Michigan Math. J. 52 (2004)

Extremal Bases and Hölder Estimates for $\bar\partial$ on Convex Domains of Finite Type

Torsten Hefer

1. Introduction

Let $D \subset\subset \mathbb{C}^n$ be a smoothly bounded convex domain of finite type in the sense of D'Angelo [D'A]. The main object of this paper is to compare two different pseudo-distances and related geometric constructions describing the complex geometry of the boundary bD of D. These constructions originated with the work of McNeal on complex geometry and on the Bergman kernels of convex domains of finite type [Mc1; Mc2]; today they have taken an important place also in the study of other function-theoretic questions on convex domains of finite type, especially for problems in which the geometric shape of the domain is to be precisely measured and translated into quantitative function-theoretic information. Prominent examples of such problems are the characterization of the zero sets of functions in the Nevanlinna class as well as finding optimal estimates for the inhomogeneous Cauchy–Riemann equation $\bar\partial u = f$ in D, where f is a $\bar\partial$-closed differential form of bidegree $(0, q+1)$ belonging to some function space—for example, to $L^p_{0,q+1}(D)$, $1 \le p \le \infty$. Our results should be seen in this context, although some of them can also be viewed as purely geometric statements about smooth convex domains of finite type that may be interesting in their own right.

The main application that we will be concerned with in this paper is the problem of (optimal) Hölder estimates for the $\bar\partial$-equation with essentially bounded right-hand side. We do not attempt to give complete references for the other problems already mentioned, instead referring the interested reader to the work of McNeal [Mc1; Mc2], McNeal and Stein [McS], Bruna, Charpentier, and Dupain [BCDu], Diederich and Mazzilli [DM], and Cumenge [Cu3]. A different approach to Hölder estimates for the $\bar\partial$-operator from the one pursued here is described in the work of Cumenge [Cu1; Cu2].

The starting point to motivate our investigations is the following theorem of Diederich, Fischer, and Fornæss [DFiFo] and Cumenge [Cu1].

1.1. Theorem. *Let $D \subset\subset \mathbb{C}^n$ be a convex domain with $\mathcal{C}^\infty$-smooth boundary of finite type m. Then there exist bounded linear integral operators*

Received May 28, 2003. Revision received November 27, 2003.
Partially supported by the *Sonderforschungsbereich 611,* Bonn.

$$T_q\colon L^\infty_{0,q+1}(D) \to \Lambda^{1/m}_{0,q}(D) \quad \textit{for } q = 0,\dots,n-1$$

such that $\bar\partial T_q f = f$ *for all* $f \in L^p_{0,q+1}(D)$ *with* $\bar\partial f = 0$.

Here, for $0 < \alpha < 1$, $\Lambda^\alpha_{0,q}(D)$ denotes the space of $(0,q)$-forms on D having coefficients that are Hölder continuous with exponent α. The result in this theorem is optimal in the sense that, for every even integer m and every $q \in \{0,\dots,n-2\}$, there exist a domain D and a $\bar\partial$-closed $(0,q)$-form f such that no solution v of the equation $\bar\partial v = f$ belongs to a Hölder space Λ^α for $\alpha > 1/m$. The optimal gain of Hölder regularity is thus expressed by a geometric invariant—namely, by the maximal possible order of contact of one-dimensional complex varieties with bD. This beautiful result generalizes the classical Hölder-$\frac{1}{2}$-estimate for $\bar\partial$ on strictly pseudoconvex domains by Kerzman [Ke] and Henkin and Romanov [HR]. In a subsequent paper, Fischer [Fi1] proved optimal type-dependent estimates for $\bar\partial$ also in L^p-spaces for $p < \infty$. Based on these results and using an idea due to Fleron [F], we then proved optimal Hölder and L^p-estimates for the equation $\bar\partial u = f$ that incorporated also the degree $(0,q+1)$ of the form $f \in L^p_{0,q+1}(D)$ (see [He2]). These optimal estimates show a precise connection between the degree and the multitype of the boundary of the domain. (The multitype is a biholomorphic invariant originally defined by Catlin [Ca1], who used it to study regularity of the $\bar\partial$-Neumann problem on pseudoconvex domains of finite type; we will discuss it in greater detail below.) The statement is as follows.

1.2. THEOREM. *Let* $D \subset\subset \mathbb{C}^n$ *be a convex domain with* $\mathcal{C}^\infty$*-smooth boundary of finite type* m_n. *Let* $M(bD) = (m_1,\dots,m_n)$ *be the multitype of* bD (*cf. Definition 2.1*), *and let* $\lambda_q := m_{n-q}(n-q) + 2q + 2$. *Then there exist bounded linear integral operators*

$$\hat S_q\colon L^p_{0,q+1}(D) \to L^{s(p,q)}_{0,q}(D) \quad \textit{for } q = 0,\dots,n-1$$

such that $\bar\partial \hat S_q f = f$ *for all* $f \in L^p_{0,q+1}(D)$ *with* $\bar\partial f = 0$; *the numbers* $s(p,q)$ *are defined by*

$$\frac{1}{s(p,q)} := \frac{1}{p} - \frac{1}{\lambda_q},$$

and the usual convention $s(p,q) = \infty$ *is applied if the right-hand side of this inequality is negative. If the right-hand side is zero, then* $s(p,q)$ *may be any finite number* ≥ 1. *Furthermore, there exist solution operators* S_q *that satisfy the following Hölder estimates*: *There are constants* $C_p > 0$ *such that, if* $p > \lambda_q$, *we have*

$$\|S_q f\|_{\Lambda^{\alpha(p,q)}_{0,q}(D)} \leq C_p \|f\|^p_{L_{0,q+1}(D)}$$

for

$$\alpha(p,q) := \frac{1}{m_{n-q}}\left(1 - \frac{\lambda_q}{p}\right) \quad \textit{and for} \quad \alpha(\infty,q) := \frac{1}{m_{n-q}}$$

with $q < n-1$. *In the special case where* $p = \infty$ *and* $q = n-1$, *the value of* $\alpha(\infty,n-1)$ *can be chosen to be* $1-\varepsilon$ (*which coincides with* $= 1/m_1 - \varepsilon$) *for any* $\varepsilon \in (0,1)$.

This result has the following geometric interpretation. McNeal [Mc1] has shown that, on smooth convex domains of finite type, the line type and the variety type coincide (a simpler proof of this fact was later given by Boas and Straube [BoSt]); this was generalized by Yu, who proved in [Yu] that the entry m_{n-q+1} of the multitype $(m_1, \dots, m_n)$ of bD at a point p equals the maximal order of contact of q-dimensional complex manifolds through p with bD, which in turn equals the maximal order of contact with bD of q-dimensional complex hyperplanes through p. Thus, our theorem states that the maximal order of contact with bD of $(q+1)$-dimensional complex manifolds (or hyperplanes) passing through points of bD determines the optimal Hölder and L^p estimates for solutions of the $\bar{\partial}$-equation on $(0, q+1)$-forms. The optimality of these estimates was shown, for example, in [F] and [He1] on certain model domains (essentially, all optimality proofs for the $\bar{\partial}$-equation are based on a general idea by Stein, already used in Krantz's paper [K1]).

More recently, Fischer [Fi2] has reconsidered the problem of finding Hölder estimates for the $\bar{\partial}$-operator on convex domains of finite type, noting that the estimates for the solution operators T_q constructed in [DFiFo] can be better adapted to the geometric situation if they are expressed in terms of the anisotropic pseudo-distance on D, which was first defined and studied by McNeal. Geometrically speaking, Fischer showed that there is a solution u of $\bar{\partial} u = f \in L^{\infty}_{0,q+1} \cap \ker \bar{\partial}$ that has higher Hölder regularity in all directions except the normal direction to bD, whereas in the normal direction the optimal Hölder exponent is the one from the isotropic estimate for $(0,1)$-forms. This result is very much in spirit of the results of Krantz (see [K1; K2; K3]) on nonisotropic Hölder estimates for $\bar{\partial}$ in strictly pseudoconvex domains (the method of proof is, of course, different).

In the strictly pseudoconvex case, however, all directions in the tangent space at a point are equivalent with respect to the estimates, whereas here the geometric situation is more complicated, which makes it much more difficult to rigorously express the fact that solutions to the $\bar{\partial}$-equation behave differently in different tangential directions. For this, the concept of McNeal's extremal bases and the related pseudo-distance (see Definitions 1.3 and 1.5 to follow) are very useful.

The precise formulation of Fischer's result is given in Theorem 1.6. Before we can state this theorem, we need to review McNeal's geometric constructions. The role they play in the theory of estimates for the $\bar{\partial}$-equation will become clearer in the last section; essential is the fact that the holomorphic support function constructed by Diederich and Fornæss [DFo], which is a key ingredient in the construction of explicit solution operators for the $\bar{\partial}$-equation on convex domains of finite type, can be estimated in terms of the extremal bases and pseudo-distances defined in what follows.

First, let us recall from [He2] that the defining function r of a smooth convex domain D may be chosen in such a way that all the level sets

$$bD_\zeta := \{z : r(z) = r(\zeta)\}$$

have the same multitype as bD for ζ close to bD (and are, in fact, homothetic to one another with respect to a chosen inner point of D). This is a simple consequence of the implicit function theorem. For simplicity we will always assume that r is such a function, and when speaking of a *sufficiently small* compact neighborhood K of bD we will always assume that all level sets of r that intersect K are homothetic to bD.

1.3. DEFINITION. Let $D = \{r < 0\} \subset\subset \mathbb{C}^n$ be a smooth convex domain of finite type m, and let r be a defining function of the special type just described. For $\zeta \in K$ with K a sufficiently small compact neighborhood of bD and for $v \in \mathbb{C}^n$ and $\varepsilon > 0$, we define the *ε-distance in direction v* by

$$\tau(\zeta, v, \varepsilon) := \sup\{c : |r(\zeta + \lambda v) - r(\zeta)| \leq \varepsilon \text{ for all } \lambda \in \mathbb{C}, \ |\lambda| \leq c\}.$$

If the point ζ and $\varepsilon > 0$ are given, a (not necessarily unique) *ε-extremal basis* (or *McNeal ε-basis*) $(w_1, \dots, w_n)$ at ζ is defined as follows: Let w_1 be the unit vector in the direction of the real gradient of the defining function r at ζ; let p_1^+ be the point of intersection of the real line

$$\lambda \mapsto \zeta + \lambda w_1, \quad \lambda \in \mathbb{R}, \ \lambda > 0,$$

with the level set $bD_{\zeta,\varepsilon} := \{z : r(z) = r(\zeta) + \varepsilon\}$. Then choose a point p_2^+ with $r(p_2^+) = r(\zeta) + \varepsilon$ such that the vector $\tilde{w}_2 := p_2^+ - \zeta$ is a solution of the constrained extremal problem

$$|\tilde{w}_2| = \text{maximum}$$

subject to the conditions

$$\tilde{w}_2 \perp \langle w_1 \rangle \quad \text{and} \quad r(\zeta + e^{i\vartheta}\tilde{w}_2) \leq r(\zeta) + \varepsilon \ \text{ for all } \vartheta \in [0, 2\pi],$$

where $\langle w_1 \rangle$ denotes the complex linear span of w_1, and orthogonality is measured by the standard hermitian inner product of $\mathbb{C}^n$. Then $w_2 := \tilde{w}_2/|\tilde{w}_2|$ belongs to the complex tangent space to bD_ζ at the point ζ, and $\tau(\zeta, w_2, \varepsilon) = |\tilde{w}_2|$ is maximal among all $\tau(\zeta, w, \varepsilon)$ for $|w| = 1$, with w in the complex tangent space to bD_ζ at ζ. Note that in the point p_2^+ the normal direction of the surface $\langle w_1 \rangle^\perp \cap bD_{\zeta,\varepsilon}$ is given by w_2, so in this point the gradient of r is a linear combination of w_1 and w_2. (Alternatively, we could have defined $\tilde{w}_2$ as any vector in the complex tangent space where the continuous function $v \mapsto \tau(\zeta, v, \varepsilon)$ attains a maximum, then taking p_2^+ as any point on the intersection of the circle $\{\zeta + e^{i\vartheta}\tilde{w}_2 : \vartheta \in [0, 2\pi]\}$ with the level set $bD_{\zeta,\varepsilon}$.)

Next, choose a point p_3^+ such that $\tilde{w}_3 = p_3^+ - \zeta$ solves the extremal problem

$$|\tilde{w}_3| = \text{maximum}$$

under the conditions

$$\tilde{w}_3 \perp \langle w_1, w_2 \rangle \quad \text{and} \quad r(\zeta + e^{i\vartheta}\tilde{w}_3) \leq r(\zeta) + \varepsilon \ \text{ for all } \vartheta \in [0, 2\pi].$$

Then letting $w_3 := \tilde{w}_3/|\tilde{w}_3|$, it follows that $\tau(\zeta, w_3, \varepsilon) = |\tilde{w}_3|$ is maximal among all $\tau(\zeta, w, \varepsilon)$ for $|w| = 1$, $w \perp \langle w_1, w_2 \rangle$. We continue this process until we finally end up with a basis $w := (w_1, \dots, w_n)$ and with n "maximal" points $p_1^+, \dots, p_n^+$. The *ε-extremal coordinates* (or *McNeal coordinates*) $(z_1, \dots, z_n)$ of a point z with respect to ζ are then defined by choosing parameterizations of the complex lines through ζ in direction w_j such that (a) $z_j = 0$ corresponds to the point ζ and (b) the point p_j^+ corresponds to the point

$$(0, \ldots, 0, |p_j^+ - \zeta|, 0, \ldots, 0)$$

on the positive real axis in the z_j-plane.

If $(w_1, w_2, \ldots, w_n)$ is an ε-extremal basis at ζ, then we will call the reordered tuple $(w_1, w_n, w_{n-1}, \ldots, w_2)$ an *ε-maximal basis at ζ*. This definition will be useful because certain estimates can be expressed in a more unified manner if this reordering is used.

Note that in McNeal's original definition the first ε-extremal direction was chosen differently; here, for technical reasons, we use the slight modification due to [BCDu]. McNeal's idea behind this definition was to use the extremal properties of the points p_j^+ for the construction of bounded plurisubharmonic functions on D having large Hessians near a given point ζ of $\bar{D}$ (where the terms *large* and *near* are quantified by the ε-distances $\tau(\zeta, w_j, \varepsilon)$). This was made possible by a careful analysis of derivatives of the defining function r of the domain D at the points p_j^+ and by an explicit construction involving r and the p_j^+. The plurisubharmonic functions thus obtained were then used by McNeal in various applications.

In the proof of Theorem 1.2 it was necessary to use a different kind of "extremal" basis at a point ζ close to bD, the so-called ε-minimal basis. Essentially, it is defined by the same construction as in Definition 1.3 but with the word *maximal* replaced by *minimal*. Actually, the construction is even a little simpler in this case, because the complication of fitting whole circles into sublevel sets of r does not arise.

1.4. Definition. Let $D = \{r < 0\} \subset\subset \mathbb{C}^n$ be a smooth convex domain of finite type m, let r be as before, and let $\zeta \in \bar{D}$ be a point close to bD. If $\varepsilon > 0$ is given (ε sufficiently small), then an *ε-minimal basis* $(v_1, \ldots, v_n)$ *at* ζ is defined as follows. Let v_1 be the unit vector in the direction of the real gradient of the defining function r at ζ, and let p_1^- be the point of intersection of the real line $\lambda \mapsto \zeta + \lambda v_1$ ($\lambda \in \mathbb{R}$) with the level set $bD_{\zeta,\varepsilon}$. Then choose a point p_2^- such that $|p_2^- - \zeta|$ is *minimal* under the conditions

$$r(p_2^-) = r(\zeta) + \varepsilon \quad \text{and} \quad (p_2^- - \zeta) \perp \langle v_1 \rangle.$$

Let v_2 be the unit vector in the direction $p_2^- - \zeta$. Then $\tau(\zeta, v_2, \varepsilon)$ is minimal among all $\tau(\zeta, w, \varepsilon)$ for $|w| = 1$, where w is in the complex tangent space to bD at ζ. Next, choose a point p_3^- such that $|p_3^- - \zeta|$ is minimal under the conditions

$$r(p_3^-) = r(\zeta) + \varepsilon \quad \text{and} \quad (p_3^- - \zeta) \perp \langle v_1, v_2 \rangle.$$

Let v_3 be the unit vector in the direction $p_3^- - \zeta$. We continue this process until we obtain a basis $v := (v_1, \ldots, v_n)$ (and n "minimal" points $p_1^-, \ldots, p_n^-$). The ε-minimal coordinates $(z_1, \ldots, z_n)$ of a point z with respect to ζ are then defined analogously to Definition 1.3.

Note that if we had used Definition 1.4 with the word *minimal* replaced by *maximal* as a definition for the ε-extremal basis in Definition 1.3, then the ε-extremal basis $(w_1, \dots, w_n)$ and the corresponding extremal points $(p_1^+, \dots, p_n^+)$ at ζ would, in general, not satisfy the estimate

$$|p_j^+ - \zeta| \lesssim \varepsilon^{1/m},$$

where m is the type of the domain, or the estimate

$$\tau(\zeta, w_2, \varepsilon) \gtrsim \tau(\zeta, w_3, \varepsilon) \gtrsim \cdots \gtrsim \tau(\zeta, w_n, \varepsilon);$$

nor could we assure that any of the $\tau(\zeta, w_j, \varepsilon)$ is the maximum of all possible $\tau(\zeta, w, \varepsilon)$ for w varying in the complex tangent space at ζ. Thus, such a basis would not properly reflect the complex geometry of bD.

We will now review McNeal's definitions of polydisks and pseudo-distances linked to the foregoing geometric constructions. Note that all the constructions depend strongly on the point ζ, on ε, and on the choice of the defining function r.

1.5. DEFINITION. For ζ close to bD and $\varepsilon > 0$ sufficiently small such that an ε-maximal basis $(v_1^+, \dots, v_n^+)$ and an ε-minimal basis $(v_1^-, \dots, v_n^-)$ are defined near ζ, we denote by $z_{k,\zeta,\varepsilon}^+$ and $z_{k,\zeta,\varepsilon}^-$ the components of a vector z with respect to the ε-maximal and the ε-minimal coordinates, respectively. Let $p_j^\pm$ be the extremal points as in Definitions 1.3 and 1.4. Then we set

$$\tau_j^\pm(\zeta, \varepsilon) := |p_j^\pm - \zeta| = \tau(\zeta, v_j^\pm, \varepsilon).$$

Furthermore, for a constant $C \in \mathbb{R}^+$, we define the polydisks $CP_\varepsilon^\pm(\zeta)$ as

$$CP_\varepsilon^\pm(\zeta) := \{z \in \mathbb{C}^n : |z_{k,\zeta,\varepsilon}^\pm| < C\tau_k^\pm(\zeta, \varepsilon) \text{ for } k = 1, \dots, n\}.$$

Finally, we define the pseudo-distances d_+ and d_- with respect to these polydisks as

$$d_\pm(z, \zeta) := \inf\{\varepsilon : z \in P_\varepsilon^\pm(\zeta)\}.$$

(Outside of a sufficiently small compact neighborhood of the diagonal of $bD \times bD$, these pseudo-distances can simply be defined by the usual Euclidean distance and then patched to the preceding definitions by a smooth nonnegative cutoff function. Then, the pseudo-distances $d_\pm$ are globally defined in $\bar{D} \times \bar{D}$. As we are mostly interested in the case that ζ and z are both close to bD and close to each other, we will always use the definition given above.)

All these constructions will soon be discussed in more detail. But first, let us give the statement of Fischer's theorem.

1.6. THEOREM. *Let $D \subset\subset \mathbb{C}^n$ be a smooth convex domain of finite type m, and let T_q be the solution operator for $\bar\partial$ constructed in* [DFiFo]. *Let d be any of the pseudo-distances d_+ or d_-. Then, for every $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that $T_q f$ satisfies the estimate*

$$|T_q f(z_1) - T_q f(z_0)| \leq C_\varepsilon \|f\|_{L^\infty(D)} \max\{d(z_0, z_1)^{1/m}, |z_0 - z_1|^{1-\varepsilon}\}.$$

This estimate is optimal in the sense that, for every even integer m, there are examples of domains D of type m and forms f for which no better estimate of this type can be obtained.

In this paper we will begin by clarifying the relation of the two ε-extremal bases used in the definition just given for the different pseudo-distances, showing some interesting properties of these bases as ε tends to zero. As a consequence, we will show that $d_+ \sim d_-$, so the two estimates obtained in Theorem 1.6 are in fact equivalent. (The notation $A \lesssim B$ is used to state that there is a constant C, independent of the parameters used in the expressions A and B, such that $A \le CB$. We let $A \sim B$ signify $A \lesssim B$ and $B \lesssim A$. We will always be precise about the dependence on parameters of constants appearing in such estimates if there is ambiguity.) This seems to be an interesting supplement to Fischer's theorem as well as an interesting result on the geometry of convex domains in its own right. In the course of the proof, we will also show that the directional pseudo-distances $\tau(\zeta, v_j, \varepsilon)$ with respect to both the ε-maximal and the ε-minimal basis can be estimated from above and below by the same powers of ε involving Catlin's multitype of the domain D. (In our earlier article [He2] we were able to show only *upper* estimates for the minimal basis and *lower* estimates for the maximal basis; in fact, this was the reason for introducing the minimal bases, because the upper estimates were needed in the estimates for the solution operators of the $\bar\partial$-equation.) The proof of these results seems to be more complicated than one would perhaps expect at first sight; in particular, it is difficult to show that all constants involved in the estimates can be chosen to be independent of the point ζ where the constructions are carried out.

A little bit out of the main line of this paper, we will then give an illustrative example showing that, in general, neither the ε-minimal nor the ε-maximal basis at a point $\zeta \in bD$ can be chosen to depend continuously on ζ. This is certainly known to most experts in the field, but it may still be interesting to have a very simple counterexample at hand. However, our counterexample depends on the choice of defining function, so it does not really settle the question of whether, after a suitable change of defining function, the extremal bases can always be chosen to be continuous with respect to ζ, at least locally.

Finally, we will use methods developed in [He2] to show that Theorem 1.6 (Fischer's theorem) can also be extended to take into account the degree of the form f, thus giving very precise anisotropic Hölder estimates depending on the multitype of the domain D. Again, it is the order of contact of $(q+1)$-planes with the boundary that determines the best possible estimates on $(0, q+1)$-forms. Our main results are summarized in the following theorem.

1.7. THEOREM. *Let $D \subset\subset \mathbb{C}^n$ be a smooth convex domain of finite type m_n, and let $(m_1, m_2, \dots, m_n)$ be its multitype.*

(1) *The pseudo-distances d_+ and d_- are equivalent.*
(2) *If K is a sufficiently small compact neighborhood of bD, if $\zeta \in K$, and if $(m_1(\zeta), \dots, m_n(\zeta))$ is the multitype of $bD_\zeta = \{z : r(z) = r(\zeta)\}$ at the point ζ, then there are constants $c, C > 0$ depending only on K (and on the fixed*

defining function of D) such that

$$c\varepsilon^{1/m_j(\zeta)} \le \tau_j^{\pm}(\zeta,\varepsilon) \le C\varepsilon^{1/m_j(\zeta)} \quad \textit{for } j = 1,\dots,n$$

and such that

$$cP_\varepsilon^{+}(\zeta) \subseteq P_\varepsilon^{-}(\zeta) \subseteq CP_\varepsilon^{+}(\zeta).$$

(3) *Let T_q be the solution operator for $\bar\partial$ on bounded $(0,q+1)$-forms used in* [DFiFo] *and* [Fi2], *and let d be one of the* (*equivalent*) *pseudo-distances d_+ or d_-. Then, for every $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that T_qf satisfies the estimate*

$$|T_qf(z_1) - T_qf(z_0)| \le C_\varepsilon \|f\|_{L^\infty(D)} \max\{d(z_0,z_1)^{1/m_{n-q}}, |z_0 - z_1|^{1-\varepsilon}\}.$$

In particular, the estimates for $(0,q+2)$-forms are better than those for $(0,q+1)$-forms if the multitype $(m_1,\dots,m_n)$ of bD has the property $m_{n-q-1} < m_{n-q}$. It is also interesting to note that Theorem 1.7 implies (stated slightly imprecisely) that an almost Lipschitz estimate for T_qf always holds in at least $q+1$ complex tangential directions at each point.

The balance of this paper is organized as follows. In Section 2 we will review some definitions and well-known facts about the geometry of smooth convex domains of finite type. We will then reconsider the ε-minimal basis and show (a) that the directional pseudo-distances defined using either this basis or McNeal's ε-maximal basis satisfy certain upper and lower estimates related to the multitype of the domain and (b) that the asymptotic behavior of these bases as ε tends to zero can be described using concepts described in [Yu] and [He2]. The equivalence of the two pseudo-distances will emerge as a by-product in showing these properties of the extremal bases. We will then construct the counterexample mentioned previously regarding the continuous choice of extremal bases.

In Section 3 we will show how to extend Fischer's theorem to incorporate the multitype of D into the estimates. Since this is rather a straightforward application of ideas already described in [Fi2] and [He2], this last section is very brief.

2. Comparison of Extremal Bases

We start with the definition of the multitype as given by Catlin [Ca1]. Let $D \subset\subset \mathbb{C}^n$ be a smoothly bounded domain, not necessarily convex, defined by a real-valued smooth function r, and let p be a point in the boundary of D. Let Γ_n be the set of all n-tuples $\Lambda = (\lambda_1,\dots,\lambda_n)$ of elements of the closed real line such that:

(1) $-\infty < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le +\infty$; and
(2) for each k, either λ_k is infinite or there exists a set of nonnegative integers $\{a_1,\dots,a_k\}$ (depending on k) with $a_k > 0$ such that

$$\sum_{j=1}^{k} \frac{a_j}{\lambda_j} = 1.$$

The set Γ_n is called the set of *weights*; it can be ordered lexicographically. A weight $\Lambda \in \Gamma_n$ is called *distinguished* if there exist holomorphic coordinates $(z_1, \dots, z_n)$ about p such that p is mapped to the origin and such that

$$\sum_{j=1}^{n} \frac{\alpha_j + \beta_j}{\lambda_j} < 1 \implies D^\alpha \bar{D}^\beta r(p) = 0,$$

where D^α and $\bar{D}^\beta$ denote (respectively) the differential operators

$$\frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} \quad \text{and} \quad \frac{\partial^{|\beta|}}{\partial \bar{z}_1^{\beta_1} \cdots \partial \bar{z}_n^{\beta_n}}.$$

2.1. Definition. The *multitype* $M(bD, p)$ of bD at the point p is defined to be the lexicographically smallest weight

$$M = (m_1(p), \dots, m_n(p)) \in \Gamma_n$$

such that $M \geq \Lambda$ for every distinguished weight Λ. The multitype $M(bD)$ of bD is defined as the n-tuple

$$M(bD) := (\sup m_1(p), \dots, \sup m_n(p)),$$

where each supremum is taken over all boundary points of D.

Since the first entry $m_1(p)$ of the multitype at a point is always equal to 1, this is also true for the first entry of the multitype. The second entry of the multitype is always an integer. On a bounded domain of finite type, the multitype takes on only a finite number of (rational) values, and it is a lexicographically upper-semicontinuous function of the point p.

On smooth *convex* domains of finite type, there is a direct relation between the multitype at a point and the q-types at this point as defined in [D'A]. From now on, we shall always assume that D is smooth and convex.

2.2. Definition. The variety 1-*type* (or simply the *type*) $\Delta_1(bD, p)$ of bD at p is defined as

$$\Delta_1(bD, p) := \sup_z \frac{\nu(z^* r)}{\nu(z)},$$

where the supremum is over all germs of nontrivial one-dimensional complex varieties $z\colon \Delta \to \mathbb{C}^n$ from the unit disc Δ into $\mathbb{C}^n$ satisfying $z(0) = p$; here $\nu(f)$ denotes the order of vanishing of the function $f(x) - p$ in $x = 0$, and $z^* r$ is the pullback of the defining function r of D to Δ.

The variety q-type $\Delta_q(bD, p)$ at the point p is defined as

$$\Delta_q(bD, p) := \inf_H \Delta_1(bD \cap H, p),$$

where the infimum is taken with respect to all $(n - q + 1)$-dimensional complex hyperplanes H passing through p.

Finally, the q-regular type $\Delta_q^r(bD, p)$ at the point p is the maximal order of contact of q-dimensional complex manifolds M with bD at p; that is,

$$\Delta_q^r(bD, p) := \sup_M \sup\left\{a \in \mathbb{R}^+ : \lim_{z\to p,\, z\in M} \frac{\operatorname{dist}(z, bD)}{|z-p|^a} \text{ exists}\right\}.$$

The following theorem states some of the main results (due to [Ca1; Mc1; BoSt; Yu]) about the relation between the several notions of type on smoothly bounded convex domains.

2.3. THEOREM. *Let D be a smoothly bounded convex domain of finite type, and let $M(bD, p) = (k_1, \dots, k_n)$ be its multitype at the point $p \in bD$. Then $k_1 = 1$ and $k_2 \le k_3 \le \cdots \le k_n$ are even integers. Moreover, for $1 \le q \le n$ we have*

$$k_{n-q+1} = \Delta_q(bD, p) = \Delta_q^r(bD, p),$$

and there exist $n-1$ complex lines $L_2, \dots, L_n$ orthogonal to each other in the complex tangent space to bD at p such that the order of contact of L_j with bD is equal to k_j for $j = 2, \dots, n$.

In particular, the type of a boundary point ζ is not only locally bounded, it is also an upper-semicontinuous function of ζ on convex domains. Recall that we assume the defining function r of D to be of a special type such that these results are also valid for all level sets of r close to bD.

It will be helpful in the sequel to have at hand some even more precise information on the multitype, which is summarized in Proposition 2.5 (due to [Yu]). All the results depend strongly on the convexity of the domain D.

2.4. DEFINITION. For any $m \in \mathbb{N}_0$, let

$$S_m(p) := \{z \in \mathbb{C}^n : \nu(z^* r) \ge m + 1\},$$

where z is identified with the complex line $z : \mathbb{C} \to \mathbb{C}^n$ defined by $\lambda \mapsto p + \lambda z$ and r is the (fixed) smooth defining function of D.

2.5. PROPOSITION.

(1) *For all $m \ge 0$, $S_m(p)$ is a complex linear subspace of $\mathbb{C}^n$.*
(2) *$S_0(p) = \mathbb{C}^n$, and $S_1(p)$ is the complex tangent space $T_p^{\mathbb{C}}(bD)$.*
(3) *$S_{m'}(p) \subseteq S_m(p)$ if $m' \ge m$.*
(4) *There exist integers $m_1, \dots, m_k$ such that the following statements hold.*
   (a) *$1 = m_1 < m_2 < \cdots < m_k$, and $m_2, \dots, m_k$ are even.*
   (b) *$S_{m_k}(p) \subset S_{m_{k-1}}(p) \subset \cdots \subset S_0(p)$ (the inclusions are strict).*
   (c) *$S_{m_l} = S_{m_l+1} = S_{m_l+2} = \cdots = S_{m_{l+1}-1}$ for $l = 1, \dots, k-1$.*
   (d) *$S_{m_k} = S_{m_k+j}$ for all $j \ge 0$; if D is of finite type, then $S_{m_k} = \{0\}$.*
   (e) *Choose a basis $(e_1, \dots, e_n)$ of $\mathbb{C}^n$ such that, for each j, the space S_{m_j} is spanned over $\mathbb{C}$ by $(e_i, \dots, e_n)$ for $i = n + 1 - d_j$, where $d_j := \dim S_{m_j}$. Then the multitype of bD at p is equal to the weight $\Lambda = (k_1, \dots, k_n)$ defined by*

$$k_j := \begin{cases} m_l & \text{if } e_j \in S_{m_{l-1}} - S_{m_l} \text{ for } 1 \le l \le k, \\ \infty & \text{if } e_j \in S_{m_k}. \end{cases}$$

   Thus we have

$$M(bD,p) = (1, \underbrace{m_2,\ldots,m_2}_{n-d_2-1 \text{ times}}, \underbrace{m_3,\ldots,m_3}_{d_2-d_3 \text{ times}}, \ldots, \underbrace{m_k,\ldots,m_k}_{d_{k-1} \text{ times}})$$

if D is of finite type.

We suggest the following definition for the special bases that have the property (4)(e) of Proposition 2.5.

2.6. Definition. Let $p \in bD$ be given. Any basis $(e_1,\ldots,e_n)$ of $\mathbb{C}^n$ that satisfies the conditions given in part (4)(e) of Proposition 2.5 relative to the point p is called a *multitype-basis* (or *Yu-basis*) at p.

Note that the first vector e_1 of an orthonormal multitype-basis at p can always be chosen to be the outer unit normal n_p at p; the existence of orthonormal multitype-bases at every point of bD is evident from Proposition 2.5. We have thus associated to every point $p \in bD$ a certain basis that exactly reflects the geometric meaning of the multitype, in the sense that each entry of the multitype is the order of contact with bD of a complex line in the direction of one of the vectors in the basis. If an orthonormal multitype-basis $(e_1,\ldots,e_n)$ is given and if the lines defined by the vectors $e_\kappa, e_{\kappa+1},\ldots,e_{\kappa+\lambda}$ all have the same order of contact with bD, then any (unitary) change of basis affecting only $e_\kappa,\ldots,e_{\kappa+\lambda}$ will still result in an (orthonormal) multitype-basis.

We will now recall the most important properties of the ε-minimal and ε-maximal bases.

2.7. Proposition. *For any smoothly bounded convex domain $D \subset\subset \mathbb{C}^n$ of finite type m, we have the following statements.* (*For the whole statement of the proposition let $P_\varepsilon^\pm$ be exactly one of P_ε^+ or P_ε^-, as there are no claims involving both polydisks in the same instance here; the same goes for the pseudo-distances $d_\pm$.*)

(1) *For each constant K there exist constants c_K and C_K depending only on K such that*
$$P^\pm_{c_K\varepsilon}(\zeta) \subseteq KP^\pm_\varepsilon(\zeta) \subseteq P^\pm_{C_K\varepsilon}(\zeta)$$
and
$$c_K P^\pm_\varepsilon(\zeta) \subseteq P^\pm_{K\varepsilon}(\zeta) \subseteq C_K P^\pm_\varepsilon(\zeta)$$
for ζ near bD and ε small enough.

(2) *There are constants $C_1 > 1$, $c_2 < 1$, and c_3, independent of ζ and ε, such that*:
$$C_1 P^\pm_{\varepsilon/2}(\zeta) \supseteq \tfrac{1}{2} P^\pm_\varepsilon(\zeta) \quad \forall \zeta, \varepsilon;$$
$$C_1 P^\pm_t(\zeta) \subseteq P^\pm_\varepsilon(\zeta) \quad \forall t < c_2\varepsilon, \zeta, \varepsilon;$$
$$c_3 P^\pm_{|r(\zeta)|}(\zeta) \subseteq D \quad \forall \zeta \in D.$$

(3) *If $v = \sum_{j=1}^n a_j v_j$, where $(v_1,\ldots,v_n)$ is an ε-minimal* (*or, respectively, an ε-maximal*) *basis at ζ, then*
$$\frac{1}{\tau(\zeta,v,\varepsilon)} \sim \sum_{j=1}^n \frac{|a_j|}{\tau_j^\pm(\zeta,\varepsilon)}.$$

(4) *For every* $z \in P^{\pm}_{\varepsilon}(\zeta)$ *we have* $\tau(z, v, \varepsilon) \sim \tau(\zeta, v, \varepsilon)$.

(5) *We have* $\tau_1(\zeta, \varepsilon) \sim \varepsilon$ *and* $\tau(\zeta, v, \varepsilon) \lesssim \varepsilon^{1/m}$ *for any unit vector* v. *If* v *is a unit vector in complex tangential direction, then we also have the estimate* $\varepsilon^{1/2} \lesssim \tau(\zeta, v, \varepsilon)$.

(6) *Let* v *be a unit vector and let*

$$a_{ij}(z, v) := \frac{\partial^{i+j}}{\partial\lambda^i \partial\bar{\lambda}^j} r(z + \lambda v)|_{\lambda=0}.$$

Then we have

$$\sum_{1 \le i+j \le m} |a_{ij}(z, v)| \tau(z, v, \varepsilon)^{i+j} \sim \varepsilon$$

uniformly for all z, v, *and* ε.

(7) *Let* w *be any orthonormal coordinate system centered at* z, *and let* b_j *be the unit vector in the* w_j*-direction. Then we have*

$$\left| \frac{\partial^{|\alpha+\beta|}}{\partial w^\alpha \partial \bar{w}^\beta} r(z) \right| \lesssim \frac{\varepsilon}{\prod_j \tau(z, b_j, \varepsilon)^{\alpha_j + \beta_j}}$$

for all multi-indices α *and* β *with* $|\alpha + \beta| \ge 1$.

(8) *There is a constant* $C_3 > 0$ *independent of* ζ *and* z (*sufficiently close to the boundary of* D) *and independent of* $\varepsilon > 0$ *such that, if* $P^{\pm}_{\varepsilon}(\zeta) \cap P^{\pm}_{\varepsilon}(z) \ne \emptyset$, *then*

$$P^{\pm}_{\varepsilon}(\zeta) \subseteq C_3 P^{\pm}_{\varepsilon}(z) \quad \textit{and} \quad P^{\pm}_{\varepsilon}(z) \subseteq C_3 P^{\pm}_{\varepsilon}(\zeta).$$

The pseudo-distances $d_+(z, \zeta)$ *and* $d_-(z, \zeta)$ *satisfy the properties*

$$d_{\pm}(z, \zeta) \sim d_{\pm}(\zeta, z),$$
$$d_{\pm}(z, \zeta) \lesssim d_{\pm}(z, w) + d_{\pm}(w, \zeta).$$

(9) *If* $\pi(z)$ *is the projection of a point* z *to the boundary* bD, *then we have the estimate* $d_{\pm}(z, \pi(z)) \sim |r(z)|$; $z \in P^{\pm}_{\varepsilon}(\zeta)$ *implies* $d_{\pm}(z, \zeta) \le \varepsilon$. *On the other hand,* $z \notin P^{\pm}_{\varepsilon}(\zeta)$ *implies* $d_{\pm}(z, \zeta) \gtrsim \varepsilon$; $d_{\pm}(z, \zeta) \le \varepsilon$ *implies* $z \in P^{\pm}_t(\zeta)$ *for all* $t \gtrsim \varepsilon$, *and* $d_{\pm}(z, \zeta) \ge \varepsilon$ *implies* $z \notin P^{\pm}_t(\varepsilon)$ *for all* $t \lesssim \varepsilon$.

Proof. These results are due to McNeal ([Mc1; Mc2]; cf. [BCDu]). Here we have transcribed almost literally the summary given in [DFiFo]; for the ε-minimal basis, see [He2, Prop. 2.2]. □

As a first comparison result, let us mention the following theorem on estimates for the directional pseudo-distances relative to the different kinds of extremal bases.

2.8. THEOREM. *Let* ζ *be sufficiently close to* bD, *and let* $(k_1, \ldots, k_n)$ *be the multitype of* bD_ζ *at the point* ζ. *Then, for any sufficiently small value of* $\varepsilon > 0$, *it follows that any* ε*-minimal basis* $(v_1, \ldots, v_n)$ *at* ζ *satisfies the estimates*

$$\tau(\zeta, v_j, \varepsilon) \lesssim \varepsilon^{1/k_j} \quad \textit{for } j = 1, \ldots, n,$$

$$\varepsilon = \varepsilon^{1/k_1} \sim \tau(\zeta, v_1, \varepsilon) \lesssim \tau(\zeta, v_2, \varepsilon) \le \tau(\zeta, v_3, \varepsilon) \le \cdots \le \tau(\zeta, v_n, \varepsilon).$$

Any ε-extremal basis $(u_1, \ldots, u_n)$ at ζ satisfies the estimates

$$\tau(\zeta, u_1, \varepsilon) \sim \varepsilon = \varepsilon^{1/k_1}, \qquad \tau(\zeta, u_j, \varepsilon) \gtrsim \varepsilon^{1/k_{n-j+2}} \quad \textit{for } j = 2, \ldots, n.$$

This can also be expressed by saying that any ε-maximal basis $(w_1, w_2, \ldots, w_n)$ at ζ satisfies

$$\tau(\zeta, w_j, \varepsilon) \gtrsim \varepsilon^{1/k_j} \quad \textit{for } j = 1, \ldots, n,$$

$$\varepsilon \sim \tau(\zeta, w_1, \varepsilon) \lesssim \tau(\zeta, w_2, \varepsilon) \leq \tau(\zeta, w_3, \varepsilon) \leq \cdots \leq \tau(\zeta, w_n, \varepsilon).$$

Proof. See Theorem 2.2 and Theorem 2.3 in [He2]. □

Looking at these estimates, it becomes clear why it is convenient, if we are interested in comparison of $\tau_j^{\pm}(\zeta, \varepsilon)$ with powers of ε, to use the reordered ε-maximal bases instead of McNeal's original ε-extremal bases.

After these preparations, we can proceed to our first new result. Let $D \subset\subset \mathbb{C}^n$ be as before, let ζ be a point close to bD, and let $M(bD_\zeta, \zeta) = (k_1, \ldots, k_n)$ be the multitype of the level surface bD_ζ at ζ. Furthermore, let $\kappa_2 := 2$ and let $\kappa_3, \ldots, \kappa_{l+1} \leq n$ and l be integers such that

$$k_{\kappa_j} = k_{\kappa_j+1} = \cdots = k_{\kappa_{j+1}-1} \quad \text{and} \quad k_{\kappa_{j+1}-1} < k_{\kappa_{j+1}}$$

for $j = 2, \ldots, l$ and such that $M(bD_\zeta, \zeta)$ is of the form

$$(k_1, k_2, \ldots, k_n) = (1, \underbrace{\mu_2, \mu_2, \ldots, \mu_2}_{(\kappa_3 - \kappa_2)}, \underbrace{\mu_3, \ldots, \mu_3}_{(\kappa_4 - \kappa_3)}, \ldots, \underbrace{\mu_l, \ldots, \mu_l}_{(\kappa_{l+1} - \kappa_l)}).$$

Assume that $(y_1, y_2, \ldots, y_n)$ is a multitype-basis at ζ. In particular, if we set

$$M_j := \{\kappa_j, \kappa_j + 1, \ldots, \kappa_{j+1} - 1\}$$

then, for each $\kappa \in M_j$, the complex line defined by y_κ has order of contact exactly equal to $\mu_j = k_{\kappa_j}$ with bD. Therefore, every unitary coordinate change inside one of the spaces generated by $\{y_\kappa\}_{\kappa \in M_\nu}$ for a fixed ν transforms the basis (y_j) into another multitype-basis. For every $\varepsilon \in (0, 1]$, we assume that we are given an ε-minimal basis $(v_{1,\varepsilon}, \ldots, v_{n,\varepsilon})$. We may also assume that $v_{1,\varepsilon} = y_1$ for all ε. Then, since $(v_{j,\varepsilon})$ and (y_j) are orthonormal bases, there exist unitary matrices $A_\varepsilon := (a_{j\kappa,\varepsilon})$ such that

$$y_\kappa = \sum_{j=1}^{n} a_{j\kappa,\varepsilon} v_{j,\varepsilon} \quad \text{for } \kappa = 1, \ldots, n.$$

Note also that each $V_\varepsilon := (v_{1,\varepsilon}, \ldots, v_{n,\varepsilon})$ can be interpreted as an element of the unitary group $U(n)$, and we will do so without further mention.

2.9. THEOREM. *Let $\varepsilon_\nu \to 0$ as $\nu \to \infty$ be any sequence such that V_{ε_ν} and A_{ε_ν} are convergent sequences in $U(n)$. Then the limit $A_0 := (a_{j\kappa,0}) := \lim_{\nu \to \infty} A_{\varepsilon_\nu}$ is a unitary diagonal block matrix of the form*

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & A_l \end{pmatrix}$$

with $A_j \in U(\kappa_{j+1} - \kappa_j)$ for $j = 2, \ldots, l$. So, each diagonal block corresponds to one of the different entries μ_j of the multitype at ζ and has the size of the multiplicity of that entry. In particular, every convergent sequence of ε_ν-minimal bases converges to a multitype-basis, and by replacing the chosen multitype-basis by the basis

$$y'_j := \sum_{\kappa=1}^{n} a'_{\kappa j,0} y_\kappa \quad \text{for } j = 1, \ldots, n,$$

where $(a'_{\kappa j,0}) = A_0^{-1}$, we can assume that $(y_j)_j$ has been chosen such that A_0 is the identity matrix $I := (\delta_{j\kappa})_{j,\kappa}$.

Furthermore, $\tau_j^-(\zeta, \varepsilon_\nu)$ satisfies the estimate

$$\tau_j^-(\zeta, \varepsilon_\nu) \gtrsim \varepsilon_\nu^{1/k_j} \quad \text{for } j = 1, \ldots, n.$$

The constant appearing in this estimate may depend on the chosen sequence (ε_ν) (and thus also on the point ζ).

Proof. By Proposition 2.7 and since (y_j) is a multitype-basis, for each κ we have

$$\varepsilon_\nu^{-1/k_\kappa} \sim \frac{1}{\tau(\zeta, y_\kappa, \varepsilon_\nu)} \sim \sum_{j=1}^{n} \frac{|a_{j\kappa,\varepsilon_\nu}|}{\tau_j(\varepsilon_\nu)},$$

and all the involved constants can be chosen to be independent of ζ in a compact neighborhood of bD. If j is an index such that $a_{j\kappa,0} \neq 0$, we obtain

$$\varepsilon_\nu^{-1/k_\kappa} \gtrsim \frac{|a_{j\kappa,\varepsilon_\nu}|}{2\tau_j^-(\zeta, \varepsilon_\nu)}$$

for all sufficiently large ν. Using $\tau_j^-(\zeta, \varepsilon) \lesssim \varepsilon^{1/k_j}$, this implies

$$\varepsilon_\nu^{1/k_j} \gtrsim \varepsilon_\nu^{1/k_\kappa}$$

as $\nu \to \infty$; but this is only possible if $k_j \geq k_\kappa$. It follows that $j \geq \kappa_s$, where s is chosen such that $k_\kappa = k_{\kappa_s}$. Therefore, A_0 is a lower triagonal block matrix of the form

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & * & A_3 & \cdots & 0 \\ \vdots & * & * & \ddots & \vdots \\ 0 & * & * & * & A_l \end{pmatrix},$$

where the sizes of the diagonal blocks $A_2, \ldots, A_l$ are given as in the statement of the theorem. Since A_0 is also unitary, it is obvious that all the entries marked by asterisks must be zero, so A_0 has the claimed form.

Now we assume that the multitype-basis has been transformed such that A_0 equals the identity matrix I (observe that we do not change the minimal bases (v_{j,ε_ν}), of course). Then, using the representation

$$y_\kappa = \sum_{j=1}^{n} a_{j\kappa,\varepsilon_\nu} v_{j,\varepsilon_\nu},$$

we can assume that, for all sufficiently large ν, the coefficient $a_{\kappa\kappa,\varepsilon_\nu}$ satisfies

$$|a_{\kappa\kappa,\varepsilon_\nu}| > \frac{1}{2},$$

so we obtain the estimate

$$\varepsilon_\nu^{-1/k_\kappa} \sim \frac{1}{\tau(\zeta, y_\kappa, \varepsilon_\nu)} \gtrsim \frac{1}{2\tau_\kappa^-(\zeta,\varepsilon_\nu)}$$

just as before. The constants do not depend (directly) on ζ but possibly on ε_{ν_0}, where ν_0 is an index such that for $\nu \ge \nu_0$ the estimate $|a_{\kappa\kappa,\varepsilon_\nu}| > \frac{1}{2}$ is valid. This will not pose a problem later. Thus, each τ_κ^- satisfies

$$\tau_\kappa^-(\zeta,\varepsilon_\nu) \gtrsim \varepsilon_\nu^{1/k_\kappa},$$

and the theorem is proved. □

2.10. Corollary. *If $M(bD_\zeta, \zeta) = (k_1, \ldots, k_n)$, then there is a constant $c_\zeta > 0$ and a constant $c > 0$ (independent of ζ) such that, for all $\varepsilon \in (0,1]$,*

$$c_\zeta \varepsilon^{1/k_j} \le \tau_j^-(\zeta,\varepsilon) \le c\varepsilon^{1/k_j} \quad \textit{for } \; j = 1, \ldots, n.$$

Proof. The upper estimate is known from [He2] (see Theorem 2.8). If the lower estimate is not satisfied, then there is a sequence $\varepsilon_\nu \to 0$ such that

$$\tau_j^-(\zeta,\varepsilon_\nu) < \frac{1}{\nu}\varepsilon_\nu^{1/k_j}.$$

Choose a subsequence such that the corresponding matrices A_{ε_ν} and V_{ε_ν} converge in $U(n)$. By Theorem 2.9 we then get an estimate for τ_j^-, contradicting our assumption as ν tends to infinity. □

We will later show that c_ζ can, in fact, be chosen independent of ζ. We will also need the following consequence of Theorem 2.9 in the comparison of the polydisks $P_\varepsilon^+(\zeta)$ and $P_\varepsilon^-(\zeta)$. If we assume that we have chosen a sequence $\varepsilon_\nu \to 0$ such that the corresponding sequences of maximal and minimal bases W_{ε_ν} and V_{ε_ν} both converge to multitype-bases W_0 and V_0 (respectively), then we will need to have an exact quantitative estimate for the rate of convergence. The case of the minimal bases is settled in the following corollary.

2.11. COROLLARY. *Let $M(bD_\zeta, \zeta) = (k_1, \dots, k_n)$, let (ε_ν) be a sequence converging to zero, and let $A_{\varepsilon_\nu} = (a_{j\kappa,\varepsilon_\nu}) \to A_0 = I$ and $V_{\varepsilon_\nu} \to V_0$ as in Theorem 2.9. Then there is a constant $C = C_\zeta > 0$ such that*

$$|a_{j\kappa,\varepsilon_\nu}| \le C_\zeta \varepsilon_\nu^{|1/k_\kappa - 1/k_j|} \quad \textit{for all } \; j \in \{1, \dots, n\}.$$

Proof. Let $\kappa \in M_s = \{\kappa_s, \kappa_s+1, \dots, \kappa_{s+1}-1\}$. For $j \in M_s$ the assertion is obvious, so we need only consider $j \in \{1, \dots, n\} - M_s$. First, let $j < \kappa_s$. By Theorem 2.9 and Corollary 2.11, we have

$$1 \sim \sum_{\mu \in M_s} |a_{\mu\kappa,\varepsilon_\nu}| + \sum_{\mu < \kappa_s} |a_{\mu\kappa,\varepsilon_\nu}| \varepsilon_\nu^{1/k_\kappa - 1/k_\mu} + \sum_{\mu \ge \kappa_{s+1}} |a_{\mu\kappa,\varepsilon_\nu}| \varepsilon_\nu^{1/k_\kappa - 1/k_\mu},$$

so the sum

$$\sum_{\mu < \kappa_s} |a_{\mu\kappa,\varepsilon_\nu}| \varepsilon_\nu^{1/k_\kappa - 1/k_\mu}$$

must be bounded above by a constant as $\nu \to \infty$. Thus, each term in the sum must be bounded above. This gives the result for $j < \kappa_s$. For $j \ge \kappa_{s+1}$ we proceed as follows. Since $A_{\varepsilon_\nu} \in U(n)$, we have $a'_{j\kappa,\varepsilon_\nu} = \overline{a_{\kappa j,\varepsilon_\nu}}$ for $A_{\varepsilon_\nu}^{-1} = (a'_{j\kappa,\varepsilon_\nu})$. Now, if $\lambda \in M_\sigma$ for a given σ then, using

$$v_{\lambda,\varepsilon_\nu} = \sum_{\mu=1}^{n} a'_{\mu\lambda,\varepsilon_\nu} y_\mu,$$

we see that

$$1 \sim \frac{\varepsilon_\nu^{1/k_\lambda}}{\tau_\lambda^-(\zeta, \varepsilon_\nu)} \sim \sum_{\mu \in M_\sigma} |a'_{\mu\lambda,\varepsilon_\nu}| + \sum_{\mu < \kappa_\sigma} |a'_{\mu\lambda,\varepsilon_\nu}| \varepsilon_\nu^{1/k_\lambda - 1/k_\mu} + \sum_{\mu \ge \kappa_{\sigma+1}} |a'_{\mu\lambda,\varepsilon_\nu}| \varepsilon_\nu^{1/k_\lambda - 1/k_\mu}.$$

The same argument as before implies that, for $\mu < \kappa_\sigma$,

$$|a'_{\mu\lambda,\varepsilon_\nu}| \lesssim \varepsilon_\nu^{1/k_\mu - 1/k_\lambda}.$$

Choosing $\lambda := j$ and $\mu := \kappa \in M_s$ with $j \ge \kappa_{s+1}$, we obtain

$$|a_{j\kappa,\varepsilon_\nu}| = |a'_{\mu\lambda,\varepsilon_\nu}| \lesssim \varepsilon_\nu^{1/k_\mu - 1/k_\lambda} = \varepsilon_\nu^{|1/k_\kappa - 1/k_j|},$$

which was to be shown. □

We will now demonstrate that statements similar to that of Theorem 2.9 and its corollaries are true for ε-maximal bases at a point ζ. For this we assume that, for every $\varepsilon \in (0, 1]$, an ε-maximal basis $(w_{j,\varepsilon})$ has been chosen at ζ close to bD; we also assume that, for a fixed multitype-basis (y_j),

$$y_\kappa = \sum_{j=1}^{n} b_{j\kappa,\varepsilon} w_{j,\varepsilon} \quad \text{for } \; \kappa = 1, \dots, n.$$

We denote $W_\varepsilon := (w_{j,\varepsilon})_j \in U(n)$ and $B_\varepsilon := (b_{j\kappa,\varepsilon})_{j,\kappa} \in U(n)$. The sets M_j $(j = 2, \dots, l)$ are defined as before. Note that the proof of the following theorem is *not* exactly analogous to the proof of Theorem 2.9.

2.12. THEOREM. *Let $\varepsilon_\nu \to 0$ as $\nu \to \infty$ be any sequence such that W_{ε_ν} and B_{ε_ν} are convergent sequences in $U(n)$. Then the limit $B_0 := (b_{j\kappa,0}) := \lim_{\nu\to\infty} B_{\varepsilon_\nu}$ is a unitary diagonal block matrix of the form*

$$B_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & B_2 & 0 & \cdots & 0 \\ 0 & 0 & B_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & B_l \end{pmatrix}$$

with $B_j \in U(\kappa_{j+1} - \kappa_j)$ for $j = 2, \dots, l$. Again, by replacing the chosen multi-type-basis by the basis

$$y_j' := \sum_{\kappa=1}^{n} b'_{\kappa j,0} y_\kappa \quad \text{for } j = 1, \dots, n,$$

where $(b'_{\kappa j,0}) = B_0^{-1}$, we can assume that $(y_j)_j$ has been chosen such that B_0 is the identity matrix. Moreover, $\tau_j^+(\zeta, \varepsilon_\nu)$ satisfies the estimate

$$\tau_j^+(\zeta, \varepsilon_\nu) \lesssim \varepsilon_\nu^{1/k_j} \quad \text{for } j = 1, \dots, n.$$

The constant appearing in this estimate may depend on the chosen sequence (ε_ν) and on the point ζ.

Proof. Again, we use Proposition 2.7 and the fact that (y_j) is a multitype-basis to derive

$$\varepsilon_\nu^{-1/k_\kappa} \sim \frac{1}{\tau(\zeta, y_\kappa, \varepsilon_\nu)} \sim \sum_{j=1}^{n} \frac{|b_{j\kappa,\varepsilon_\nu}|}{\tau_j^+(\zeta, \varepsilon_\nu)}.$$

For $\kappa = 1$, the estimate $\tau_1^+(\zeta, \varepsilon) \sim \varepsilon$ (with constants that are independent even of ζ) is known. Now, for any $\kappa \geq 2$, let $j_\kappa \geq 2$ be the *minimal* index such that $b_{j_\kappa \kappa,0} \neq 0$. Then the right-hand side in the previous estimate is bounded above by a constant times $\varepsilon_\nu^{-1/k_{j_\kappa}}$ as ν tends to infinity, so

$$\varepsilon_\nu^{-1/k_\kappa} \lesssim \varepsilon_\nu^{-1/k_{j_\kappa}},$$

which implies $k_\kappa \geq k_{j_\kappa}$. Starting with $\kappa = 2$ and with j_2 the minimal index belonging to $\kappa = 2$, this implies $k_2 = k_{j_2}$, so $j_2 \in M_2 = \{\kappa_2, \kappa_2 + 1, \dots, \kappa_3 - 1\}$. Since j_2 is minimal, we obtain (for all sufficiently large ν) the estimate

$$\varepsilon_\nu^{-1/k_2} \sim \frac{1}{\tau(\zeta, y_2, \varepsilon_\nu)} \lesssim \frac{1}{\tau_{j_2}^+(\zeta, \varepsilon_\nu)}$$

because $\tau_2^+(\zeta, \varepsilon_\nu) \leq \tau_3^+(\zeta, \varepsilon_\nu) \leq \cdots \leq \tau_n^+(\zeta, \varepsilon_\nu)$. Therefore,

$$\tau_{j_2}^+(\zeta, \varepsilon_\nu) \lesssim \varepsilon_\nu^{1/k_2} = \varepsilon_\nu^{1/k_{j_2}}.$$

Now let

$$\tilde{y}_{3,\varepsilon_\nu} := y_3 - \frac{b_{j_2 3,\varepsilon_\nu}}{b_{j_2 2,\varepsilon_\nu}} y_2.$$

The vectors $\tilde{y}_{3,\varepsilon_\nu}$ are well-defined by the foregoing choice of j_2 if ν is sufficiently large; and, in the representation of $\tilde{y}_{3,\varepsilon_\nu}$ by the basis (w_{j,ε_ν}), the coefficient of w_{j_2,ε_ν} is equal to zero for all sufficiently large ν. Then, each $\tilde{y}_{3,\varepsilon_\nu}$ lies in the space $\langle y_2, y_3\rangle$ generated over $\mathbb{C}$ by y_2 and y_3 and has a norm uniformly bounded above and below (away from zero) as $\nu \to \infty$. Therefore, we obtain the estimate

$$\varepsilon_\nu^{-1/k_3} \lesssim \frac{1}{\tau(\zeta, \tilde{y}_{3,\varepsilon_\nu}, \varepsilon_\nu)} \sim \sum_{j=2}^{n} \frac{|\gamma_{j3,\varepsilon_\nu}|}{\tau_j^+(\zeta, \varepsilon_\nu)},$$

where

$$\gamma_{j3,\varepsilon_\nu} := b_{j3,\varepsilon_\nu} - \frac{b_{j_2 3,\varepsilon_\nu}}{b_{j_2 2,\varepsilon_\nu}} b_{j2,\varepsilon_\nu}$$

(and $\gamma_{j_2 3,\varepsilon_\nu} = 0$). By the same reasoning as before, the minimal j_3 satisfying $\lim \gamma_{j_3 3,\varepsilon_\nu} \neq 0$ (observe that this sequence *is* convergent) must satisfy $j_3 \neq j_2$ and $j_3 \in M_\lambda$, where λ is such that $k_3 = k_{\kappa_\lambda}$. To see this, distinguish the cases $k_3 > k_2$ (implying $j_3 > j_2 = 2$ and $k_3 \geq k_{j_3}$) and $k_3 = k_2$ (implying $j_2 \neq j_3$ and $j_2, j_3 \in M_\lambda$ for the same λ). Then

$$\tau_{j_3}^+(\zeta, \varepsilon_\nu) \lesssim \varepsilon_\nu^{1/k_3} = \varepsilon_\nu^{1/k_{j_3}}.$$

Continuing this procedure for $\kappa = 4, \ldots, n$, we see that in fact every index j must appear exactly once as a minimal index and that

$$\tau_j^+(\zeta, \varepsilon_\nu) \lesssim \varepsilon_\nu^{1/k_j} \quad \text{for } j = 1, \ldots, n.$$

Arguing as in the proof of Theorem 2.9, this easily implies that B_0 is a unitary block matrix as claimed. The proof is complete. □

2.13. COROLLARY. *If $M(bD_\zeta, \zeta) = (k_1, \ldots, k_n)$, then there is a constant $C_\zeta > 0$ and a constant $C > 0$ (independent of ζ) such that*

$$C\varepsilon^{1/k_j} \leq \tau_j^+(\zeta, \varepsilon) \leq C_\zeta \varepsilon^{1/k_j} \quad \textit{for } j = 1, \ldots, n.$$

Proof. The proof is almost identical to the proof of Corollary 2.10. The lower estimate is known from [He2]. If the upper estimate is not satisfied, then there is a sequence $\varepsilon_\nu \to 0$ such that

$$\tau_j^+(\zeta, \varepsilon_\nu) > \nu \varepsilon_\nu^{1/k_j}.$$

Choose a subsequence such that the corresponding matrices B_{ε_ν} and W_{ε_ν} converge in $U(n)$. By Theorem 2.12, we then get an estimate for τ_j^+ contradicting the assumption as ν tends to infinity. □

2.14. COROLLARY. *Let $M(bD_\zeta, \zeta) = (k_1, \ldots, k_n)$, let (ε_ν) be a sequence converging to zero, and let $B_{\varepsilon_\nu} = (b_{j\kappa,\varepsilon_\nu}) \to B_0 = I$ and $W_{\varepsilon_\nu} \to W_0$ as in Theorem 2.12. Then there is a constant $C_\zeta > 0$ such that*

$$|b_{j\kappa,\varepsilon_\nu}| \leq C_\zeta \varepsilon_\nu^{|1/k_\kappa - 1/k_j|} \quad \textit{for all } j \in \{1, \ldots, n\}.$$

Proof. The proof is exactly the same as that of Corollary 2.11. □

Next, we come to the problem of comparing the polydisks $P_\varepsilon^{\pm}(\zeta)$ defined with respect to the different extremal bases at a fixed point ζ.

2.15. THEOREM. *Let $P_\varepsilon^+ := P_\varepsilon^+(\zeta)$ and $P_\varepsilon^- := P_\varepsilon^-(\zeta)$ be the maximal and minimal polydisks defined by the bases $(w_{j,\varepsilon})$ and $(v_{j,\varepsilon})$, respectively. Then there is a constant $c = c_\zeta > 0$ independent of ε such that, for all $\varepsilon \in (0, 1]$,*

$$cP_\varepsilon^+ \subseteq P_\varepsilon^- \quad \textit{and} \quad cP_\varepsilon^- \subseteq P_\varepsilon^+.$$

Proof. If $\varepsilon_\nu \to 0$ is a fixed sequence, then we can extract a subsequence (w.l.o.g. the whole sequence) such that $W_{\varepsilon_\nu} = (w_{j,\varepsilon_\nu})$ and $V_{\varepsilon_\nu} = (v_{j,\varepsilon_\nu})$ converge to unitary matrices W_0 and V_0 having a block structure (as described previously) with respect to a fixed multitype-basis. Of course, W_0 and V_0 also describe multitype-bases, so we can assume without loss of generality that W_0 is the identity matrix and that our fixed basis $y_1, \dots, y_n$ coincides with the standard basis of $\mathbb{C}^n$. We may also assume that $\zeta = 0$ is the origin. For any positive K, we then define the polydisks

$$KP_\varepsilon := \left\{ z = \sum_{j=1}^n z_j y_j : |z_j| < K\varepsilon^{1/k_j} \right\}$$

with *fixed axes* $y_1, \dots, y_n$ independent of ε. As before, we have sequences of unitary matrices A_{ε_ν} and B_{ε_ν} such that, for $\kappa = 1, \dots, n$,

$$y_\kappa = \sum_{j=1}^n a_{j\kappa,\varepsilon_\nu} v_{j,\varepsilon_\nu} = \sum_{j=1}^n b_{j\kappa,\varepsilon_\nu} w_{j,\varepsilon_\nu},$$

$$v_{\kappa,\varepsilon_\nu} = \sum_{j=1}^n a'_{j\kappa,\varepsilon_\nu} y_j,$$

$$w_{\kappa,\varepsilon_\nu} = \sum_{j=1}^n b'_{j\kappa,\varepsilon_\nu} y_j.$$

Let $z \in \gamma P_{\varepsilon_\nu}^-$ for some constant γ. This implies

$$|\langle z, v_{\kappa,\varepsilon_\nu}\rangle| < \gamma\tau_\kappa^-(0, \varepsilon_\nu) \le \gamma C_1 \varepsilon_\nu^{1/k_\kappa}$$

for $\kappa = 1, \dots, n$ and a constant $C_1 > 0$ independent of γ. Here $\langle\cdot,\cdot\rangle$ denotes the standard inner product of $\mathbb{C}^n$. Then we obtain from Corollary 2.11 that

$$\begin{aligned}
|\langle z, y_\kappa\rangle| &= \left|\left\langle z, \sum_{j=1}^n a_{j\kappa,\varepsilon_\nu} v_{j,\varepsilon_\nu}\right\rangle\right| \\
&\le \sum_{j=1}^n |a_{j\kappa,\varepsilon_\nu}||\langle z, v_{j,\varepsilon_\nu}\rangle| \\
&< C_2 \varepsilon_\nu^{|1/k_j - 1/k_\kappa|} \gamma C_1 \varepsilon_\nu^{1/k_j} \\
&\le C_2 \gamma C_1 \varepsilon_\nu^{1/k_\kappa}
\end{aligned}$$

for a positive constant C_2, so $z \in \gamma C_1 C_2 P_{\varepsilon_\nu} \subseteq P_{\varepsilon_\nu}$ if γ is chosen small enough.

On the other hand, if $z \in \delta P_{\varepsilon_\nu}$ for some positive δ, then

$$
\begin{aligned}
|\langle z, v_{\kappa,\varepsilon_\nu}\rangle| &= \left|\left\langle z, \sum_{j=1}^{n} a'_{j\kappa,\varepsilon_\nu} y_j \right\rangle\right| \\
&\le \sum_{j=1}^{n} |a'_{j\kappa,\varepsilon_\nu}||\langle z, y_j\rangle| \\
&< C_3 \varepsilon_\nu^{|1/k_j - 1/k_\kappa|} \delta \varepsilon_\nu^{1/k_j} \\
&\le C_3 \delta \varepsilon_\nu^{1/k_\kappa}
\end{aligned}
$$

for a sufficiently large constant C_3. Hence there are positive constants c_4, C_4 such that

$$c_4 P^-_{\varepsilon_\nu} \subseteq P_{\varepsilon_\nu} \subseteq C_4 P^-_{\varepsilon_\nu}$$

for all ν. The same arguments can be applied to $P^+_{\varepsilon_\nu}$, showing that the statement of the theorem holds at least for all ε in the fixed sequence ε_ν.

Now suppose that the statement of the theorem does not hold for general $\varepsilon > 0$. Then there is a sequence $\varepsilon_\nu \to 0$ such that, for all ν, we have

$$\frac{1}{\nu} P^+_{\varepsilon_\nu} \not\subseteq P^-_{\varepsilon_\nu}.$$

We extract a subsequence such that the foregoing assumptions on (w_{j,ε_ν}) and (v_{j,ε_ν}) are satisfied, giving us a constant $c > 0$ such that

$$c P^+_{\varepsilon_\nu} \subseteq P^-_{\varepsilon_\nu},$$

clearly a contradiction to our indirect assumption, because the inclusion

$$\frac{1}{\nu} P^+_{\varepsilon_\nu} \subseteq c P^+_{\varepsilon_\nu}$$

is valid for all sufficiently large ν. The proof of the inclusion $cP^-_\varepsilon \subseteq P^+_\varepsilon$ is exactly the same, so we do not elaborate upon it. $\square$

Finally, using Theorem 2.15, we can show that all the constants c_ζ and C_ζ appearing in our various theorems and corollaries so far can be chosen to be independent of ζ for all ζ in a compact neighborhood of bD. This will be a by-product of our main objective, the comparison of the pseudo-distances d_+ and d_- defined in the Introduction. Note that in the proof of the uniformity of the constants for the ε-maximal basis we actually use results on the ε-minimal basis, and vice versa.

Suppose that K is a compact neighborhood of bD, chosen so small that for $\zeta \in K$ the level sets bD_ζ are all homothetic to bD. Suppose further that $(z_0, \zeta_0) \in K \times K - \Delta(K)$, where $\Delta(K)$ denotes the diagonal in $K \times K$. We consider an open product neighborhood $U \times V$ of (z_0, ζ_0) with $\bar{U} \cap \bar{V} = \emptyset$ and such that $\bar{U}$ and $\bar{V}$ are compact. Since both d_+ and d_- are pseudo-distances, there are constants $0 < c < C < \infty$ (depending on U and V) satisfying

$$c d_+(z,\zeta) \le d_-(z,\zeta) \le C d_+(z,\zeta) \quad \text{for all } (z,\zeta) \in \bar{U} \times \bar{V}.$$

Now let $(\zeta_0, \zeta_0) \in \Delta(K)$ and let U be an open neighborhood of ζ_0 with closure contained in K. For all $\zeta \in \bar{U}$, let

$$\gamma_\zeta := \inf\{\gamma > 0 : d_+(z,\zeta) \le \gamma d_-(z,\zeta) \text{ for all } z \in \bar{U}\}.$$

By Theorem 2.15, each $\gamma_\zeta < \infty$. We want to show that γ_ζ is a locally bounded function of ζ. We know that, for all $z \in U$ and all $\delta > 0$,

$$d_+(z,\zeta) \le (\gamma_\zeta + \delta) d_-(z,\zeta).$$

On the other hand, by Proposition 2.7 there exists a constant c_3, independent of z, ζ, such that

$$d_+(\zeta,z) \le c_3 d_+(z,\zeta) \quad \text{and} \quad d_-(z,\zeta) \le c_3 d_-(\zeta,z).$$

Thus, we see that

$$d_+(\zeta,z) \le c_3^2(\gamma_\zeta + \delta) d_-(\zeta,z)$$

for all $(z,\zeta) \in \bar{U} \times \bar{U}$ and all $\delta > 0$. This implies $\gamma_z \le c_3^2 \gamma_\zeta$, so $\zeta \mapsto \gamma_\zeta$ is locally bounded. Finally, a simple compactness argument shows that there is a constant $C < \infty$, depending only on K, such that

$$d_+(z,\zeta) \le C d_-(z,\zeta) \quad \text{for all } (z,\zeta) \in K \times K.$$

Because the roles of d_+ and d_- can be reversed in this proof, we have actually shown the following.

2.16. THEOREM. *Let $D \subset\subset \mathbb{C}^n$ and $K \supset bD$ be as before. Then there exist constants $c_K, C_K > 0$ depending only on K such that, for all $(z,\zeta) \in K \times K$, the estimate*

$$c_K d_+(z,\zeta) \le d_-(z,\zeta) \le C_K d_+(z,\zeta)$$

holds. In particular, Fischer's estimate in Theorem 1.6 is (up to constants) independent of the choice of pseudo-distance d_+ or d_-.

There are some interesting corollaries to the results obtained so far.

2.17. COROLLARY. *The constant c_ζ with $c_\zeta P_\varepsilon^{\mp}(\zeta) \subseteq P_\varepsilon^{\pm}(\zeta)$ can be chosen to be independent of the point ζ in a compact neighborhood of bD.*

Proof. In fact, if this were not the case then we could find a sequence ζ_ν in K converging to some $\zeta \in K$ such that the *maximal* constants c_{ζ_ν} satisfying

$$c_{\zeta_\nu} \overline{P_\varepsilon^{+}(\zeta_\nu)} \subseteq \overline{P_\varepsilon^{-}(\zeta_\nu)} \quad \text{for all } \varepsilon \in (0,1]$$

tend to zero as $\nu \to \infty$. We can find $\varepsilon_\nu \to 0$ such that

$$2c_{\zeta_\nu} P_{\varepsilon_\nu}^{+}(\zeta_\nu) \not\subseteq P_{\varepsilon_\nu}^{-}(\zeta_\nu),$$

so there is a sequence ξ_ν also converging to ζ with

$$\xi_\nu \in 2c_{\zeta_\nu} P_{\varepsilon_\nu}^{+}(\zeta_\nu) - P_{\varepsilon_\nu}^{-}(\zeta_\nu).$$

But that is a contradiction, since by Theorem 2.16 there exist constants c_K, C_K such that

$$c_K d_+(\xi_\nu,\zeta_\nu) \le d_-(\xi_\nu,\zeta_\nu) \le C_K d_+(\xi_\nu,\zeta_\nu),$$

whereas by our assumption and by Proposition 2.7,

$$d_+(\xi_\nu,\zeta_\nu) \le c_3 c_\nu \varepsilon_\nu \quad \text{and} \quad d_-(\xi_\nu,\zeta_\nu) \ge c_3\varepsilon_\nu$$

for some constant c_3 (independent of ν) and a sequence $c_\nu \to 0$. □

2.18. COROLLARY. *The constants c_ζ and C_ζ in Corollaries 2.10 and 2.13 (respectively) can be chosen to be independent of the point ζ in a compact neighborhood K of bD, so we have*

$$\tau_j^\pm(\zeta,\varepsilon) \sim \varepsilon^{1/m_j(\zeta)},$$

where $(m_1(\zeta),\dots,m_n(\zeta))$ denotes the multitype of bD_ζ at the point ζ.

Proof. We will show that c_ζ can be chosen to be independent of ζ; the proof for C_ζ is the same. Suppose there is a sequence $\zeta_\nu \to \zeta$ in K such that the sequence of the *maximal* c_{ζ_ν} satisfying

$$c_{\zeta_\nu}\varepsilon^{1/m_j(\zeta_\nu)} \le \tau_j^-(\zeta_\nu,\varepsilon) \quad \text{for all } \varepsilon, j$$

tends to zero. Then we can find one index j and a subsequence $\varepsilon_\nu \to 0$ such that

$$\tau_j^-(\zeta_\nu,\varepsilon_\nu) < 2c_{\zeta_\nu}\varepsilon_\nu^{1/m_j(\zeta_\nu)}. \tag{2.19}$$

On the other hand, by Corollary 2.18, there is a constant c such that $cP^+_{\varepsilon_\nu}(\zeta_\nu) \subseteq P^-_{\varepsilon_\nu}(\zeta_\nu)$ for all ν. But this is a contradiction, for by Theorem 2.13 there is a constant c independent of ζ_ν and ε_ν such that the Euclidean volume of $P^+_{\varepsilon_\nu}(\zeta_\nu)$ satisfies

$$\operatorname{vol} P^+_{\varepsilon_\nu}(\zeta_\nu) \ge c\varepsilon_\nu^{2(1/m_1(\zeta_\nu)+\cdots+1/m_n(\zeta_\nu))},$$

whereas by (2.19) and Theorem 2.10 there is a constant C independent of ζ_ν and ε_ν such that

$$\operatorname{vol} P^-_{\varepsilon_\nu}(\zeta_\nu) \le |c_{\zeta_\nu}|^2 C\varepsilon_\nu^{2(1/m_1(\zeta_\nu)+\cdots+1/m_n(\zeta_\nu))}.$$ □

2.20. COROLLARY. *The constants C_ζ appearing in the estimates in Corollaries 2.11 and 2.14 can be chosen to be independent of ζ in a sufficiently small compact neighborhood of bD.*

Proof. The proofs of Corollaries 2.11 and 2.14 show that the constants C_ζ are in fact independent of ζ if this is true for the constants appearing in Corollaries 2.10 and 2.13 (respectively), so the result follows from Corollary 2.18. □

2.21. COROLLARY. *If K is a sufficiently small compact neighborhood of bD, then there exist constants $0 < c < C < \infty$ such that, for all $\zeta \in K$ and all $\varepsilon \in (0,1]$, the Euclidean volumes of the polydisks $P^+_\varepsilon(\zeta)$ and $P^-_\varepsilon(\zeta)$ can be estimated by*

$$c\varepsilon^{2(1/m_1(\zeta)+\cdots+1/m_n(\zeta))} \le \operatorname{vol} P^\pm_\varepsilon(\zeta) \le C\varepsilon^{2(1/m_1(\zeta)+\cdots+1/m_n(\zeta))},$$

where $(m_1(\zeta),\dots,m_n(\zeta))$ denotes the multitype of bD_ζ at the point ζ.

The following theorem is a simple but interesting and useful consequence of our preceding results. It states that the directional pseudo-distances $\tau(z,\gamma,\varepsilon)$ can be expressed in terms of the coefficients of γ with respect to a multitype-basis at z close to bD.

2.22. THEOREM. *Let $(k_1, \dots, k_n)$ be the multitype at a point z close to bD, let (y_κ) be a multitype-basis at z, and suppose that $\gamma = \sum_{j=1}^n \gamma_j y_j$ with $|\gamma| = 1$. Then*

$$\frac{1}{\tau(z, \gamma, \varepsilon)} \sim \sum_{j=1}^{n} |\gamma_j| \varepsilon^{-1/k_j}.$$

Proof. Let $(v_{j,\varepsilon})$ be a set of ε-minimal bases at z for each sufficiently small $\varepsilon > 0$. As before, we write

$$y_j = \sum_\kappa a_{\kappa j,\varepsilon} v_{\kappa,\varepsilon},$$

and we know that $|a_{\kappa j,\varepsilon}| \lesssim \varepsilon^{|1/k_\kappa - 1/k_j|}$. Therefore, if $\gamma = \sum_{\kappa=1}^n \gamma_{\kappa,\varepsilon} v_{\kappa,\varepsilon}$, we see that

$$\frac{1}{\tau(z, \gamma, \varepsilon)} \sim \sum_\kappa \frac{|\gamma_{\kappa,\varepsilon}|}{\tau_\kappa(z, \varepsilon)} \sim \sum_\kappa |\gamma_{\kappa,\varepsilon}| \varepsilon^{-1/k_\kappa} \le \sum_{\kappa,j} |a_{\kappa j,\varepsilon} \gamma_j| \varepsilon^{-1/k_\kappa}$$
$$\lesssim \sum_{\kappa,j} |\gamma_j| \varepsilon^{|1/k_\kappa - 1/k_j| - 1/k_\kappa} \lesssim \sum_j |\gamma_j| \varepsilon^{-1/k_j}.$$

The proof for the reverse estimate follows along the same lines but using the matrices $(a_{\kappa j,\varepsilon})^{-1}$. $\square$

Before we proceed to the Hölder estimates announced in Theorem 1.7, we present an example showing that, in general, it is impossible even locally to choose the ε-extremal bases at ζ to depend continuously on the base point ζ. So, whereas all the estimates for the relevant geometric quantities given here behave nicely with respect to the base point, we cannot (a priori) expect to have continuously varying families of extremal bases; the problem appears most clearly in the vicinity of those boundary points where the defining function behaves symmetrically along different lines in the complex tangent space. This is the basic idea of the following.

EXAMPLE. Let $r(\zeta) := |\zeta|^2 - 1$ be the defining function of the unit ball in $\mathbb{C}^n$ for $n > 2$, and let $\zeta_j = \xi_j + i\eta_j$. Our example domain will be a small perturbation of the unit ball. For $\alpha \in \mathbb{R}^+$, let

$$r_\alpha(\zeta) := r(\zeta) + \alpha \xi_n^2.$$

For $\alpha > 0$ chosen sufficiently small, the set

$$D_\alpha := \{z \in \mathbb{C}^n : r_\alpha(z) < 1\} \subset\subset \mathbb{C}^n$$

is a strictly convex, smoothly bounded domain. Let

$$\zeta_0 := (0, \dots, 0, -i) \in bD_\alpha$$

and

$$\zeta_\delta^{(j)} := \left(0, \dots, 0, \delta, 0, \dots, 0, \, -i\sqrt{1-\delta^2}\right) \in bD_\alpha,$$

where $\delta \in [0, 1]$ stands in the jth coordinate. Of course, $\zeta_\delta^{(j)} \to \zeta_0$ for $\delta \to 0$. The complex tangent space $T_{\zeta_0}^{\mathbb{C}}(bD_\alpha)$ equals $\mathbb{C}^{n-1} \times \{0\}$, whereas

$$T^{\mathbb{C}}_{\zeta_\delta^{(j)}}(bD_\alpha) = \left\{t \in \mathbb{C}^n : \delta t_j + i\sqrt{1-\delta^2}\,t_n = 0\right\}.$$

It is then easy to see that, for all $t \in T^{\mathbb{C}}_{\zeta_0}(bD_\alpha)$ with $\|t\| = 1$ and all $\lambda \in \mathbb{C}$, we have

$$r_\alpha(\zeta_0 + \lambda t) = |\lambda|^2,$$

so *every* orthonormal basis of $T^{\mathbb{C}}_{\zeta_0}(bD_\alpha)$ forms, together with the normal direction, an ε-minimal (or ε-maximal) basis at ζ_0. Now let $\gamma := \gamma(\delta) := i\sqrt{1-\delta^2}$ and

$$v_j := \frac{e_j - \frac{\delta}{\gamma} e_n}{\left\| e_j - \frac{\delta}{\gamma} e_n \right\|},$$

where $(e_1, \dots, e_n)$ denotes the standard basis of $\mathbb{C}^n$. Thus, v_j is a unit vector in $T^{\mathbb{C}}_{\zeta_\delta^{(j)}}(bD_\alpha)$. At the point $\zeta_\delta^{(j)}$ (for $\delta > 0$), the complex tangent space splits into an orthogonal sum

$$T^{\mathbb{C}}_{\zeta_\delta^{(j)}}(bD_\alpha) = T_1 \oplus T_2,$$

where

$$T_1 = \langle v_j \rangle \quad \text{and} \quad T_2 = \langle e_1, \dots, e_{j-1}, e_{j+1}, \dots, e_{n-1} \rangle.$$

If $t \in T_2$ is a unit vector, then $r_\alpha(\zeta_\delta^{(j)} + \lambda t) = |\lambda|^2$, so all directions in T_2 are equivalent with respect to the growth of r_α along complex tangent lines. Suppose now that $t \in T_2$ is a unit vector and that $a, b \in \mathbb{R}$ are positive numbers with $a^2 + b^2 = 1$. We want to show that, for fixed modulus $|\lambda|^2 = c^2$ for some $c > 0$ and fixed $\delta > 0$, that (i) the function

$$f(\lambda, a, b) := r_\alpha(\zeta_\delta^{(j)} + \lambda(at + bv_j))$$

always attains its maximum in a point with $a = 0$ and (ii) this maximum is larger than $|\lambda|^2$. A simple calculation (using $a^2 + b^2 = 1$ and $\delta^2 + |\gamma|^2 = 1$) shows that

$$f(\lambda, a, b) = |\lambda|^2 + \alpha(\operatorname{Im}(\lambda))^2 \delta^2 b^2.$$

If $|\lambda| = c$ and if $\delta > 0$ is fixed, then the maximal possible value of this expression is obviously greater than $|\lambda|^2$ and is attained for $b = 1$ and $\lambda = \pm ic$; thus, $a = 0$. This shows that v_j is the *unique* ε-minimal direction in the point $\zeta_\delta^{(j)}$ for any $\varepsilon > 0$. By considering $j = 1$ and $j = 2$ (for $n \geq 3$) and by letting $\delta \to 0$, it is then obvious that there is no continuous choice of ε-minimal basis near ζ_0. The same type of example also settles the case of the ε-maximal bases.

However, it remains conceivable that, by a small perturbation of the defining function, the problem could be avoided in this case.

3. Nonisotropic Hölder Estimates

As an application of the geometric constructions carried out so far, we will now show how to extend the Hölder estimates that were obtained in [Fi2] to get the estimates we claimed in Theorem 1.7. In order to do so, we will use the estimates of the last section in the integral estimates that are given in [Fi2] and then suitably modify some of Fischer's ideas. In [Fi2] it was shown that, to obtain nonisotropic estimates for the $\bar\partial$-equation, it is enough to prove certain estimates for the following four types of integrals:

$$I_a := \int_{bD} \frac{|(Q \wedge (\bar\partial_\zeta Q)^k \wedge \beta)^T|}{|S|^{k+1}|\zeta - z|^{2n-2k-2}}\, d\sigma_{2n-1}; \tag{1}$$

$$I_b := \int_{bD} \frac{|(\delta_\gamma Q \wedge (\bar\partial_\zeta Q)^k \wedge \beta)^T|}{|S|^{k+1}|\zeta - z|^{2n-2k-3}}\, d\sigma_{2n-1}; \tag{2}$$

$$I_c := \int_{bD} \frac{|(Q \wedge \delta_\gamma(\bar\partial_\zeta Q)(\bar\partial_\zeta Q)^{k-1} \wedge \beta)^T|}{|S|^{k+1}|\zeta - z|^{2n-2k-3}}\, d\sigma_{2n-1}; \tag{3}$$

$$I_d := \int_{bD} \frac{|(Q \wedge (\bar\partial_\zeta Q)^k \wedge \beta)^T||\delta_\gamma S|}{|S|^{k+2}|\zeta - z|^{2n-2k-3}}\, d\sigma_{2n-1}. \tag{4}$$

Here $(\dots)^T$ denotes the bD-tangential part of a form, $Q = \sum_{j=1}^n Q_j d\zeta_j$ is a suitable decomposition of the Diederich–Fornæss support function S of the domain D such that $\langle Q(z,\zeta), \zeta - z\rangle = S(z,\zeta)$, and β denotes (bounded) differential forms that give the expressions in the denominators the bidegree $(n, n-1)$; $d\sigma_{2n-1}$ is the surface element of bD, and δ_γ denotes the directional derivative in direction γ. If we are interested in solution operators on $(0, q+1)$-forms, the index k runs from 0 to $n - q - 2$.

In [Fi2] it is shown that the solution operator T_q can be represented as $T_q = R_q + B_q$, where B_q is the Bochner–Martinelli operator (satisfying an almost Lipschitz estimate, i.e., an isotropic Hölder estimate of order $1 - \varepsilon$ for any $\varepsilon \in (0,1)$) and where $\delta_\gamma R_q f(z)$ can be estimated by the integrals $I_a, \dots, I_d$ just listed. (The solution operators are constructed using the Cauchy–Fantappiè calculus as described, e.g., in [Ra].)

3.1. Theorem. *Let $D = \{r < 0\} \subset\subset \mathbb{C}^n$ be a smooth, convex domain of finite type with multitype $M(bD) = (\mu_1, \dots, \mu_n)$. Let $0 \le q < n$, $\mu := \mu_{n-q}$, and $\varrho := |r(z)|$ for $z \in D$ sufficiently close to bD. Suppose that the integrals $I_a, \dots, I_d$ defined previously satisfy the following estimates for $k = 0, \dots, n-q-2$ and all $z \in D$.*

(i) *For all $\sigma \in (0,1]$, there exist $C_\sigma > 0$ such that $|I_a| \le C_\sigma \varrho^{\sigma(1/\mu - 1)}$.*
(ii) *$|I_b|, |I_c|, |I_d| \lesssim \varrho^{1/\mu}/\tau(z,\gamma,\varrho) + |\log \varrho|$. If z and γ are such that $\tau(z,\gamma,\varepsilon) \sim \varepsilon$ (for constants independent of z, γ, ε), then we suppose there is a constant C_0 such that $|I_{b,c,d}| \le C_0 \varrho^{1/\mu - 1}$.*

Then the anisotropic Hölder estimate in Theorem 1.7 holds.

Proof. In [Fi2], Fischer shows that the two estimates in this theorem always hold for $\mu = m_n$ (the type of the domain) and $k = 0, \dots, n-2$, and from this he derives the nonisotropic estimate stated in Theorem 1.6 (see the proof of [Fi2, Thm. 1.2]). In fact, Fischer uses estimate (i) to obtain an estimate similar to (ii) for $|I_a|$ (with $\mu = \mu_n$ in both cases). We must proceed differently and consider more cases, but we will use the notation from [Fi2].

Let $u(z) := R_q f(z)$ and $A := d(z_0, z_1)$, where $d = d_+$ or d_-. Let $\gamma = (z_1 - z_0)/|z_1 - z_0|$ and let ν be the inward normal direction at $\zeta_0 = \pi(z_0)$ (projection of z_0 to bD). Moreover, define $\tilde{z}_j := z_j + A\nu$ for $j = 0, 1$. Then, we obviously have

$$\begin{aligned}|u(z_0) - u(z_1)| &\le |u(z_0) - u(\tilde{z}_0)| + |u(\tilde{z}_0) - u(\tilde{z}_1)| + |u(\tilde{z}_1) - u(z_1)| \\ &\le \int_{[z_0,\tilde{z}_0]} |\delta_\nu u(t)|\,dt + \int_{[\tilde{z}_0,\tilde{z}_1]} |\delta_\gamma u(t)|\,dt + \int_{[\tilde{z}_1,z_1]} |\delta_\nu u(t)|\,dt.\end{aligned}$$

The first and third integral are bounded by a constant times $\int_0^A s^{1/\mu-1}\,ds \lesssim A^{1/\mu}$ if the estimates (i) and (ii) hold, because the function $s \mapsto s^{1/\mu-1}$ is decreasing on the interval $(0, \infty)$ and we always have $|I_{a,b,c,d}| \lesssim \varrho^{1/\mu-1}$ on the lines $[z_0, \tilde{z}_0]$ and $[z_1, \tilde{z}_1]$.

For the second integral, let m_κ be the order of contact with $bD_{\tilde{z}_0}$ of the line through $\tilde{z}_0$ in direction γ, and suppose that the index κ is chosen to be minimal. Here $(m_1, \ldots, m_n)$ is the multitype at $\tilde{z}_0$. By the results of Section 2, if $\gamma = \sum_{k=\kappa}^n \gamma_k y_k$ is the representation of γ by a multitype-basis (y_k) then there are global constants C and c such that, for all sufficiently small $\alpha > 0$,

$$c \sum_{k=\kappa}^{n} |\gamma_k| \alpha^{-1/m_k} \le \frac{1}{\tau(\tilde{z}_0, \gamma, \alpha)} \le C \sum_{k=\kappa}^{n} |\gamma_k| \alpha^{-1/m_k}. \tag{3.2}$$

We therefore have $\alpha^{1/\mu}/\tau(\tilde{z}_0, \gamma, \alpha) \lesssim \alpha^{1/\mu - 1/m_\kappa}$, with constants independent of $\tilde{z}_0$ and the specific choice of γ. If $m_\kappa \le \mu$ then the function $\alpha \mapsto \alpha^{1/\mu}/\tau(\tilde{z}_0, \gamma, \alpha)$ is (pseudo-)decreasing. Thus, by (ii), for the line integrals of $|I_b|$, $|I_c|$, and $|I_d|$, we obtain

$$\begin{aligned}\int_{[\tilde{z}_0,\tilde{z}_1]} |I_{b,c,d}|\,ds &\lesssim \int_0^{\tau(\tilde{z}_0,\gamma,A)} \frac{A^{1/\mu}}{\tau(\tilde{z}_0, \gamma, A)}\,ds + \int_0^{|\tilde{z}_0-\tilde{z}_1|} |\log(A)|\,ds \\ &\lesssim A^{1/\mu} + |\tilde{z}_0 - \tilde{z}_1| |\log|\tilde{z}_0 - \tilde{z}_1|| \le A^{1/\mu} + C_\varepsilon |z_0 - z_1|^{1-\varepsilon}.\end{aligned}$$

If $m_\kappa > \mu$ then the estimates for the line integrals of $|I_b|$, $|I_c|$, and $|I_d|$ become simpler, since $m_\kappa \ge \mu$ implies $\alpha^{1/\mu}/\tau(\tilde{z}_0, \gamma, \alpha) \lesssim 1$ for all α near 0. In order to estimate $\int_{[\tilde{z}_0,\tilde{z}_1]} |I_a|\,ds$, assume that $\varepsilon > 0$ is given; then we can choose $\sigma \in (0, 1]$ so small that $|z_1 - z_0|^\varepsilon d(z_0, z_1)^{\sigma(1/\mu-1)}$ remains bounded as $z_1 \to z_0$. On the line $[\tilde{z}_0, \tilde{z}_1]$ we have $\varrho \gtrsim A$, so $\varrho^{\sigma(1/\mu-1)} \lesssim A^{\sigma(1/\mu-1)}$. Therefore, estimate (i) implies

$$\int_{[\tilde{z}_0,\tilde{z}_1]} |I_a|\,ds \lesssim |z_1 - z_0|^\varepsilon A^{\sigma(1/\mu-1)} |z_1 - z_0|^{1-\varepsilon} \lesssim |z_1 - z_0|^{1-\varepsilon},$$

with constants depending only on ε.

Since $T_q = R_q + B_q$ and since the Bochner–Martinelli transformation maps L^∞ continuously into $\Lambda^{1-\varepsilon}$ for any $\varepsilon \in (0, 1)$, we can conclude that the Hölder estimate stated in Theorem 1.7 holds. □

Thus, in order to prove Theorem 1.7, we need only show that the estimates (i) and (ii) in Theorem 3.1 hold. This will again be done by modifying Fischer's original proof and inserting our estimates from the previous section. This modification is straightforward, so we will only briefly indicate how it is done. First, we will quote the necessary estimates for the integrands of the integrals $I_a, \dots, I_d$.

3.3. LEMMA. *Let $\zeta, z \in \bar{D}$, $\zeta_0 := \pi(z)$, and $\zeta \in P_\varepsilon^{\pm}(\zeta_0)$ for some $\varepsilon > 0$. Then the term*

$$|(Q \wedge (\bar{\partial}_\zeta Q)^k \wedge \beta)^T|$$

appearing in the integrands of $I_a, \dots, I_d$ can be estimated by a sum of terms of the form

$$E_{\mu\nu}^k := \frac{\varepsilon^k}{\prod_{l=1}^k \tau_{\mu_l}^{\pm}(\zeta_0, \varepsilon)\tau_{\nu_l}^{\pm}(\zeta_0, \varepsilon)},$$

where all the μ_l, ν_l exceed unity and each index appears at most once in each of the sets $\{\mu_l\}, \{\nu_l\}$. If δ_γ is the derivative with respect to z in direction γ, then the terms

$$|(\delta_\gamma Q \wedge (\bar{\partial}_\zeta Q)^k \wedge \beta)^T| \quad \textit{and} \quad |(Q \wedge (\delta_\gamma \bar{\partial}_\zeta Q) \wedge (\bar{\partial}_\zeta Q)^{k-1} \wedge \beta)^T|$$

can be estimated by a sum of terms of the form $E_{\mu\nu}^k / \tau(\zeta_0, \gamma, \varepsilon)$.

Proof. This is a special case of [Fi2, Lemma 3.8], which is sufficient for our purposes. □

By the proof of [Fi2, Lemma 2.1], it is easy to see that we need only replace Fischer's Lemma 3.14 by the following statement (in fact, that lemma states exactly the estimates of our Lemma 3.4 with m_n instead of m_{k+2}). This is because the procedure of exhausting a neighborhood of a fixed boundary point ζ_0 by poly-annuli—which was used in [Fi2] (and previously in [DFiFo; Fi1]) to incorporate the powers of the support function $|S|$ appearing in the denominators of the integrands in $I_a, \dots, I_d$ into the estimates—always goes through as soon as sufficiently good estimates for the integrals in the following lemma are known.

The exhaustion procedure alluded to here is based on good estimates for the support function S and also on the following. Let (y_κ) be a multitype-basis at $\tilde{z}_0$, let $\gamma = \sum \gamma_k y_k$ be a unit vector, and set $\tau(\alpha) := \tau(\tilde{z}_0, \gamma, \alpha)$. Now, if $i_0 \sim -\log_2 \varrho$, then Theorem 2.22 implies that

$$\begin{aligned}\sum_{j=0}^{i_0} \frac{(2^{-j})^{1/\mu}}{\tau(2^{-j})} &\lesssim \sum_{k=1}^{n} \sum_{j=0}^{i_0} |\gamma_k| (2^{-j})^{1/\mu - 1/m_k} \\ &\lesssim \sum_{k=1}^{n} |\gamma_k| \varrho^{1/\mu - 1/m_k} + |\log \varrho| \\ &\lesssim \frac{\varrho^{1/\mu}}{\tau(\varrho)} + |\log \varrho|.\end{aligned}$$

If γ is such that $\tau(\tilde{z}_0, \gamma, \varepsilon) \sim \varepsilon$, then clearly

$$\sum_{j=0}^{i_0} \frac{(2^{-j})^{1/\mu}}{\tau(\tilde{z}_0, \gamma, 2^{-j})} \lesssim \frac{\varrho^{1/\mu}}{\tau(\tilde{z}_0, \gamma, \varrho)}.$$

All these estimates can also be proved without the use of Theorem 2.22.

3.4. LEMMA. *Let $\zeta_0 = \pi(z)$ for $z \in D$ close to the boundary. Then there are constants independent of z and ε such that the following estimates hold.*

(1) $$\int_{bD\cap P_\varepsilon^-(\zeta_0)} \frac{|(Q\wedge(\bar\partial_\zeta Q)^k\wedge\beta)^T|}{|\zeta-z|^{2n-2k-3+1-\sigma}}\,d\sigma_{2n-1} \lesssim \varepsilon^{\sigma/m_{k+2}+k+1} \quad \textit{for all } \sigma\in(0,1].$$

(2) $$\int_{bD\cap P_\varepsilon^-(\zeta_0)} \frac{|(\delta_\gamma Q\wedge(\bar\partial_\zeta Q)^k\wedge\beta)^T|}{|\zeta-z|^{2n-2k-3}}\,d\sigma_{2n-1} \lesssim \frac{\varepsilon^{1/m_{k+2}+k+1}}{\tau(\zeta_0,\gamma,\varepsilon)}.$$

(3) $$\int_{bD\cap P_\varepsilon^-(\zeta_0)} \frac{|(Q\wedge(\delta_\gamma\bar\partial_\zeta Q)\wedge(\bar\partial_\zeta Q)^{k-1}\wedge\beta)^T|}{|\zeta-z|^{2n-2k-3}}\,d\sigma_{2n-1} \lesssim \frac{\varepsilon^{1/m_{k+2}+k+1}}{\tau(\zeta_0,\gamma,\varepsilon)}.$$

Proof. Let $w_j = s_j + it_j$ be the ε-minimal coordinates at the point $\zeta_0 \in bD$, and let $\tau_j(\varepsilon) := \tau_j^-(\zeta_0,\varepsilon)$. By Lemma 3.3, the first integral can then be estimated by

$$I := \int_{|t_1|<\tau_1(\varepsilon)}\int_{|w_2|<\tau_2(\varepsilon)}\cdots\int_{|w_n|<\tau_n(\varepsilon)} \frac{\varepsilon^k\,dt_1\,dV(w_2,\dots,w_n)}{\prod_{l=1}^k \tau_{\mu_l}(\varepsilon)\tau_{\nu_l}(\varepsilon)\big(\sum_{j=1}^n|w_j|\big)^{2n-2k-3+1-\sigma}},$$

where all the μ_l,ν_l exceed unity and each index appears at most once in each of the sets $\{\mu_l\},\{\nu_l\}$. Since $\tau_1(\varepsilon)\lesssim\tau_2(\varepsilon)\le\cdots\le\tau_n(\varepsilon)$, the worst case that can appear is the integral

$$J := \int_{|t_1|<\tau_1(\varepsilon)}\int_{|w_2|<\tau_2(\varepsilon)}\cdots\int_{|w_n|<\tau_n(\varepsilon)} \frac{\varepsilon^k\,dt_1\,dV(w_2,\dots,w_n)}{\prod_{l=2}^{k+1}\tau_l^2(\varepsilon)\big(\sum_{j=1}^n|w_j|\big)^{2n-2k-3+1-\sigma}}.$$

Integrating over $dt_1\,dV(w_2,\dots,w_{k+1})$ and observing that $\tau_1(\varepsilon)\sim\varepsilon$ and

$$\int_{|w_j|<\tau_j(\varepsilon)} \frac{dV(w_j)}{\tau_j^2(\varepsilon)} \lesssim 1,$$

we see that J can be estimated above by a constant multiplied by

$$J_k := \int_{|w_{k+2}|<\tau_{k+2}(\varepsilon)}\cdots\int_{|w_n|<\tau_n(\varepsilon)} \frac{\varepsilon^{k+1}\,dV(w_{k+2},\dots,w_n)}{\big(\sum_{j=k+2}^n|w_j|\big)^{2n-2k-3+1-\sigma}}.$$

For this integral, we use polar coordinates in the variables $(w_{k+3},\dots,w_n)$, leaving out only the direction w_{k+2} corresponding to the minimal entry of the multitype of which we can still dispose; this enables us to derive the estimate

$$\begin{aligned}
J_k &\lesssim \int_{|w_{k+2}|<\tau_{k+2}(\varepsilon)}\int_0^1 \frac{\varepsilon^{k+1}r^{2n-2k-5}\,dr\,dV(w_{k+2})}{(|w_{k+2}|+r)^{2n-2k-3+1-\sigma}}\\
&\lesssim \int_{|w_{k+2}|<\tau_{k+2}(\varepsilon)}\int_0^\infty \frac{\varepsilon^{k+1}s^{2n-2k-5}\,ds\,dV(w_{k+2})}{|w_{k+2}|^{2-\sigma}(1+s)^{2n-2k-3+1-\sigma}}\\
&\lesssim \int_{|w_{k+2}|<\tau_{k+2}(\varepsilon)} \frac{\varepsilon^{k+1}\,dV(w_{k+2})}{|w_{k+2}|^{2-\sigma}} \lesssim \varepsilon^{k+1}\tau_{k+2}^\sigma(\varepsilon) \lesssim \varepsilon^{\sigma/m_{k+2}+k+1},
\end{aligned}$$

which was to be shown. The remaining estimates follow in exactly the same way from Lemma 3.3 by setting $\sigma = 1$ in the preceding estimates. □

We observe that, in the solution operator acting on $(0, q+1)$-forms, no terms appear except those that can be estimated as here for $0 \le k \le n-q-2$; hence it is clear that the worst possible estimate in this case is controlled by the entry m_{n-q} of the multitype of D, not by m_n. The rest of Fischer's original proof can be used without any change to complete the proof of Theorem 1.7.

References

[BoSt] H. P. Boas and E. J. Straube, *On equality of line type and variety type of real hypersurfaces in* $\mathbb{C}^n$, J. Geom. Anal. 2 (1992), 95–98.

[BCDu] J. Bruna, Ph. Charpentier, and Y. Dupain, *Zero varieties for the Nevanlinna class in convex domains of finite type in* $\mathbf{C}^n$, Ann. of Math. (2) 147 (1998), 391–415.

[Ca1] D. Catlin, *Boundary invariants of pseudoconvex domains,* Ann. of Math. (2) 120 (1984), 529–586.

[Ca2] ———, *Subelliptic estimates for the* $\bar\partial$*-Neumann problem on pseudoconvex domains,* Ann. of Math. (2) 126 (1987), 131–191.

[Cu1] A. Cumenge, *Estimées Lipschitz optimales dans les convexes de type fini,* C. R. Acad. Sci. Paris Sér. I Math. 325 (1997), 1077–1080.

[Cu2] ———, *Sharp estimates for* $\bar\partial$ *in convex domains of finite type,* Ark. Mat. 39 (2001), 1–25.

[Cu3] ———, *Zero sets of functions in the Nevanlinna or the Nevanlinna–Drjbachian classes,* Pacific J. Math. 199 (2001), 79–92.

[D'A] J. P. D'Angelo, *Real hypersurfaces, orders of contact, and applications,* Ann. of Math. (2) 115 (1982), 615–637.

[DFiFo] K. Diederich, B. Fischer, and J. E. Fornæss, *Hölder estimates on convex domains of finite type,* Math. Z. 232 (1999), 43–61.

[DFo] K. Diederich and J. E. Fornæss, *Support functions for convex domains of finite type,* Math. Z. 230 (1999), 145–164.

[DM] K. Diederich and E. Mazzilli, *Zero varieties for the Nevanlinna class on all convex domains of finite type,* Nagoya Math. J. 163 (2001), 215–227.

[Fi1] B. Fischer, L^p *Estimates on convex domains of finite type,* Math. Z. 236 (2001), 401–418.

[Fi2] ———, *Nonisotropic Hölder estimates on convex domains of finite type,* Michigan Math. J. 52 (2004), 219–239.

[F] J. F. Fleron, *Sharp Hölder estimates for* $\bar\partial$ *on ellipsoids and their complements via order of contact,* Proc. Amer. Math. Soc. 124 (1996), 3193–3202.

[He1] T. Hefer, *Optimale Regularitätssätze für die* $\bar\partial$*-Gleichung auf gewissen konkaven und konvexen Modellgebieten,* Bonner Math. Schriften 301 (1997), 1–74.

[He2] ———, *Hölder and* L^p *estimates for* $\bar\partial$ *on convex domains of finite type depending on Catlin's multitype,* Math. Z. 242 (2002), 367–398.

[HR] G. M. Henkin and A. V. Romanov, *Exact Hölder estimates for the solutions of the* $\bar\partial$*-equation,* Izv. Akad. Nauk SSSR Ser. Mat. 35 (1971), 1171–1183.

[Ke] N. Kerzman, *Hölder and* L^p *estimates for solutions of* $\bar\partial u = f$ *in strongly pseudoconvex domains,* Comm. Pure Appl. Math. 24 (1971), 301–379.

[Kl] S. G. Krantz, *Optimal Lipschitz and* L^p *regularity for the equation* $\bar\partial u = f$ *on strongly pseudo-convex domains,* Math. Ann. 219 (1976), 233–260.

[K2] ———, *Intrinsic Lipschitz classes on manifolds with applications to complex function theory and estimates for the $\bar\partial$ and $\bar\partial_b$ equations,* Manuscripta Math. 24 (1978), 351–378.

[K3] ———, *Correction to my paper "Intrinsic Lipschitz classes on manifolds with applications to complex function theory and estimates for the $\bar\partial$ and $\bar\partial_b$ equations",* Manuscripta Math. 27 (1979), 433–434.

[Mc1] J. D. McNeal, *Convex domains of finite type,* J. Funct. Anal. 108 (1992), 361–373.

[Mc2] ———, *Estimates on the Bergman kernels of convex domains,* Adv. Math. 109 (1994), 108–139.

[McS] J. D. McNeal and E. M. Stein, *Mapping properties of the Bergman projection on convex domains of finite type,* Duke Math. J. 73 (1994), 177–199.

[Ra] R. M. Range, *Holomorphic functions and integral representations in several complex variables,* Grad. Texts in Math., 108, Springer-Verlag, New York, 1986.

[Yu] J. Yu, *Multitypes of convex domains,* Indiana Univ. Math. J. 41 (1992), 837–849.

Universität Bonn
Mathematisches Institut
Beringstraße 1
D-53115 Bonn
Germany

hefer@math.uni-bonn.de