$$N_1(X/Y) := Z_1(X/Y)_{\mathbb{Q}}/\equiv,$$

where the *numerical equivalence* $\equiv$ is by definition the smallest equivalence relation that makes N^1 and N_1 into dual spaces.

Inside $N_1(X/Y)$ there is a distinguished cone of effective 1-cycles,

$$\mathrm{NE}(X/Y) = \left\{ Z \;\middle|\; Z \equiv \sum a_i C_i \text{ with } a_i \in \mathbb{Q}_{\geq 0} \right\} \subset N_1(X/Y).$$

A subcone $F \subset \mathrm{NE}(X/Y)$ is said to be *extremal* if $u, v \in \mathrm{NE}(X/Y)$ and $u + v \in F$ together imply $u, v \in F$. The cone F is also called an *extremal face* of $\mathrm{NE}(X/Y)$. A one-dimensional extremal face is called an *extremal ray*.

We define the *relative Picard number* $\rho(X/Y)$ by

$$\rho(X/Y) := \dim_{\mathbb{Q}} N^1(X/Y) < \infty.$$

An element $D \in N^1(X/Y)$ is called *f-nef* if $D \geq 0$ on $\mathrm{NE}(X/Y)$.

If X is complete and Y is a point, write $\mathrm{NE}(X)$ and $\rho(X)$ for $\mathrm{NE}(X/Y)$ and $\rho(X/Y)$, respectively. We note that $N_1(X/Y) \subset N_1(X)$ and that $N^1(X/Y)$ is the corresponding quotient of $N^1(X)$.

2.2. *Singularities of Pairs*

In this section we quickly review the definitions of singularities that we use in the MMP (see e.g. [KoMo, Sec. 2.3] for details). We recommend that the reader skip this section on first reading.

2.8. Let us recall the definitions of the singularities for pairs.

DEFINITION 2.9 (Discrepancies and Singularities of Pairs). Let X be a normal variety and $D = \sum d_i D_i$ a $\mathbb{Q}$-divisor on X, where the D_i are distinct and irreducible such that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $f\colon Y \to X$ be a proper birational morphism from a normal variety Y. Then we can write

$$K_Y = f^*(K_X + D) + \sum a(E, X, D)E,$$

where the sum runs over all the distinct prime divisors $E \subset Y$ and where $a(E, X, D) \in \mathbb{Q}$. This $a(E, X, D)$ is called the *discrepancy* of E with respect to (X, D). We define

$$\mathrm{discrep}(X, D) := \inf_E \{a(E, X, D) \mid E \text{ is exceptional over } X\}.$$

From now on, we assume that $0 \le d_i \le 1$ for every i. We say that (X, D) is

$$\begin{cases} \text{terminal} \\ \text{canonical} \\ \text{klt} \\ \text{plt} \\ \text{lc} \end{cases} \quad \text{if } \operatorname{discrep}(X, D) \quad \begin{cases} > 0, \\ \ge 0, \\ > -1 \text{ and } \llcorner D \lrcorner = 0, \\ > -1, \\ \ge -1. \end{cases}$$

Here klt is an abbreviation for *Kawamata log terminal,* plt for *purely log terminal,* and lc for *log-canonical*.

Suppose there exists a log resolution $f\colon Y \to X$ of (X, D). That is, suppose Y is nonsingular, the exceptional locus $\operatorname{Exc}(f)$ is a divisor, and $\operatorname{Exc}(f) \cup f^{-1}(\operatorname{Supp} D)$ is a simple normal crossing divisor. Assume that $a(E_i, X, D) > -1$ for every exceptional divisor E_i on Y. Then the pair (X, D) is said to be dlt, which is an abbreviation for *divisorial log terminal*.

For details on dlt, see [KoMo, Def. 2.37 & Thm. 2.44]. The following results are well known to the experts; see, for example, [Fj1, Lemma 5.2] or [Ma, Prop. 14-3-2].

Proposition 2.10. *Let X be a toric variety and let D be the complement of the big torus regarded as a reduced divisor. Then (X, D) is log-canonical. Let $D = \sum_i D_i$ be the irreducible decomposition of D. We assume that $K_X + \sum_i a_i D_i$ is $\mathbb{Q}$-Cartier, where $0 \le a_i < 1$ (resp. $0 \le a_i \le 1$) for every i. Then $\left(X, \sum a_i D_i\right)$ is Kawamata log terminal (resp. log-canonical).*

3. Toric Mori Theory

Throughout this section we will work over an algebraically closed field k of arbitrary characteristic. We begin by explaining the MMP for toric varieties.

3.1 (Minimal Model Program for Toric Varieties). We start with a projective toric morphism $f\colon X \to Y$; that is, f is induced by a map of lattices, where $X =: X_0$ is a $\mathbb{Q}$-factorial toric variety and where a $\mathbb{Q}$-divisor $D_0 := D$ on X. The aim is to set up a recursive procedure that creates intermediate $f_i\colon X_i \to Y$ and D_i on X_i. After finitely many steps, we obtain final objects $\tilde{f}\colon \tilde{X} \to Y$ and $\tilde{D}$ on $\tilde{X}$. Assume that we have already constructed $f_i\colon X_i \to Y$ and D_i with the following properties:

(i) X_i is $\mathbb{Q}$-factorial, and f_i is projective;
(ii) D_i is a $\mathbb{Q}$-divisor on X_i.

If D_i is f_i-nef, then we set $\tilde{X} := X_i$ and $\tilde{D} = D_i$. If D_i is not f_i-nef, then we can take an extremal ray R of $\operatorname{NE}(X_i/Y)$ such that $R \cdot D_i < 0$ (see Theorem 4.1). Thus we have a contraction morphism $\varphi_R\colon X_i \to W_i$ over Y (see Theorem 4.5). If $\dim W_i < \dim X_i$ (in which case we call φ_R a *Fano contraction*), then we set $\tilde{X} := X_i$ and $\tilde{D} := D_i$ and stop the process. If φ_R is birational and contracts a divisor

(we call this a *divisorial contraction*), we put $X_{i+1} := W_i$ and $D_{i+1} := \varphi_{R*}D_i$ and then repeat this process. In the case where φ_R is small (we call this a *flipping contraction*), then there exists a log-flip $\psi : X_i \dashrightarrow X_i^+$ over Y. Here, a log-flip means an *elementary transformation* with respect to D_i (see Theorem 4.8). Note that ψ is an isomorphism in codimension 1. We put $X_{i+1} := X_i^+$ and $D_{i+1} := \psi_* D_i$ and then repeat this process. By counting the relative Picard number $\rho(X_i/Y)$, divisorial contractions can occur finitely many times (see Theorem 4.5). By Theorem 4.9, every sequence of log-flips terminates after finitely many steps. Hence, this process always terminates and we obtain $\tilde{f}\colon \tilde{X} \to Y$ and $\tilde{D}$. We call this process the (D-)*minimal model program over* Y, where D is a divisor used in the process. When we apply the minimal model program we say that, for example, we *run the MMP over* Y *with respect to the divisor* D.

REMARK 3.2 (Toric Mori Theory vs. the General MMP). The general (log) MMP is still conjectural in dimension ≥ 4 [KoMo, Sec. 3.7]. As we shall see in Section 4, the MMP for toric varieties is fully established and works for *any* $\mathbb{Q}$-divisor D. A generalization of the MMP for non-$\mathbb{Q}$-factorial toric varieties is treated in [Fj2, Sec. 2].

REMARK 3.3 (Original Toric Mori Theory). In the general MMP, if we assume that Y is projective, X has only terminal singularities, f is birational, and $D = K_X$ is the canonical divisor of X, then we can recover the original toric Mori theory of [R, Thm. (0.2)].

REMARK 3.4. In 3.1, it is sufficient to assume that D is an $\mathbb{R}$-divisor. We do not treat the $\mathbb{R}$-generalization here because this generalization is obvious for experts and we do not need $\mathbb{R}$-divisors in this paper. We leave the details to the reader.

REMARK 3.5. In general, the assumption that X is $\mathbb{Q}$-factorial is not crucial in the toric Mori theory. By [Fj1, Cor. 5.9], there always exists a small projective $\mathbb{Q}$-factorialization. Namely, for any toric variety X, there exists a small projective toric morphism $\tilde{X} \to X$ such that $\tilde{X}$ is $\mathbb{Q}$-factorial (see also [Fj1, Sec. 5], which is a baby version of this paper). Observe that we can remove the assumption that X is complete in [Fj1, Thm. 5.5 & Prop. 5.7] by the results in Sections 3 and 4 of this paper.

REMARK 3.6. Let X be a toric variety and D a $\mathbb{Q}$-divisor on X. The assumption that D is $\mathbb{Q}$-Cartier can be removed in some cases. In fact, by replacing X with its small projective $\mathbb{Q}$-factorialization, we can assume that D is $\mathbb{Q}$-Cartier. See the proof of Corollary 5.8.

4. Proof of Toric Mori Theory

In this section, we give a simple and noncombinatorial proof of the toric Mori theory.

THEOREM 4.1 (Cone Theorem). *Let* $f\colon X \to Y$ *be a proper toric morphism. Then the cone*

$$\mathrm{NE}(X/Y) \subset N_1(X/Y)$$

is a polyhedral convex cone. Moreover, if f is projective then the cone is strongly convex.

Proof. By taking the Stein factorization of f, we may assume that f is surjective with connected fibers. We consider $V(\sigma) \subset Y$ for some cone σ. Then $f^{-1}(V(\sigma))$ is a union of $V(\tau) \subset X$ for some cones τ, since f is a proper toric morphism. We divide Y into a finite disjoint union of tori $Y = \amalg_i Y_i$. We put $V_i := f^{-1}(Y_i) \to Y_i$. Let $\amalg_j V_{ij}$ be the normalization of V_i. Then we can check that V_{ij} is a toric variety for every i, j by using this fact—that $f^{-1}(V(\sigma))$ is a union of orbit closures—inductively on $\dim V(\sigma)$. We note that V_{ij} is dominant onto Y_i for every i, j since Y_i is a torus. Thus we obtain a collection of proper surjective toric morphisms with connected fibers: $\{V_{ij} \to Y_i\}_{i,j}$. By changing the notation V_{ij}, we write $\{f_i \colon X_i \to Y_i\}_i$ for $\{V_{ij} \to Y_i\}_{i,j}$. Note that $i \neq i'$ does not imply $Y_i \neq Y_{i'}$ in this notation. Since Y_i is a torus, $X_i \simeq F_i \times Y_i$ for every i, where F_i is a complete toric variety (cf. the exercise on p. 41 of [Fl]).

Claim. *We have the commutative diagram*

$$\begin{array}{ccc} N_1(F_i) & \simeq & N_1(X_i/Y_i) \\ \cup & & \cup \\ \mathrm{NE}(F_i) & \simeq & \mathrm{NE}(X_i/Y_i) \end{array}$$

for every i. In particular, $\mathrm{NE}(X_i/Y_i)$ is a polyhedral convex cone for every i.

Proof. We consider the cycle map $Z_1(F_i) \to Z_1(X_i/Y_i)$ that is induced by the inclusion $F_i \simeq F_i \times \{\text{a point of } Y_i\} \subset F_i \times Y_i \simeq X_i$. It induces

$$\varphi_i \colon N_1(F_i) \longrightarrow N_1(X_i/Y_i).$$

Let $0 \neq v \in N_1(F_i)$. Then there exists $\mathcal{L} \in \mathrm{Pic}(F_i)$ such that $\mathcal{L} \cdot v \neq 0$. Let $p_i \colon X_i \to F_i$ be the first projection. Then $p_i{}^*\mathcal{L} \cdot \varphi_i(v) = \mathcal{L} \cdot v \neq 0$ by the projection formula. Therefore, φ_i is injective. Since Y_i is a torus and $X_i \simeq F_i \times Y_i$, it is obvious that φ_i is surjective. Since $\mathrm{NE}(F_i)$ is well known to be a polyhedral convex cone (cf. [Fl, Prop., p. 96]), the other parts are obvious. □

We now resume our proof of the theorem. Consider the following commutative diagram:

$$\begin{array}{ccc} \bigoplus_i N_1(X_i/Y_i) & \twoheadrightarrow & N_1(X/Y) \\ \cup & & \cup \\ \bigoplus_i \mathrm{NE}(X_i/Y_i) & \twoheadrightarrow & \mathrm{NE}(X/Y), \end{array}$$

and observe that $\bigoplus_i Z_1(X_i/Y_i) \to Z_1(X/Y)$ is surjective. So, by combining this with the previous claim, we obtain the required cone theorem for $\mathrm{NE}(X/Y) \subset N_1(X/Y)$. The last part follows from Kleiman's criterion. □

We give another proof of Theorem 4.1 that works by assuming the characteristic of k to be zero. This assumption is required only in the following proof.

Proof of Theorem 4.1 in Characteristic Zero. Assume that the characteristic of k is zero. We further assume that X is quasi-projective and $\mathbb{Q}$-factorial. Let T be the big torus of X. We put $D = \sum_i D_i = X \setminus T$ regarded as a reduced divisor. We can take an ample $\mathbb{Q}$-divisor $L = \sum_i a_i D_i$ with $0 < a_i < 1$ for every i. Then

$$-\left(K_X + \sum_i (1 - a_i) D_i\right) \sim \sum_i a_i D_i$$

is ample (obviously, f-ample) and $\left(X, \sum_i (1 - a_i) D_i\right)$ is klt by Proposition 2.10. Hence, the well-known relative cone theorem (see e.g. [KMM, Thm. 4-2-1] or [KoMo, Thm. 3.25]) implies that $\mathrm{NE}(X/Y)$ is a rational polyhedral convex cone. Next, by Chow's lemma and the desingularization theorem, we can take a proper birational toric morphism $X' \to X$ from a nonsingular quasi-projective toric variety X'. As a result, the general case follows from the foregoing special case. Details are left to the reader. □

REMARK 4.2. In Theorem 4.1, if X is complete then every extremal ray of $\mathrm{NE}(X/Y)$ is spanned by torus-invariant curves on X. Related topics are treated in the first author's paper [Fj1]. If X is *not* complete then $\mathrm{NE}(X/Y)$ is not necessarily spanned by a torus invariant curve, as the following example shows.

EXAMPLE 4.3. Let Y be a one-dimensional (not necessarily complete) toric variety. We put $X = Y \times \mathbb{P}^1$. Let $f : X \to Y$ be the first projection. Then $\mathrm{NE}(X/Y)$ is a half-line. When Y is a one-dimensional torus, there are no torus-invariant curves in the fibers of f. If $Y \simeq \mathbb{P}^1$ or $\mathbb{A}^1$, then $\mathrm{NE}(X/Y)$ is spanned by a torus-invariant curve in a fiber of f.

The following remark is obvious, since a torus is a connected linear algebraic group (cf. [Su, Lemma 5]). We present it here for the reader's convenience.

REMARK 4.4. Let T be the big torus of X. Then T acts on $\mathrm{NE}(X/Y)$. Let R be an extremal ray of $\mathrm{NE}(X/Y)$. Then there exists a nef torus-invariant Cartier divisor D on X such that $D \cdot [C] = 0$ if and only if $[C] \in R$. Therefore, R is T-invariant and so T acts on $\mathrm{NE}(X/Y)$ trivially. Hence, the action of T on $N_1(X/Y)$ is trivial as well.

As a consequence, an extremal ray R of $\mathrm{NE}(X/Y)$ does not necessarily contain a torus-invariant curve even though it is torus invariant.

THEOREM 4.5 (Contraction Theorem). *Let $f : X \to Y$ be a projective toric morphism, and let F be an extremal face of* $\mathrm{NE}(X/Y)$. *Then there exists a projective surjective toric morphism*

$$\varphi_F : X \longrightarrow Z$$

over Y with the following properties.

(i) *Z is a toric variety that is projective over Y.*
(ii) *φ_F has connected fibers.*
(iii) *Let C be a curve in a fiber of f; then $[C] \in F$ if and only if $\varphi_F(C)$ is a point.*

Furthermore, if F is an extremal ray R and if X is $\mathbb{Q}$-factorial, then Z is $\mathbb{Q}$-factorial and $\rho(Z/Y) = \rho(X/Y) - 1$ if φ_R is not small.

Proof. Since $\mathrm{NE}(X/Y)$ is a polyhedral convex cone, we can take an f-nef Cartier divisor D such that $D \cdot [C] \geq 0$ for every $[C] \in \mathrm{NE}(X/Y)$ and $D \cdot [C] = 0$ if and only if $[C] \in F$. Replacing D with a linearly equivalent divisor, we may assume that D is a torus-invariant Cartier divisor on X. We put $\varphi_F : X \to Z$ as in Proposition 4.6. Then $\varphi_F : X \to Z$ has the required properties. The latter part of the contraction theorem is well known (see e.g. [KoMo, Prop. 3.36]). □

PROPOSITION 4.6. *Let $f : X \to Y$ be a proper surjective toric morphism between toric varieties. Let D be an f-nef torus-invariant Cartier divisor on X. Then D is f-free, that is, $f^*f_*\mathcal{O}_X(D) \to \mathcal{O}_X(D)$ is surjective. Moreover, we have a projective toric morphism $\varphi : X \to Z$ over Y such that*

(i) *φ has only connected fibers, and*
(ii) *for any irreducible curve on X and with $f(C)$ a point, $\varphi(C)$ is a point if and only if $D \cdot C = 0$.*

Proof. Let

$$f : X \xrightarrow{g} \tilde{Y} \xrightarrow{h} Y$$

be the Stein factorization of f. For part (i), we may assume that Y is affine (hence so is $\tilde{Y}$). It is sufficient to prove that D is g-free. We can apply the argument in [Fl, Prop., p. 68] with minor modifications. See also [N, Ch. IV, 1.8 Lemma (2)] and [Ma, Lemma 14-1-11]. For part (ii), $\varphi : X \to Z := \mathrm{Proj}_{\tilde{Y}} \bigoplus_{m\geq 0} g_*\mathcal{O}_X(mD)$ is equivariant by construction. When $\tilde{Y}$ is a point, it is well known that Z is a projective toric variety constructed from a suitable polytope. Let $T \subset \tilde{Y}$ be the big torus. Then $g^{-1}(T) \simeq T \times F$ for some complete toric variety F. Hence Z contains a torus as a nonempty Zariski open set by the previous case (where $\tilde{Y}$ is a point). It is obvious that Z is normal and has a suitable torus action by construction. Therefore, Z is the required toric variety. □

THEOREM 4.7 (Finitely Generatedness of Divisorial Algebra). *Let $f : X \to Y$ be a proper birational toric morphism and D a torus invariant Cartier divisor on X. Then*

$$\bigoplus_{m\geq 0} f_*\mathcal{O}_X(mD)$$

is a finitely generated $\mathcal{O}_Y$-algebra.

Proof. We may assume that Y is affine. It is thus sufficient to show that

$$\bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(mD))$$

is a finitely generated k-algebra. We put $X = X(\Delta)$; that is, X is a toric variety associated to a fan Δ in N. Let ψ_D be the support function of D. We put

$$P_D = \{u \in M_{\mathbb{R}} \mid u \geq \psi_D \text{ on } |\Delta|\},$$
$$P_{aD} = \{u \in M_{\mathbb{R}} \mid u \geq a\psi_D \text{ on } |\Delta|\},$$

where $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ is the dual lattice of N and $|\Delta|$ stands for the *support* of the fan Δ. We define

$$C = \{(u, a) \in M_{\mathbb{R}} \times \mathbb{R}_{\geq 0} \mid u \in P_{aD}\}$$

and $C_{\mathbb{Z}} = C \cap (M \times \mathbb{Z}_{\geq 0})$. The k-algebra $\bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mD))$ is the semi-group ring associated to $C_{\mathbb{Z}}$. We can easily check that C is a finite intersection of half-spaces, which are defined over the rational numbers, in $M_{\mathbb{R}} \times \mathbb{R}$. Hence C is a rational polyhedral convex cone and so $C_{\mathbb{Z}}$ is a finitely generated semi-group. This implies that $\bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mD))$ is a finitely generated k-algebra. □

We will generalize Theorem 4.7 in Corollary 5.8 to follow.

THEOREM 4.8 (Elementary Transformation). *Let $\varphi\colon X \to W$ be a small toric morphism and let D be a torus-invariant $\mathbb{Q}$-Cartier divisor on X such that $-D$ is φ-ample. Let l be a positive integer such that lD is Cartier. Then there exists a small projective toric morphism*

$$\varphi^+\colon X^+ := \mathrm{Proj}_W \bigoplus_{m \geq 0} \varphi_* \mathcal{O}_X(mlD) \longrightarrow W$$

such that D^+ is a φ^+-ample $\mathbb{Q}$-Cartier divisor, where D^+ is the proper transform of D on X^+. The commutative diagram

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow \quad \swarrow & \\ & W & \end{array}$$

is called the elementary transformation (with respect to D).

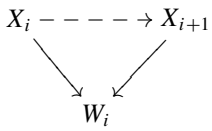
Moreover, if X is $\mathbb{Q}$-factorial and $\rho(X/W) = 1$, then likewise X^+ is $\mathbb{Q}$-factorial and $\rho(X^+/W) = 1$.

Proof. The first part is obvious by the previous theorem and the construction of $\varphi^+\colon X \to W$. See, for example, [Ko+, Prop. 4.2] or [KoMo, Lemma 6.2]. The latter part is well known (see e.g. [KoMo, Prop. 3.37]). □

THEOREM 4.9 (Termination of Elementary Transformations). *Let*

$$\begin{array}{ccccccccc} X_0 & \dashrightarrow & X_1 & \dashrightarrow & X_2 & \dashrightarrow & \cdots \\ & \searrow \quad \swarrow & & \searrow \quad \swarrow & & & \\ & W_0 & & W_1 & & & \end{array}$$

be a sequence of elementary transformations of toric varieties with respect to a fixed $\mathbb{Q}$-Cartier divisor D. More precisely, the commutative diagram

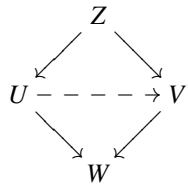


is the elementary transformation with respect to D_i, where $D_0 = D$ and D_i is the proper transform of D on X_i for every i (see Theorem 4.8). Then the sequence terminates after finitely many steps.

Proof. Suppose there exists an infinite sequence of elementary transformations, and let Δ be the fan corresponding to X_0. Since the elementary transformations do not change one-dimensional cones of Δ, there exist numbers $k < l$ such that the composition $X_k \dashrightarrow X_{k+1} \dashrightarrow \cdots \dashrightarrow X_l$ is an identity. This contradicts the negativity in Lemma 4.10. □

The following result is easy but very important. The proof is well known (see e.g. [KoMo, Lemma 3.38]).

LEMMA 4.10 (Negativity Lemma). *Consider a commutative diagram*



and $\mathbb{Q}$-Cartier divisors D and D' on U and V, respectively, where:

(1) *$f: U \to W$ and $g: V \to W$ are birational morphisms between varieties*;
(2) *$f_*D = g_*D'$*;
(3) *$-D$ is f-ample and D' is g-ample*; *and*
(4) *$\mu: Z \to U$ and $\nu: Z \to V$ are common resolutions.*

Then $\mu^*D = \nu^*D' + E$, where E is an effective $\mathbb{Q}$-divisor and E is exceptional over W. Moreover, if f or g is nontrivial then $E \neq 0$.

5. On the Zariski Decomposition

In this section, we treat the Zariski decomposition on toric varieties. Since the MMP works for any divisors, it is obvious that the Zariski decomposition holds with no extra assumptions.

There are many variants of the Zariski decomposition. Here we adopt the following definition (cf. [KMM, Def. 7-3-5]).

DEFINITION 5.1 (Zariski Decomposition). Let $f: X \to Y$ be a proper surjective morphism of normal varieties. An expression $D = P + N$ with $\mathbb{R}$-Cartier divisors D, P, and N on X is called the *Zariski decomposition of D relative to f*

in the sense of Cutkosky–Kawamata–Moriwaki (we write CKM for short) if the following conditions are satisfied:

(1) P is f-nef,
(2) N is effective, and
(3) the natural homomorphisms $f_*\mathcal{O}_X(\llcorner mP \lrcorner) \to f_*\mathcal{O}_X(\llcorner mD \lrcorner)$ are bijective for all $m \in \mathbb{N}$.

The divisors P and N are said to be the *positive* and *negative part* of D, respectively.

DEFINITION 5.2 (Pseudo-Effective Divisors). Let $f: X \to Y$ be a projective morphism between varieties and let D be a $\mathbb{Q}$-Cartier divisor on X. Then D is *f-pseudo-effective* if there is an f-big (see [KoMo, Def. 3.22]) Cartier divisor A on X such that $nD + A$ is f-big for every $n \geq 0$.

REMARK 5.3. Let $f: X \to Y$ be a projective morphism between varieties and let D be a $\mathbb{Q}$-Cartier divisor on X. It is not difficult to see that, if D is f-pseudo-effective, then $nD + A$ is f-big for every $n \geq 0$ and any f-big Cartier divisor A (cf. [Mo, (11.3)]). In particular, if D is an effective divisor on X, then D is f-pseudo-effective. More generally, if there exists an $m > 0$ such that $f_*\mathcal{O}_X(mD) \neq 0$, then D is f-pseudo-effective.

The following theorem is a slight generalization of [K, Prop. 5]. Related topics are in [N, Ch. IV, Sec. 1]. Both [K] and [N] showed how to subdivide a given fan.

THEOREM 5.4 (cf. [K, Prop. 5]). *Let $f: X \to Y$ be a projective surjective toric morphism and let D be a $\mathbb{Q}$-Cartier divisor on X. Assume that D is f-pseudo-effective. Then there exists a projective birational toric morphism $\mu: Z \to X$ such that μ^*D has a Zariski decomposition relative to $f \circ \mu$ in the sense of CKM whose positive part is $f \circ \mu$-semi-ample* (*see* [KMM, Def. 0-1-4]).

REMARK 5.5. If $f_*\mathcal{O}_X(mD) \neq 0$ for some positive integer m, then it is easy to check that $(f_k)_*\mathcal{O}_{X_k}(mD_k) \neq 0$ for every k; here $f_k: X_k \to Y$ is as in the following proof. Hence, that proof works without any changes even if we replace the assumption that D is f-pseudo-effective with a slightly stronger one that $f_*\mathcal{O}_X(mD) \neq 0$ for some positive integer m. Thus, it may not be necessary to introduce the notion of f-pseudo-effective divisors. See Corollary 5.6 and the proof of Corollary 5.8.

Proof of Theorem 5.4. By taking a resolution of singularities, we may assume without loss of generality that X is nonsingular. Run the MMP on X over Y with respect to D. We obtain a sequence of divisorial contractions and elementary transformations over Y:

$$X =: X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \cdots \dashrightarrow X_k \dashrightarrow X_{k+1} \dashrightarrow \cdots .$$

Since D_k is pseudo-effective over Y for every k, there exists an l such that D_l is nef over Y (for the definition of D_k, see 3.1). The reader may verify that relative pseudo-effectivity of D is preserved in each step by Lemma 4.10 and Remark 5.3.

Take a nonsingular quasi-projective toric variety Z with proper birational toric morphisms $\mu\colon Z \to X$ and $\mu_i\colon Z \to X_i$ for every $0 < i \leq l$. Then we obtain that $\mu^*D = \mu_l^*D_l + E$, where E is an effective $\mathbb{Q}$-divisor by the negativity in Lemma 4.10. This decomposition is the Zariski decomposition of D in the sense of CKM. □

The next corollary is obvious by Theorem 5.4.

COROLLARY 5.6. *Let $f\colon X \to Y$ be a projective surjective toric morphism and let D be a $\mathbb{Q}$-Cartier divisor on X. Then D is f-pseudo-effective if and only if $f_*\mathcal{O}_X(mD) \neq 0$ for some positive integer m.*

REMARK 5.7. There exist various generalizations of Theorem 5.4. We do not pursue such generalizations here. For example, Theorem 5.4 holds for a (not necessarily $\mathbb{R}$-Cartier) $\mathbb{R}$-divisor D, with suitable modifications. We leave the details to the reader.

The following result is a generalization of Theorem 4.7.

COROLLARY 5.8 (Finitely Generatedness of Divisorial Algebra II). *Let $f\colon X \to Y$ be a proper surjective toric morphism, and let D be a (not necessarily $\mathbb{Q}$-Cartier) Weil divisor on X. Then*

$$\bigoplus_{m\geq 0} f_*\mathcal{O}_X(mD)$$

is a finitely generated $\mathcal{O}_Y$-algebra.

Proof. By Remarks 3.5 and 3.6, we may assume that X is $\mathbb{Q}$-factorial. Hence, D is $\mathbb{Q}$-Cartier. By replacing X birationally, we may assume that f is projective. If $f_*\mathcal{O}_X(mD) = 0$ for every $m > 0$, then the claim is obvious. Therefore, we may assume that $f_*\mathcal{O}_X(mD) \neq 0$ for some $m > 0$, that is, D is f-pseudo-effective. Since (by Theorem 5.4) there exists a projective birational toric morphism $\mu\colon Z \to X$ such that μ^*D has a Zariski decomposition with $f \circ \mu$-semi-ample positive part, it follows that $\bigoplus_{m\geq 0} f_*\mathcal{O}_X(mD)$ is finitely generated. □

6. Application to Hypersurface Singularities

In this section we apply toric Mori theory to the study of singularities, and we shall recover Ishii's results [I1]. We work over the complex number field $\mathbb{C}$ throughout this section.

Let us briefly recall the notion of *nondegenerate* hypersurface singularities. For the details, see [I1].

DEFINITION 6.1 (Nondegenerate Polynomials). For a polynomial

$$f = \sum_{\mathbf{m}} a_{\mathbf{m}} x^{\mathbf{m}} \in \mathbb{C}[x_0, x_1, \ldots, x_n],$$

where $x^{\mathbf{m}} = x_0^{m_0} x_1^{m_1} \cdots x_n^{m_n}$ for $\mathbf{m} = (m_0, m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0}^{n+1}$, and a face γ of the Newton polytope $\Gamma_+(f)$ of f, denote $\sum_{\mathbf{m} \in \gamma} a_{\mathbf{m}} x^{\mathbf{m}}$ by f_γ. A polynomial f is said to be *nondegenerate* if, for every compact face γ of $\Gamma_+(f)$, the $\partial f_\gamma / \partial x_i$ $(i = 0, \ldots, n)$ have no common zero on $(\mathbb{C}^\times)^{n+1}$.

The following definitions are due to Ishii; see the introduction of [I1]. For the definitions of the singularities, see Definition 2.9.

DEFINITION 6.2 (Minimal and Canonical Models). Let $(x \in X)$ be a germ of normal singularity on an algebraic variety. We call a morphism $\varphi\colon Y \to X$ *a minimal* (resp. *the canonical*) *model* of $(x \in X)$ if

(1) φ is a proper and birational morphism,
(2) Y has at most terminal (resp. canonical) singularities, and
(3) K_Y is φ-nef (resp. φ-ample).

Obviously, if a canonical model exists then it is unique up to isomorphisms over X.

The next theorem is [I1, Thm. 2.3].

THEOREM 6.3. *Let $X \subset \mathbb{C}^{n+1}$ be a normal hypersurface defined by a nondegenerate polynomial f. Then $(0 \in X)$ has both a minimal model and a canonical model.*

Proof. Take a projective birational toric morphism $g\colon V \to \mathbb{C}^{n+1}$ such that V is a nonsingular toric variety and the proper transform X' of X on V is nonsingular (see e.g. [I1, Prop. 2.2]). Run the MMP over $\mathbb{C}^{n+1}$ with respect to $K_V + X'$. Then we obtain $\tilde{\varphi}\colon (\tilde{V}, \tilde{X}) \to \mathbb{C}^{n+1}$ such that $K_{\tilde{V}} + \tilde{X}$ is $\tilde{\varphi}$-nef. We note that the pair $(\tilde{V}, \tilde{X})$ has canonical singularities and that $\tilde{V}$ has at most terminal singularities; hence $\tilde{V}$ is nonsingular in codimension 2. Thus we obtain $K_{\tilde{X}} = (K_{\tilde{V}} + \tilde{X})|_{\tilde{X}}$. Therefore, $K_{\tilde{X}}$ is nef over X. It is not difficult to check that $\tilde{X}$ has at most terminal singularities. Hence, this $\tilde{X}$ is a minimal model of $(0 \in X)$ (see Definition 6.2). By using the relative base point–free theorem (see e.g. [KMM, Thm. 3-1-1 & Rem. 3-1-2(1)] or [KoMo, Thm. 3.2.4]) we obtain the canonical model of $(0 \in X)$. □

DEFINITION 6.4 (Dlt and Log-Canonical Models). Let $(x \in X)$ be as in Definition 6.2. We call a morphism $\varphi\colon Y \to X$ *a dlt* (resp. *the log-canonical*) *model* of $(x \in X)$ if:

(1) φ is proper birational;
(2) (Y, E) is dlt (resp. log-canonical), where E is the reduced exceptional divisor of φ; and
(3) $K_Y + E$ is φ-nef (resp. φ-ample).

Clearly, if a log-canonical model exists then it is unique up to isomorphisms over X. The notion of dlt models is new.

The next result is a slight generalization of [I1, Thm. 3.1]. The arguments in the following proof are more or less known to experts of the MMP.

THEOREM 6.5. *Let $X \subset \mathbb{C}^{n+1}$ be a normal hypersurface defined by a nondegenerate polynomial f. Then $(0 \in X)$ has both a minimal model and a log-canonical model.*

Proof. Take a projective birational toric morphism $f_0 \colon V_0 \to \mathbb{C}^{n+1}$ such that (a) V_0 is a nonsingular toric variety and (b) the proper transform X_0 of X on V_0 is nonsingular. We may assume that the reduced exceptional divisor E_0 of f_0 intersects X_0 transversally, that is, $E_0 \cup X_0$ is a simple normal crossing divisor on V_0 (see e.g. [I1, Prop. 2.2]). We note that f_0 is an isomorphism outside E_0. Run the MMP over $\mathbb{C}^{n+1}$ with respect to $K_{V_0} + X_0 + E_0$. Then we obtain a sequence of divisorial contractions and elementary transformations

$$V_0 \dashrightarrow V_1 \dashrightarrow V_2 \dashrightarrow \cdots \dashrightarrow V_k \dashrightarrow V_{k+1} \dashrightarrow \cdots,$$

and the final object $\tilde{f} \colon \tilde{V} \to \mathbb{C}^{n+1}$ has the property that $K_{\tilde{V}} + \tilde{X} + \tilde{E}$ is $\tilde{f}$-nef, where $\tilde{X}$ is the proper transform of X_0 on $\tilde{V}$ and $\tilde{E}$ is the reduced $\tilde{f}$-exceptional divisor. We note that the exceptional locus of $V_i \to \mathbb{C}^{n+1}$ is of pure codimension 1 for every i, since $\mathbb{C}^{n+1}$ is nonsingular. Therefore, $\tilde{f}$ is an isomorphism outside $\tilde{E}$. Since $(\tilde{V}, \tilde{X} + \tilde{E})$ is dlt (cf. [KoMo, Cor. 3.44]), $\tilde{V}$ is nonsingular in codimension 2 around $\tilde{X} \cap \tilde{E}$ (cf. [KoMo, Cor. 5.55]). Thus, we obtain that $K_{\tilde{X}} + \tilde{E}|_{\tilde{X}} = (K_{\tilde{V}} + \tilde{X} + \tilde{E})|_{\tilde{X}}$ and that $(\tilde{X}, \tilde{E}|_{\tilde{X}})$ is dlt (cf. [KoMo, Prop. 5.59]). We have to check that $\tilde{E}|_{\tilde{X}}$ is a reduced $\tilde{f}|_{\tilde{X}}$-exceptional divisor. Let $\tilde{E} = \sum_i \tilde{E}_i$ be the irreducible decomposition. It is sufficient to show that $\tilde{f}(\tilde{E}_i) \subset \mathrm{Sing}(X)$, where $\mathrm{Sing}(X)$ is the singular locus of X. We write $K_{\tilde{V}} + \tilde{X} + \sum a_i \tilde{E}_i = \tilde{f}^*(K_{\mathbb{C}^{n+1}} + X)$. Hence, $\sum(1 - a_i)\tilde{E}_i$ is $\tilde{f}$-nef. If $\tilde{f}(\tilde{E}_i) \not\subset \mathrm{Sing}(X)$, then $a_i < 1$. We note that $(\mathbb{C}^{n+1}, X)$ is plt outside $\mathrm{Sing}(X)$ (for the definition of plt, see Definition 2.9). This implies that $\tilde{f}(\tilde{E}_i) \subset \mathrm{Sing}(X)$ for every i by [KoMo, Lemma 3.39]. Therefore, $(\tilde{X}, \tilde{E}|_{\tilde{X}})$ is a dlt model of $(0 \in X)$. By construction, $\tilde{f}|_{\tilde{X}}$ is an isomorphism outside $\tilde{E}|_{\tilde{X}}$. Since $K_{\tilde{V}} + \tilde{X} + \tilde{E}$ is nef over $\mathbb{C}^{n+1}$, we obtain a contraction morphism $\tilde{V} \to V'$ over $\mathbb{C}^{n+1}$ with respect to the divisor $K_{\tilde{V}} + \tilde{X} + \tilde{E}$ (cf. Theorem 4.5). Let $X^\heartsuit$ be the normalization of the proper transform of $\tilde{X}$ on V'. We put $E^\heartsuit := \mu_*(\tilde{E}|_{\tilde{X}})$, where $\mu \colon \tilde{X} \to X^\heartsuit$. Then it is not difficult to see that $K_{X^\heartsuit} + E^\heartsuit$ is ample over X, $E^\heartsuit$ is the reduced exceptional divisor of $X^\heartsuit \to X$, and $K_{\tilde{X}} + \tilde{E}|_{\tilde{X}} = \mu^*(K_{X^\heartsuit} + E^\heartsuit)$. Thus, $(X^\heartsuit, E^\heartsuit)$ is the required log-canonical model of $(0 \in X)$. □

REMARK 6.6 (Ishii's Constructions). Answering our questions, Ishii informed us that the definition of $\bar{E}$ in Claim 3.8 in the proof of [I1, Thm. 3.1] is not correct. She told us that $\bar{E}$ should be defined as $\nu^*(K_{T_N(\Sigma_2)} + X(\Sigma_2) + E) - K_{\bar{X}}$, and then all discussions go well in the proof of that theorem. Though we did not check the proof according to her corrected definition of $\bar{E}$, we see that our models coincide with her models. From now on, we freely use the notation in [I1].

Let F be the complement of the big torus of $T_N(\Sigma)$. Then the pair $(T_N(\Sigma), X(\Sigma))$ (resp. $(T_N(\Sigma), X(\Sigma) + F)$) has only canonical (resp. log-canonical) singularities. Therefore, by the arguments in Claim 2.8 of [I1], it is not difficult to see that $m_{\mathbf{q}} \geq 0$ (resp. $m_{\mathbf{q}} \geq -1$) for every $\mathbf{q} \in \tilde{\Sigma}[1] \setminus \Sigma_0[1]$ (resp. $\mathbf{q} \in \tilde{\Sigma}[1] \setminus \Sigma_2[1]$)

without the assumption that $D_{\mathbf{q}} \cap X(\tilde{\Sigma}) \neq \emptyset$ in Claim 2.8 (resp. Claim 3.5) in [I1]. Thus, we see that $(T_N(\Sigma_0), X(\Sigma_0))$ (resp. $(T_N(\Sigma_2), X(\Sigma_2) + E)$) has only canonical (resp. log-canonical) singularities. Ishii told us that she did not know the notion of singularities of pairs when she wrote [I1]. Therefore, $T_N(\Sigma_0) \simeq \mathrm{Proj}_{\mathbb{C}^{n+1}} \bigoplus_{m\geq 0} g_*\mathcal{O}_V(m(K_V + X'))$, where g, V, and X' are as in the proof of Theorem 6.3, and $T_N(\Sigma_2) \simeq \mathrm{Proj}_{\mathbb{C}^{n+1}} \bigoplus_{m\geq 0} f_{0*}\mathcal{O}_X(m(K_{V_0} + X_0 + E_0))$, where V_0, X_0, E_0, and f_0 are as in the proof of Theorem 6.5. Hence it is not difficult to see that the models constructed in [I1] coincide with ours. Details are left to the reader.

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